**Functions of bounded variation** $\alpha : [a,b] \to \mathbb{R}$ is in $BV[a,b]$ if $V_a^b\alpha = \sup_{P\ partition} V_a^b(\alpha, P) = \sup_P \sum_{i=1}^n |\alpha(x_i) - \alpha(x_{i-1})| < \infty$

**Theorem 1.** *(Jordan's theorem:) $\alpha \in BV[a,b]$ if and only if $\alpha$ can be written as $\alpha = \beta - \gamma$ with $\beta, \gamma$non-decreasing.*

*Proof.* $\Leftarrow$
Assume $\alpha = \beta - \gamma$ with $\beta, \gamma : \xrightarrow{non\ decreasing} \mathbb{R}$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$.

$$V_a^b(\alpha, P) = \sum_{i=1}^n |\alpha(x_i) - \alpha(x_{i-1})| = \sum_{i=1}^n |[\beta(x_i) - \beta(x_{i-1})] - [\gamma(x_i) - \gamma(x_{i-1})]|$$

$$\leq \sum_{i=1}^n |\beta(x_i) - \beta(x_{i-1})| + \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})| = V_a^n(\beta, P) + V_a^b(\gamma, P) \leq V_a^b\beta + V_a^b\gamma$$

$$= \underbrace{[\beta(b) - \beta(a)] + [\gamma(b) - \gamma(a)]}_{independent\ of\ P} < \infty$$

Hence,

$$V_a^b\alpha = \sup V_a^b(\alpha, P) \leq [\beta(b) - \beta(a)] + [\gamma(b) - \gamma(a)] < \infty$$

**Exs:** If $f, g \in BV[a,b]$, then $V_a^b(f \pm g) \leq V_a^b f + V_a^b g$
$\Rightarrow$: Assume that $\alpha \in BV[a,b]$, need to find $\beta, \gamma$ such that $\alpha = \beta - \gamma(\gamma = \beta - \alpha)$. Introduce the variation function $v(x) = V_a^x\alpha$=total variation of $\alpha$ on $[a,x]$.
**Exs:**$V(x)$ is well-defined because $V_a^x\alpha \geq V_c^d\alpha$ for any interval $[c,d] \subseteq [a,b]$.
**Question:** Is $v$ non-decreasing? Yes! Let $x < y$, need to verify if

$$\underbrace{v(x)}_{=V_a^x\alpha} \leq \underbrace{v(y)}_{V_a^y\alpha}.$$

True by Exs above since $[a,b] \subseteq [a,y]$
**Question**: Is $V(x_i) - \alpha(x_i)$ non-decreasing? Let $x < y$, need to verify if $v(x) - \alpha(x) \leq v(y) - \alpha(y) \Leftrightarrow \alpha(y) - \alpha(x) \leq \underbrace{v(y) - v(x)}_{V_a^y\alpha - V_a^x\alpha = V_x^y\alpha = V_x^y\alpha = \sup_{P partition\ of\ [x,y]} V_x^y(\alpha, P)}$
Note:$|\alpha(y) - \alpha(x)| = V_x^y(\alpha, P_0)$, where $P_0 = \{x, y\}$. Thus, $\alpha(y) - \alpha(x) \leq$

$|\alpha(y)-\alpha(x)| = V_x^y(\alpha, P_0) \leq V_x^y \alpha = v(y) - v(x)$.
**Exs:** Check that for any $c \in (a,b)$ and any $f \in BV[a,b]$,

$$V_a^c f + V_c^a f = V_a^b f$$

Set $\beta(x) = v(x), \gamma(x) = v(x) - \alpha(x)$. Clearly, $\alpha = \beta - \gamma$ $\square$

**Remark:**Jordan's theorem suggests that $BV[a,b]$ could be a good source of Riemann-Stieltjes integrators, with respect to which continuous functions functions on $[a,b]$ can be integrated

$$\int fd\underbrace{\alpha}_{\in BV} = \int fd(\beta-\gamma) \stackrel{?}{=} \int fd\beta - \int fd\gamma$$

**Interchanging integrals & integrators**
Recall:

$$\int_a^b \left(\underbrace{udv}_{u(x)v'(x)dx} + \underbrace{vdu}_{v'(x)u'(x)dx}\right) = u(b)v(b) - u(a)v(a) \to integration\ by\ parts$$

**Integration by parts for Riemann-Stieltjes integrals:** Let $f, \alpha : [a,b] \to \mathbb{R}$ be arbitrary functions. Then $f \in \mathcal{R}_\alpha[a,b] \Leftrightarrow \alpha \in \mathcal{R}_f[a,b]$ ($f \in \mathcal{R}_\alpha[a,b] \Leftrightarrow \exists I \to, \forall \varepsilon > 0, \exists$ a partition $P_0$, s.t., $|S_\alpha(f,P,T) - I| < \varepsilon$, for $P \supseteq P_0$, and any selection of points subordinate to $P$)
In either case:

$$\int_a^b fd\alpha + \int_a^b \alpha df = \alpha(b)f(b) - \alpha(a)f(a)$$

**Corollary 1.** *$\mathcal{C}[a,b] \subseteq \mathcal{R}_\alpha[a,b]$ for all $\alpha \in BV[a,b]$*

*Proof.* Use the Jordan's theorem to write $\alpha = \beta - \gamma$ where $\beta$ and $\gamma$ are non-decreasing. Note by a previous theorem, $\mathcal{C}[a,b] \subseteq \mathcal{R}_\beta[a,b] \cap \mathcal{R}_\gamma[a,b]$. This implies if $f \in \mathcal{C}[a,b]$,then

$$\int_a^b fd\beta \& \int_a^b fd\gamma are\ well-defined$$

By IBP,

$$\int_a^b \beta df \& \int_a^b \gamma df are\ well-defined\ too.$$

i.e., $\beta, \gamma \in \mathcal{R}_f[a,b]$. But it is easy to see that $\mathcal{R}_f[a,b]$ is a vector space, with

$$\int_a^b (\beta \mp \gamma) df$$

This implies $\beta \mp \gamma \in \mathcal{R}_f[a,b]$. In particular, $\alpha = \beta - \gamma \in \mathcal{R}_f[a,b]$. Use IBP again, get $f \in \mathcal{R}_\alpha[a,b]$ $\square$