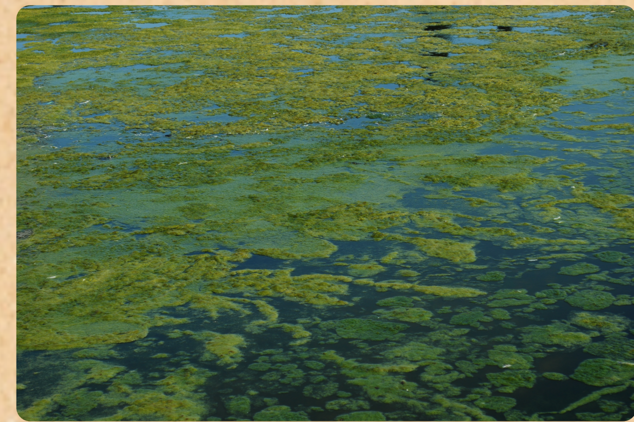
Transient Dynamical Indicators of Critical Transitions:

Toward an Intensity-Based Understanding

Grace Zhang

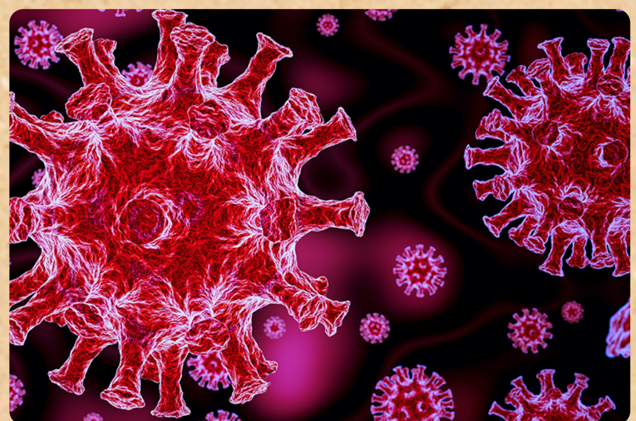
desertification



lake eutrophication

species extinction

financial market shifts



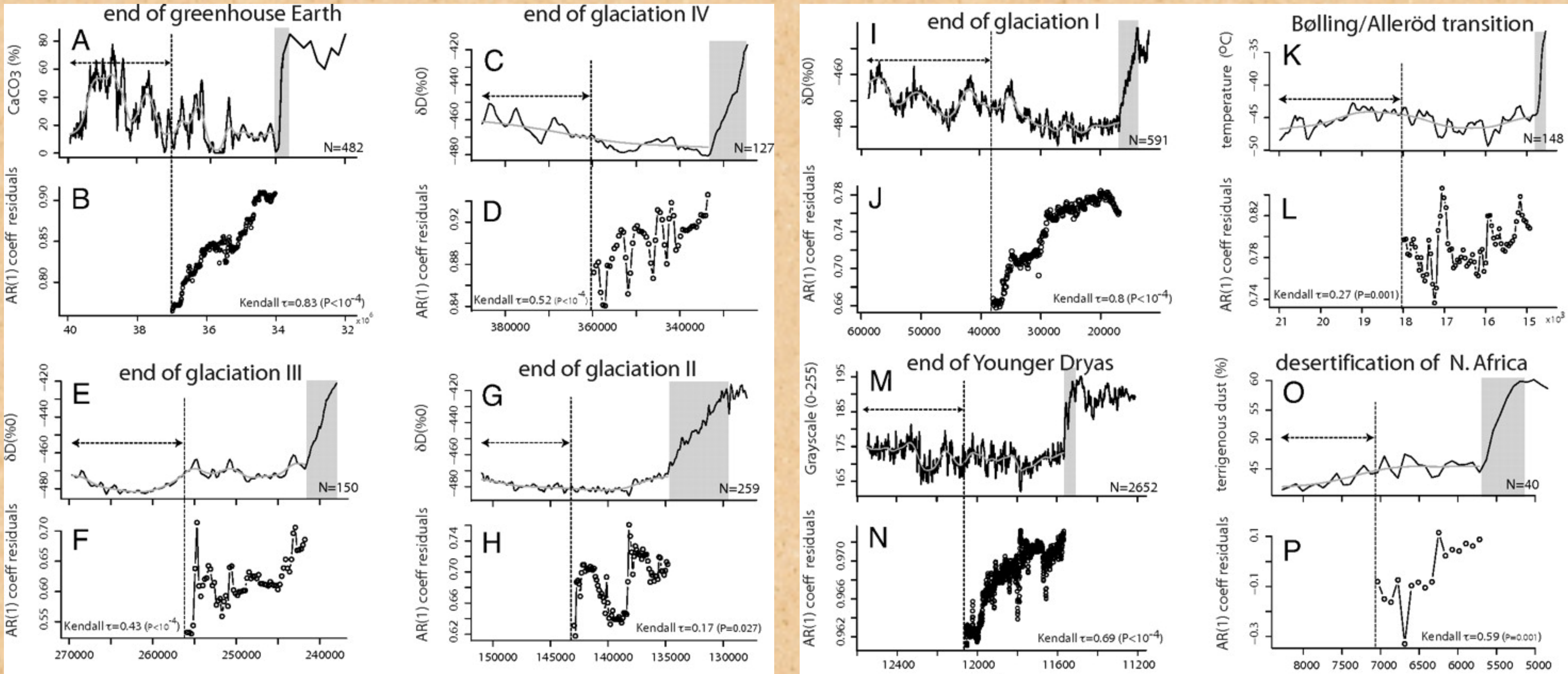
emerging infectious diseases

shifts in brain activity

"**Complex dynamical systems**, ranging from ecosystems to financial markets and the climate, can have **tipping points**...

Although predicting such critical points before they are reached is extremely difficult, work... is now suggesting the existence of generic **early-warning signals** that may indicate for a wide class of systems if a critical threshold is approaching."

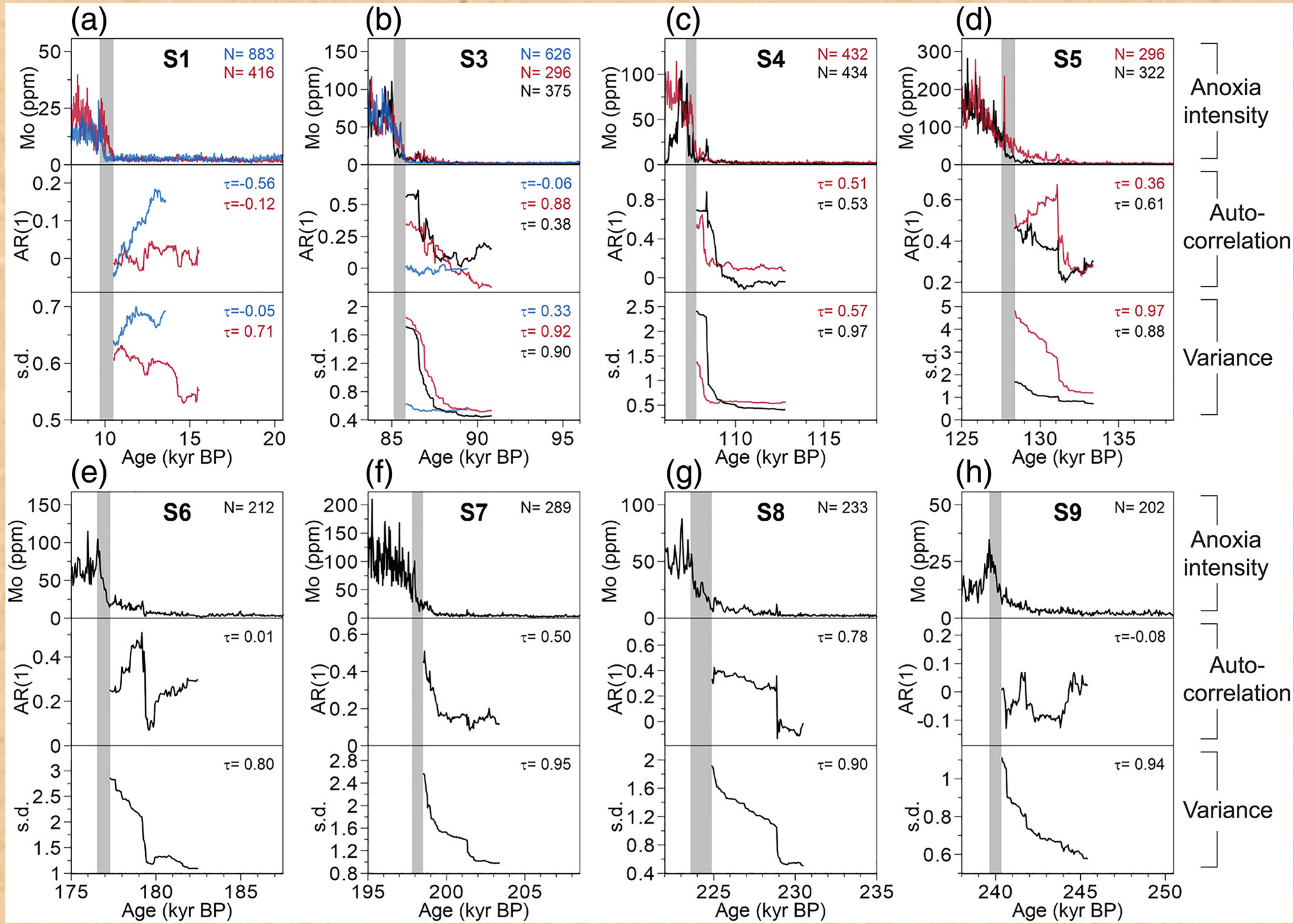
Scheffer, M., Bascompte, J., Brock, W. et al. Early-warning signals for critical transitions. Nature 461, 53–59 (2009).



Autocorrelation increased prior to 8 prehistoric climate change events.

(Data from various sediment cores and ice cores.)

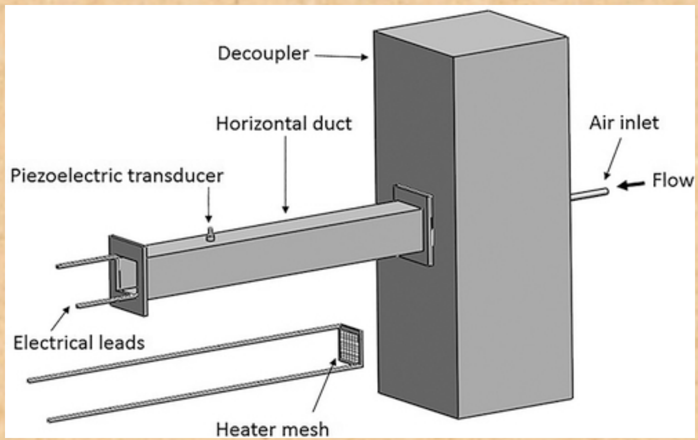
(Dakos et. al. 2008)

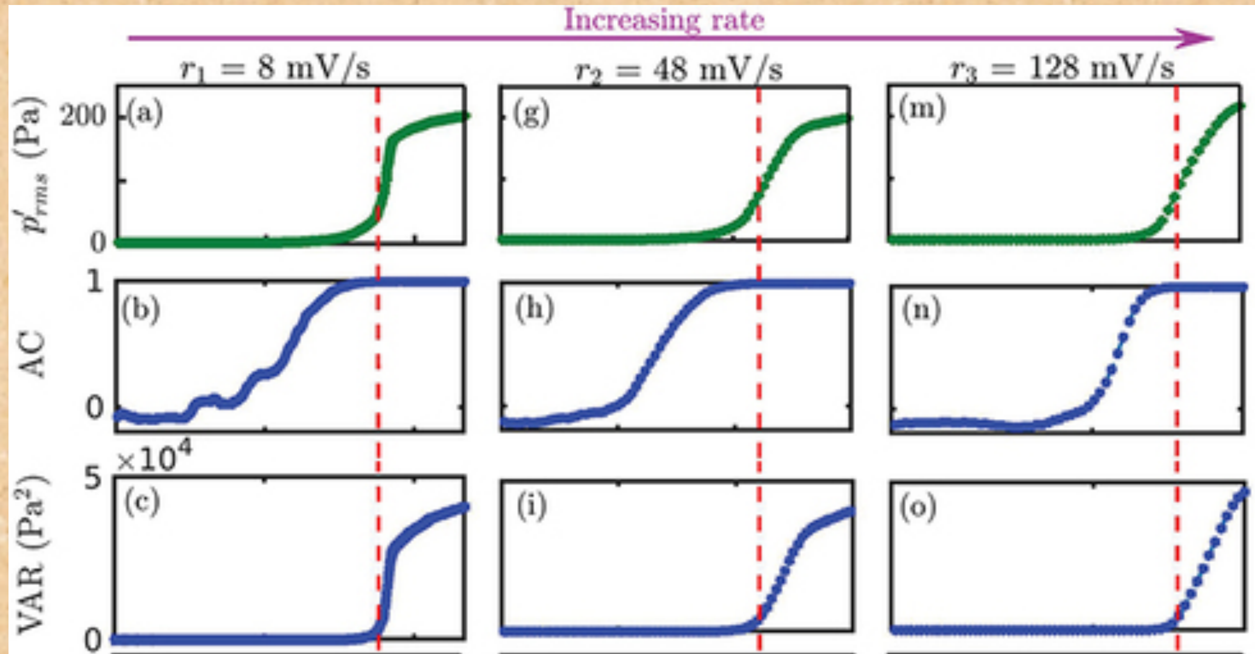


Autocorrelation and variance typically increased prior to marine anoxic transitions in the Mediterranean Sea.

(Colors for 3 sediment cores. Time is measured before present.)

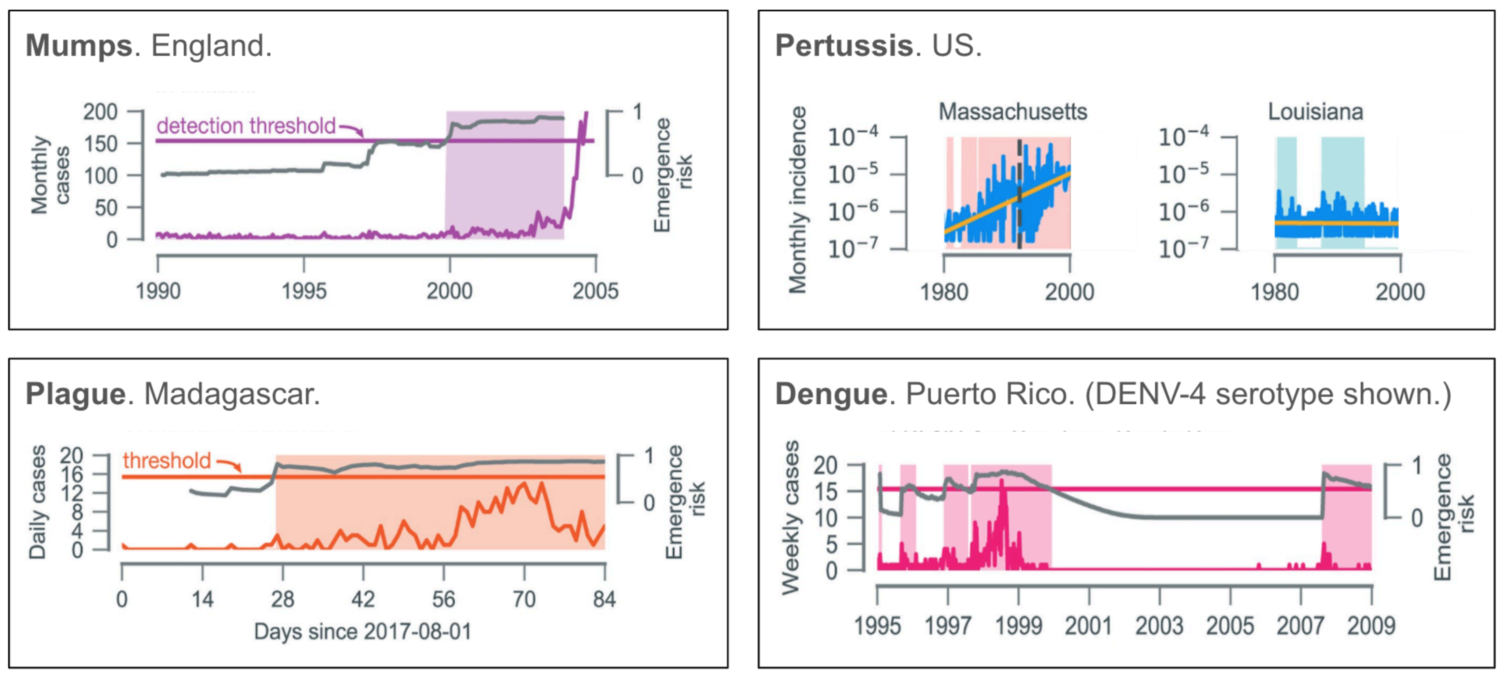
(Hennekam et. al. 2020.)





Autocorrelation increases prior to transition to thermoacoustic instability in a prototypical laboratory set-up (a horizontal Rijke tube).

(Pavithran and Sujith. 2021.)



Patterns in time series statistics, including autocorrelation & variance, allowed a simple machine learning algorithm to anticipate disease outbreaks.

(Brett and Rohani. 2020.)

Critical Slowing Down

&

Early Warning Indicators

local bifurcation

Roughly: As the system approaches a tipping point, its resilience drops, and this leads to detectable statistical signals.

asymptotic resilience

increased variance and autocorrelation

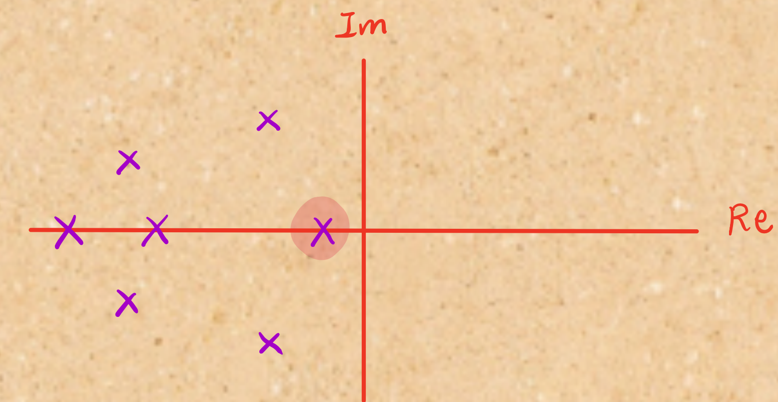
Asymptotic Resilience

$$x' = f(x)$$

$f : U \subset \mathbb{R}^n \to \mathbb{R}^n$
$f \in C^1$

Assume x_* is an attracting, hyperbolic equilibrium, and let $A = Df(x_*)$.

spec(A)

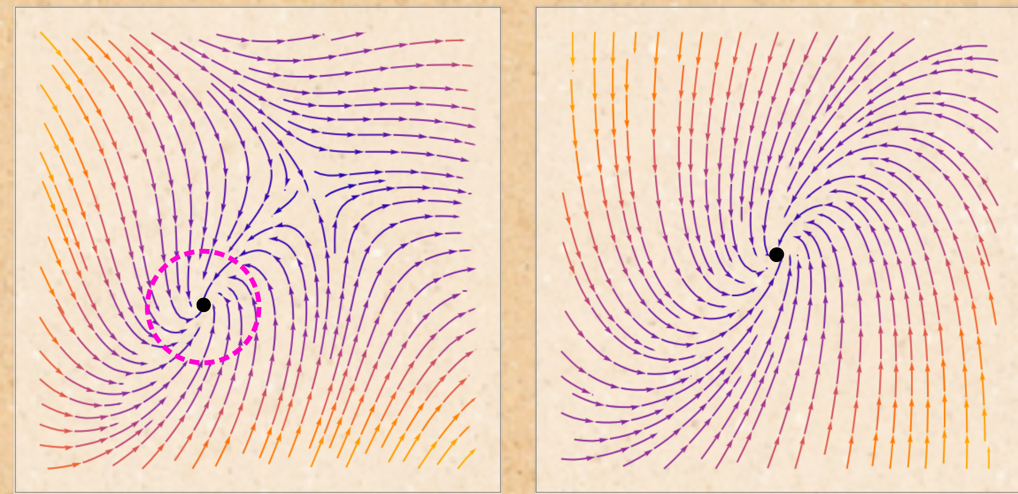


Definition: The asymptotic resilience of x_* is:

$$-\mathrm{Re}(\lambda_1)$$

where λ_1 is an eigenvalue of A with maximal real part.

Asymptotic Resilience measures the long-term return rate to equilibrium after a small perturbation.



Linear ODE: Long term decay estimate: $\|\varphi(t, x_0)\| \leq e^{-\lambda t} \|x_0\|$ for all x_0, for any $0 < \lambda < -\mathrm{Re}(\lambda_1)$ with t sufficiently large.

Nonlinear ODE: Stable Manifold Theorem gives same estimate, but only for x_0 in a local neighborhood $V \subset \mathbb{R}^n$ containing x_*.

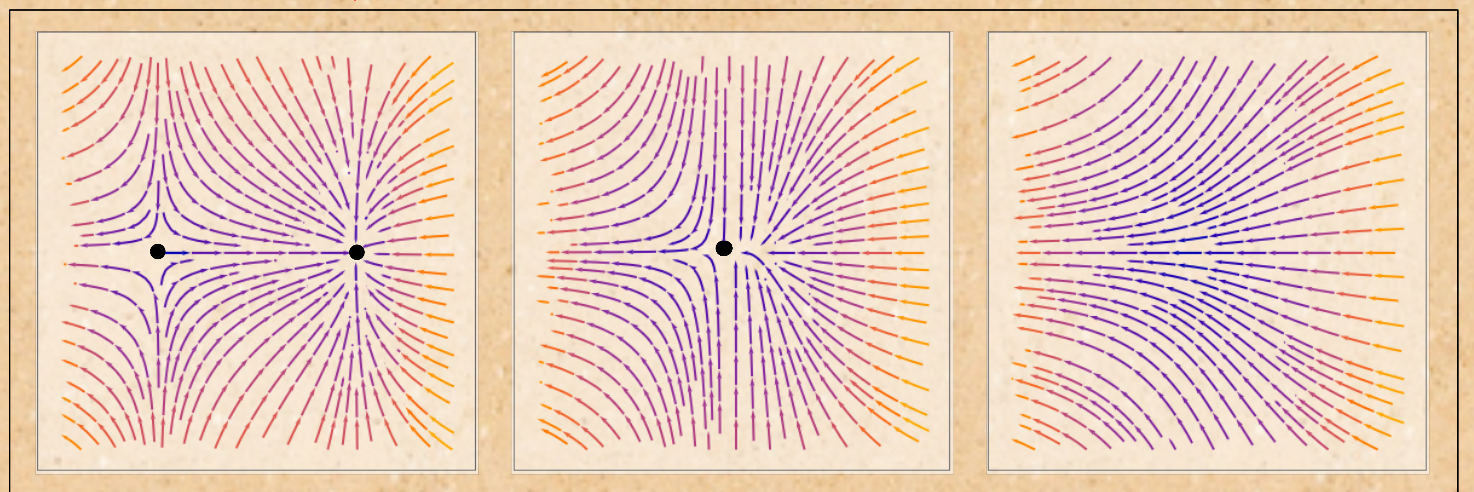
Local One-Parameter Bifurcation

Parameterized family of ODEs: $x' = f(x,p)$ $\quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $f \in C^1$

Assume there is an equilibrium $f(x_*, p_*) = 0$. Let $A = D_x f(x_*, p_*)$.

Fact: a sudden change to local topology near x_* when p_* is adjusted slightly is only possible if x_* is non-hyperbolic. (e.g. there could be a change to / loss of stability, or the equilibrium could disappear.)

Example Saddle-node bifurcation.



Local bifurcation is characterized by the real part of the dominant eigenvalue approaching and then passing through 0.

$\Rightarrow$ asymptotic resilience decreases prior to bifurcation.

$\Rightarrow$ long term recovery rate decreases prior to bifurcation.

Why might this lead to increased autocorrelation & variance in empirical time series data leading up to the critical transition?

Early Warning Indicators

Assume there is a repeated disturbance (additive noise on state variable) after each period Δt.

Assume that the return to equilibrium in between is precisely exponential with rate $\lambda = \mathrm{Re}(\lambda_1)$.

This can be represented by an auto-regressive model:

$$\boxed{\underbrace{x_{n+1} - x_*}_{y_{n+1}} = \underbrace{e^{\lambda \Delta t}}_{:=\alpha} \underbrace{(x_n - x_*)}_{y_n} + \sigma \underbrace{\varepsilon_n}}$$

σ: standard deviation

ε_n: random number from standard normal distribution

$$\Rightarrow y_{n+1} = \alpha y_n + \sigma \varepsilon_n$$

$$\boxed{\begin{gathered} y_{n+1} = \alpha y_n + \sigma \varepsilon_n \\ \alpha = e^{\lambda \Delta t} \\ \lambda = \mathrm{Re}(\lambda_1) \end{gathered}}$$

$\alpha \to 1$ as $\lambda \to 0^-$

variance:

$$\mathrm{Var}[y_{n+1}] = \mathrm{Var}[\alpha y_n] + \mathrm{Var}[\sigma \varepsilon_n] = \alpha^2 \mathrm{Var}[y_n] + \sigma^2$$

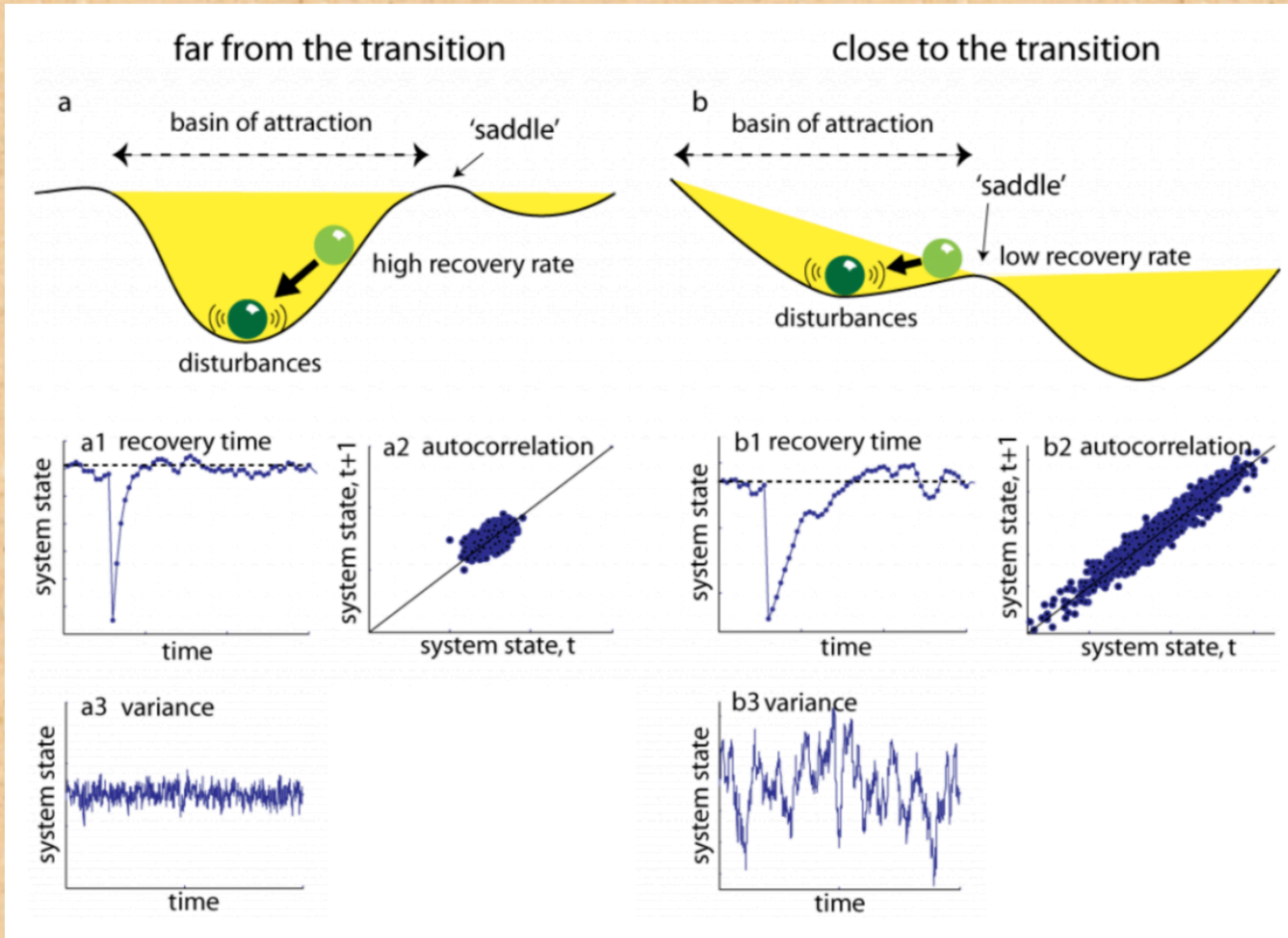
$$\Rightarrow \mathrm{Var}[y_n] = \frac{\sigma^2}{1-\alpha^2}$$ $\longrightarrow \infty$ as $\lambda \to 0^-$

autocorrelation

$$\mathrm{Corr}[y_{n+1}, y_n] = \frac{\mathrm{Cov}[y_{n+1}, y_n]}{\mathrm{Var}[y_n]} = \alpha$$ $\longrightarrow 1$ as $\lambda \to 0^-$

since
$$\begin{aligned} \mathrm{Cov}[y_{n+1}, y_n] = E[y_{n+1} \cdot y_n] &= E[(\alpha y_n + \sigma \varepsilon_n) y_n] \\ &= E[\alpha y_n^2] + 0 \\ &= \alpha \mathrm{Var}[y_n] \end{aligned}$$

$\Rightarrow$ variance & autocorrelation are larger for λ closer to 0.



(Image: early-warning-signals.org)

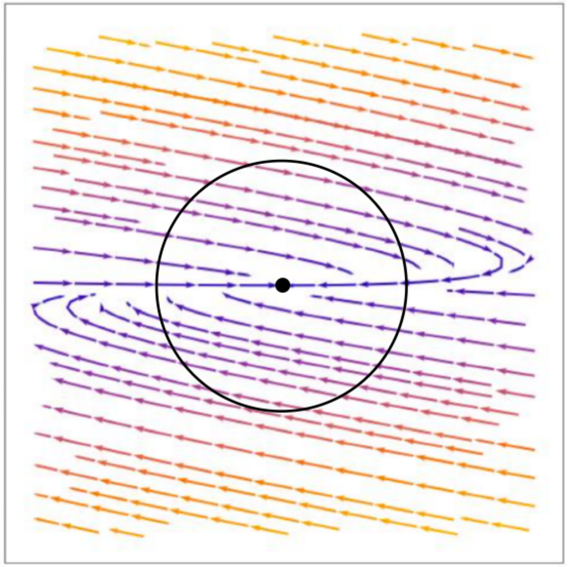
Intuitive idea: slower recovery implies that

1) state stays away from mean longer.
2) current state is more similar to previous state.

- Drawback 1: relies on <u>long-term</u> approximation.

$$A = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}$$

$$x' = Ax$$



"Reactivity"

Assuming precise exponential return is unrealistic.

(Neubert and Caswell. 1997.)

- Drawback 2: relies on <u>local</u> approximation.

- However, in reality, perturbations may be large & drive state far away from equilibrium.

ECOLOGICAL APPLICATIONS
ECOLOGICAL SOCIETY OF AMERICA

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Regime shift indicators fail under noise levels commonly observed in ecological systems

Charles T. Perretti, Stephan B. Munch

First published: 01 September 2012 | https://doi.org/10.1890/11-0161.1 | Citations: 41

There is a need for better understanding of how transient dynamics affect the presence and behavior of indicators prior to critical transitions

short-term dynamics

dynamics far away from equilibrium

want to study a different measure of resilience.

Intensity of Attraction

Intensity of Attraction

- Introduced by McGehee for discrete maps (1988).
- Extended to continuous vectorfields by Meyer (2019).

- Measures resilience for any attractor, not just rest points.
- Takes into account the wider basin of attraction, not just a local neighborhood.
- Relies on metric information, not just a topological (i.e. linear) approximation.
- Formalizes the idea of perturbation differently, to allow for possibly large perturbations.

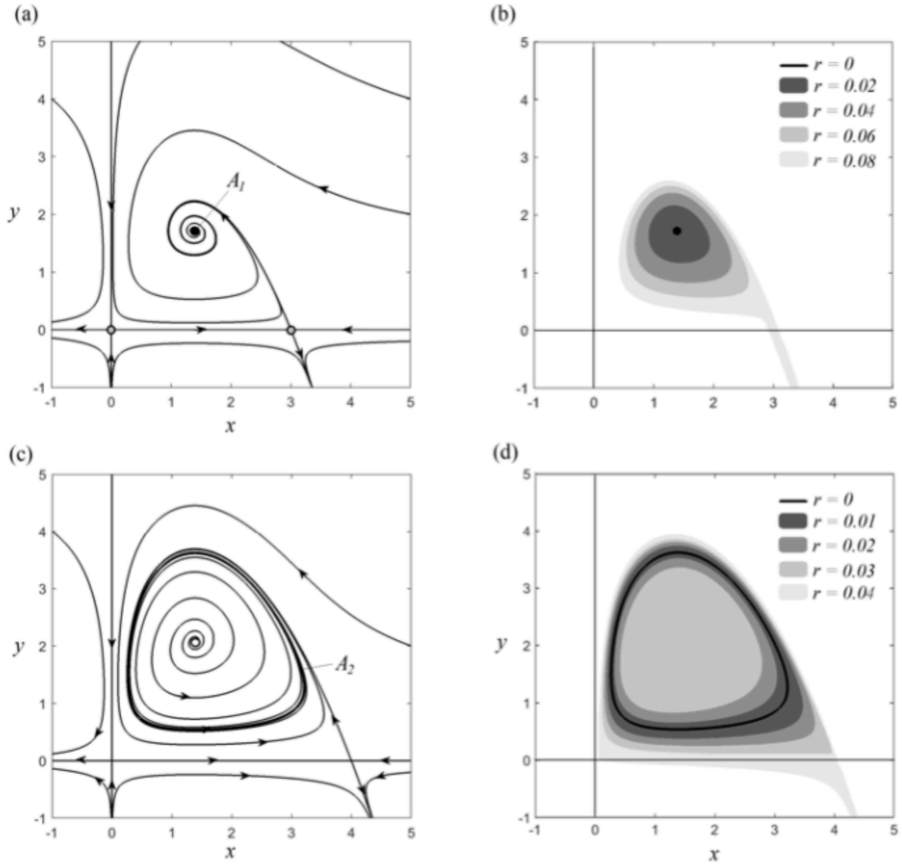
Perturbation
- a function $g(t)$.
- may be discontinuous.
- added to the vectorfield.

represents perturbations like environmental forces or human pressure.

K=3

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{K}\right) - ky(1 - e^{-cx})$$
$$\frac{dy}{dt} = -by + \beta y(1 - e^{-fx}).$$

$a = 1, k = 0.5,\ c = 1.5,\ b = 0.5,\ \beta = 1,\ f = 0.5$



K=4

Example Two predator-prey systems.

(Meyer. 2019.)

Attractors

Defn A set S is *invariant* under the flow φ, if $\varphi_t(S) \subset S$ for all $t \in \mathbb{R}$.

Defn The *omega limit set* of a set N is

$$\omega(N) = \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \varphi_t(N)}$$

Defn An *attractor* A is a compact, non-empty, invariant set which is the omega limit set $\omega(N)$ of some neighborhood $N \supset A$ of itself.

Defn The *basin of attraction* of an attractor A is

$$\{x \mid \omega(x) \subset A \text{ and } \omega(x) \neq 0\}.$$

Intensity of Attraction

Bounded control system:

$$x' = f(x) + g(t)$$

$f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, Lipschitz
$g \in L^\infty(I \subset \mathbb{R}, \mathbb{R}^n)$

L^∞: space of essentially bounded (i.e. bounded except on a set of measure 0) measurable functions, where the norm is: $\|g\|_\infty = \inf\{C \geq 0 : \|g(x)\| \leq C \text{ for almost all } x \in I\}$

Define: $B_r := \{g : \|g\|_\infty < r\}$ set of r-bounded control functions

$$R_r(S) := \bigcup_{g \in B_r} \bigcup_{x_0 \in S} \bigcup_{t \geq 0} \varphi_g(t, x_0)$$ reachable set under r-bounded control.

The intensity of attraction of an attractor A is:

$$\sup\{r \geq 0 \mid R_r(A) \subset K \subset \text{basin}(A) \text{ for some compact } K\}$$

K=3

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{K}\right) - ky(1 - e^{-cx})$$
$$\frac{dy}{dt} = -by + \beta y(1 - e^{-fx}).$$

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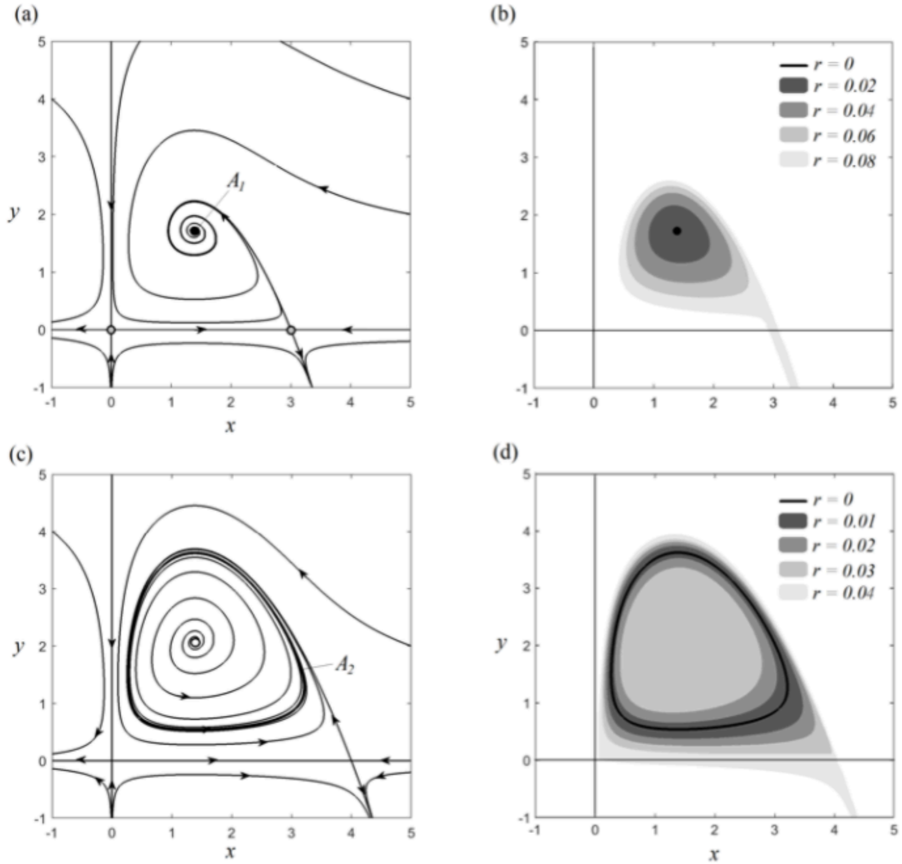

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Example Two predator-prey systems.

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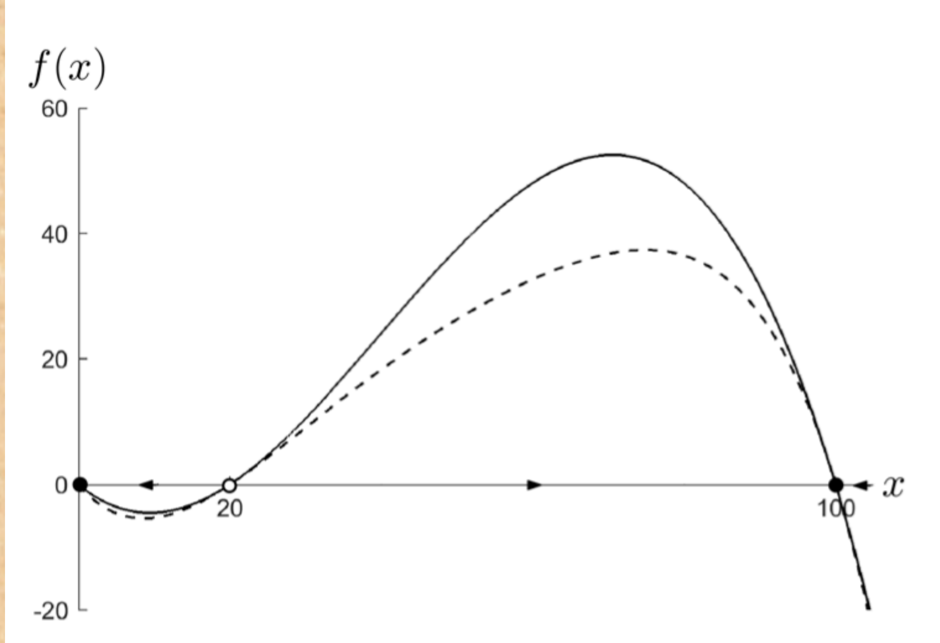


$$\frac{dx}{dt} = x\left(1 - \frac{x}{100}\right)\left(\frac{x}{20} - 1\right)$$ (solid)

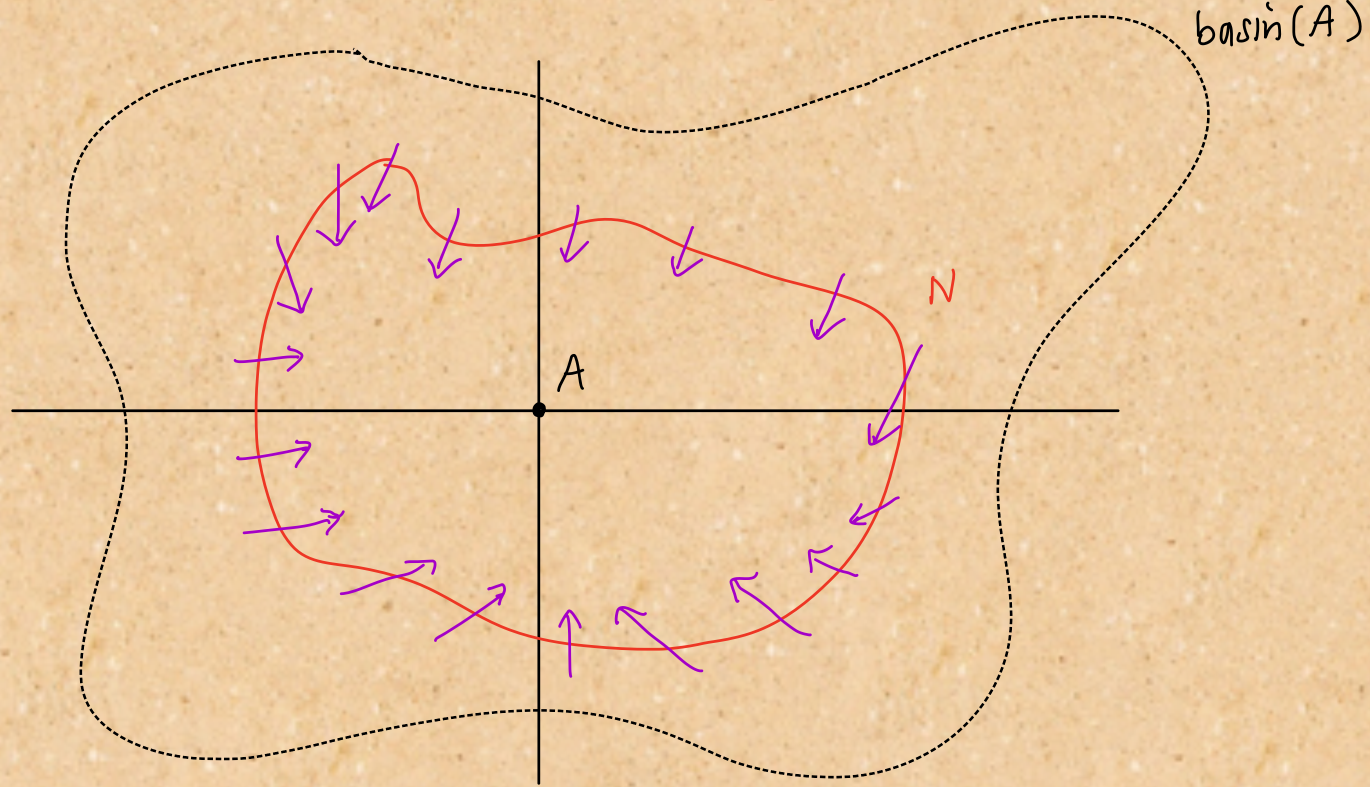
$$\frac{dx}{dt} = x\left(1 - \frac{x}{100}\right)\left(\frac{x}{20} - 1\right)(0.0002x^2 - 0.024x + 1.4)$$ (dashed)

Example Two vector fields with the same equilibria. At $x=100$, the asymptotic resilience is identical, but the intensity of attraction is different.

(Meyer. 2019.)

Basin Steepness

Conjecture. If there is a neighborhood N of the attractor whose closure is within its basin of attraction, such that the inward normal component of the vector field at every point on the boundary of N is at least k, then the intensity of the attractor is at least k.



Future Prospects

1) Relate intensity to local bifurcations.

2) Relate intensity to tipping behavior due to forces pushing state variables across the boundary between alternative basins of attraction.

3) Answer assorted theoretical questions about intensity.

Intensity Through Local Bifurcations

- How does intensity behave when passing through local bifurcations?

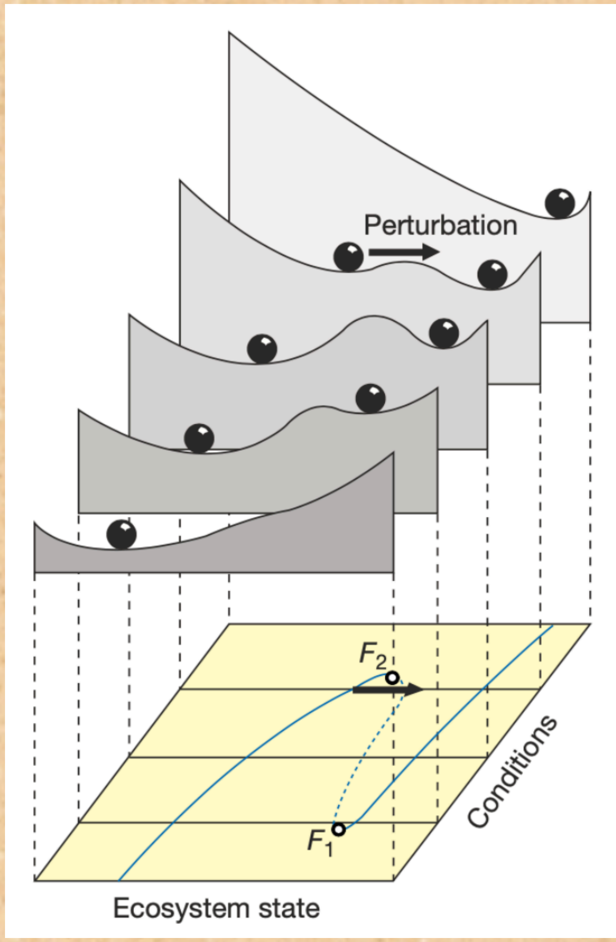
- Does it display any systematic change, and under what conditions?

- If there is a systematic change, does it lead to any expected data signatures?

(Data from examples in beginning of presentation are publicly available.)

Tipping Across Basin Boundaries

- How does intensity measure the reversibility of hysteretic transitions?



(Image: Scheffer et. al. 2001)

Open Theoretical Questions

- Proof of conjecture about basin steepness. Also provides a possibly useful way to estimate a bound on intensity without numerically simulating reachable sets.

- Critical reachable set:

  What is the smallest set $A \subset N \subset \text{basin}(A)$ so that if $N \subset R_r(A)$ then $\text{intensity}(A) \leq r$?

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Thank You.

"Critical slowing down, as an idea, can be traced back at least as far as the 1950s, when physicists theorized that it would arise… near a phase change. But… [its] potential usefulness went unrecognized until a boozy conversation in 2003 at a restaurant-bar… where [limnologist Stephen Carpenter] and several colleagues had gathered for a conference.

Crawford "Buzz" Holling, an eminent Canadian theoretical ecologist, had begun reminiscing about a celebrated explanation of [spruce budworm] outbreaks that he and two collaborators had developed… but there was one aspect of the model that Holling said he had never understood: Before an outbreak… the insect population would start to vary more and more erratically from one place to another across the forest.

Sitting across the table was William "Buz" Brock, a mathematical economist specializing in dynamical systems at Madison. Brock knew right away why the variance in the insect population had increased near the brink of an outbreak. He whipped out a yellow legal pad, and, over a couple of bottles of wine, explained critical slowing down to his ecologist companions.

Carpenter said he realized "immediately" that the phenomenon could serve as an ecological warning signal. It turned out the German ecologist Christian Wissel had made the same point 20 years earlier, but hardly anyone had noticed."

- Quanta Magazine. "Nature's Critical Warning System." 2015.