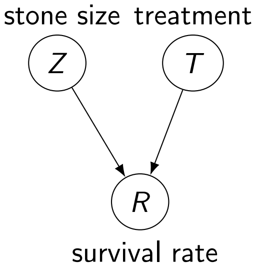
On the Relationship Between Equivariant Predictive Models and Structural Causal Model Identification

Grace Yin
Department of Statistics
University of British Columbia
MSC Presentation

April 19, 2022

Introduction

Structural Causal Model (SCM)



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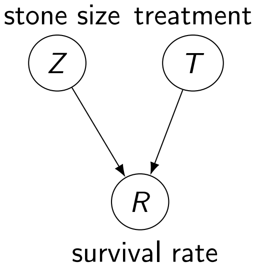

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Structural Causal Model (SCM) [2]

Definition

A structural causal model (SCM) $\mathfrak{C} := (G, S, P_N)$ consists of a collection S of d (structural) assignments

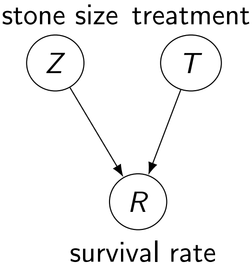
$$X_j := f_j(PA_j, N_j),\ j = 1, \cdots, d$$

where $PA_j \subset \{X_1, \cdots, X_d\} \setminus \{X_j\}$ are parents of X_j and $P_N = P_{N_1, \cdots, N_d}$ is the joint (product) distribution over the noise variables which are assumed to be jointly independent. $G = (V, E)$ is the graph contains vertices and edges.

Structural Causal Models (SCM)

intervention: $do(\cdots)$: sets the variable value, without changing other nodes

treatment: $T = A$ or $T = B$



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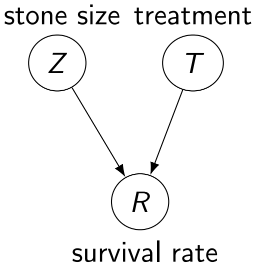

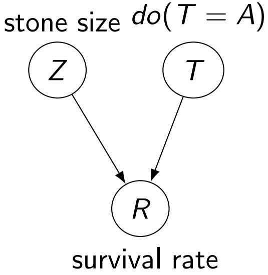

Introduction

Question: Suppose there are two candidate SCMs, how can we identify which one is correct?

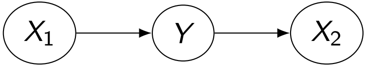

Figure: SCM 1

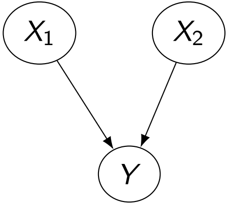

Figure: SCM 2

Introduction

- Potential solution: conditional independence test
  - High computational costs [1]
- $X \rightarrow Y$
  - the functional assignment: $Y := f(X, E) = \beta_0 + \beta_1 X + E$
  - has constant risk for $X' \leftarrow X + a, a \in \mathbb{R}(+)$
- Our approach: apply constant risk theorem
  - across interventions described by the action of a group

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Background and Notation

joint distribution of (X, Y):

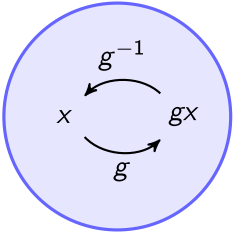
$$\tilde{P} = P \otimes Q_x,$$

P: marginal distribution on $(\mathbf{X}, \mathcal{X})$
Q_x: conditional distribution (Markov kernel) from $(\mathbf{X}, \mathcal{X})$ into $(\mathbf{Y}, \mathcal{Y})$
risk function is defined as

$$R(\tilde{P}, \rho) = \int_{\mathcal{X} \times \mathcal{Y} \times Z} P(dx) Q_x(dy) \rho_x(dz) L(y, z).$$

ρ : decision procedure $\mathbf{X} \times \mathcal{Z} \to [0, 1]$ where $(\mathbf{Z}, \mathcal{Z})$ is the decision space
L: loss function $\mathcal{Y} \times \mathcal{Z} \to [-\infty, \infty)$.

Background and Notation



- $\mathcal{G}$: a group acting measurably on **X** and **Y**
- $g \in \mathcal{G}$: a group action, $\Phi(g, x) = gx$
- **conditional shift**: $g_x \tilde{P} = (P \circ g_x^{-1}) \otimes Q_x$ for $g_x \in \mathcal{G}$

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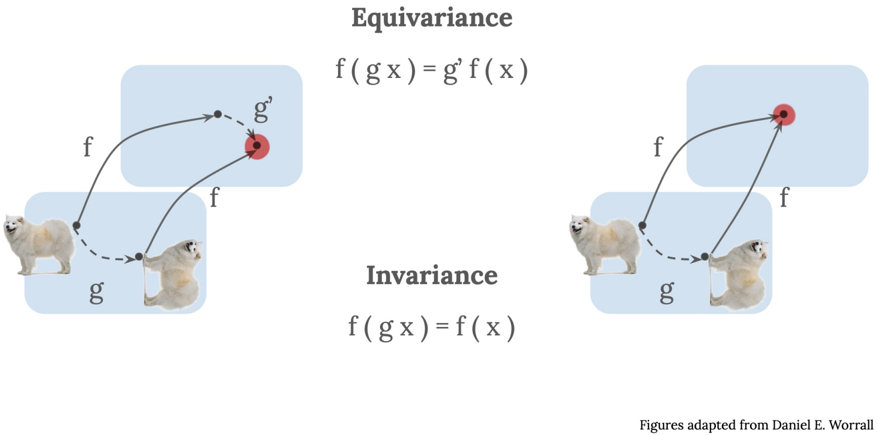


Figure: Equivariance and Invariance map [3]

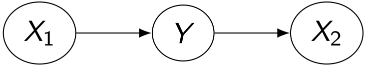
Constant Risk Theorem

- equivariant markov kernel:
$$Q_{gx} = Q_x \circ g^{-1}$$
- invariant loss function:
$$L(gy, gz) = L(y, z)$$
- equivariant decision procedure ρ:
$$\rho_{gx} = \rho \circ g^{-1}$$

Theorem

For an invariant loss function L, the risk of a decision procedure ρ is constant under the conditional shift $g_x \tilde{P}$ for any group action $g_x \in \mathcal{G}$, if ρ is equivariant and the kernel Q_x is equivariant. i.e.,
$$\forall g_x \in \mathcal{G},\ R(\rho, g_x \tilde{P}) = R(\rho, \tilde{P})$$

Identify the Structure of SCM



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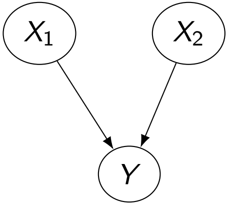

Figure: SCM 2

$$\begin{cases} X_1 \sim \mathcal{N}(0, \sigma^2) \\ Y := \beta_1 X_1 + \varepsilon_y \\ X_2 := \beta_2 Y + \varepsilon_2 \end{cases}$$

with $\varepsilon_2 \sim \mathcal{N}(0,1)$, $\varepsilon_y \sim \mathcal{N}(0, \sigma^2)$

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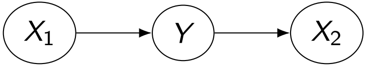
Simulation Experiments and Discussion



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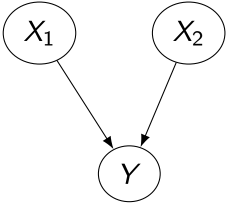

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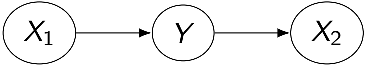

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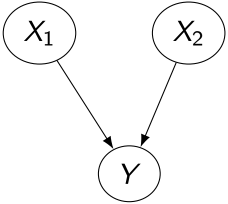

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Simulation experiments:

- simulate data from SCM 1
- simulate estimated coefficients in shifting environment 0
- compute risk for three models in different shifting environments

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loss function: least square loss function

risk function:

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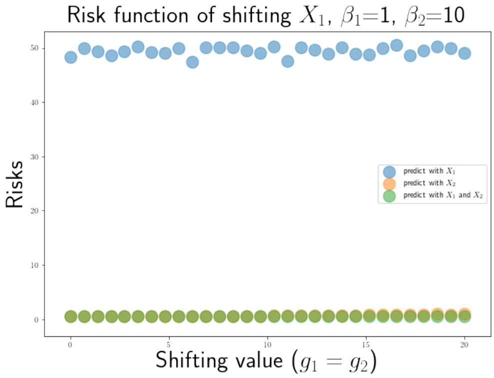
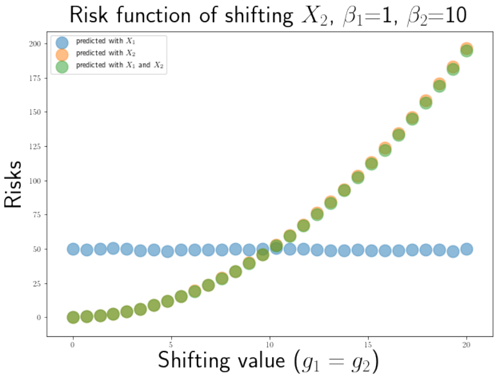
Simulation Experiments and Discussion

The simulation results verified our constant risk theorem.

- shifting X_1: constant risk for all three predictive models
- shifting X_2: only ℓ_1 has constant risk

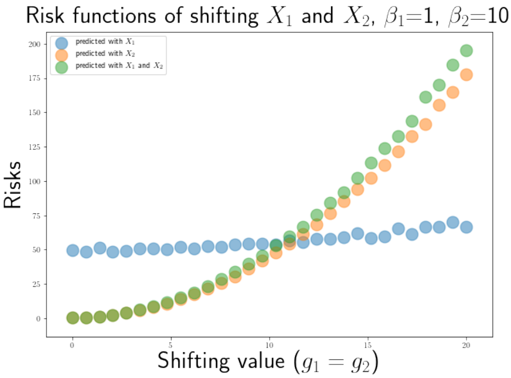




Simulation Experiments and Discussion

Differences between constant risk approach and risk minimization approach

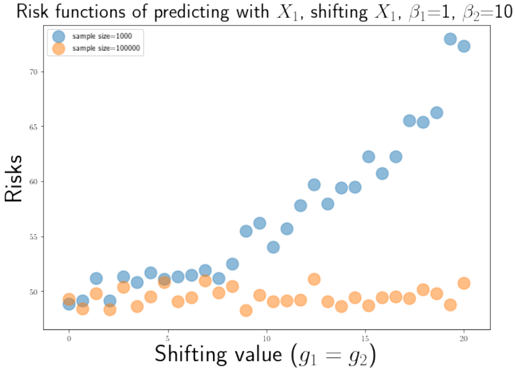
- a crossover among three predictive models when shifting X_1 and X_2



Simulation Experiments and Discussion

Sample size can influence the constant risk results:

- simulated $\hat{\beta}_0$ and $\hat{\beta}_1$ for ℓ_1
- sample size $n_1 = 1000$ vs sample size $n_2 = 100000$
- empirical risks: $\hat{R}(g) \propto (\beta_0 - \hat{\beta}_0)^2 + (x_{1,i} + g)^2(\beta_1 - \hat{\beta}_1)^2$



Potential Extension

Potential directions:

- Apply on other examples of SCMs
- Identify the theoretical interconnection between the risk function among linear regression models
- Implement algorithms
- Explore non-linear predictive models

Reference

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[2] J. Peters, D. Janzing, and B. Schlkopf.
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Acknowledgement

Special thanks to :

- my supervisor: Prof. Ben Bloem-Reddy
- the various members of the UBC Department of Statistics

Github QR code:

Thank You for Listening!