Toward Transient Dynamical Indicators of Critical Transitions

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Contents

1 Introduction

A **tipping point** or **critical transition** occurs in a dynamical system when a small perturbation to system conditions causes an abrupt overall shift in qualitative behavior. Empirically, tipping points have been studied in contexts as diverse as Earth's climate [1, 2], emerging infectious diseases [3], aquatic and land ecosystems [4,5], the onset of medical health states [6,7], financial markets [8], and more [9–11]. Since critical transitions often represent a shift into an undesirable or catastrophic regime, and since such transitions may not be easily or at all reversible, it is of pressing interest to anticipate them before they occur, in order to inform management strategies and improve the odds of prevention. Unfortunately, in complex real world systems, the conditions under which a critical transition occurs, and the underlying mechanisms driving the approach to transition, are usually extremely difficult to characterize.

As a result, there is particular interest in generic mathematical signals that can warn of impending tipping in a wide variety of systems without reference to specific underlying mechanisms. Such **early warning signals** have been most commonly studied as precursors of local codimension-1 bifurcations of ODEs, where they are based on the phenomenon of **critical slowing down** [10]. Roughly speaking, as the bifurcation parameter gradually nears its critical value, the resilience of the system drops (becoming slower to recover from perturbations), and this produces certain detectable statistical trends over time. In the context of critical slowing down, the term resilience refers specifically to what is known in the ecology literature as **asymptotic resilience**. In Section (2), I review two different measures of resilience, including asymptotic resilience and another known as **intensity of attraction**). In Section (3), I summarize the theory of critical slowing down.

Early warning signals derived from asymptotic resilience and critical slowing down are a powerful tool for anticipating critical transitions, and their usefulness has already been demonstrated in numerous empirical contexts including, for instance, early detection of emerging infectious diseases [3] and retrospective analyses of prehistoric climate change events [2]. But a major limitation is the assumption that the system experiences only small, infrequent perturbations, which do not drive the system state very far from equilibrium and which leave sufficient time for recovery in between disturbances. In particular, there is a neglect of transient behavior within the larger domain of attraction. Such transient states can result from large, closely repeated, or continual disturbances, as are common in real world systems. A second shortcoming arises from the fact that critical slowing down relates only to one specific category of tipping behavior – local bifurcations. In contrast, a dynamical system might tip due to large perturbations pushing the system state across a boundary between domains of alternative attractors. (Many other dynamical behaviors also correspond to tipping, and are not considered here, including global bifurcations, rate-induced tipping, and transitions to chaotic regimes.)

Early warning signals derived from transient dynamics are a research area that demand future development. In Section (4), the thesis proposal, I consider the possibility for transient indicators to arise from intensity of attraction, and suggest preliminary steps toward developing an understanding of such indicators.

2 Resilience Quantification

Loosely, resilience refers to the capacity for a system to retain its overall qualitative structure in the face of disturbances. Its precise definition differs between authors and disciplines; an abundance of approaches to quantifying resilience have been proposed. In this section, I define asymptotic resilience and intensity of attraction. For a review of other mathematical definitions of resilience that I do not cover, see [12].

2.1 Preliminaries

Let $U \subset \mathbb{R}^n$ be open, and assume that $f : U \to \mathbb{R}^n$ is locally Lipschitz continuous. Consider a system of ODEs

$$x' = f(x) \tag{1}$$

and let $\varphi : D \subset \mathbb{R} \times U \to U$ be the associated local flow, so that $\varphi(t, x_0) = x(t)$ solves the ODE (i.e. $x'(t) = f(x(t))$), with initial condition $x(0) = x_0$.

In many contexts, f is globally Lipschitz continuous, in which case trajectories are defined for as long as they remain within U. If no trajectories leave U, then φ is a global flow defined for all $t \in \mathbb{R}$. In this paper we will assume for the sake of simplicity that the flow is defined on any time domain of interest.

There are also a couple of notational conveniences we will take. For a flow $\varphi : \mathbb{R} \times U \to U$, we will denote the time t map as $\varphi_t : U \to U$. We will also naturally extend its definition to allow set-valued inputs $S \subset U$ as follows:

$$\varphi_t(S) = \{x \in U \mid \varphi_t(x_0) = x \text{ for some } x_0 \in S\}.$$

Two central objects of study in this paper is are attractors and their associated domains, or basins of attraction. An attractor characterizes the eventual behavior approached by the system over time (or at least the part of the system lying within its basin). In order to define attractors and basins, we must first formalize some aspects of long-term behavior.

Definition 1. Consider a subset $S \subset U$. S is **forward invariant** under the flow φ if it contains all its forward images in time: $\varphi_t(S) \subset S$ for all $t \in \mathbb{R}^+$. Similarly, S is **backward invariant** if $\varphi_t(S) \subset S$ for all $t \in \mathbb{R}^-$. S is **invariant** if it is both forward and backward invariant. □

Intuitively, an invariant set is one which always stays within itself. The next definition describes where an arbitrary set limits toward in the long run.

Definition 2. The **omega limit set** of $S \subset U$ is

$$\omega(S) = \bigcap_{T>0} \overline{\bigcup_{t>T} \varphi_t(S)}.$$

□

Now we can define an attractor and its basin.

Definition 3. An **attractor** $A \subset U$ is a non-empty, compact, invariant set which is the omega limit set $\omega(N)$ of some neighborhood N of itself. Its **basin of attraction** is $basin(A) = \{x \in U \mid \omega(x) \subset A, \omega(x) \neq \emptyset\}$. □

While attractors may have interesting structures – periodic or chaotic, for instance – we will begin with the simplest type of attractor: an attracting rest point.

Definition 4. x_* is a **rest point** or **equilibrium** of the ODE (1) if $f(x_*) = 0$.

The following proposition is standard theory.

Proposition 5. *If all eigenvalues of linearization at the rest point x_* have negative real part, that is,*

$$Re(\lambda) < 0 \text{ for all } \lambda \in spec(\boldsymbol{D}f(x_*))$$

then x_* is an attractor. Note: we also call such an x_* a ***stable rest point****.*

what is the best terminology here?

2.2 Asymptotic Resilience

Throughout this subsection, we will assume that x_* is a stable rest point of an ODE. Probably the most classical mathematical definition of resilience, originally developed by theoretical ecologists, represents long-term return rates to x_*, and is measured by the dominant eigenvalue at linearization. citation

Definition 6. Let $\mathbf{A} = Df(x_*)$ denote the Jacobian, and recall that all eigenvalues of $\mathbf{A}$ have negative real part. Let $\lambda_1(\mathbf{A})$ be the eigenvalue with largest (closest to 0) real part.

The **asymptotic resilience** of the system at the stable rest point is $Re(\lambda_1(\mathbf{A}))$.

Note: we will refer to λ_1 as the **dominant eigenvalue** of $\mathbf{A}$. □

For the linearized system $x' = \mathbf{A}x$, asymptotic resilience estimates the rate at which trajectories approach the equilibrium. The following theorem is standard theory for linear ODEs. See for example (Chicone p. 175) do citation

Theorem 7. *If all eigenvalues of an $n \times n$ matrix $\mathbf{A}$ have negative real part, and if $Re(\lambda) < L < 0$ for all eigenvalues λ of A, then there is some constant $C > 0$ such that for all $x \in \mathbb{R}^n$ and $t \geq 0$,*

$$|e^{tA}x| \leq Ce^{Lt}|x|.$$

□

Note the expression $e^{tA}x$ in left hand side is exactly the flow for $x' = \mathbf{A}x$. So the theorem says that trajectories must decay to the origin at an exponential rate with a bound on that rate governed by λ_1. For nonlinear systems, a similar bound on decay rate is justified by the Stable Manifold Theorem.

Definition 8. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Define the **stable eigenspace** of $\mathbf{A}$ to be the subspace of $\mathbb{R}^n$ spanned by those eigenvectors of $\mathbf{A}$ with negative real part. Similarly, define the **unstable eigenspace** of $\mathbf{A}$ to be the subspace of $\mathbb{R}^n$ spanned by those eigenvectors of $\mathbf{A}$ with positive real part. □

Note in the case that $\mathbf{A}$ has no eigenvalues with real part equal to 0, it can be decomposed into a direct sum of linear subspaces $\mathbf{A} = E_s \oplus E_u$. In this case we call $\mathbf{A}$ **hyperbolic**.

Definition 9. Let $x' = f(x)$ be an ODE with $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, local flow ϕ_t, and a rest point at x_0. The **stable manifold** M^s of x_0 is

$$M^s = \{x \in U \mid \lim_{t \to \infty} \phi_t(x) = x_0\}$$

□

Theorem 10. *(Stable Manifold Theorem) Consider a non-linear system*

$$x' = \mathbf{A}(x) + h(x),$$

where $\mathbf{A}, h : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathbf{A}$ linear. Let ϕ_t be the local flow. Assume $h(0) = Dh(0) = 0$ so that there is a rest point at the origin. Let M^s be the stable manifold of the origin. Also suppose $\mathbf{A}$ has no eigenvalues with real part equal to 0. Let E_s and E_u be the stable and unstable eigenspaces of $\mathbf{A}$, respectively, so that $\boldsymbol{A} = E_s \oplus E_u$. Let $P_s : \mathbb{R}^n \to E_s$ be the linear projection operator onto the stable eigenspace. Also let λ_1 be the dominant eigenvalue of E_s.

*Then there exists a ball $B_\delta(0) \subset \mathbb{R}^n$ about the origin, an $\epsilon > 0$, and a function $\alpha : B_\delta \cap E_s \to E_u$ with $\alpha(0) = D\alpha(0) = 0$ so that its graph $M^s_{loc} = \{(x, \alpha(x)) \in \mathbb{R}^n : x \in B_\delta \cap E_s\}$ is a **local stable manifold** of the origin. That is,*

$$M^s_{loc} = \{x \in U : |P^s(x)| \in B_\delta(0)\} \cap M^s$$

Furthermore, for any $Re(\lambda_1) < L < 0$, there exists $C > 0$ such that for all $x \in M^s_{loc}$, $t \geq 0$,

$$|\phi_t(x)| \leq Ce^{Lt}|x|$$

□

For a stable rest point, $\mathbf{A} = E_s$, because all eigenvalues have negative real part, so the local stable manifold M^s_{loc} is just a δ-neighborhood of the rest point. Hence, any trajectory beginning sufficiently close to equilibrium decays toward it at an exponential rate with a bound on that rate governed by λ_1.

Note that trajectories need not decay monotonically in distance to the rest point, not even for linear systems. For instance, a perturbation may initially amplify in magnitude − a phenomenon termed **reactivity** by Neubert and Caswell in [13] (Figure 1). However, with some large enough choice of C, the Stable Manifold Theorem still implies an exponentially decaying bound on the rate of return to equilibrium.

(a) $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$ (b) $A_2 = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}$

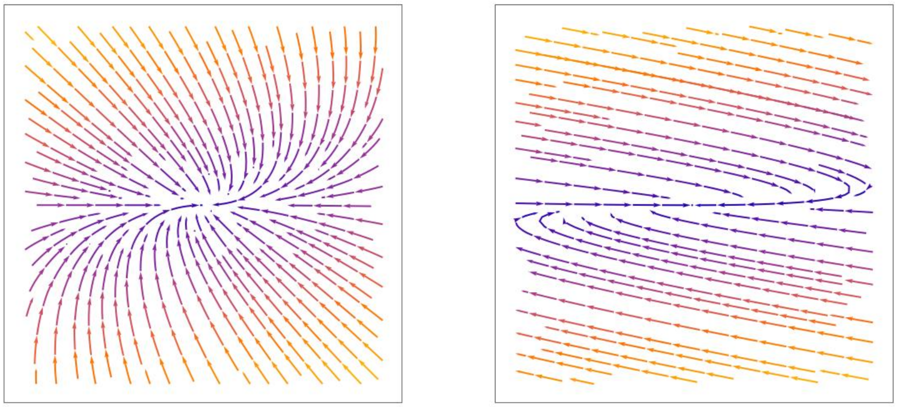

Figure 1: Phase portraits of two linear systems $x' = \mathbf{A}x$ with the same eigenvalues $\lambda = -1, -2$. (a) All trajectories decay monotonically in magnitude. (b) There are trajectories beginning arbitrarily close to the origin which initially increase in magnitude. Example reproduced from [13].

2.3 Intensity of Attraction

Asymptotic resilience notably relies on linearizing at a point attractor. In contrast, intensity of attraction, introduced by Katherine Meyer in her PhD thesis [14], measures resilience not only for points but also for any other type of attractor. Even more importantly, it captures metric information across the entire basin of attraction rather than simplifying the phase space to a locally topologically equivalent approximation. We now review the necessary background and define intensity of attraction.

First, the idea of perturbation is represented by a control function added to an underlying vector field. We assume that the control function

$$g : I \subset \mathbb{R} \to \mathbb{R}^n$$

is taken from the space of essentially bounded (i.e. bounded except on a set of measure 0) measurable functions $L^\infty(I, \mathbb{R}^n)$, where the norm is

$$||g||_\infty = \inf\{C \geq 0 : ||g(x)|| \leq C \text{ for almost every } x \in I\}.$$

We also assume g is locally integrable (i.e. integrable on every compact subset of its domain I).

Definition 11. A **bounded control system** is a non-autonomous ODE

is this ok terminology?

$$x' = f(x) + g(t) \tag{2}$$

where $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, $g \in L^\infty(I, \mathbb{R}^n)$ is locally integrable, and its norm $||g||_\infty$ is finite. □

how can g have infinite norm and be locally integrable? do i need this as a separate condition?

Because the right hand side $f(x) + g(t)$ may be a discontinuous function, solutions $x(t)$ of the ODE are considered in an extended sense, which is that $x'(t) = f(x) + g(t)$ almost everywhere. The conditions on g are enough to guarantee local existence and uniqueness of solutions in such a sense. First, the hypotheses of Caratheodory's theorem are satisfied, establishing existence. Second, boundedness of g guarantees Lipschitz continuity (local if f is locally Lipschitz, global if f is globally Lipschitz), thereby implying uniqueness.

As a result of well-defined solutions, we can extend the standard local flow notation to the bounded control setting. Fixing an underlying vector field f, we will denote as follows the flow obtained by applying a choice of perturbation g.

Definition 12. $\varphi_g(t, x_0) : D \subset \mathbb{R} \times U \to U$ is the local flow defined by

$$\varphi_g(t, x_0) = x(t)$$

where $x(t)$ solves in the extended sense the ODE (2), with initial condition $x(0) = x_0$. □

Next, intensity of attraction considers not just one single control function, but entire families of control functions, where every function in a family is bounded by the same constant r.

Definition 13. Denote by $B_r \subset L^\infty[I, \mathbb{R}^n]$ the set of control functions bounded above by r:

$$B_r = \{g : ||g||_\infty < r\}$$

□

Supposing that vector field perturbations, such as environmental forces, or human-designed control, are limited by some ceiling on magnitude, B_r can be thought of as a collection of all possible perturbations.

This leads into the notion of all possible states reachable in forward time, under control bounded by r, and beginning from some arbitrary initial set.

Definition 14. Consider $S \subset U$. The **reachable set** of S under r-bounded control is

$$R_r(S) = \bigcup_{g \in B_r} \bigcup_{x_0 \in S} \bigcup_{t \geq 0} \varphi_g(t, x_0)$$

□

Finally, we are ready to define intensity of attraction, which captures the idea of the smallest magnitude of control necessary in order to escape from (all compact subsets of) a basin of attraction:

why compact subsets?

Definition 15. If A is an attractor of $x' = f(x)$, then its **intensity of attraction** is

$$intensity(A) = \sup\{r \geq 0 \mid R_r(A) \subset K \subset basin(A), \text{ for some compact } K\}$$

□

add exampl?

3 Critical Slowing Down

Asymptotic resilience helps determine a bound on the rate of return to equilibrium after a small perturbation to the system. Because local bifurcation is characterized by $Re(\lambda_1)$ passing through zero, recovery typically becomes slower when closer to the verge of bifurcation. This is the core idea of critical slowing down.

First, we briefly review the necessary background on (one parameter a.k.a. co-dimension one) local bifurcations. For a detailed treatment of local bifurcation theory, see a reference such as .

cite Chicone

3.1 Local Bifurcation

Consider a parameterized family of ODEs

$$x' = f(x, p) \tag{3}$$

$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. Here, x represents state variables, while p represents a parameter.

Assume that there is a rest point $f(x_0, p_0) = 0$ for some value of the parameter $p = p_0$. Also suppose the Jacobian $\mathbf{A} = Df(x_0, p_0)$ is hyperbolic. Because all its eigenvalues are nonzero, $\mathbf{A}$ is an invertible transformation, and the Implicit Function Theorem applies to f. This produces a curve $p \to c(p)$ in $\mathbb{R}^n$, defined locally for $p \in [p_0 - \epsilon, p_0 + \epsilon]$, such that $c(p_0) = x_0$ and $f(c(p), p) = 0$.

In other words, if the value of the parameter is adjusted slightly, there is still a rest point at $c(p)$. Further, since the eigenvalues of $\mathbf{A}$ are continuous with respect to x and p, then the rest point $(c(p), p)$ for p sufficiently close to p_0 is stable if and only if (x_0, p_0) was stable. Hence, it is clear that the only way for a very small change in parameter value to lead to a loss of a stable rest point is if $\mathbf{A}$ is non-hyperbolic. Let us focus on that case.

Definition 16. A **local bifurcation** occurs at (x_0, p_0) if the Jacobian matrix $\mathbf{D}f(x_0, p_0)$ has an eigenvalue with zero real part. Note this could either be a real eigenvalue $\lambda = 0$ or it could be a pair of imaginary eigenvalues $\lambda = \pm\omega i$. If $\mathbf{A}$ has a real eigenvalue $\lambda = 0$, a **saddle-node** or **fold** bifurcation occurs. If $\mathbf{A}$ has a pair of imaginary eigenvalues $\lambda = \pm\omega i$, a **Hopf** bifurcation occurs. □

Note that the definition of saddle-node bifurcation here includes what are commonly termed transcritical and pitchfork bifurcations.

clarify the relationship and the definitions here.

Examples of bifurcation types, normal forms

Subcritical and supercritical.

3.2 Critical Slowing Down

From the Stable Manifold Theorem (section), we know that trajectories beginning sufficiently close to a stable rest point approach it at a rate bounded exponentially. In particular, for any $Re(\lambda_1) < L < 0$ there is some constant $C > 0$ such that

ref

$$|\phi_t(x)| \leq Ce^{Lt}|x|$$

Formal derivation (how to handle the constant C in front of the recovery rate bound?)

Example: normal form for transcritical or saddle node bifurcation

3.3 Early Warning Signals

Roughly, when an ODE approaches a tipping point (specifically, a local bifurcation) where an attracting rest point destabilizes, the rest point gradually loses asymptotic resilience, becoming slower to recover from small perturbations. Since real world systems are naturally perturbed all the time, this loss of asymptotic resilience ought to be empirically observable – indeed, variance and auto-correlation in the system state tend to increase leading up to tipping. Intuitively speaking, this is because a slow-recovering system stays far away from the mean longer, so variance increases; and because the current state of a slowly moving system tends to stay more similar to its next state, so auto-correlation increases as well (Figure).

ref figure

insert figure

Formal derivation of variance and auto-correlation after a small shock of size epsilon? (how?)

4 Thesis Proposal

Example where intensity might be useful, more so than asymptotic resilience and classical critical slowing down.

Propose some avenues that may help build toward a theory of early warning indicators brought about by changes in intensity during the time leading up to a critical transition.

Want to pursue analytic results as well as simulate application-specific examples.

Might be that intensity only provides useful warning indicators in a specific class of critical transitions, or for a specific class of ODEs.

4.1 Estimates of Intensity

One basic limitation of using intensity of attraction in any application right now: numerical computations of intensity (which currently use set-valued Euler methods on a fixed grid) are too time-intensive.

Want to develop other tools that can be used for estimating intensity or estimating bounds on intensity. (Also there is need for improved numerical methods, but not focus of this thesis proposal.)

For instance, prove that intensity can be bounded by drawing a neighborhood of the attractor on whose boundary the vector field always points inward/outward with a minimum normal component.

Idea of "basin steepness" but with caveats that there's not necessarily any potential function.

4.2 Intensity Through Local Bifurcations

One question is how intensity of attraction behaves when passing through a local bifurcation, and whether it always displays a systematic change, similar to the way that asymptotic resilience changes systematically by passing through zero.

First step may be to prove continuity of intensity with respect to parameter changes. Conjecture (McGehee or Meyer?)

Then investigate one dimensional saddle-node, transcritical, and pitchfork bifurcations.

Maybe nothing interesting?

Critical widening? Not explained by critical slowing down with state variable perturbation. But can be explained by decreasing intensity leading up to bifurcation with vector field perturbation. Since we see this in data, suggests that

Numerical computations of intensity across application-specific examples of bifurcation.

4.3 Tipping Across Basin Boundaries

Critical slowing down pertains only to local bifurcations. But another class of tipping behavior occurs when perturbations push a state variable into an alternative basin of attraction. Intensity measures how difficult it is for this type of tipping to occur under bounded control types of perturbations.

4.4 Reversibility of Hysteretic Transitions

Define hysteresis and give example of hysteresis.

At least two types: cubic type or parabolic plus additional steady state type (e.g. KIausmeier equation)

Intensity of the alternative attractors describes whether the basin boundary tends to be cross before the bifurcation point or not.

Define intensity of the repeller in between?

4.5 Further Possibilities

Machine learning based early warning signals? Possible connection between machine-learning based and analytical theory based early warning signals? i.e. using theory to inform ML design.

Validating on actual data from somewhere?

Connections to Flow-Kick systems?

Further connections to Multiflows?

5 Conclusion

Mention reactivity

Mention papers where critical transitions occur with no lead warning.

Mention flickering?

As pressures exerted by modern day anthropogenic practices on the Earth grow in magnitude and complexity, threatening physical, ecological, and social systems on all scales with unprecedented forms of change, this goal becomes even more pressing.

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