Transient Dynamical Indicators of Critical Transitions

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Contents

1 Introduction

A **tipping point** or **critical transition** occurs in a dynamical system when a small perturbation to system conditions causes an abrupt overall shift in qualitative behavior. Empirically, tipping points have been studied in contexts as diverse as Earth's climate [1, 2], emerging infectious diseases [3], aquatic and land ecosystems [4, 5], the onset of medical health states [6, 7], socio-economic systems [8], and more [9–11]. Since critical transitions often represent a shift into an undesirable or catastrophic regime, and since such transitions may not be easily or at all reversible [12–14], it is of pressing interest to anticipate them before they occur, in order to inform management strategies and possibly improve the odds of prevention. Unfortunately, in complex real world systems, the conditions under which a critical transition occurs, and the underlying mechanisms driving the approach to transition are usually extremely difficult to characterize.

As a result, there is particular interest in generic mathematical signals that can warn of impending tipping in a wide variety of systems without reference to specific underlying mechanisms. Such **early warning signals** have been most commonly studied as precursors of local codimension-1 bifurcations of ODEs, where they are based on the phenomenon of **critical slowing down** [10]. Roughly speaking, as the bifurcation parameter gradually nears its critical value, the resilience of the system drops (becoming slower to recover from perturbations), and this produces certain detectable statistical trends over time. In the context of critical slowing down, the term resilience refers specifically to what is known in the ecology literature as **asymptotic resilience**. In Section 2, I review asymptotic resilience, and also two other quantitative measures of resilience (**reactivity** and **intensity of attraction**). In Section 3, I summarize the theory of critical slowing down.

Early warning signals derived from asymptotic resilience and critical slowing down are a powerful tool for anticipating critical transitions, and their usefulness has already been demonstrated in numerous empirical contexts, including . But a major limitation is the assumption that the system experiences only small, infrequent perturbations, which do not drive the system state very far from equilibrium and which leave sufficient time for recovery in between disturbances. In particular, there is a neglect of transient behavior within the larger domain of attraction. Such transient states can result from large, closely repeated, or continual disturbances, as are common in real world systems.

citations

Early warning signals derived from certain transient dynamics have been developed recently in the infectious disease literature [15]. In Section 4 I first review transient indicators arising from reactivity. Then, I consider the possibility for other transient indicators to arise from intensity of attraction, an idea further developed in Section 5 into the thesis proposal.

2 Resilience Quantification

The concept of resilience differs between authors and disciplines, and an abundance of quantification approaches have been proposed. Loosely, resilience refers to the capacity for a system to retain its overall qualitative structure in the face of disturbances. In this section, I define asymptotic resilience, reactivity, and intensity of attraction. For some other definitions of resilience that I do not cover, see [16].

Let $U \subset \mathbb{R}^n$ be open, and suppose that $f : U \to \mathbb{R}^n$. Consider a system of ordinary differential equations

$$x' = f(x) \tag{1}$$

and let $\varphi : \mathbb{R} \times U \to U$ be the associated local flow, so that $\varphi(t, x_0) = x(t)$ solves the ODE with initial condition $x(0) = x_0$.

2.1 Asymptotic Resilience

The most commonly used definition of resilience in theoretical ecology represents long-term return rates to a stable point equilibrium, and is measured by the dominant eigenvalue of linearization.

Definition 1. Suppose that x_* is a stable rest point of the ODE (1). That is, $f(x_*) = 0$, and $Re(\lambda) < 0$ for all $\lambda \in spec(\mathbf{A})$, where $\mathbf{A} = Df(x_*)$ is the Jacobian. Let $\lambda_1(\mathbf{A})$ be the eigenvalue with largest (closest to 0) real part.

The **asymptotic resilience** of the system at the stable rest point is $Re(\lambda_1(\mathbf{A}))$.

□

For the linearized system $x' = \mathbf{A}x$, asymptotic resilience governs the exponential rate at which trajectories approach the equilibrium. explain more

For nonlinear systems, the Stable Manifold Theorem implies that for any α such that $Re(\lambda_1) < \alpha < 0$, there exists a constant C and a neighborhood $V \ni x_*$ such that $|\varphi(t, x_0)| \leq Ce^{\alpha t}$ for all $x_0 \in V$. maybe show how to derive this from Stable Manifold Theorem

Hence, asymptotic resilience bounds the rate of return to equilibrium after a small perturbation to the system. Because local bifurcation is characterized by $Re(\lambda_1)$ passing through zero, the system recovers slower when nearer to bifurcation. This is the core idea of critical slowing down, which will be explained further in Section 3.

2.2 Reactivity

Asymptotic resilience governs the long-term rate of recovery. However, in the short term, perturbations can initially be amplified before eventually decaying to the stable equilibrium (Figure 1). Motivated by this transient behavior, an alternative measure of system response to perturbations was introduced by Neubert and Caswell in [17].

(a) $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$ (b) $A_2 = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}$

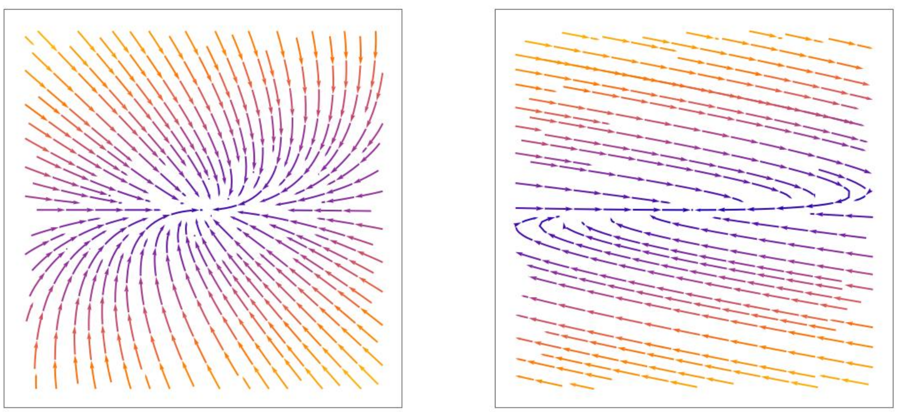

Figure 1: Phase portraits of two linear systems $x' = \mathbf{A}x$ with the same eigenvalues $\lambda = -1, -2$. (a) All trajectories decay monotonically in magnitude. (b) Trajectories may initially increase in magnitude. Example reproduced from [17].

Definition 2. Suppose that x_* is a stable rest point of the ODE (1). Let $\mathbf{A} = Df(x_*)$ be the Jacobian, and let $\mathbf{H} = \frac{\mathbf{A}+\mathbf{A}^T}{2}$ be its symmetric part. Since $\mathbf{H}$ is a real symmetric matrix, it has real eigenvalues. Let $\lambda_1(\mathbf{H})$ be the maximum eigenvalue.

The **reactivity** of the system at the stable rest point is $\lambda_1(\mathbf{H})$.
If this number is positive, the system is called **reactive**.

□

Reactivity measures the maximum possible relative rate of initial amplification. The following proposition, which relies on elementary linear algebra, shows this result for linear systems.

Proposition 3 (Neubert and Caswell)**.** *For the linear system* $x' = \mathbf{A}x$, $\lambda_1(\mathbf{H}) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{||x||'}{||x||}$.

Proof.

$$\begin{aligned} \frac{||x||'}{||x||} &= \frac{1}{||x||}\frac{d}{dt}\left(x^T x\right)^{1/2} \\ &= \frac{1}{2||x||^2}(x^T x' + (x')^T x) \\ &= \frac{1}{2||x||^2}(x^T \mathbf{A} x + x^T \mathbf{A}^T x) \\ &= \frac{x^T \mathbf{H} x}{||x||^2} \end{aligned}$$

This expression is a scale invariant function of x, so to maximize it, we only need to consider unit vectors.

$$\max_{||x||=1} x^T \mathbf{H} x$$

$\mathbf{H}$ is real symmetric, hence diagonalizable with an orthogonal change of basis. Let $\{\lambda_1, \lambda_2, \ldots \lambda_n\} = spec(\mathbf{H})$, in order from largest to smallest.

$$\begin{aligned} x^T \mathbf{H} x &= x^T (\mathbf{B}\mathbf{D}\mathbf{B}^T) x \\ &= (x^T \mathbf{B}) \mathbf{D} (\mathbf{B}^T x) \\ &= y^T \mathbf{D} y, \quad \text{where } y = \mathbf{B}^T x \text{ is also unit.} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2. \end{aligned}$$

The maximum of this expression over all $||y|| = 1$ is clearly λ_1, when $y = (y_1, \ldots, y_n) = (1, 0, \ldots, 0)$.

$\square$

to do: relation between nonlinear and linearized systems

2.3 Intensity of Attraction

The definitions of asymptotic resilience and reactivity both require linearizing at a point attractor. In contrast, intensity of attraction, introduced by Katherine Meyer in her PhD thesis [18], measures resilience for any attractor, and captures metric information across its entire basin of attraction rather than replacing the nonlinear dynamics with a local approximation.

Define attractor

Define control

Define reachable sets

Define intensity of attraction

3 Critical Slowing Down

this results in certain detectable statistical trends over time – in particular, gradually increasing variance and auto-correlation in the system state

3.1 Local Bifurcation

3.2 Critical Slowing Down

3.3 Early Warning Signals

3.4 Limitations

Early warning signals derived from critical slowing down are a powerful tool for anticipating critical transitions, and their usefulness has already been demonstrated in numerous empirical contexts, including . citations However, they have at least a few significant limitations. First, being based on a linear approximation at a stable equilibrium, they are relevant only to small perturbations that do not move the system state very far from equilibrium. Second, being a measure of long term rates of return to equilibrium, they (1) may overlook short term behavior that occurs immediately after the perturbation and (2) are relevant only to infrequent perturbations, so that the system has enough time to recover in between disturbances. In particular, they are not reliable in cases of closely repeating or continual disturbances, as are common in real world ecological systems. Third, they specifically precede local bifurcations, while the informal tipping point concept may correspond to other dynamical behaviors such as global bifurcations, perturbations pushing a state variable across the boundary between two basins of attraction, or rate-induced tipping behavior.

4 Transient Dynamical Indicators of Critical Transitions

4.1 Indicators from Reactivity

4.2 Possibility for Indicators from Intensity of Attraction

5 Thesis Proposal

5.1 Continuity of Intensity of Attraction

5.2 Intensity through Critical Transitions

5.3 Further Possibilities

6 Conclusion

Machine learning based early warning signals? Possible connection between machine-learning based and analytical theory based early warning signals? i.e. using theory to inform ML design.

Mention papers where critical transitions occur with no lead warning.

Mention flickering?

As pressures exerted by modern day anthropogenic practices on the Earth grow in magnitude and complexity, threatening physical, ecological, and social systems on all scales with unprecedented forms of change, this goal becomes even more pressing.

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