Toward Transient Dynamical Indicators of Critical Transitions

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Contents

1 Introduction

A **tipping point** or **critical transition** occurs in a dynamical system when a small perturbation to system conditions causes an abrupt overall shift in qualitative behavior. Empirically, tipping points have been studied in contexts as diverse as Earth's climate [1, 2], emerging infectious diseases [3], aquatic and land ecosystems [4,5], the onset of medical health states [6,7], financial markets [8], and more [9–11]. Since critical transitions often represent a shift into an undesirable or catastrophic regime, and since such transitions may not be easily or at all reversible, it is of pressing interest to anticipate them before they occur, in order to inform management strategies and improve the odds of prevention. Unfortunately, in complex real world systems, the conditions under which a critical transition occurs, and the underlying mechanisms driving the approach to transition, are usually extremely difficult to characterize.

As a result, there is particular interest in generic mathematical signals that can warn of impending tipping in a wide variety of systems without reference to specific underlying mechanisms. Such **early warning signals** have been most commonly studied as precursors of local codimension-1 bifurcations of ODEs, where they are based on the phenomenon of **critical slowing down** [10]. Roughly speaking, as the bifurcation parameter gradually nears its critical value, the resilience of the system drops (becoming slower to recover from perturbations), and this produces certain detectable statistical trends over time. In the context of critical slowing down, the term resilience refers specifically to what is known in the ecology literature as **asymptotic resilience**. In Section (2), I review two different quantifications of resilience, asymptotic resilience and another known as **intensity of attraction**). In Section (3), I summarize the theory of critical slowing down.

Early warning signals derived from asymptotic resilience and critical slowing down are a powerful tool for anticipating critical transitions, and their usefulness has already been demonstrated in numerous empirical contexts including, for instance, early detection of emerging infectious diseases [3] and retrospective analyses of prehistoric climate change events [2]. But a major simplification is the assumption that the system experiences only small, infrequent perturbations, which do not drive the system state very far from equilibrium and which leave sufficient time for recovery in between disturbances. In particular, there is a neglect of transient behavior within the larger domain of attraction. Such transient states can result from large, closely repeated, or continual disturbances, as are common in real world systems. A second shortcoming arises from the fact that critical slowing down relates only to one specific category of tipping behavior – local bifurcations. In contrast, another way a dynamical system might tip is due to perturbations pushing the system state across a boundary between domains of alternative attractors. (Many other dynamical behaviors also correspond to tipping, and are not considered here, including global bifurcations, rate-induced tipping, and transitions to chaotic regimes.)

Early warning signals derived from transient dynamics are a research area that demand future development. In Section (4), the thesis proposal, I consider the possibility for transient indicators to arise from intensity of attraction, and suggest preliminary steps toward developing an understanding of such indicators.

2 Resilience Quantification

Loosely, resilience refers to the capacity for a system to retain its overall qualitative structure in the face of disturbances. Its precise definition differs between authors and disciplines; an abundance of approaches to quantifying resilience have been proposed. In this section, I define asymptotic resilience and intensity of attraction. For a review of some other mathematical definitions of resilience that I do not cover, see [12].

2.1 Preliminaries

Let $U \subset \mathbb{R}^n$ be an open set, and assume that $f : U \to \mathbb{R}^n$ is a locally Lipschitz continuous function which is continuously differentiable. Consider a system of ordinary differential equations (ODEs)

$$x' = f(x) \tag{1}$$

The Lipschitz condition guarantees well-defined solutions, but only for sufficiently short intervals of time; hence we call the solutions **local**. We will use flow notation to collect all solution trajectories into one convenient object, called the **local flow** $\varphi : D \subset \mathbb{R} \times U \to U$, which is defined such that $\varphi(t, x_0) = x(t)$ is a solution to the initial value problem

$$x'(t) = f(x(t)), \qquad x(0) = x_0.$$

Depending on context, f may be globally Lipschitz continuous, in which case trajectories are defined for as long as they remain within the domain U. If no trajectories leave U, then φ is a **global flow**, meaning it is defined for all time. In this paper we will simply assume for convenience that the flow is defined on any time domain of interest.

Additionally, we will be taking the following notational conveniences. For a flow φ, we will denote the time-t map as $\varphi_t : U \to U$. We will naturally extend this notation to allow set-valued inputs $S \subset U$:

$$\varphi_t(S) = \{x \in U \mid \varphi_t(x_0) = x \text{ for some } x_0 \in S\}.$$

Intuitively, the map φ_t outputs the location of any input point after it flows for t units of time. If the input is a set, then the output is also a set, consisting of all the locations reached at time t.

Two central objects of study in this paper are attractors and their associated basins of attraction. Attractors characterize the system's behavior as $t \to \infty$, by pulling trajectories toward them – at least, those trajectories which begin within their basin of attraction. Tipping behavior often comes down to either an abrupt shift in the nature of an attractor or an abrupt switch from one attractor to an alternative attractor. When we talk about resilience in this paper, we are referring to the resilience *of an attractor*.

In order to define attractors and basins, we must first formalize some aspects of long-term behavior.

Definition 1. Consider a subset $S \subset U$. S is **invariant** under the flow φ if it contains all its own images in time: $\varphi_t(S) \subset S$ for all $t \in \mathbb{R}$. □

Intuitively, an invariant set is one which always stays put. The next definition describes where an arbitrary set ends up, or at least limits toward, in the long tun.

Definition 2. The **omega limit set** of $S \subset U$ is

$$\omega(S) = \bigcap_{T>0} \overline{\bigcup_{t>T} \varphi_t(S)}.$$

□

Now we have the vocabulary to formally define attractors and basins.

Definition 3. An **attractor** $A \subset U$ is a non-empty, compact, invariant set which is the omega limit set $\omega(N)$ of some neighborhood N of itself. Its **basin of attraction**, also called its **domain**, is

$$basin(A) = \{x \in U \mid \omega(x) \subset A, \omega(x) \neq \emptyset\}.$$

□

As mentioned, attractors are the fixed structures in a system which are approached by solution trajectories in the long run. Each attractor has a certain dominion of rule – those trajectories beginning within its basin are the ones that end up drawn toward it. While attractors may have interesting structures – periodic or chaotic, for instance – we will begin with the simplest type of attractor: an **attracting rest point**. These attracting rest points, also called **stable rest points**, capture the intuitive idea of a "steady state."

The next definition says that a rest point is any unmoving point, while the subsequent proposition, which is standard theory, gives conditions under which a rest point is an attracting one.

Definition 4. x_* is a **rest point** or **equilibrium** of the ODE (1) if $f(x_*) = 0$.

Proposition 5. *If all eigenvalues of linearization at the rest point x_* have negative real part, that is,*

$$Re(\lambda) < 0 \textit{ for all } \lambda \in spec(\boldsymbol{D}f(x_*)),$$

then x_* is an attractor. $\square$

Finally, we give useful terminology to classes of rest points which do not fall into the above category.

Definition 6. If all eigenvalues of linearization at the rest point x_* have non-zero real part, then x_* is called **hyperbolic**. Otherwise, at least one eigenvalue has zero real part, and we call x_* **non-hyperbolic**. $\square$

Definition 7. If x_* is hyperbolic, and at least one eigenvalue of linearization at the rest point x_* has positive real part, then x_* is called **unstable**. $\square$

Hyperbolic rest points can be thought of as "nice" rest points, ones near which the dynamics are predictable in some sense. Unstable rest points match the intuitive notion of unstable states – around them, nearly all trajectories are repelled away. At non-hyperbolic points the behavior is unpredictable – the point may be stable, unstable, or neither.

This concludes our set up of the preliminary framework, and we continue next to the definitions of asymptotic resilience and intensity of attraction.

2.2 Asymptotic Resilience

Throughout this subsection, we will assume that x_* is an attracting rest point of an ODE. Probably the most commonly used mathematical definition of resilience, originating in theoretical ecology, represents long-term return rates to x_*, and is measured by (the real part of) the dominant eigenvalue at linearization.

citation

Definition 8. Let $\mathbf{A} = Df(x_*)$ denote the Jacobian, and recall that all eigenvalues of $\mathbf{A}$ have negative real part. Let $\lambda_1(\mathbf{A})$ be the eigenvalue with largest (closest to 0) real part. The **asymptotic resilience** of the system at the stable rest point is equal to the negative of that real part,

$$-Re(\lambda_1(\mathbf{A})).$$

Note: we will refer to λ_1 as the **dominant eigenvalue** or the **slow eigenvalue** of $\mathbf{A}$. $\square$

For the linearized system $x' = \mathbf{A}x$, asymptotic resilience estimates the rate at which trajectories approach the equilibrium. The following theorem is standard theory for linear ODEs.

Theorem 9. *For an $n \times n$ matrix $\mathbf{A}$, if $Re(\lambda) < L < 0$ for all eigenvalues λ of $\mathbf{A}$, then there is some constant $C > 0$ such that for all $x \in \mathbb{R}^n$ and $t \geq 0$,*

$$|e^{t\mathbf{A}}x| \leq Ce^{Lt}|x|.$$

Further, in the long term C can be taken to equal 1. That is, there is some $T \geq 0$ such that

$$|e^{t\mathbf{A}}x| \leq e^{Lt}|x| \quad \textit{for all } t \geq T.$$

$\square$

TO DO: Is there a converse to the inequality? how to say that this is a good bound? i.e. for almost all trajectories, in the limit as $t \to \infty$, decay is exactly exponential with rate constant $Re\lambda_1$. I feel like this is true, but I can't find a statement of it in a book.

to do

Note the operator $e^{t\mathbf{A}}$ in the left hand side is exactly the flow φ_t for the linear system $x' = \mathbf{A}x$. So this theorem says that, in the long term, trajectories must decay to the origin at an exponential rate. Further, a bound on that rate is governed by the asymptotic resilience.

For nonlinear systems, similar bounds on decay rate is justified by the Stable Manifold Theorem, a fundamental result in dynamical systems theory which says that, at sufficiently nice rest points, the linear approximation is a good approximation. A special case of the Stable Manifold Theorem is stated here, while a full version can be found in any standard text.

Theorem 10. *(Stable Manifold Theorem, for attracting rest points) Consider a non-linear system*

$$x' = \mathbf{A}(x) + h(x),$$

where $\mathbf{A}, h : \mathbb{R}^n \to \mathbb{R}^n$ *with* $\mathbf{A}$ *linear. Let* ϕ_t *be the local flow. Assume there is an attracting rest point at the origin. Let* λ_1 *be the dominant eigenvalue of* $\mathbf{A}$*. Then there exists a neighborhood* $N \ni 0$ *which is a* ***local stable manifold*** *of the origin. That is, for all* $x \in N$, $\lim_{t\to\infty} \phi_t(x) = 0$*.*

Furthermore, for any $Re(\lambda_1) < L < 0$*, there exists* $C > 0$ *such that for all* $x \in N$, $t \geq 0$,

$$|\phi_t(x)| \leq Ce^{Lt}|x|,$$

and for some $T \geq 0$, C *can be taken to equal* 1

$$|\phi_t(x)| \leq e^{Lt}|x| \quad \text{for } t \geq T.$$

□

does the long term $C = 1$ statement hold? Can't find this in a book but I feel like it should be true. Also how about a converse to the inequality?

The bound on decay rate stated in the last line of the theorem implies that any trajectory beginning sufficiently close to equilibrium decays toward equilibrium at an exponential rate, and a long term bound on that rate is governed by the asymptotic resilience. Since the equilibrium represents a steady state, then any point which is very close to, but not quite at, the equilibrium represents a slightly perturbed state. Hence, the rate of decay can be thought of as the recovery rate from small perturbations.

(how to say that the exponential bound on asymptotic return rate is a good bound? i.e. most trajectories do eventually decay at that rate, and not much faster than it.)

to do

Remark 11. Note that trajectories need not decay monotonically in distance to the rest point, not even for linear systems. For instance, a trajectory can initially amplify in magnitude – a phenomenon termed **reactivity** by Neubert and Caswell in [13] (Figure 1). However, with some large enough choice of T, the Stable Manifold Theorem implies that this short term growth negligibly affects the long term decay.

(a) $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$ (b) $A_2 = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}$

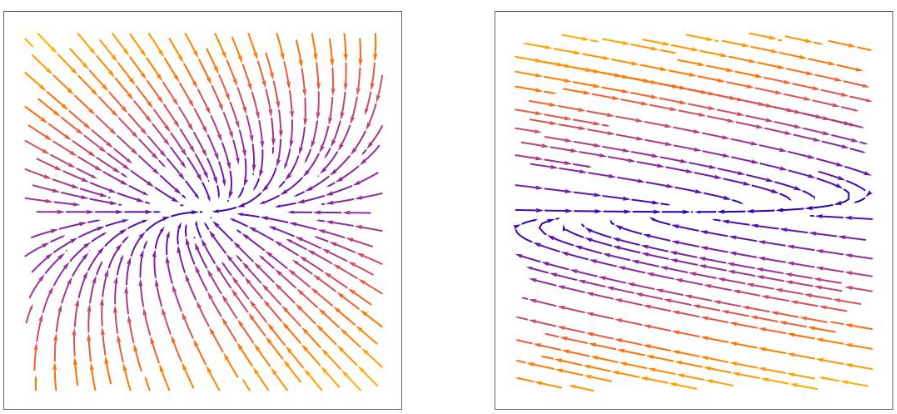

Figure 1: Phase portraits of two linear systems $x' = \mathbf{A}x$. (a) All trajectories decay monotonically in magnitude. (b) There are trajectories beginning arbitrarily close to the origin which initially increase in magnitude. Notice that both matrices have the same eigenvalues $\lambda = -1, -2$; hence asymptotic resilience cannot tell whether an equilibrium is reactive. Example reproduced from [13].

2.3 Intensity of Attraction

Asymptotic resilience notably relies on linearizing at a point attractor. In contrast, intensity of attraction, originally introduced by Richard McGehee for discrete maps [14], and extended to the continuous case by Katherine Meyer [15], measures resilience not only for rest points but also for any other type of attractor. Even more importantly, it captures metric information across the entire basin of attraction rather than simplifying to a topologically equivalent approximation within a limited neighborhood. We now review the necessary background in order to define intensity of attraction.

First of all, the idea of perturbation will now be represented by what is known as a **control function** added to an underlying vector field. This construction allows for perturbations which are not necessarily small and isolated, but possibly large and continuous, reflecting important types of perturbation that commonly occur in ecological and other applied settings, such as environmental forces or human-driven pressure (intentional or unintentional) on an ecosystem. We assume that the control function

$$g : I \subset \mathbb{R} \to \mathbb{R}^n$$

is taken from the space of essentially bounded (i.e. bounded except on a set of measure 0) measurable functions $L^\infty(I, \mathbb{R}^n)$, where the norm is

$$||g||_\infty = \inf\{C \geq 0 : ||g(x)|| \leq C \text{ for almost every } x \in I\}.$$

We also assume g is locally integrable (i.e. integrable on every compact subset of its domain I). These mild assumptions will ensure that g is nice enough for our perturbed system to remain well-defined. Next, we formalize how the perturbation is added to an underlying system.

Definition 12. A **bounded control system** is a non-autonomous ODE

is this ok terminology?

$$x' = f(x) + g(t) \tag{2}$$

where $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, $g \in L^\infty(I, \mathbb{R}^n)$ is locally integrable, and its norm $||g||_\infty$ is finite. □

how can g have infinite norm and be locally integrable? do i need this as a separate condition?

Here, the underlying system is thought of as an ODE $x' = f(x)$; but it is altered by adding a perturbation $g(t)$ to the vector field $f(x)$ on the right hand side. The effect of $g(t)$ is to adjust, at every point in time, the path of solutions somewhat away from what would have been their original trajectory.

It remains to be justified whether this construction produces a well-defined system. Because the right hand side $f(x)+g(t)$ may be a discontinuous function, solutions $x(t)$ of the ODE must be considered in an extended sense, which is that $x'(t) = f(x) + g(t)$ almost everywhere. Fortunately, the conditions on g are enough to guarantee local existence and uniqueness of solutions in such an extended sense. Briefly: (1) the hypotheses of Caratheodory's theorem are satisfied, establishing existence, and (2) boundedness of g guarantees Lipschitz continuity (local if f is locally Lipschitz, global if f is globally Lipschitz), thereby implying uniqueness.

So we have well-defined solutions, and can therefore extend the standard local flow notation to the bounded control setting. Fixing an underlying vector field f, we will denote as follows the flow obtained by applying a choice of perturbation g.

Definition 13. $\varphi_g(t, x_0) : D \subset \mathbb{R} \times U \to U$ is the local flow defined by

$$\varphi_g(t, x_0) = x(t)$$

where $x(t)$ solves in the extended sense the ODE (2), with initial condition $x(0) = x_0$. □

Intensity of attraction considers not just one single control function, but entire families of control functions – specifically, those where every function is bounded by some maximum magnitude r. The next definition gives a notation for these families.

Definition 14. Denote by $B_r \subset L^\infty[I, \mathbb{R}^n]$ the set of control functions bounded above by r:

$$B_r = \{g : ||g||_\infty < r\}$$

□

This leads, next, into the notion of all possible states reachable in forward time, under the family of all possible control functions bounded by r, and beginning from some arbitrary initial set of states.

Definition 15. Consider $S \subset U$. The **reachable set** of S under r-bounded control is the set

$$R_r(S) = \bigcup_{g \in B_r} \bigcup_{x_0 \in S} \bigcup_{t \geq 0} \varphi_g(t, x_0)$$

□

Finally, we are ready to define intensity of attraction, which captures the folllowing idea: what is the smallest magnitude of control necessary in order to escape from (all compact subsets of) a basin of attraction?

why compact subsets?

Definition 16. If A is an attractor of $x' = f(x)$, then its **intensity of attraction** is

$$\mathit{intensity}(A) = \sup\{r \geq 0 \mid R_r(A) \subset K \subset \mathit{basin}(A), \text{ for some compact } K\}$$

□

Another way to understand intensity of attraction is through the idea of "basin steepness" – what is the steepest part of the basin that must be overcome in order to escape the influence of the attractor? For systems whose state x is one dimensional, this intuition is precise: the vector field $f : \mathbb{R} \to \mathbb{R}$ is always integrable, producing a potential function, and the maximum steepness of that potential on the basin determines intensity of attraction.

Proposition 17. *To do: 1D basin steepness*

Proof. content... □

Add figure with 1D example

Unfortunately, for two and higher dimensional systems, no potential function necessarily exists, complicating the landscape analogy. Still, the next conjecture formalizes a sense in which intensity equals basin steepness. This conjecture has not yet been proven in the continuous case, but in McGehee's original conception of intensity for discrete maps, an analogous statement is true.

Conjecture 18. *If there is a neighborhood N of the attractor whose closure is within its basin of attraction, such that the inward normal component of the vector field f at every point on the boundary of N is at least k, then the intensity of the attractor is at least k.*

Add picture

3 Critical Slowing Down

Critical slowing down and associated early warning signals of critical transitions have been seen in a variety of empirical contexts; in each case, certain detectable statistical trends in a time series appear leading up to a critical transition. A theoretical basis for these leading indicators is rooted in bifurcation theory.

It should be remarked that another set of early warning signals for certain critical transitions, not discussed further here, are the spatial pattern-formation mechanisms (e.g. systematic changes in vegetation patterns preceding desertification) .

citation

3.1 Local Bifurcation

For a detailed treatment of local bifurcation theory, see any standard reference. For the purposes of this paper, we include only an extremely brief overview.

Consider a parameterized family of ODEs

$$x' = f(x, p) \tag{3}$$

$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. Here, x represents state variables, while p represents a parameter.

Conceptually, a local bifurcation is a critical state at which a very small change in the parameter value p causes a drastic change in the qualitative nature of a rest point. In particular, its local topology changes – for example, the rest point could switch between being stable and unstable, or it could disappear altogether. It is well known that such a change in local topology is only possible at non-hyperbolic rest points.

Definition 19. A **local bifurcation** occurs at (x_0, p_0) if the Jacobian matrix $\mathbf{D}f(x_0, p_0)$ has an eigenvalue with zero real part. Note this could either be a real eigenvalue $\lambda = 0$ or it could be a pair of imaginary eigenvalues $\lambda = \pm\omega i$. In the former case, a **saddle-node** or **fold** bifurcation occurs. In the latter, a **Hopf** bifurcation occurs. □

For simplicity, the term saddle-node bifurcation used here subsumes what are commonly termed transcritical and pitchfork bifurcations, while usage may vary between different authors. However, the distinction between the three categories is useful from an applied point of view, because they reflect different behavior. Briefly, a true **saddle-node/fold bifurcation**, which is neither of the other two types, involves the simultaneous creation or destruction of two equilibria – it has also been called a "blue-sky" bifurcation, because two points appear out of or disappear into the blue sky. A **pitchfork bifurcation**, is similar, but there is already an existing rest point, so that the system switches locally between having one or three rest points; pitchfork bifurcations only occur in sufficiently symmetrical systems, a scenario that does not typically occur in ecological applications. Finally, a **transcritical bifurcation** involves no new birth or death of rest points; instead, a stable and an unstable rest point collide and exchange stabilities with each other.

A **Hopf bifurcation** differs from all the others in that a periodic orbit is created or destroyed.

3.2 Critical Slowing Down and Early Warning Signals

From the previous subsection, we know that local bifurcations are characterized by an eigenvalue's real part approaching and passing through zero. For a stable rest point, all of whose eigenvalues' real parts are negative, this necessarily means it is the dominant eigenvalue's real part which passes through zero. Hence, local bifurcation involving an abrupt change to a steady state is characterized by asymptotic resilience going to zero. We also know that asymptotic resilience provides a bound on long term recovery rates from small perturbations. So, leading up to the bifurcation, recovery rates tend to slow, and this phenomenon is termed critical slowing down.

Since real world systems frequently face natural perturbations, this slowing down of recovery rate ought to be empirically observable – indeed, variance and auto-correlation in the system state tend to increase leading up to tipping. Intuitively speaking, this is because a slow-recovering system stays far away from the mean longer, so variance increases; and because the current state of a slowly moving system tends to stay more similar to its next state, so auto-correlation increases as well (Figure 2).

Formal derivations of the existence of these early warning signals (increased variance and increased auto-correlation) were difficult for me to find in the literature. An approach has been taken in [16], approximating the decay by a discrete auto-regressive process with random additive noise applied after each period Δt,

$$|x_{n+1} - x^*| = e^{\lambda \Delta t}|x_n - x^*| + \sigma\epsilon_n.$$

However, it requires the simplifying assumption that return between disturbances is precisely exponential. Another approach has been taken in [17] where the trajectory is modelled as a continuous stochastic mean-reverting process (an Ornstein-Uhlenbeck process), which is the continuous analog of the auto-regressive process of the previously mentioned authors. But this has the same underlying issue of a fundamental simplifying approximation.

Neubert and Caswell in [13] emphasize that asymptotic resilience crucially ignores short term behavior. While decay rates are exponential in the limit, perturbations could actually amplify in the short term, and this amplification could happen for arbitrarily small perturbations (also see the end of subsection 2.2). As a result, simplifying assumptions such as precise exponential decay cannot always be applied reasonably.

Typically, discussions of early warning indicators are framed from within applied point of view. Empirically, increased variance and increased auto-correlation have indeed been observed in many example systems, and this has often been taken as evidence of their existence in general. Yet, one drawback to the informal treatment is that the precise conditions under which these early warning signals arise, and hence their reliability across different systems and circumstances, remains unclear.

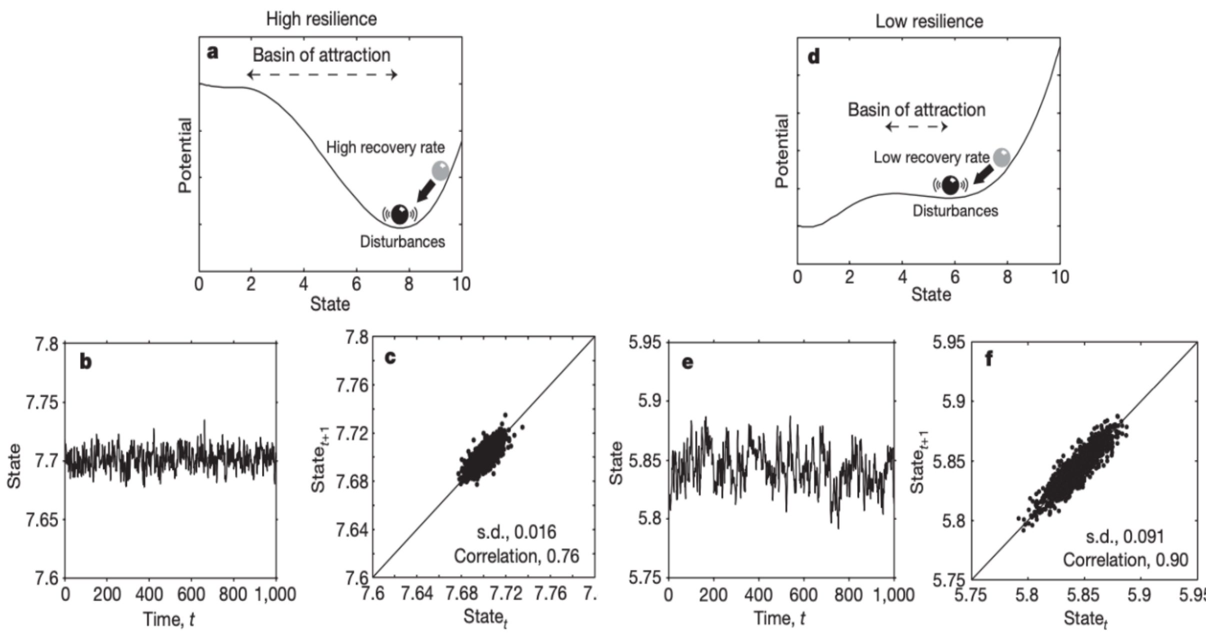

Figure 2: Critical slowing down and early warning signals. Figure reproduced from [10].

4 Thesis Proposal

The formal understanding of early warning signals of critical transitions requires future clarification; specifically, the current theory relies on long-term and linear approximations while ignoring transient behavior.

One line of research seeks to clarify the effect of short term non-exponential behavior. For example, reactivity has recently been shown to change systematically leading up to bifurcation in a certain class of infectious disease models [18].

On the other hand, I propose to clarify the effect of behavior far away from the attractor – far enough that it is not within the descriptive bounds of a linear approximation. I suggest investigating the relationship between intensity of attraction and critical transitions.

Toward this goal, I propose to relate intensity to at least three different aspects of tipping behavior: local bifurcations, tipping across basin boundaries, and reversibility of hysteretic transitions. As a supplementary pursuit, I would also like to improve the general theoretical understanding of intensity of attraction, by considering various open questions related to it. Finally,

Perhaps this work may eventually help build toward a theory of early warning indicators brought about by changes in intensity during the time leading up to a critical transition.

TO DO: Example where intensity might be useful, more so than asymptotic resilience and classical critical slowing down.

4.1 Intensity Through Local Bifurcations

How does intensity of attraction behave when passing through a local bifurcation? Does it display a systematic change, similar to the way that asymptotic resilience? Here, a first step may be to prove continuity of intensity with respect to parameter changes.

Conjecture 20. *(McGehee or Meyer?) Intensity is continuous with respect to parameter changes. To do: phrase this formally.*

Next, I would investigate one dimensional saddle-node, transcritical, and pitchfork bifurcations.

Maybe nothing interesting?

Critical widening? Not explained by critical slowing down with state variable perturbation. But can be explained by decreasing intensity leading up to bifurcation with vector field perturbation. Since we see this in data, suggests that

Numerical computations of intensity across application-specific examples of bifurcation.

4.2 Tipping Across Basin Boundaries

Critical slowing down pertains only to local bifurcations. But another class of tipping behavior occurs when perturbations push a state variable into an alternative basin of attraction. Intensity measures how difficult it is for this type of tipping to occur under bounded control types of perturbations.

4.3 Reversibility of Hysteretic Transitions

Define hysteresis and give example of hysteresis.

At least two types: cubic type or parabolic plus additional steady state type (e.g. KIausmeier equation)

Intensity of the alternative attractors describes whether the basin boundary tends to be crossed before the bifurcation point or not.

Define intensity of the repeller in between?

4.4 Other Open Questions About Intensity

4.4.1 Estimates of Intensity

One basic limitation of using intensity of attraction in any application right now: numerical computations of intensity (which currently use set-valued Euler methods on a fixed grid) are too time-intensive. Instead, analytic tools should be developed that can be used for estimating intensity. (There is also a need for improved numerical methods, although this important avenue is not the focus of my thesis proposal.)

For instance, I would start by pursuing a proof of Conjecture 18, which may provide a tractable way to estimate a lower bound on intensity without requiring a full numerical computation.

4.4.2

4.5 Further Possibilities

Machine learning based early warning signals? Possible connection between machine-learning based and analytical theory based early warning signals? i.e. using theory to inform ML design.

Validating on actual data from somewhere?

Connections to Flow-Kick systems?

Further connections to Multiflows?

5 Conclusion

Mention reactivity

Mention papers where critical transitions occur with no lead warning.

Mention flickering?

As pressures exerted by modern day anthropogenic practices on the Earth grow in magnitude and complexity, threatening physical, ecological, and social systems on all scales with unprecedented forms of change, this goal becomes even more pressing.

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