

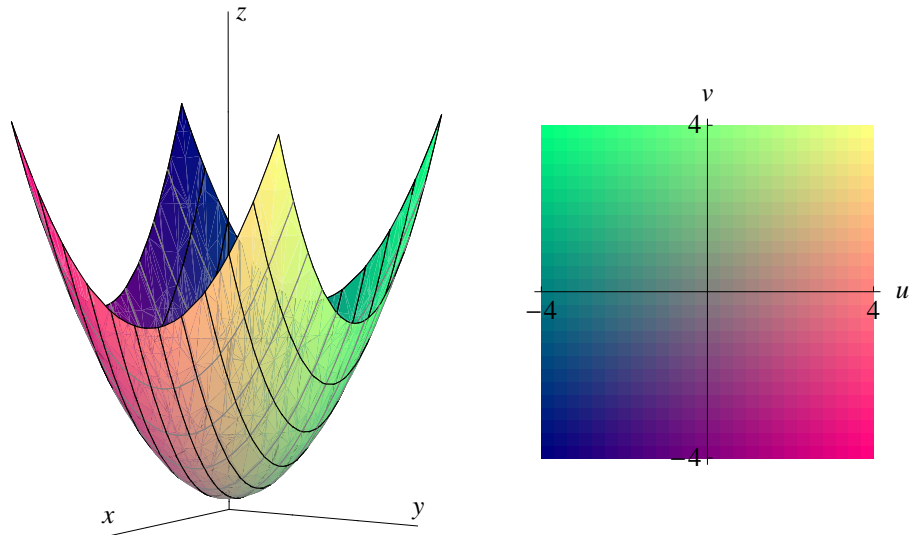
Parametric Surfaces

1. (a) *Parameterize the elliptic paraboloid $z = x^2 + y^2 + 1$. Sketch the grid curves defined by your parameterization.*

Solution. There are several ways to parameterize this. Here are a few.

- i. One way to think of parameterizing is simply that we want to describe the surface using 2 variables. This amounts to describing x , y , and z using just 2 variables. In this case, z is already written in terms of x , y , and z , so we can describe the surface just using x and y . That is, we can use the parameterization $\vec{r}(x, y) = \langle x, y, x^2 + y^2 + 1 \rangle$. We often use the variables u and v as the parameters (just as we usually used t for the parameter when parameterizing curves), so we could also write this as $\vec{r}(u, v) = \langle u, v, u^2 + v^2 + 1 \rangle$. (It is certainly not necessary to use u and v though.)

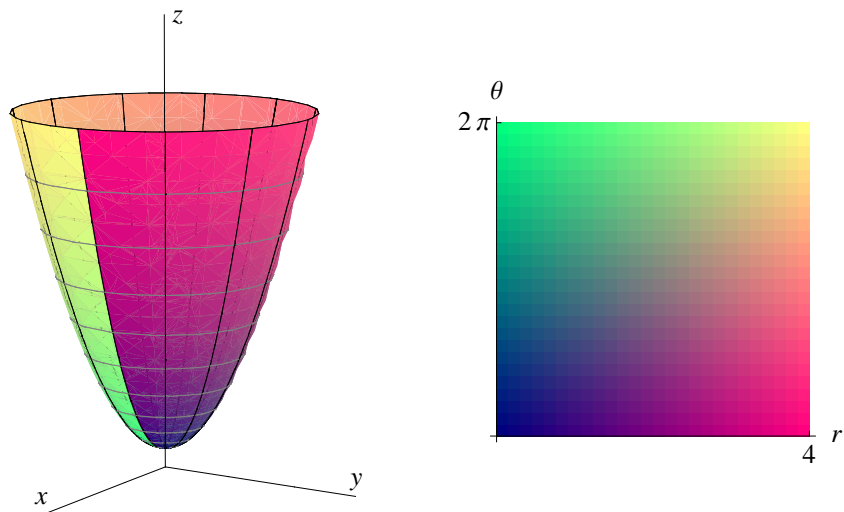
Here is a picture of this parameterization:



The gray curves are where u is constant, and the black curves are where v is constant. The right picture shows a way of coloring the uv -plane; the paraboloid is colored according to the corresponding u and v value at each point. For instance, from the right picture, we see that $u = 4$, $v = -4$ is colored pink. Therefore, the point $\vec{r}(4, -4) = \langle 4, -4, 17 \rangle$ on the paraboloid is colored pink.

- ii. Another possibility is to use cylindrical coordinates to rewrite the surface as $z = r^2 + 1$. Then, every point can be described in terms of r and θ since $z = r^2 + 1$. Converting back to Cartesian coordinates, $x = r \cos \theta$, $y = r \sin \theta$, and $z = r^2 + 1$, so we have the parameterization $\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r^2 + 1 \rangle$.

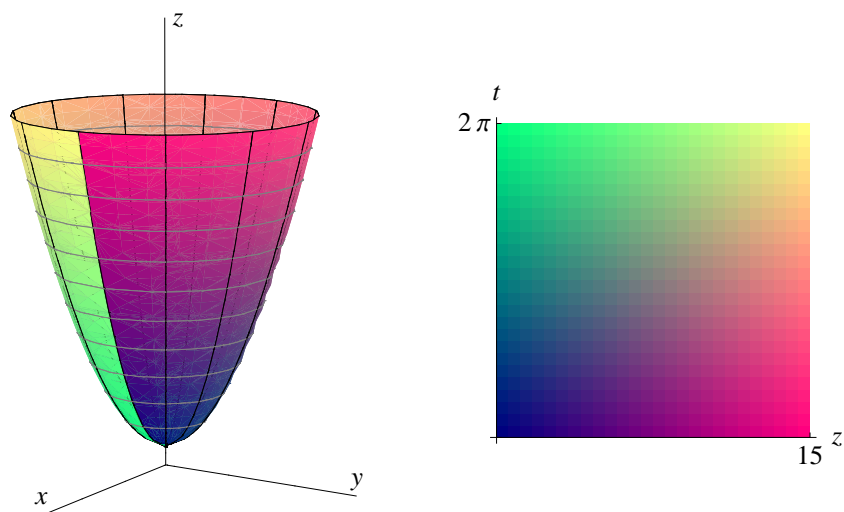
Here is a picture of this parameterization:



The gray curves are where r is constant, and the black curves are where θ is constant.

- iii. Yet another possibility is to think of slicing (or taking cross-sections or traces). This approach is slightly more difficult than the previous one, but it's also more flexible. Looking at the surface, we know that taking traces in $z = k$ gives us circles, and circles are curves that we know how to parameterize. If we imagine slicing at a particular z -value, then the slice is the circle $x^2 + y^2 = z - 1$, which is a circle centered at $(x, y) = (0, 0)$ with radius $\sqrt{z - 1}$. Therefore, we know that x and y can be described by $x = \sqrt{z - 1} \cos t$, $y = \sqrt{z - 1} \sin t$. This gives the parameterization $\vec{r}(z, t) = \langle \sqrt{z - 1} \cos t, \sqrt{z - 1} \sin t, z \rangle$.

Here is a picture of this parameterization:



The gray curves are where z is constant, and the black curves are where t is constant.

- (b) If we only want to parameterize the part of the elliptic paraboloid under the plane $z = 10$, what restrictions would you place on the parameters you used in (a)?

Solution.

- i. For the parameterization $\vec{r}(u, v) = \langle u, v, u^2 + v^2 + 1 \rangle$, we need to restrict u and v . Since we want z (the last component) to be less than 10, we need $u^2 + v^2 < 9$.
- ii. For the parameterization $\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r^2 + 1 \rangle$, we need to restrict r and θ . Since the paraboloid was written as $z = r^2 + 1$ in cylindrical coordinates and we want $z < 10$, we need $r < 3$. We know that θ can be anything, so our restrictions are $0 \leq r < 3, 0 \leq \theta < 2\pi$.
- iii. For the parameterization $\vec{r}(z, t) = \langle \sqrt{z-1} \cos t, \sqrt{z-1} \sin t, z \rangle$, we need to restrict z and t . We already know that we want $z < 10$. Looking at the paraboloid, we also want $z \geq 1$.⁽¹⁾ Looking back, we used t to parameterize a circle, and the parameterization we chose means $0 \leq t < 2\pi$ is a good restriction. So, for this parameterization, we have $1 \leq z < 10, 0 \leq t < 2\pi$.

2. Parameterize the plane that contains the 3 points $P(1, 0, 1)$, $Q(2, -2, 2)$, and $R(3, 2, 4)$.

Solution. One way to parameterize the plane is to let the vectors \overrightarrow{PQ} and \overrightarrow{PR} define our grid. We can think of P as an “origin” for the plane and the vectors \overrightarrow{PQ} and \overrightarrow{PR} as a set of “axes” for the plane. That is, we can reach any point in the plane by starting at P , going in the direction of \overrightarrow{PQ} for a while, and then going in the direction of \overrightarrow{PR} for a while.

In this case, $\overrightarrow{PQ} = \langle 1, -2, 1 \rangle$ and $\overrightarrow{PR} = \langle 2, 2, 3 \rangle$, so our parameterization is

$$\vec{r}(u, v) = \langle 1, 0, 1 \rangle + u\langle 1, -2, 1 \rangle + v\langle 2, 2, 3 \rangle = \langle 1 + u + 2v, -2u + 2v, 1 + u + 3v \rangle$$

(You can think of this as saying: start at $P(1, 0, 1)$, go off in the direction of $\overrightarrow{PQ} = \langle 1, -2, 1 \rangle$ for a bit — how long is determined by u — and then go off in the direction of $\overrightarrow{PR} = \langle 2, 2, 3 \rangle$ for a bit.)

Alternatively, you could find the equation of the plane (see the worksheet “Lines and Planes”) — it is $8x + y - 6z = 2$. Then, we can write any one of the variables in terms of the other two and use those other two as parameters. For instance, $y = 2 - 8x + 6z$ expresses y in terms of x and z . If we want to use u and v as our parameters, then we can just have $x = u$, $z = v$, and $y = 2 - 8u + 6v$, which gives the parameterization $\vec{r}(u, v) = \langle u, 2 - 8u + 6v, v \rangle$.

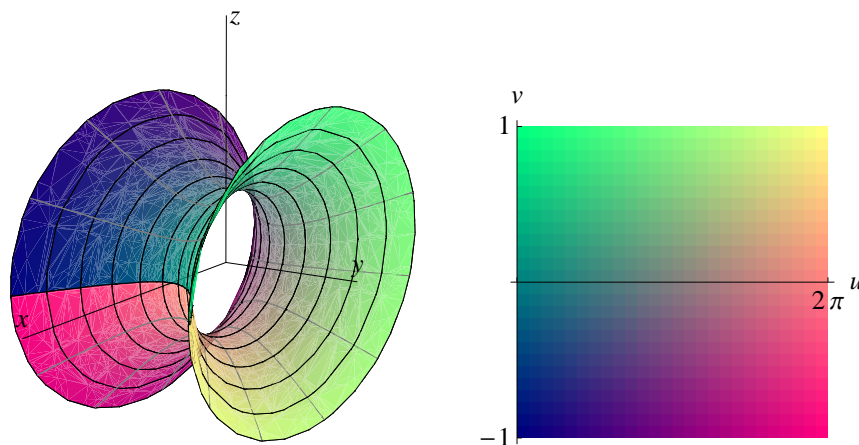
Of course, there are many other parameterizations. One way to check whether your parameterization is reasonable is to remember that you are supposed to be parameterizing $8x + y - 6z = 2$. So, however you parameterize, this relationship should be satisfied. For instance, in our first parameterization $\vec{r}(u, v) = \langle 1 + u + 2v, -2u + 2v, 1 + u + 3v \rangle$, you can easily check that $8(1 + u + 2v) + (-2u + 2v) - 6(1 + u + 3v) = 2$.

3. Parameterize the hyperboloid $x^2 - 4y^2 + z^2 = 1$.

Solution. The traces in $y = k$ of this surface will be circles. In particular, since $x^2 + z^2 = 1 + 4y^2$, the trace in $y = k$ will be a circle centered at $(x, z) = (0, 0)$ with radius $\sqrt{1 + 4y^2}$. We can parameterize this by taking $x = \sqrt{1 + 4y^2} \cos u$, $z = \sqrt{1 + 4y^2} \sin u$ with $0 \leq u < 2\pi$. Our other parameter is just y ; if we rename it v , then we have the parameterization $\vec{r}(u, v) = \langle \sqrt{1 + 4v^2} \cos u, v, \sqrt{1 + 4v^2} \sin u \rangle$ with $0 \leq u < 2\pi$ (v can be anything).

⁽¹⁾Notice that this is implied in our parameterization since $\sqrt{z-1}$ is not defined if $z < 1$.

Here is a picture of the parameterization.

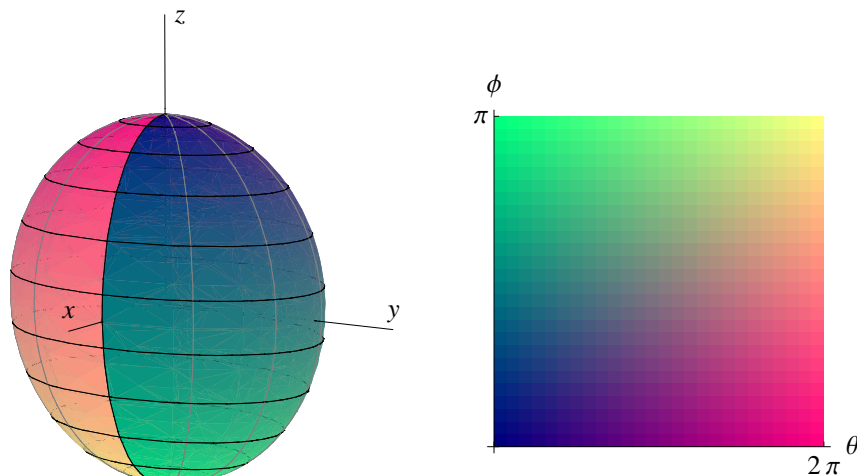


The gray curves are where u is constant, and the black curves are where v is constant.

4. Parameterize the ellipsoid $9x^2 + 4y^2 + z^2 = 36$.

Solution. There is not much work to do here if we take a clever approach. Let's rewrite the given equation as $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{6}\right)^2 = 1$. We know that $X^2 + Y^2 + Z^2 = 1$ can be parameterized as $X = \sin \phi \cos \theta$, $Y = \sin \phi \sin \theta$, and $Z = \cos \phi$, and if we think of X as being $\frac{x}{2}$, Y as being $\frac{y}{3}$, and Z as being $\frac{z}{6}$, then we get $x = 2 \sin \phi \cos \theta$, $y = 3 \sin \phi \sin \theta$, $z = 6 \cos \phi$. That is, our parameterization is $\boxed{\vec{r}(\theta, \phi) = \langle 2 \sin \phi \cos \theta, 3 \sin \phi \sin \theta, 6 \cos \phi \rangle}$ with $\boxed{0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi}$.

Here is a picture of the parameterization.

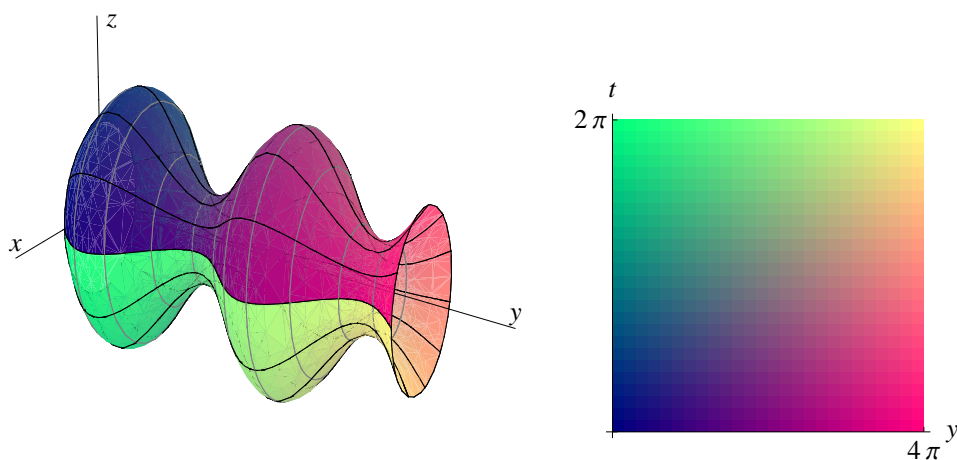


The gray curves are where θ is constant, and the black curves are where ϕ is constant.

5. Consider the curve $z = 2 + \sin y$, $0 \leq y \leq 4\pi$ in the yz -plane. Let S be the surface obtained by rotating this curve about the y -axis. Find a parameterization of S .

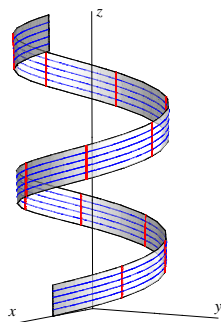
Solution. The traces in $y = k$ of this surface will be circles. In particular, if we look at the trace in $y = k$, we see a circle centered at $(x, z) = (0, 0)$ with radius $2 + \sin y$. We can parameterize this by taking $x = (2 + \sin y) \cos t$, $z = (2 + \sin y) \sin t$ with $0 \leq t < 2\pi$. Thus, a parameterization of the surface is $\vec{r}(y, t) = \langle (2 + \sin y) \cos t, y, (2 + \sin y) \sin t \rangle$ with $0 \leq y \leq 4\pi$, $0 \leq t < 2\pi$.

Here is a picture of the parameterization.

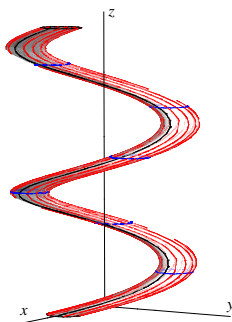


The gray curves are where y is constant, and the black curves are where t is constant.

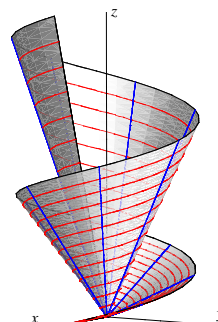
6. Here are three surfaces.



(I)



(II)



(III)

Match each function with the surface it parameterizes. Which curves are where u is constant and which curves are where v is constant?

(a) $\vec{r}(u, v) = \left\langle \frac{\cos u}{4} + \cos v, \frac{\sin u}{4} + \sin v, v \right\rangle, 0 \leq u \leq 2\pi, 0 \leq v \leq 4\pi.$

Solution. If u is a constant, then $\vec{r}(u, v)$ has the form $\langle C_1 + \cos v, C_2 + \sin v, v \rangle$ where C_1 and C_2 are constants. You should recognize this as a helix (remember $\langle \cos t, \sin t, t \rangle$), shifted.

On the other hand, if v is a constant, then $\vec{r}(u, v)$ has the form $\langle \frac{1}{4} \cos u + C_1, \frac{1}{4} \sin u + C_2, C_3 \rangle$ where C_1, C_2 , and C_3 are constants. You should recognize this as parameterizing a circle (which is parallel to the xy -plane since the z -component does not vary with u).

The surface which has helices as one set of grid curves and circles as the other is (II). The grid curves with u constant are shown in red; the grid curves with v constant are shown in blue.

(b) $\vec{r}(u, v) = \left\langle \cos u, \sin u, u + \frac{v}{4} \right\rangle, 0 \leq u \leq 4\pi, 0 \leq v \leq 2\pi.$

Solution. If u is a constant, then $\vec{r}(u, v)$ has the form $\langle C_1, C_2, C_3 + \frac{v}{4} \rangle$ where C_1, C_2 , and C_3 are constants. This simply parameterizes a vertical line segment (of length $\frac{\pi}{2}$ since v varies between 0 and 2π).

On the other hand, if v is a constant, then $\vec{r}(u, v)$ has the form $\langle \cos u, \sin u, u + C \rangle$ where C is a constant. This parameterizes a helix.

The surface which has vertical line segments as one set of grid curves and helices as the other is (I). The grid curves with u constant are shown in red; the grid curves with v constant are shown in blue.

(c) $\vec{r}(u, v) = \langle u \cos v, u \sin v, uv \rangle, 0 \leq u \leq 2\pi, 0 \leq v \leq 4\pi.$

Solution. If u is a constant, then $\vec{r}(u, v)$ has the form $\langle C \cos v, C \sin v, Cv \rangle = C \langle \cos v, \sin v, v \rangle$ where C is a constant. This parameterizes a helix. Notice that, unlike in the two previous parts, the helices here can be different sizes.

On the other hand, if v is a constant, then $\vec{r}(u, v)$ has the form $\langle C_1 u, C_2 u, C_3 u \rangle$ where C_1, C_2 , and C_3 are constants. You should recognize this as parameterizing a line segment; since u starts at 0, this line segment always starts at the origin.

The matching surface is (III). The grid curves with u constant are shown in red; the grid curves

with v constant are shown in blue.