Live Session 1 - Discrete Response Models

Professor Jeffrey Yau

Agenda

1. Welcome to w271 and an overivew of topics covere in this lecture (5 minutes)
2. Discussion of the analysis of Binary variables (25 minutes)
   - Confidence Inference for the Probability of Success
3. Instructor and Students Introdution (30 minutes)
4. A discussion of the weekly workflow, tips for success in this course, and live-session workflow (30 minutes)
5. Questions

Topics covered in this lecture and Required Readings

- Required Readings: BL2015: Ch. 1 (Skip Sections 1.2.6 and 1.2.7), Appendix B.3 and B.5
- The order it was covered in the async video:

1.1 Weekly Overview

1.2 Introduction to Categorical Data, and the Bernoulli and Binomial Probability Models

```
1.2.1 Computing Probabilities of Binomial Probability Model
1.2.2 Simulating a Binomial Probability Model
1.2.3 Maximum Likelihood Estimation Page
1.2.4 Maximum Likelihood Estimation (2)
```

1.3 Wald Confidence Interval 1.3.1 Alternative Confidence Intervals and True Confidence Level 1.3.2 Hypothesis Tests for Pi Page (5 mins)

1.4 Introduction to Multiple Binary Variables Page

1.5 Formulation of Contingency Table and Confidence Interval of Two Binary Variables

1.6 Relative Risk

1.7 Odds Ratios

Note: For the first five lectures of the course, I follow the book very closely to allow for multiple touch points - async video lectures, assigned readings, live session examples, and homework on the same concepts and techniques.

Topics cover (in more details)

- An introduction to categorical data, Bernoulli probability model, and Binomial probability model
- Computing the probability of binomial probability model
- Simulating a binomial probability model
- Estimating the Binomial probability model using maximum likelihood estimation (MLE)
- Confidence intervals:
  - Wald confidence interval
  - Alternative confidence intervals
- Hypothesis test for the probability of success
- The case of two binary variable
  - Contingency tables
  - The notions of relative risks, odds, and odds ratios
- Two Binary variables
  - Contingency table
  - MLE
  - C.I.s for the difference of two probabilities
  - Relative Risks
  - Odds
  - Odds ratios (OR)
  - log(OR)
  - Estimation and inference

- This week's lecture starts with the simplest case of discrete response modeling, the Binomial probability model, covering both parameter estimation and statistical inference. In fact, it points out the importance of proposing the appropriate mathematical framework to model the variable of interest based on the "type" of the variable. In this case, the "type" of the variable of interest is "discrete", randering continuous probability model not appropriate.
  - This is the simplest case of modeling categorical variables, as there are only two categories. Therefore, we do not have to worry about whether or not the categories are ordered (or ranked) or count (such as the number of customers arriving at a particular store within a specific hour)
  - You have already studied both the Bernoulli and Binomial probability models in w203 and perhaps other statistics courses elsewhere. However, many aspects of this model are typically not obvious are are overlooked are the underlying assumptions of the model; note that all probability and statistical models come with assumptions. In this course, we will study how to apply theoretical statistical models in practice.
  - The Binomial probability model is a reasonable model for distribution of an event happened (or "success") in a given number of trails, so long as the assumptions listed on page 3 and 4 of our textbook *"Analysis of Categorical Data with R"* are not violated.
  - The book lists several examples and discusses in each cases whether each of the assumptions is reasonable
  - Another (perhaps) new aspect of studying Bernoulli and Binominal probability models is to model it in R using the R built-in *dbinom(n,size,prob)* function
  - We also study how to simulate a probability model and use simulation to evaluate how well procedures perform when these assumptions are violated. (Page 7 and 8 of the book describes it)
- A lot of time is spent on discussing the confidence interval of this model, a discussion typically not covered in the elementary statistics courses.
- As you will see, we use simulation extensively in this course, as simulation is one of the best ways to "get a feel" of a probability model and gain an understanding of its behavior
- A single binomial probability model is then extended two two binomial probability model, introducing the notion of contingency table. Also introduced is the formulation of a likelihood function and the method of maximum likelihood, a version of which was taught in the parameter estimation module in w203.
- With more than one binomial random variables, the concept of relative risk, odds, and odds ratio become very powerful, as the meaning of the difference between probability of success, $\pi_1 - \pi_2$, is a function of the magnitude of π_1 and π_2.
- All of these concepts are introduced as the preparation for the study of the regression model of categorical response data, with binary response being the simplest case.

In the live session today, I want to focus on (1) the statistical analysis workflow, (2) confidence intervals, and (3) the concepts of odds ratios.

Statistical Analysis Workflow Revisit

- Postulate a statistical model that conform with the underlying (business, policy, scientific, etc.) question being asked
- Estimate the parameter of the statistical model
- Check model assumptions
- Conduct statistical inference

In the case of the success parameter π from the Bernoulli distribution, we use MLE:

$$
\begin{aligned}
L(\pi|y_1 \cdots y_n) &= P(Y_1 = y_1, \ldots, Y_n = y_n) \\
&= P(Y_1 = y_1) \times \cdots \times P(Y_n = y_n) \\
&= \prod_{i=1}^{n} P(Y_i = y_i) \\
&= \prod_{i=1}^{n} \pi_i^{y_i}(1 - \pi_i)^{(1-y_i)}
\end{aligned}
$$

The MLE is $\hat{\pi} = \frac{w}{n}$, where w denotes the number of successes in n trials.

The variance of the estimate is $\hat{Var}(\hat{\pi}) = \frac{\hat{\pi}(1-\hat{\pi})}{n}$

A Summary of the Confidence Intervals for the Probability of Success π

Recall that in w203, we've learned that the *typical form* of a confidence interval for a parameter of a probability model, θ, is:

estimator $\pm$ (distributional value) $\times$ (standard deviation of estimator)

- *What is the frequentist interpretation of a (1-α)100% confidence interval (as we learned in w203)?*

In fact, if we assume that $\hat{pi}$ follow the Normal Distribution, that is, $\hat{\pi} \sim N(\pi, \hat{Var}(\hat{\pi}))$,

we have the **Wald confidence interavl** takes the following form:

1. Wald Confidence Interval:

$$\hat{\pi} \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$$

What "assumption" is required for one to construct the Wald confidence interval?

While one of MLE's properties is that it is asymptotically normal, an implicit assumption to construct this confidence interval is that the normal approximation is a good approximation.

What are the problems with this confidence interval?

Regardless of whether or not the above assumption is satisfied, this confidence interval suffers from some problems:

1. The interval an exceed the (0,1) interval, which would not be sensible for probability.
2. When $w = 0$ or $w = 1$, the confidence interval degenerates into a single point equal to $\hat{\pi}$
3. The coverage is not necessarily equal to the stated level $1 - \alpha$

Why does a confidence interval method not actually achieve its stated confidence level of binomial random variable?

Example: Computation of Wald Confidence Interval

Let's pause for a minute before writing codes.

Suppose we have a Bernoulli probability model with $\pi = 0.6$. In addition, suppose that we "think" that the sample of observations (that is, our data) come from this (theoretical) distribution.

```
set.seed(23951)
n_sim = 1
n_trials = 10
p = 0.6
bin <- rbinom(n=n_sim, size = n_trials, prob = p)
bin
```

```
## [1] 6
```

- I need to pause here to ensure that the students understand this logic (as well as why I use simulation here to get the data), as this is a very important thought process in statistical analysis.

From the simulation, we get a total of 6 successes from 10 draws. Our hypothetical value of the true probability of success is $\pi = 0.6$. Let's use this information to derive the Wald confidence interval.

Recall the Wald Confidence Interval's formula before writing done the code.

$$\hat{\pi} \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$$

$w = 6$ and $n = 10$

```
w = 6
n = 10
alpha = 0.05

pi.hat = w/n # Recall that this is an MLE for pi in Bernoulli probability model
var.wald = pi.hat*(1-pi.hat)/n

wald.CI_lower.bound = pi.hat - qnorm(p = 1-alpha/2)*sqrt(var.wald)
wald.CI_upper.bound = pi.hat + qnorm(p = 1-alpha/2)*sqrt(var.wald)

round(data.frame(pi.hat, var.wald, wald.CI_lower.bound,wald.CI_upper.bound),4)
```

```
##   pi.hat var.wald wald.CI_lower.bound wald.CI_upper.bound
## 1    0.6    0.024              0.2964              0.9036
```

Is this a "good" confidence interval or not? What do we mean by "good"? Does it achieve its stated confidence level?

- we have never really questioned the validity of confidence interval in elemenary statistics courses as well as w203? Why now?
- As it turns out, Wald confidence interval does not perform well when it comes to achieving the true confidence level, say $95\% = 1 - \alpha$ where $\alpha = 0.05$. Let's examine the reason behind it.
- Binomial random variables are discrete random variables, meaning that it can only take a finite number of values; given the number of trails n, there are only $n + 1$ possible intervals, corresponding to $w \in 0, 1, \ldots, n$.
- Given the true parameter value π, some of these $n + 1$ intervals contain π, obviously, and some don't.
- More formally, the true confidence level at a given parameter value π, called it $C(\pi)$, following the book's notation, is teh sume of teh binomial probabilities for all the $n+1$ intervals that actually contain π:

$$C(\pi) = \sum_{i=1}^{n} I(w) \binom{n}{w} \pi^w (1-\pi)^{n-w}$$

where $I(w) = 1$ if the interval contains π and 0 otherwise.

Let's do some computation to solidify our intuition. Let's continue to use the numbers from the example above: $n = 10$ and $\pi = 0.6$

```
pi = 0.6
alpha = 0.05
n = 10
w = 0:n

wald.CI.true.coverage = function(pi, alpha=0.05, n) {

  w = 0:n

  pi.hat = w/n
  pmf = dbinom(x=w, size=n, prob=pi)
```

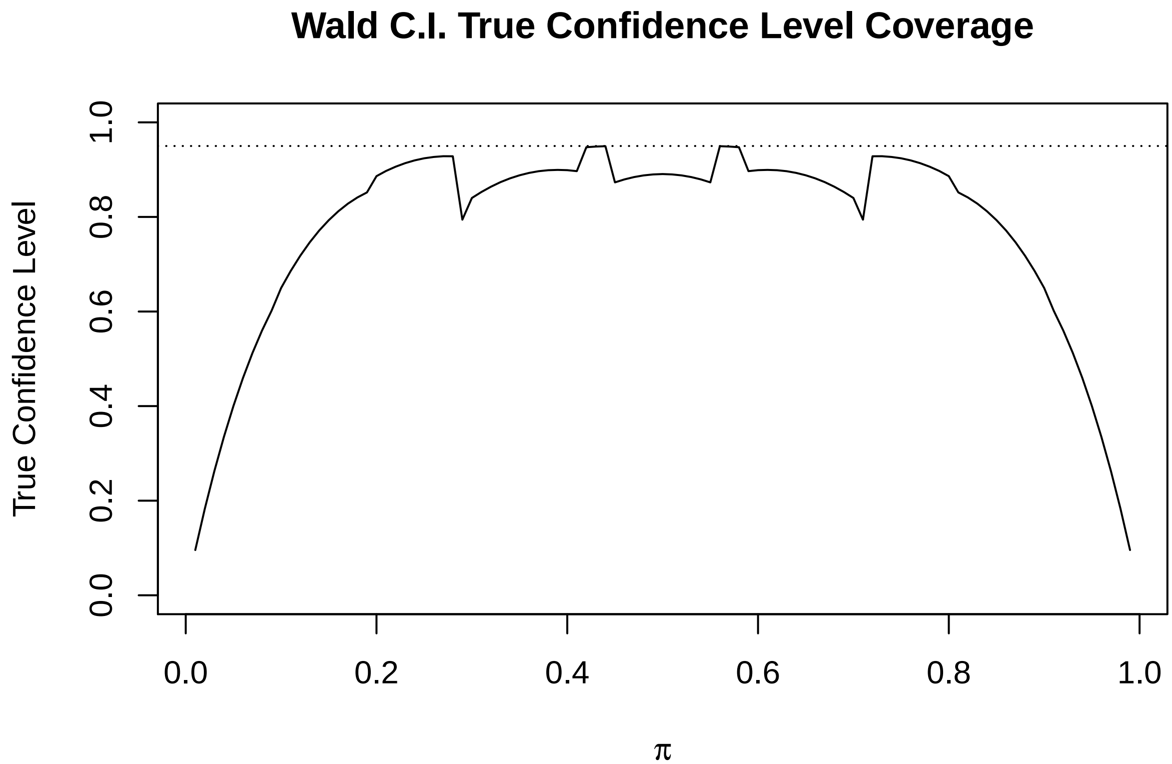
```
  var.wald = pi.hat*(1-pi.hat)/n
  wald.CI_lower.bound = pi.hat - qnorm(p = 1-alpha/2)*sqrt(var.wald)
  wald.CI_upper.bound = pi.hat + qnorm(p = 1-alpha/2)*sqrt(var.wald)

  covered.pi = ifelse(test = pi>wald.CI_lower.bound, yes = ifelse(test = pi<wald.CI_upper.bound, yes=1,

  wald.CI.true.coverage = sum(covered.pi*pmf)

  wald.df = data.frame(w, pi.hat, round(data.frame(pmf, wald.CI_lower.bound,wald.CI_upper.bound),4), co

  return(wald.df)
}

wald.df = wald.CI.true.coverage(pi=0.6, alpha=0.05, n=10)
wald.CI.true.coverage.level = sum(wald.df$covered.pi*wald.df$pmf)

# Let's compute the ture coverage for a sequence of pi
pi.seq = seq(0.01,0.99, by=0.01)
wald.CI.true.matrix = matrix(data=NA,nrow=length(pi.seq),ncol=2)
counter=1
for (pi in pi.seq) {
    wald.df2 = wald.CI.true.coverage(pi=pi, alpha=0.05, n=10)
    #print(paste('True Coverage is', sum(wald.df2$covered.pi*wald.df2$pmf)))
    wald.CI.true.matrix[counter,] = c(pi,sum(wald.df2$covered.pi*wald.df2$pmf))
    counter = counter+1
}
str(wald.CI.true.matrix)
```

```
##  num [1:99, 1:2] 0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.1 ...
```

```
wald.CI.true.matrix[1:5,]
```

```
##      [,1]   [,2]
## [1,] 0.01 0.0956
## [2,] 0.02 0.1828
## [3,] 0.03 0.2624
## [4,] 0.04 0.3347
## [5,] 0.05 0.4002
```

```
# Plot the true coverage level (for given n and alpha)
plot(x=wald.CI.true.matrix[,1],
     y=wald.CI.true.matrix[,2],
     ylim=c(0,1),
     main = "Wald C.I. True Confidence Level Coverage", xlab=expression(pi),
     ylab="True Confidence Level",
     type="l")
abline(h=1-alpha, lty="dotted")
```



There has been a lot of research on finding an interval for π, the earliest of these studies began in the early 20^{th} century. However, the mathematical details go outside of the scope of this course. Interested readers can refer to the papers referenced in Chapter 1 of the book. Here, we will only summarize some of the findings from this literature. In practice, these intervals can be computed using functions from various R libraries. Even in that case, it is instructive to construt these intervals "manually", for not all statistical or programming languages come with these readily-available functions.

Alternatives: For $n < 40$, use *Wilson interval*. For $n \geq 40$, use *Agresti-Coull interval*. Note that even for $n < 40$, the *Agresti-Coull interval* is still generally better than the *Wald interval*.

2. Wilson "score" Interval:

$$\tilde{\pi} \pm \frac{Z_{1-\frac{\alpha}{2}} n^{1/2}}{n + Z^2_{1-\frac{\alpha}{2}}} \sqrt{\hat{\pi}(1-\hat{\pi}) + \frac{Z^2_{1-\frac{\alpha}{2}}}{4n}}$$

where $\tilde{\pi} = \frac{w + \frac{1}{2} Z^2_{1-\frac{\alpha}{2}}}{n + Z^2_{1-\frac{\alpha}{2}}}$, which can be considered as an "adjusted" estimate of π.

One of the advantages of this interval is that it is bounded between 0 and 1.

3. Agresti-Coull interval:

$$\tilde{\pi} \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\tilde{\pi}(1-\tilde{\pi})}{n + Z^2_{1-\frac{\alpha}{2}}}}$$

- Agresti and Caffo (2000), based on their examination of various types of CIs, recommended that adding one success and one failure for each group results in an interval that does a reasonable job.

Introduction (30 minutes, depending on the number of students attending the sessions)

1. Instructor's self introduction
2. Students' self introduction: each student takes turn introducing himself/herself (3 minutes each), addressing the questions below A. Which is your cohort? B. What courses have you taken? C. What company are you working for, and what's your role? D. Do you use machine learning or statistic modeling in your current work? If so, what techniques do you use? E. Why do you take this course? F. What's your biggest apprehension, if any, for this course?
3. Course Overview, Other Reminders, Q&A

Quick Introduction for this course (10 minutes)

- Professor Yau to give an overivew of this course, addressing various perspectives though this course primarily focuses on statistical models for various types of response variales:

  1. Statistics perspective

  2. Machine learning perspective

- This course focuses on statistical model building used in data science, covering steps from defining a (business, policy, scientific, etc) problem that can be addressed using data at hand, conducting EDA as a pre-model-building step (a step whose importance is often underestimated and that is often mis-understood and mis-used), understading the underlying statistical assumptions of a model under consideration, specifying a model, engineering features, estimating a model using functions in R, testing hypothesis, evaluating a model, conducting model diagnostics, and testing model assumptions. On occasion basis, I will also compare models covered in this course with machine learning techniques that can be used to solve similar problems.

- **The design of this course: I designed this course to study each model as mathematically rigorous as practically possible.** Each model is introduced with mathematical formulation, assumptions examined, estimation (in R) illustrated, statistical inference extensively studied, and practical topics, such as feature engineering (aka variable creation and transformation as termed in the statistical literature), model performance measurement, model interpretation, and model selection from data science perspective, extensively discussed.

- **Multiple pillars to learn difficult and abstract concepts:** async lectures, assigned readings, live sessions, and exercises, of learning the materials. Each of these pillars are tightly integrated, with a lot of repetitions among different pillars. The main reason of this design is that the materials covered in this course are conceptually difficult, and the various forms of learning (i.e. from watching asyn lectures - passive learning - to working through difficult analytic and empirical problems - active learning) can maximize the chance of mastering this materials. Different people absorb materials differently, but regardless of how

Weekly Workflow (10 minutes)

A typical week of the course proceeds as follows:

- Before live session: Watch all async content, study the assigned readings, attempt some of the end-of-chapter exercises, and write down questions / comments that you want to discuss in the live session

- There will be a weekly homework, corresponding to the materials covered in that week or before. The homework is due the beginning of each live session.
- In live session: Please come to the live session prepared. Live sessions are not lectures, but I will take 30 minutes or so to discuss important and/or difficult-to-understand concepts by going through some examples. It will be hard to you to follow if you don't at least watch the async lecture and study the readings.
- After live session: Review the materials covered in the previous week and continue to attempt some of the end-of-chapter exercises. There are numerous exercises at the end of each chapter. Do as many of them as you can.
- During the live sessions, I will also share some of my professional experience ("stories") in statistical and machine learning modeling, managing data science projects, managing data science teams, and even dealing with businesses to answer business questions using data science techniques.
- Live session quiz: There is a live session quiz per week starting in week 2. It will be a few multiple choice or true/false questions.

Professors' Expectation and How to Succeed in this Class (10 minutes)

Please take out the syllabus and review some of the highlights together

Here are some strategies based on our past experience and how I designed the course

- Review materials taught in the new version of w203, especially the part on linear regression modeling.
- **On watching the async lecture videos:** Different people learn differently; as such, it is difficult to give a "general" rule of how one should watch the async lectures. That said, I think it is very rare that students can watch each lecture in one sitting and master the materials. It is more likely that you will have to watch a section; pause; read the corresponding sections in the text and work through some examples; and then rewatch the section of the lecture to ensure that you understand the materials. This is why I followed so closely the assigned textbooks, which I chose by considering many different texts. This is also why mastering the concepts and methods in this course is very time consuming.
- Study your readings. More importantly, do the exercises in the texts. This is an advanced course; it is very difficult, if not impossible, to learn statistical modeling by simply reading textbooks or watching videos; doing hands-on work is critical to learn the materials taught in this course (and any statistical modeling and machine learning course). Note that it is also impossible to cover every single concept in a ninety-minute async lecture; you will have to build upon the materials taught in the async lecture by studying the readings and doing many exercises.
- Come to the live session prepared. I expect that you should watch the async video lecture, study the assigned readings, work on the examples in the book, and attempt a at least a few end-of-chapter exercises. In the live session, we may occasionally ask questions that you can only answer if you keep up with the readings.
- Form study groups! Do it now (if you haven't done so already)!
- Do not skip the readings or async lectures. In this course, once you fall behind, it is extremely difficult to catch up, as we cover a different statistical model almost every week.
- For the lab, form groups, but do not use the **"divide-and-conquer"** strategy. No matter how many times I told the students in the past, they still used this strategy. Just keep this in mind: using "divide-and-conquer" strategy is effectively asking others

- Attend office hours to ask questions related to the concepts and techniques covered in async lectures and readings; don't just come to office hours when you have questions on the labs. **Note that for the labs, we will only answer clarifying questions in the office hours.**
- Most importantly, this is a professional masters' program, and I expect you to behave professionally.
- **Grading: let's look at the syllabus together ...**

Live Session Work-flow

1. Live session quiz and go over quiz solution (15 minutes)
2. Go over homework; may have students present part of their solutions (15 - 20 minutes)
3. Professor discusses / highlights important or difficult-to-understand concepts, perhpas through an extended example with R-code (20 - 25 minutes)
4. Break-out sessions + whole-class discussions to study/review concepts learned this week (30 - 40 minutes)
5. Q&A?

Q&A