

# Ecological Complexity and Feedback Control in a Prey–Predator System with Holling Type III Functional Response

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*This article deals with a bioeconomic model of prey–predator system with Holling type III functional response. The dynamical behavior of the system is extensively discussed. Continuous type gestational delay of predators is incorporated in the system to study delay induced instability. It is observed that the system undergoes singularity induced bifurcation at interior equilibrium point when net economic revenue of the system increases through zero. State feedback controller is designed to stabilize the system at positive economic profit. Time delay is considered as a bifurcation parameter to prove the occurrence of Hopf bifurcation phenomenon in the neighborhood of the coexisting equilibrium point. Finally, some numerical simulations are carried out to verify the analytical results and the system is analyzed through graphical illustrations. © 2015 Wiley Periodicals, Inc. Complexity 000: 00–00, 2015*

**Key Words:** bioeconomic model; functional response; differential algebraic system; time delay; Hopf bifurcation

## 1. INTRODUCTION

**R**ecently Scientists and researchers give emphasis on the interaction between mathematics and biology which initiate a new research area. The major objective of this research area is the development and analysis of mathematical models as tools for resource management and policy development. The analysis may be applied to a wide variety of resources: renewable and exhaustible

resources, terrestrial and marine resources, energy, land and soils, water resources, fisheries, and still others. Interactions of mathematics and biology can be divided into three categories. The first class involves routine application of existing mathematical techniques to biological problems. Such applications influence mathematics only when the importance to biological applications requires further developments. In other cases, however, existing mathematical methods are insufficient, but it is possible to develop new mathematics within the conventional frameworks. In the final class, some fundamental issues in biology appear to require new thoughts quantitatively or

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analytically. Nowadays biological theories are evolved drastically; therefore, we need to develop some useful mathematical models to illustrate different consequences of these biological systems. It may be noted that the formulation of the mathematical models are significantly influenced by the biological assumptions in the past and the consequent expansion of those assumptions in recent time [1–4]. In this regard, differential algebraic equations (DAEs) could be considered as an important tool to analyze such biological models.

A quantitative and qualitative understanding of the interaction of different species is crucial for the management of fisheries. These interactions are found in several forms such as competition, predation, parasitism, mutualism, and refuge. However, functional response of predators to prey density is one of the most important interactions between prey–predator relationships. It is globally accepted that the specified form of functional response plays an important role toward model predictions [5–8]. Three general forms of functional response are commonly used in ecological models: linear, hyperbolic, and sigmoidal [9]. The linear functional (Lotka–Volterra) response specifies a directly proportional relationship between the consumption rate of an individual predator and the density of its prey. Holling [10] extended this to include a “cap” or limitation (Holling’s type I), where there is an abrupt upper threshold representing predator satiation. Examples of this type of functional response are somewhat rare; they are most commonly found in herbivore plant interactions, and some invertebrate predator–prey interactions. The hyperbolic functional response, most commonly known as Holling’s type II function, incorporates search rate and predator handling time to produce a smooth asymptotic curve. The gradient at the origin is determined by search rate or success rate. Formulation of the hyperbolic-shaped function has been extended to represent other mechanistic factors such as digestive satiation, digestion as a background process introducing handling time, mutual interference of predators, spatial heterogeneity, and stochasticity [11]. Type II responses also have been described by the Ivlev equation [12], which represents the probability of feeding at the maximal rate as exponentially distributed with resource density.

There are at least three processes which cause type III functional responses: (i) when prey are rare, predator has little opportunity to learn where to find them and how to catch and kill them, (ii) predator switching prey types as function of relative prey density and (iii) prey refuge, that is, there may be a limited number of safe places, in which prey are not vulnerable to predation. The sigmoidal-shaped functional response, known also as Holling type III, was originally proposed as a model to represent a predator switching its preference from one focal prey to an alternative prey at times when the focal prey is at low densities or depleted [10]. The characteristic feature is

that at low densities of the focal prey, the predator consumes it less proportionally than is available in the environment, relative to the predators’ other prey, resulting in a “dip” at the lower end of the functional response curve.

The stability of the interacting species is an important problem in the management of exploited populations and it is important that the mathematical model of a natural interaction submit to exploitation predicts under what conditions the stability of system that represents the interaction changes. It is observed that a numerous number of research articles of the population dynamics proposed the interaction between the species and the stability analysis of the population in presence of harvesting effort but quite a few number of articles considered the bioeconomic models to investigate the dynamical behavior of the ecosystem toward the positive economic profit. The dynamical behavior of a biological economic system from economic perspective could be investigated using DAEs. The investigated results may finally ensure long run sustainability of an ecosystem. Moreover, a considerable amount of research has already been carried out toward the application of DAEs in bioeconomic [13–20].

Several multispecies models are already developed using hyperbolic (type II) functional relationship to represent how changes in prey density affect predator consumption rate, even though such an assumption could greatly overestimate the effect that generalist predators, such as whales, might have on depleted prey stocks [21]. Predator switching has a stabilizing effect on prey–predator systems [22] if a predator focuses on a more available prey [23]. In addition, the prey’s antipredator efforts may promote predator switching [24]. As most marine mammals and boreal fish species are considered to be generalist predators (including feeders, grazers, etc.), a sigmoidal (type III) functional response might be more appropriate [25]. This has been proposed as being likely for minke whales, harp seals, and cod in the Barents Sea where switching between prey species has been hypothesized [26,27]. One can also get several research articles on ratio-dependent predator–prey models with Holling type monotonic functional responses [28,29]. Sigmoidal functional responses are rarely used in complex multispecies models as they are difficult to implement [30,31].

The main objective of this article is to examine the dynamical behavior of a bioeconomic system with Holling type III or sigmoidal functional response. The continuous gestation delay of predator population is incorporated in the model. We have divided the article in two parts, in the first part, we consider the model system with zero economic profit and singularity induced bifurcation phenomenon is shown at the interior equilibrium of the system. Subsequently, state feedback controller is designed to eliminate singularity induced bifurcation. However, in the second part, we consider the system with positive

economic profit and the occurrence of Hopf bifurcation is shown at the interior equilibrium point through considering time delay as a bifurcation parameter. It is also proved that the time delay can cause a stable equilibrium to become unstable.

## 2. THE MODEL AND ITS QUALITATIVE PROPERTIES

We consider a prey–predator system with Holling type III functional response. The ecological set up of the system is based on the following assumptions:

We consider a prey–predator bioeconomic model incorporating the following assumptions:

- The growth of prey is assumed to be logistic and only prey population is harvested which is proportional to the density of the prey.
- It is considered that the predators have competition among themselves for their survival.
- The present level of predator affects instantaneously the growth of the prey, but the growth of the predator is influenced by the amount of the prey in the past according to the distributed time delay process.

Let us assume  $x$  and  $y$  are respectively the size of the prey and predator population at time  $t$ . Then basic model governed by the following two first-order ordinary differential equations:

$$\begin{aligned}\frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \frac{\alpha x^2 y}{a + x^2} - h(t), \\ \frac{dy}{dt} &= \frac{\beta y x^2}{a + x^2} - d y - \gamma y^2,\end{aligned}\quad (1)$$

where  $r$  is the intrinsic growth rate of the prey,  $K$  is the environmental carrying capacity of prey,  $\alpha$  is the maximal relative increase of predation,  $a$  is half capturing saturation constant,  $h(t)$  is the harvesting at time  $t$ ,  $d$  is the natural death rate of predator,  $\beta$  represents the conversion efficiency of consumed prey into new predators (we assume  $0 < \beta < 1$ , as the whole biomass of the prey is not transformed to the biomass of the predator). Density dependent mortality rate  $\gamma y^2$  describes either a self limitation of consumers or the influence of predation.  $\gamma$  is the intraspecific coefficient of the predator population. Self limitation can occur if there is some other factor (other than food) becomes limiting at high population densities. Conversely, predation on consumers can increase as  $\gamma y^2$  due to the fact that the higher consumer densities attract more attention from predators or if consumers become more vulnerable at higher densities [32]. Thus, the growth of the predator species  $y$ , in second equation of the system (1), is limited due to the presence of the term  $\gamma y^2$  and can be non-positive for large enough value of  $y$ , even if the size of the prey is very high.

The functional form of harvest is generally considered using the phrase catch-per-unit-effort (CPUE) hypothesis

[33] to describe an assumption that catch per unit effort is proportional to the stock level. Thus we consider

$$h(t) = qEx, \quad (2)$$

where  $E$  is the harvesting effort used to harvest prey population and  $q$  is the catchability coefficient of prey population.

The Antarctic krill-whale community is a good example of the present model. Krill is a main source of food of whales and the Antarctic krill population is being increasingly harvested. Conversely, the moratorium imposed by International Whaling Commission (IWC) on killing of whales continues. Large catches from the lower trophic level (krill) can have serious implications for production at both the lower trophic level (krill) and the higher trophic level (whale). It is, therefore, necessary to regulate harvesting at the lower trophic level.

Let us extend our model by considering the following algebraic equation

$$(pqx - c)E - s = 0, \quad (3)$$

where  $c$  is the constant fishing cost per unit effort,  $p$  is the constant price per unit biomass of landed fish and  $s$  is the net economic revenue obtained from the fishery.

Thus, using (2) and (3) system (1) becomes

$$\begin{aligned}\frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \frac{\alpha x^2 y}{a + x^2} - qEx, \\ \frac{dy}{dt} &= \frac{\beta y x^2}{a + x^2} - d y - \gamma y^2, \\ (pqx - c)E - s &= 0,\end{aligned}\quad (4)$$

### 2.1. Interior Equilibria and Local Stability Analysis

The interior equilibrium point of the system (4) is  $\bar{P}(\bar{x}, \bar{y}, \bar{E})$

where  $\bar{y} = \frac{\beta \bar{x}^2 - d(a + \bar{x}^2)}{\gamma(a + \bar{x}^2)}$ ,  $\bar{E} = \frac{s}{pq\bar{x} - c}$  and  $\bar{x}$  satisfies the following equation

$$C_0 \bar{x}^6 + C_1 \bar{x}^5 + C_2 \bar{x}^4 + C_3 \bar{x}^3 + C_4 \bar{x}^2 + C_5 \bar{x} + C_6 = 0, \quad (5)$$

where  $C_0 = pqr\gamma$ ,

$$C_1 = -cr\gamma - Kpqr\gamma,$$

$$C_2 = -dKpqr\alpha + Kpq\alpha\beta + 2apqr\gamma + cKr\gamma + Kqs\gamma,$$

$$C_3 = cdK\alpha - cK\alpha\beta - 2acr\gamma - 2apqKr\gamma,$$

$$C_4 = 2acKr\gamma + 2aKqs\gamma + a^2 pqr\gamma - apqdK\alpha,$$

$$C_5 = acdk\alpha - a^2 cr\gamma - a^2 kpqr\gamma$$

$$C_6 = kra^2 c\gamma + a^2 qsk\gamma$$

From system (4) we have the following matrix

$$N = \begin{bmatrix} \frac{2\bar{x}^3\bar{y}\alpha}{(a+\bar{x}^2)^2} - \frac{\bar{x}\bar{y}\alpha}{(a+\bar{x}^2)} - \frac{r\bar{x}}{K} + \frac{pq^2\bar{x}\bar{E}}{pq\bar{x}-c} & \frac{-\alpha\bar{x}^2}{a+\bar{x}^2} \\ \frac{2a\beta\bar{x}\bar{y}}{(a+\bar{x}^2)^2} & -\gamma\bar{y} \end{bmatrix}.$$

Thus, the characteristic polynomial of the matrix  $N$  at  $\bar{P}(\bar{x}, \bar{y}, \bar{E})$  is given by

$$\mu^2 + b_1(X, E)\mu + b_2(X, E) = 0,$$

where,  $b_1 = \frac{r\bar{x}}{K} - \frac{2\bar{x}^3\bar{y}\alpha}{(a+\bar{x}^2)^2} + \frac{\bar{x}\bar{y}\alpha}{a+\bar{x}^2} - \frac{pq^2\bar{x}\bar{E}}{pq\bar{x}-c} + \gamma\bar{y}$ ,

$$b_2 = \frac{a\bar{x}^2\bar{y}\alpha\beta}{(a+\bar{x}^2)^3} + \frac{r\bar{x}\bar{y}\gamma}{K} - \frac{2\bar{x}^3\bar{y}^2\alpha\gamma}{(a+\bar{x}^2)^2} + \frac{\bar{x}\bar{y}^2\alpha\gamma}{a+\bar{x}^2} - \frac{pq^2\bar{x}\bar{E}\bar{y}\gamma}{pq\bar{x}-c}.$$

We find that  $b_1 > 0$  and  $b_2 > 0$  if  $\frac{r}{K} + \frac{\bar{y}\alpha}{a+\bar{x}^2} > \frac{pq^2\bar{E}}{pq\bar{x}-c} + \frac{2\bar{x}^2\bar{y}\alpha}{(a+\bar{x}^2)^2}$ .

Hence, the interior equilibrium point  $\bar{P}(\bar{x}, \bar{y}, \bar{E})$  of system (4) is asymptotically stable if  $\frac{r}{K} + \frac{\bar{y}\alpha}{a+\bar{x}^2} > \frac{pq^2\bar{E}}{pq\bar{x}-c} + \frac{2\bar{x}^2\bar{y}\alpha}{(a+\bar{x}^2)^2}$ .

In particular, if we consider intraspecific coefficients of the predator population is zero, that is,  $\gamma=0$  then the interior equilibrium of the model system (4) becomes  $P(x^*, y^*, E^*)$ ,

$$\text{where } x^* = \sqrt{\frac{ad}{\beta-d}},$$

$$y^* = \frac{1}{\alpha} \left( \frac{a\beta r}{\sqrt{da(\beta-d)}} - \frac{a\beta r}{k(\beta-d)} - \frac{a\beta qs}{adpq-c\sqrt{da(\beta-d)}} \right) \text{ and}$$

$$E^* = \frac{s\sqrt{\beta-d}}{pq\sqrt{da-c}\sqrt{\beta-d}}.$$

It is noted that for the existence of the interior equilibrium point,  $P(x^*, y^*, E^*)$ , it is necessary  $\beta > d$ ,  $pqda > c\sqrt{da(\beta-d)}$  and  $\left( rk + \frac{kqs\sqrt{da(\beta-d)}}{adpq-c\sqrt{da(\beta-d)}} \right) > \frac{r\sqrt{da(\beta-d)}}{(\beta-d)}$ .

In this particular case, the characteristic polynomial of the matrix  $N$  at  $P(x^*, y^*, E^*)$ , is reduced to,

$$\mu^2 + p_1(X, E)\mu + p_2(X, E) = 0, \quad (6)$$

where  $p_1 = \frac{rx^*}{K} - \frac{2x^{*3}y^*\alpha}{(a+x^{*2})^2} + \frac{x^*y^*\alpha}{a+x^{*2}} - \frac{pq^2x^*E^*}{pqx^*-c}$ ,

$$p_2 = \frac{ax^{*3}y^*\alpha\beta}{(a+x^{*2})^3} > 0.$$

$$\text{We find that } p_1 > 0 \text{ if } \frac{r}{K} + \frac{y^*\alpha}{a+x^{*2}} > \frac{pq^2E^*}{pqx^*-c} + \frac{2x^{*2}y^*\alpha}{(a+x^{*2})^2}.$$

Hence, the interior equilibrium point  $P(x^*, y^*, E^*)$ , of system (4) is asymptotically stable if  $\frac{r}{K} + \frac{y^*\alpha}{a+x^{*2}} > \frac{pq^2E^*}{pqx^*-c} + \frac{2x^{*2}y^*\alpha}{(a+x^{*2})^2}$ .

Again, it is observed that  $p_1=0$  for  $s=s^* = \frac{r(k(\beta-d)-\sqrt{da(\beta-d)})(adpq-c\sqrt{da(\beta-d)})}{qk(\beta-d)\sqrt{da(\beta-d)}}$ ,

and hence the roots of the characteristic Eq. (6) become purely imaginary and they are conjugate to each other.

$$\text{Also we have } \frac{d}{ds} [\text{trace } N_{P(x^*, y^*, E^*)}]_{s=s^*} = \frac{pq^2\sqrt{da(\beta-d)}}{(\sqrt{adpq-c}\sqrt{\beta-d})(adpq-c\sqrt{da(\beta-d)})} \neq 0.$$

Hence, by the Hopf bifurcation theorem [34], the system (4) enters into a Hopf type small amplitude periodic solution at  $s=s^*$  in absence of  $\gamma$  near the positive interior equilibrium point  $P(x^*, y^*, E^*)$ .

### 3. QUALITATIVE ANALYSIS OF THE SYSTEM WITH TIME DELAY

It is generally recognized that some kind of time delay is inevitable in population interactions and tends to be destabilizing in the sense that longer delays may destroy the stability of positive equilibria [35,36]. Time delay due to gestation is a common example, because generally the consumption of prey by the predator throughout its past history governs the present birth rate of the predator. The effect of this kind of delay on the asymptotic behaviour of populations has been studied by a number of authors [37–40]. In a more realistic model, the delay effect should be an average over past populations. This results in an equations with a distributed delay or infinite delay. It is inherently assumed that all the metabolic energy a predator obtains through food (prey) used for its growth ultimately enhance the predator population. Here, the predator population consumes the prey population at a constant rate  $\beta$ , but the reproduction of predators after predating the prey population is not instantaneous thus it will be incorporated by some time lag required for gestation of predators. Suppose the time interval between the moments when an individual prey is killed and the corresponding biomass is added to the predator population is considered as the time delay  $\tau$ . Let us take the entire past history of prey biomass, which is to be measured by  $\frac{1}{\tau} \exp(-\frac{1}{\tau}(t-\tau_0))$  where  $\tau_0 < t$  is considered as a particular time in the past and  $t$  represents the present time [35,36].

Thus, the prey biomass in predator's equation is replaced by the following form [17,41]:

$$z(t) = \int_{-\infty}^t x(t) \frac{1}{\tau} \exp\left(-\frac{1}{\tau}(t-\tau_0)\right) d\tau_0. \quad (7)$$

where  $\frac{1}{\tau} \exp(-\frac{1}{\tau}(t-\tau_0))$  is called the normalized delay kernel,  $\tau$  is mean time delay and  $\tau_0 < t$  is in past history.

Differentiating Eq. (7) with respect to  $t$  and substituting in Eq. (4) we have the following differential algebraic system:

$$\begin{aligned}\frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \frac{\alpha x^2 y}{a+x^2} - qEx, \\ \frac{dy}{dt} &= \frac{\beta y z^2}{a+z^2} - dy - \gamma y^2, \\ \frac{dz}{dt} &= \frac{1}{\tau} (x-z), \\ (pqx-c)E &= s=0.\end{aligned}\quad (8)$$

The differential algebraic system (8) can be expressed in the following way,

$$f(X, E, \mu) = \begin{bmatrix} f_1(X, E, \mu) \\ f_2(X, E, \mu) \\ f_3(X, E, \mu) \end{bmatrix} = \begin{bmatrix} rx \left(1 - \frac{x}{K}\right) - \frac{\alpha x^2 y}{a+x^2} - qEx \\ \frac{\beta y z^2}{a+z^2} - dy - \gamma y^2 \\ \frac{1}{\tau} (x-z) \end{bmatrix}$$

$$g(X, E, \mu) = (pqx-c)E-s,$$

where  $X = (x, y, z)^T$  and  $\mu$  is the bifurcation parameter.

In matrix form the system (8) can be further expressed as:

$$A \begin{pmatrix} \dot{X} \\ \dot{E} \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \text{ where } A = \begin{pmatrix} 1000 \\ 0100 \\ 0010 \\ 0000 \end{pmatrix} \text{ is a singular matrix.}$$

The stability of the several equilibrium points of the system would be investigated in the region  $\Omega = \{ (x, y, z, E) : x \geq 0, y \geq 0, z \geq 0, E \geq 0 \}$ .

Let us now consider two separate cases with zero economic profit and with positive economic profit.

### 3.1. Equilibrium Points: Existence and Stability

The following lemma represents all possible nonnegative equilibrium points of system (8).

#### Lemma 1

System (8) has two equilibrium points  $P_0(0, 0, 0, 0)$  and  $P_1(K, 0, K, 0)$  for any positive set of parameters. The third boundary equilibrium point  $P_2(\tilde{x}, 0, \tilde{z}, \tilde{E})$  exists if and only if  $c < Kpq$  where  $\tilde{x} = \frac{c}{pq}$ ,  $\tilde{z} = \frac{c}{pq}$ ,  $\tilde{E} = \frac{-cr + Kpqr}{Kpq^2}$ . The interior equilibrium point  $P_3(x^*, y^*, z^*, E^*)$  of the system (7) exists if  $\beta > \left(d + \frac{adp^2q^2}{c^2}\right)$  and  $\frac{cr}{Kpq} > \frac{pqzc(-c^2d - adp^2q^2 + c^2\beta)(c^2 - ap^2q^2)}{(c^2 + ap^2q^2)^2\gamma}$ . When these conditions are satisfied  $x^*$ ,  $y^*$ ,  $z^*$  and  $E^*$  are given by as follows,

$$\begin{aligned}x^* &= \frac{c}{pq}, y^* = \frac{-c^2d - adp^2q^2 + c^2\beta}{(c^2 + ap^2q^2)\gamma}, z^* = \frac{c}{pq}, \\ E^* &= \frac{1}{q} \left( r - \frac{cr}{Kpq} - \frac{\alpha cpq(-c^2d - adp^2q^2 + c^2\beta)}{(c^2 + ap^2q^2)2\gamma} \right).\end{aligned}$$

To study stability behavior of the equilibrium points of differential algebraic system (8), we need to construct the matrix  $M$  in the following form:

$$\begin{aligned}M &= D_X f - D_E f (D_E g)^{-1} D_X g \\ &= \begin{bmatrix} r - \frac{2xr}{K} - \frac{2\alpha xy}{(a+x^2)^2} - qE + \frac{pq^2Ex}{pqx-c} & -\frac{\alpha x^2}{a+x^2} & 0 \\ 0 & \frac{\beta z^2}{a+z^2} - d - 2\gamma & \frac{2\alpha\beta yz}{(a+z^2)^2} \\ \frac{1}{\tau} & 0 & -\frac{1}{\tau} \end{bmatrix}.\end{aligned}$$

where  $D_X f$ , and so forth, denotes the matrix of partial derivatives of the components of  $f$  with respect to  $X$ .

The characteristic polynomial of the matrix  $M$  is given by

$$\omega^3 + a_1(X, E)\omega^2 + a_2(X, E)\omega + a_3(X, E) = 0,$$

where

$$\begin{aligned}a_1 &= d + \frac{1}{\tau} + r \left( -1 + \frac{2x}{K} \right) + \frac{cEq}{c-pqx} + \frac{2axy\alpha}{(a+x^2)^2} - \frac{\beta z^2}{a+z^2} + 2\gamma, \\ a_2 &= -\frac{r}{\tau} + \frac{2rx}{K\tau} + \frac{2axy\alpha}{\tau(a+x^2)^2} + d \left[ \frac{1}{\tau} + r \left( -1 + \frac{2x}{K} \right) + \frac{cEq}{c-pqx} + \frac{2axy\alpha}{(a+x^2)^2} \right] \\ &\quad + \frac{rz^2\beta}{a+z^2} - \frac{z^2\beta}{\tau(a+z^2)} - \frac{2rxz^2\beta}{K(a+z^2)} \\ &\quad - \frac{2axyz^2\alpha\beta}{(a+x^2)^2(a+z^2)} - 2\gamma\gamma + \frac{2\gamma}{\tau} + \frac{4rx\gamma}{K} + \frac{4axy^2\alpha\gamma}{(a+x^2)^2} \\ &\quad + \frac{cEq(a+z^2 - \tau\beta z^2 + 2a\tau\gamma + 2\tau yz^2\gamma)}{\tau(c-pqx)(a+z^2)}, \\ a_3 &= \frac{1}{\tau} \left[ d \left( r \left( -1 + \frac{2x}{K} \right) + \frac{cEq}{c-pqx} + \frac{2axy\alpha}{(a+x^2)^2} \right) + \frac{rz^2\beta}{a+z^2} - \frac{2rxz^2\beta}{K(a+z^2)} \right. \\ &\quad + \frac{2ax^2yz\alpha\beta}{(a+x^2)(a+z^2)^2} - \frac{2axyz^2\alpha\beta}{(a+x^2)^2(a+z^2)} - 2\gamma\gamma \\ &\quad \left. + \frac{4rx\gamma}{K} + \frac{4axy^2\alpha\gamma}{(a+x^2)^2} + \frac{cEq(-\beta z^2 + 2\gamma(a+z^2)\gamma)}{(c-pqx)(a+z^2)} \right]\end{aligned}$$

The conditional stability of the boundary equilibrium points  $P_0(0, 0, 0, 0)$  and  $P_1(K, 0, K, 0)$  of system (8) are provided in the following lemma.

#### Lemma 2

The boundary equilibrium point  $P_0(0, 0, 0, 0)$  is unstable and  $P_1(K, 0, K, 0)$  is locally asymptotically stable if  $\beta < d \left( 1 + \frac{a}{K^2} \right)$ .



The eigen values of the characteristic polynomial at the boundary equilibrium point  $P_0(0,0,0,0)$  are  $-d, r, \frac{1}{\tau}$ . Thus the boundary equilibrium point  $P_0(0,0,0,0)$  is unstable. Again, the eigen values of the characteristic polynomial at the boundary equilibrium point  $P_1(K,0,K,0)$  are  $-r, -\frac{1}{\tau}, \frac{-ad-dK^2+K^2\beta}{a+K^2}$ . It is clearly observed that the boundary equilibrium point  $P_1(K,0,K,0)$  is stable if  $\beta < d(1+\frac{a}{K^2})$ .

Let us now study the dynamic behavior of the differential algebraic system (8) at the boundary equilibrium point,  $P_2(\tilde{x},0,\tilde{z},\tilde{E})$  and interior equilibrium point,  $P_3(x^*,y^*,z^*,E^*)$ . This can be investigated using the singularity induced bifurcation phenomena. Here, we are interested to discuss the local stability of the model system (8) at the interior equilibrium point,  $P_3(x^*,y^*,z^*,E^*)$ .

For this purpose total economic rent  $s$  is assumed to be the bifurcation parameter, that is,  $\mu=s$ .

Consequently, we have the following theorem:

### Theorem 1

The differential algebraic system (8) has a singularity induced bifurcation at the interior equilibrium point  $P_3(x^*,y^*,z^*,E^*)$ . When the bifurcation parameter  $s$  increases through zero the stability of the interior equilibrium point  $P_3(x^*,y^*,z^*,E^*)$  changes from stable to unstable.

### Proof

It is evident that  $D_Eg=pqx-c$  has a simple zero eigen value. Thus, we can define  $\Delta(X,E,s)=D_Eg=pqx-c$ .

i. It follows from Lemma 1 that

$$\text{Trace}(D_E f \text{adj}(D_Eg)D_Xg)_{P_3} = -pq^2x^*E^* \neq 0,$$

where  $\text{Trace}(D_E f \text{adj}(D_Eg)D_Xg)_{P_3}$  is the sum of the elements of the principal diagonal of the  $3 \times 3$  order matrix  $[D_E f \text{adj}(D_Eg)D_Xg]$  at the interior equilibrium point  $P_3$  of the system (8).

i. It can be proved using Lemma 1 that

$$\begin{vmatrix} D_Xf & D_Ef \\ D_Xg & D_Eg \end{vmatrix}_{P_3} = \begin{vmatrix} r - \frac{2xr}{k} - \frac{2axxy}{(a+x^2)^2} - qE & -\frac{xx^2}{a+x^2} & 0 & -qx \\ 0 & \frac{\beta z^2}{a+z^2} - d - 2y\gamma & \frac{2a\beta yz}{(a+z^2)^2} & 0 \\ \frac{1}{\tau} & 0 & -\frac{1}{\tau} & 0 \\ pqE & 0 & 0 & pqx-c \end{vmatrix}_{P_3} = \frac{pq^2x^*y^*E^*\gamma}{\tau} \neq 0.$$

i. It can also be shown using Lemma 1 that

$$\begin{vmatrix} D_Xf & D_Ef & D_\lambda f \\ D_Xg & D_Eg & D_\lambda g \\ D_X\Delta & D_E\Delta & D_\lambda\Delta \end{vmatrix}_{P_3} = \begin{vmatrix} r - \frac{2xr}{k} - \frac{2axxy}{(a+x^2)^2} - qE & -\frac{xx^2}{a+x^2} & 0 & -qx & 0 \\ 0 & \frac{\beta z^2}{a+z^2} - d - 2y\gamma & \frac{2a\beta yz}{(a+z^2)^2} & 0 & 0 \\ \frac{1}{\tau} & 0 & -\frac{1}{\tau} & 0 & 0 \\ pqE & 0 & 0 & pqx-c & -1 \\ pq & 0 & 0 & 0 & 0 \end{vmatrix}_{P_3} = \frac{pq^2x^*y^*\gamma}{\tau} \neq 0.$$

It is observed from (i)–(iii) that all the conditions for singularity induced bifurcation [42] are satisfied. Hence the differential algebraic system (8) has a singularity induced bifurcation at the interior equilibrium point  $P_3(x^*,y^*,z^*,E^*)$  and the bifurcation value is  $s=0$ .

Again, it is noted that

$$M_1 = -\text{trace}(D_E f \text{adj}(D_Eg)D_Xg)_{P_3} = pq^2x^*E^*,$$

$$M_2 = \left[ D_\lambda\Delta - (D_X\Delta \quad D_E\Delta) \begin{pmatrix} D_Xf & D_Ef \\ D_Xg & D_Eg \end{pmatrix}^{-1} \begin{pmatrix} D_\lambda f \\ D_\lambda g \end{pmatrix} \right]_{P_3} = \frac{1}{E^*}.$$

Consequently, from Lemma 1 we have,

$$\frac{M_1}{M_2} = pq^2x^*E^{*2} > 0.$$

It is observed that when  $s$  increases through zero, one eigenvalue of the model system (8) moves from  $C^-$  (set of all complex numbers with negative real part) to  $C^+$  (set of all complex numbers with positive real part) along the real axis by diverging through  $\infty$ . Consequently, instability behavior is incorporated in the system and the system becomes unstable at the interior equilibrium point  $P_3(x^*,y^*,z^*,E^*)$  [42]. ■

### 3.2. Hopf Bifurcation Analysis

Subsequent to Theorem 1, it is clear that the differential algebraic system (8) becomes unstable when the economic interest of the harvesting is considered to be positive. Prey–predator models with constant parameters are often found to approach a steady state in which the species coexist in equilibrium. But if parameters used in the model are changed, other types of dynamical behavior

may occur and the critical parameter values at which such transitions happen are called bifurcation points. In this section, let us investigate the Hopf bifurcation of the system (8) with positive economic profit while assuming  $\tau$  as the bifurcation parameter.

Let  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  be the interior equilibrium point when economic profit is positive.

From the model system (8), we have the following matrix

$$R = \begin{bmatrix} \frac{2\alpha\hat{x}^3\hat{y}}{(a+\hat{x}^2)^2} - \frac{\alpha\hat{x}\hat{y}}{a+\hat{x}^2} - \frac{r\hat{x}}{K} + \frac{pq\hat{q}^2\hat{E}\hat{x}}{pq\hat{x}-c} - \frac{-\alpha\hat{x}^2}{a+\hat{x}^2} & 0 \\ 0 & -\hat{y}\gamma - \frac{2a\beta\hat{y}\hat{z}}{(a+\hat{z}^2)^2} \\ \frac{1}{\tau} & 0 - \frac{1}{\tau} \end{bmatrix}.$$

Thus, the characteristic polynomial of the matrix  $R$  is given by

$$\mu^3 + d_1(X, E)\mu^2 + d_2(X, E)\mu + d_3(X, E) = 0,$$

$$\text{where } d_1 = \frac{r\hat{x}}{K} - \frac{2\alpha\hat{x}^3\hat{y}}{(a+\hat{x}^2)^2} + \frac{\alpha\hat{x}\hat{y}}{a+\hat{x}^2} - \frac{pq\hat{q}^2\hat{E}\hat{x}}{pq\hat{x}-c} + \hat{y}\gamma + \frac{1}{\tau},$$

$$d_2 = \frac{r\hat{x}}{K\tau} - \frac{pq\hat{q}^2\hat{E}\hat{x}}{(pq\hat{x}-c)\tau} + \frac{r\hat{x}\hat{y}\gamma}{K} - \frac{2\alpha\hat{x}^3\hat{y}}{(a+\hat{x}^2)^2\tau} + \frac{\alpha\hat{x}\hat{y}}{(a+\hat{x}^2)\tau} - \frac{pq\hat{q}^2\hat{E}\hat{x}\hat{y}\gamma}{pq\hat{x}-c} + \frac{\hat{y}\gamma}{\tau} + \frac{\alpha\hat{x}\hat{y}^2\gamma}{a+\hat{x}^2} - \frac{2\alpha\hat{x}^3\hat{y}^2\gamma}{(a+\hat{x}^2)^2},$$

$$d_3 = \frac{2\alpha\hat{x}^2\hat{y}\hat{z}\alpha\beta}{\tau(a+\hat{x}^2)(a+\hat{z}^2)^2} + \frac{r\hat{x}\hat{y}\gamma}{K\tau} - \frac{2\alpha\hat{x}^3\hat{y}^2\gamma}{\tau(a+\hat{x}^2)^2} + \frac{\alpha\hat{x}\hat{y}^2\gamma}{\tau(a+\hat{x}^2)} - \frac{pq\hat{q}^2\hat{E}\hat{x}\hat{y}\gamma}{(pq\hat{x}-c)\tau}.$$

It is noted that  $d_1 > 0$  and  $d_3 > 0$  if  $\frac{r}{K} + \frac{\hat{y}\alpha}{a+\hat{x}^2} > \frac{pq\hat{q}^2\hat{E}}{pq\hat{x}-c} + \frac{2\alpha\hat{x}^3\hat{y}}{(a+\hat{x}^2)^2}$ .

Let us assume,  $A(\tau) = d_1d_2 - d_3$ .

Then,  $A(\tau) = \frac{1}{\tau^2}(t_1\tau^2 + t_2\tau + t_3)$ ,

where

$$\begin{aligned} t_1 &= \frac{r^2\hat{x}^2\hat{y}\gamma}{K^2} + \frac{pq\hat{q}^2\hat{E}^2\hat{x}^2\hat{y}\gamma}{(pq\hat{x}-c)^2} - \frac{2pq\hat{q}^2r\hat{E}\hat{x}^2\hat{y}\gamma}{(pq\hat{x}-c)K} - \frac{pq\hat{q}^2\hat{E}\hat{x}\hat{y}^2\gamma^2}{pq\hat{x}-c} + \frac{r\hat{x}\hat{y}^2\gamma^2}{K} \\ &\quad + \frac{4pq\hat{q}^2\hat{E}\hat{x}^4\hat{y}^2\alpha\gamma}{(a+\hat{x}^2)^2(pq\hat{x}-c)} - \frac{4r\alpha\gamma\hat{x}^4\hat{y}^2}{(a+\hat{x}^2)^2K} \\ &\quad - \frac{2pq\hat{q}^2\hat{E}\hat{x}^2\hat{y}^2\alpha\gamma}{(a+\hat{x}^2)^2(pq\hat{x}-c)} - \frac{2r\alpha\gamma\hat{x}^4\hat{y}^2}{(a+\hat{x}^2)K} + \frac{4\alpha\hat{x}^6\hat{y}^3\gamma}{(a+\hat{x}^2)^4} - \frac{2\alpha\hat{x}^3\hat{y}^3\gamma^2}{(a+\hat{x}^2)^2} \\ &\quad - \frac{4\alpha^2\hat{x}^4\hat{y}^3\gamma}{(a+\hat{x}^2)^3} + \frac{r\alpha\gamma\hat{x}^2\hat{y}^2}{(a+\hat{x}^2)K} + \frac{\alpha\hat{x}\hat{y}^3\gamma^2}{(a+\hat{x}^2)} + \frac{\alpha\hat{x}\hat{y}^3\gamma}{(a+\hat{x}^2)^2}, \\ t_2 &= \frac{pq\hat{q}^4\hat{E}^2\hat{x}^2}{(pq\hat{x}-c)^2} - \frac{2pq\hat{q}^2r\hat{E}\hat{x}^2}{(pq\hat{x}-c)K} + \frac{2r\hat{x}\hat{y}\gamma}{K} - \frac{2pq\hat{q}^2\gamma\hat{E}\hat{x}^2\hat{y}}{(pq\hat{x}-c)} + \hat{y}\gamma^2 \\ &\quad - \frac{4r\alpha\hat{x}^4\hat{y}}{(a+\hat{x}^2)^2K} + \frac{4pq\hat{q}^2\hat{E}\hat{x}^4\hat{y}\alpha}{(a+\hat{x}^2)^2(pq\hat{x}-c)} - \frac{4\hat{y}^2\alpha\gamma\hat{x}^3}{(a+\hat{x}^2)^2} \\ &\quad + \frac{4\alpha\hat{x}^6\hat{y}^2}{(a+\hat{x}^2)^4} - \frac{2pq\hat{q}^2\hat{E}\hat{x}^2\hat{y}\alpha}{(a+\hat{x}^2)(pq\hat{x}-c)} + \frac{2r\alpha\hat{x}^2\hat{y}}{k(a+\hat{x}^2)} - \frac{2\alpha\hat{x}\hat{y}^2\gamma}{a+\hat{x}^2} - \frac{4\hat{y}^2\alpha^2\gamma\hat{x}^4}{(a+\hat{x}^2)^3} \end{aligned}$$

$$\begin{aligned} &+ \frac{\alpha\hat{x}\hat{y}^2}{(a+\hat{x}^2)^2} - \frac{2\alpha\alpha\beta\hat{x}^2\hat{y}\hat{z}}{(a+\hat{x}^2)(a+\hat{z}^2)^2} + \frac{r\hat{x}^2}{k^2}, \\ t_3 &= \frac{r\hat{x}}{K} - \frac{2\alpha\hat{x}^3\hat{y}}{(a+\hat{x}^2)^2} + \frac{\alpha\hat{x}\hat{y}}{(a+\hat{x}^2)} - \frac{pq\hat{q}^2\hat{E}\hat{x}}{pq\hat{x}-c} + \hat{y}\gamma. \end{aligned}$$

Now we have the following theorem which ensures the local stability of the system (8) around interior equilibrium point,  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ .

### Theorem 2

If  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  exists with  $\frac{r}{K} + \frac{\hat{y}\alpha}{a+\hat{x}^2} > \frac{pq\hat{q}^2\hat{E}}{pq\hat{x}-c} + \frac{2\alpha\hat{x}^3\hat{y}}{(a+\hat{x}^2)^2}$  and  $t_1\tau^2 + t_2\tau + t_3 > 0$ , then  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  is locally asymptotically stable.

### Proof

The condition  $\frac{r}{K} > \frac{pq\hat{q}^2\hat{E}}{pq\hat{x}-c} + \frac{2\alpha\hat{x}^3\hat{y}}{(a+\hat{x}^2)^2} - \frac{\hat{y}\alpha}{a+\hat{x}^2}$  implies that  $d_1 > 0$  and  $d_3 > 0$ .

Finally,  $t_1\tau^2 + t_2\tau + t_3 > 0$  implies that  $A(\tau) = d_1d_2 - d_3 > 0$ . Hence by Routh Hurwitz criterion, the theorem follows. ■

### Theorem 3

If  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  exists with  $\frac{r}{K} + \frac{\hat{y}\alpha}{a+\hat{x}^2} > \frac{pq\hat{q}^2\hat{E}}{pq\hat{x}-c} + \frac{2\alpha\hat{x}^3\hat{y}}{(a+\hat{x}^2)^2}$  and  $\tau < \frac{2\gamma(a+\hat{x}^2)^3}{\alpha\hat{x}\alpha\beta}$ , then a simple Hopf bifurcation occurs at the positive unique value  $\tau = \tau^*$ .

### Proof

The characteristic equation of the model system (6) at  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  is given by

$$\mu^3 + d_1(\tau)\mu^2 + d_2(\tau)\mu + d_3(\tau) = 0, \quad (9)$$

The Eq. (9) has two purely imaginary roots if and only if  $d_1d_2 = d_3$  for a unique value of  $\tau$  (say  $\tau^*$ ) at which we have a Hopf bifurcation. Thus in the neighborhood of  $\tau^*$  the characteristic Eq. (9) cannot have real roots. For  $\tau = \tau^*$  we have  $(\mu^2 + d_2)(\mu + d_1) = 0$ . This equation has two purely imaginary roots and a real root as,

$$\mu_1 = i\sqrt{d_2}, \mu_2 = -i\sqrt{d_2} \text{ and } \mu_3 = -d_1.$$

The roots are of the following form

$$\mu_1(\tau) = p(\tau) + iq(\tau), \mu_2(\tau) = p(\tau) - iq(\tau) \text{ and } \mu_3(\tau) = -d_1(\tau).$$

To apply Hopf bifurcation theorem as stated in Liu's criterion [43] we need to verify the transversality condition,

$$\left[ \frac{dp(\tau)}{d\tau} \right]_{\tau=\tau^*} \neq 0.$$

Substituting  $\mu_1(\tau) = p(\tau) + iq(\tau)$ , in the Eq. (9) and differentiating the resulting equation w.r.t.  $\tau$ , and setting  $p(\tau) = 0$  and  $q(\tau) = \sqrt{d_2}$ , we get the transversality condition at  $\tau = \tau^*$  as

$$\left[ \frac{dp(\tau)}{d\tau} \right]_{\tau=\tau^*} = \left[ -\frac{d_2(d_1 d_2' - d_3 + d_1' d_2)}{2(d_2^2 + d_1^2 d_2)} \right]_{\tau=\tau^*}.$$

Thus from the expressions of  $d_1, d_2$  and  $d_3$  we find,

$$\begin{aligned} \left[ \frac{dp(\tau)}{d\tau} \right]_{\tau=\tau^*} &= \frac{v^2 \hat{x}^2 \tau^* + 2v(\hat{x} + \hat{x} \hat{y} \gamma \tau^*) + \hat{y} \left( 2\gamma - \frac{a \hat{x} \alpha \beta \tau^*}{(a + \hat{x}^2)(a + \hat{z}^2)^2} + \hat{y} \gamma^2 \tau^* \right)}{2 \left( v \hat{x} \hat{y} \gamma + (v \hat{x} + \hat{y} \gamma + \frac{1}{\tau})^2 + \frac{v \hat{x}}{\tau} + \frac{\hat{y} \gamma}{\tau} \right) \tau^{*3}} \\ &> 0 \text{ if } \tau < \frac{2\gamma(a + \hat{x}^2)^3}{a \hat{x} \alpha \beta}, \end{aligned}$$

$$\text{where } v = \frac{r}{K} - \frac{pq^2 \hat{E}}{pq\hat{x} - c} - \frac{2\alpha \hat{x}^3 \hat{y}}{(a + \hat{x}^2)^2} + \frac{\hat{y} \alpha}{(a + \hat{x}^2)}.$$

Thus, it can be concluded that the interior equilibrium point,  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  is locally asymptotically stable for  $\tau < \tau^*$ . Furthermore, according to the Liu's criterion a simple Hopf bifurcation occurs at  $\tau = \tau^*$  and for  $\tau > \tau^*$ , the interior equilibrium point,  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  approaches to a periodic solution. Hence the theorem follows. ■

### 3.3. Qualitative Analysis of the System with State Feedback Controller

Therefore, it is easy to understand that the system (8) remains unstable when economic interest of the harvesting is considered to be positive. However, this phenomenon is against our natural interest. Subsequently, we need to take some preventive measures to resume the sustainability of the ecosystem at positive economic interest.

Thus, to stabilize the system (8) at positive economic interest, we could have designed a suitable state feedback controller, [44] in the following form  $w(t) = u(E(t) - E^*)$ ,

where  $u$  stands for net feedback gain.

System (8) can be further expressed in the following form:

$$\begin{aligned} \frac{dx}{dt} &= rx \left( 1 - \frac{x}{K} \right) - \frac{\alpha x^2 y}{a + x^2} - qEx, \\ \frac{dy}{dt} &= \frac{\beta y z^2}{a + z^2} - dy - \gamma y^2, \\ \frac{dz}{dt} &= \frac{1}{\tau} (x - z), \\ (pqx - c)E - s + u(E(t) - E^*) &= 0. \end{aligned} \quad (10)$$

Consequently, we have the following theorem:

#### Theorem 4

The differential algebraic model system (10) is stable at the interior equilibrium point,  $P_3(x^*, y^*, z^*, E^*)$ , of the model system (8), if

$$u > \max \left[ \begin{aligned} &\frac{pq^2 x^* E^*}{\frac{rx^*}{K} - \frac{2x^{*3} y^* \alpha}{(a + x^{*2})^2} + \frac{x^* y^* \alpha}{a + x^{*2}} + y^* \gamma + \frac{1}{\tau}}, \\ &\frac{pq^2 x^* y^* E^* \gamma + \frac{pq^2 x^* E^*}{\tau}}{\frac{rx^* y^* \gamma}{K} - \frac{2x^{*3} y^{*2} \alpha \gamma}{(a + x^{*2})^2} + \frac{x^* y^{*2} \alpha \gamma}{a + x^{*2}} + \frac{rx^*}{K\tau} - \frac{2x^{*3} y^* \alpha}{(a + x^{*2})^2 \tau} + \frac{x^* y^* \alpha}{(a + x^{*2}) \tau} + \frac{y^* \gamma}{\tau}}, \\ &\frac{pq^2 x^* y^* E^* \gamma}{\frac{2ax^{*2} y^* \alpha \beta}{(a + x^{*2})(a + z^{*2})^2 \tau} + \frac{rx^* y^* \gamma}{K\tau} - \frac{2x^{*3} y^{*2} \alpha \gamma}{(a + x^{*2})^2 \tau} + \frac{x^* y^{*2} \alpha \gamma}{(a + x^{*2}) \tau}} \end{aligned} \right]$$

#### Proof

For the differential algebraic model system (10), we can obtain the following Jacobian at the interior equilibrium point  $P_3(x^*, y^*, z^*, E^*)$ , of the model system (8),

$$J_{p_3} = \begin{bmatrix} \frac{2x^{*3} y^* \alpha}{(a + x^{*2})^2} - \frac{x^* y^* \alpha}{a + x^{*2}} - \frac{rx^*}{K} + \frac{pq^2 x^* E^*}{u} & -\frac{\alpha x^{*2}}{a + x^{*2}} & 0 \\ 0 & -\gamma y^* & \frac{2a\beta y^* z^*}{(a + z^{*2})^2} \\ \frac{1}{\tau} & 0 & -\frac{1}{\tau} \end{bmatrix}.$$

Therefore, the characteristic polynomial of the matrix  $J$  is given

$$\begin{aligned} \mu^3 + w_1(X, E)\mu^2 + w_2(X, E)\mu + w_3(X, E) &= 0, \\ \text{where } w_1 &= \frac{rx^*}{K} - \frac{pq^2 x^* E^*}{u} - \frac{2x^{*3} y^* \alpha}{(a + x^{*2})^2} + \frac{x^* y^* \alpha}{a + x^{*2}} + y^* \gamma + \frac{1}{\tau}, \\ w_2 &= \frac{rx^* y^* \gamma}{K} - \frac{pq^2 x^* y^* E^* \gamma}{u} - \frac{2x^{*3} y^{*2} \alpha \gamma}{(a + x^{*2})^2} + \frac{x^* y^{*2} \alpha \gamma}{a + x^{*2}} + \frac{rx^*}{K\tau} \\ &\quad - \frac{pq^2 x^* E^*}{u\tau} - \frac{2x^{*3} y^* \alpha}{(a + x^{*2})^2 \tau} + \frac{x^* y^* \alpha}{\tau(a + x^{*2})} + \frac{y^* \gamma}{\tau}, \\ w_3 &= \frac{2ax^{*2} y^* z^* \alpha \beta}{(a + x^{*2})(a + z^{*2})^2 \tau} + \frac{rx^* y^* \gamma}{K\tau} - \frac{pq^2 x^* y^* E^* \gamma}{u\tau} \\ &\quad - \frac{2x^{*3} y^{*2} \alpha \gamma}{(a + x^{*2})^2 \tau} + \frac{x^* y^{*2} \alpha \gamma}{(a + x^{*2}) \tau} \end{aligned}$$



It is observed that the system (10) is stable at the interior equilibrium point,  $P_3(x^*, y^*, z^*, E^*)$ , if the net

feedback gain  $u$  satisfies the following condition due to Routh Hurwitz criterion:

$$u > \max \left[ \begin{aligned} & \frac{pq^2 x^* E^*}{\frac{rx^*}{K} - \frac{2x^{*3} y^* \alpha}{(a+x^{*2})^2} + \frac{x^* y^* \alpha}{a+x^{*2}} + y^* \gamma + \frac{1}{\tau}}, \\ & \frac{pq^2 x^* y^* E^* \gamma + \frac{pq^2 x^* E^*}{\tau}}{\frac{rx^* y^* \gamma}{K} - \frac{2x^{*3} y^{*2} \alpha \gamma}{(a+x^{*2})^2} + \frac{x^* y^{*2} \alpha \gamma}{a+x^{*2}} + \frac{rx^*}{K\tau} - \frac{2x^{*3} y^* \alpha}{(a+x^{*2})^2 \tau} + \frac{x^* y^* \alpha}{(a+x^{*2})\tau} + \frac{y^* \gamma}{\tau}}, \\ & \frac{pq^2 x^* y^* E^* \gamma}{\frac{2ax^{*2} y^* \alpha \beta}{(a+x^{*2})(a+z^{*2})^2 \tau} + \frac{rx^* y^* \gamma}{K\tau} - \frac{2x^{*3} y^{*2} \alpha \gamma}{(a+x^{*2})^2 \tau} + \frac{x^* y^{*2} \alpha \gamma}{(a+x^{*2})\tau}} \end{aligned} \right].$$

Thus, we can design a suitable state feedback controller to eliminate singularity induced bifurcation phenomenon. ■

#### 4. NUMERICAL SIMULATIONS AND DISCUSSION

It may be noted that as the parameters of the model are not based on real world observations, the main features described by the simulations presented in this section should be considered from a qualitative, rather than a quantitative point of view. However, numerous scenarios covering the breath of the biological feasible parameter space were conducted and the results shown above display the gamut of dynamical results collected from all the scenarios tested. For the purpose of simulation experiments we mainly use the software MATLAB and Mathematica. As the problem is not a case study, the real world data are not available for this model. We, therefore,

take here some relevant data available in literature with the sole purpose of solving the system numerically to obtain the results after numerical simulations.

This section can be classified into two categories. First category consists of the results where the total economic profit is considered to be zero. In the second category, numerical simulations are represented with positive economic profit.

##### 4.1. Singularity Induced Bifurcation

Let us consider the following numerical set parameters to verify the analytical results obtained from system (8):

$$r=1.5, K=100, \alpha=0.8, a=10, q=0.5, d=0.002, \beta=0.6, \gamma=0.05, \tau=2.5, p=12, c=0.5.$$

Interior equilibrium solution of system (8) is  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E}) = P(3.901, 3.326, 3.901, 2.5)$  when  $s=0$ . However, it is

FIGURE 1

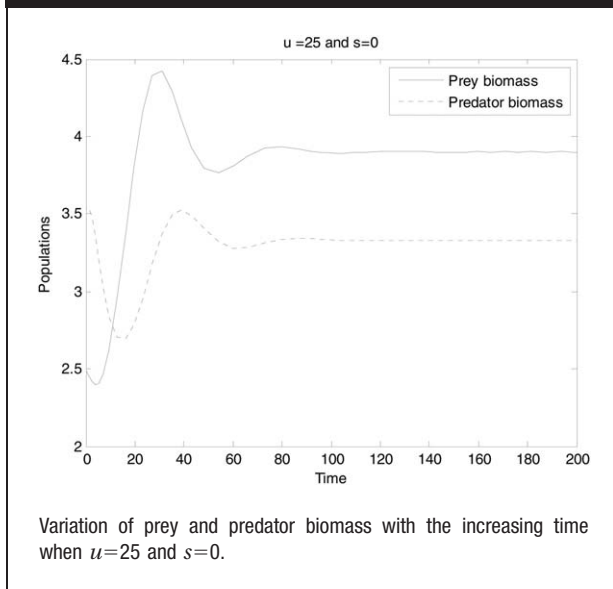
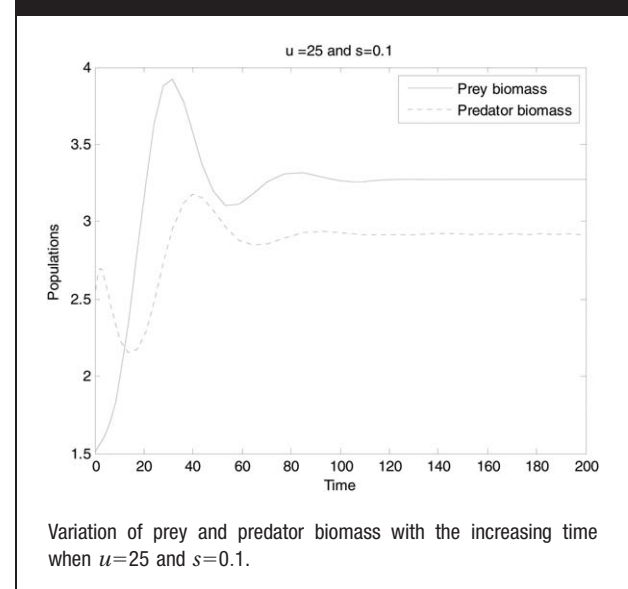
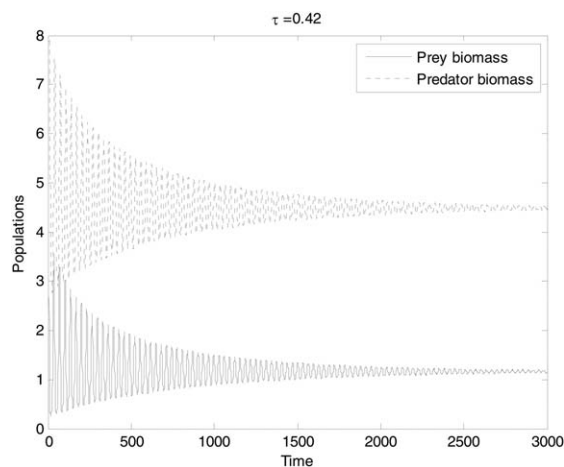


FIGURE 2



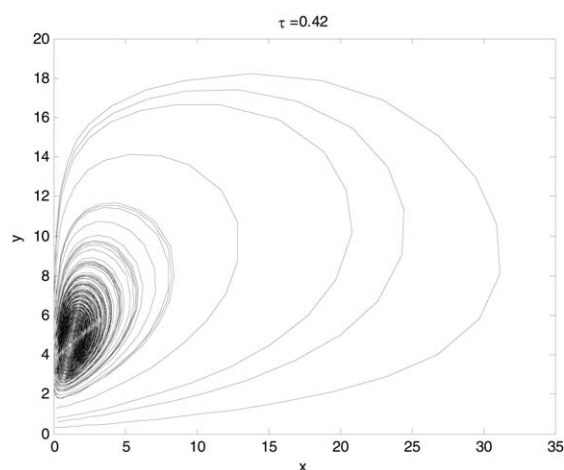
**FIGURE 3**



Variation of prey and predator biomass with the increasing time when  $\tau=0.42 < \tau^*$ .

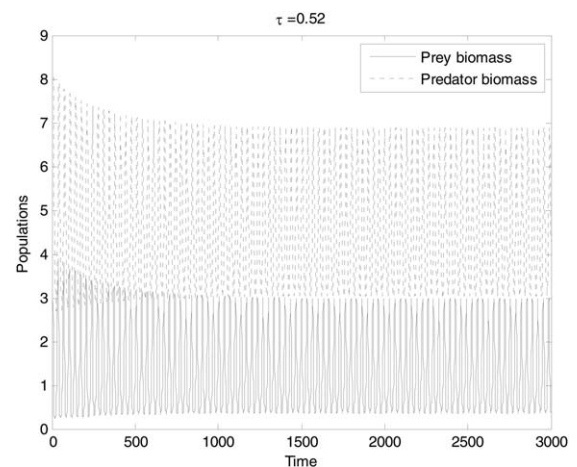
observed that, when  $s=-0.1$ , eigen values of the characteristic polynomial of system (8) are  $-67.7359, -0.4, -0.00223587$ . But, when we consider  $s=0.1$ , eigen values become  $78.7046, -0.478, -0.00240128$ . Hence, it may be numerically concluded that when  $s$  increases through zero two eigen values of the characteristic polynomial of the system (8) remain same but one eigen value of the system (8) moves from  $C^-$  to  $C^+$  along the real axis by

**FIGURE 4**



Phase space trajectories of prey and predator biomass beginning with different initial levels when  $\tau=0.42 < \tau^*$ .

**FIGURE 5**

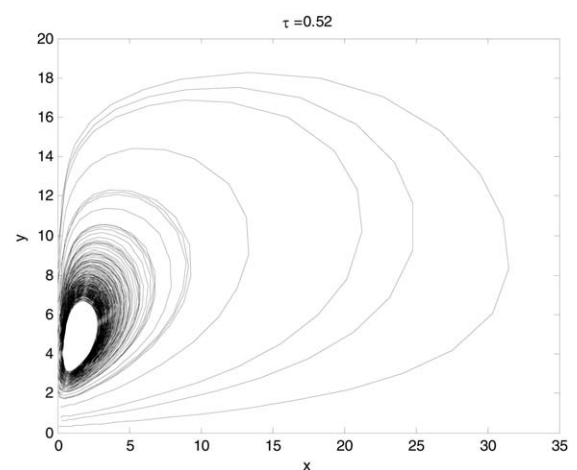


Variation of prey and predator biomass with the increasing time when  $\tau=0.52=\tau^*$ .

diverging through  $\infty$ . Consequently, an instability behavior induced in the system and the system (8) becomes unstable at the interior equilibrium point  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ .

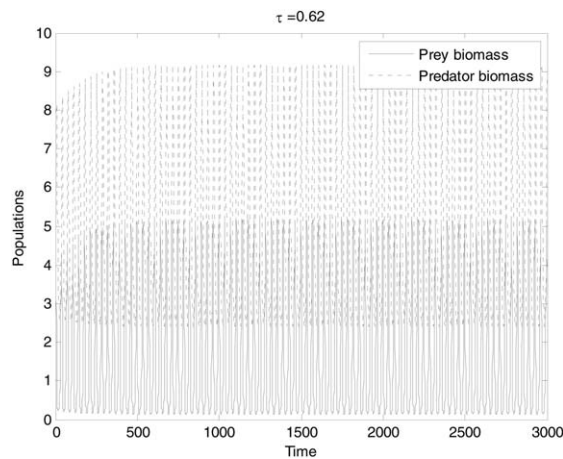
Conversely, if we want to stabilize the system (8) in case of positive economic interest, we need to introduce a state feedback controller, of the form  $w(t)=u(E(t)-2.5)$ , as we had discussed in the theory part. Subsequently, we have obtained differential algebraic system (10) as follows:

**FIGURE 6**



Phase space trajectories of prey and predator biomass beginning with different initial levels when  $\tau=0.52=\tau^*$ .

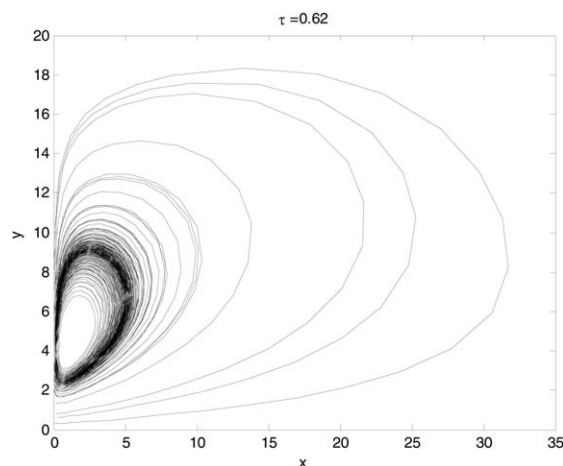
**FIGURE 7**



Variation of prey and predator biomass with the increasing time when  $\tau=0.62 > \tau^*$ .

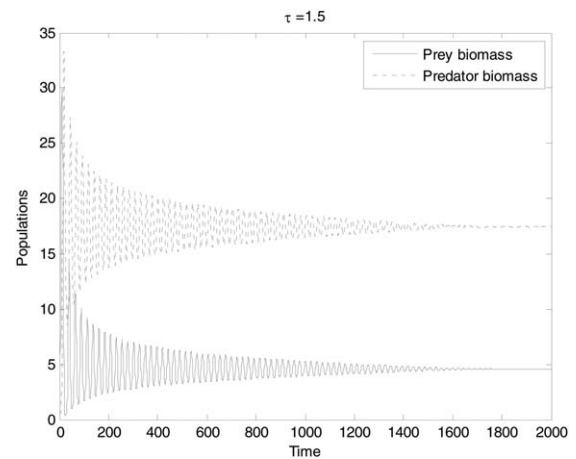
$$\begin{aligned}\frac{dx}{dt} &= 1.5x \left(1 - \frac{x}{100}\right) - \frac{0.8x^2y}{10+x^2} - 0.5Ex, \\ \frac{dy}{dt} &= \frac{0.6yz^2}{10+z^2} - 0.002y - 0.05y^2, \\ \frac{dz}{dt} &= \frac{1}{2.5}(x-z), \\ (12 \times 0.5x - 0.5)E - s + u(E(t) - 2.5) &= 0.\end{aligned}$$

**FIGURE 8**



Phase space trajectories of prey and predator biomass beginning with different initial levels when  $\tau=0.62 > \tau^*$ .

**FIGURE 9**



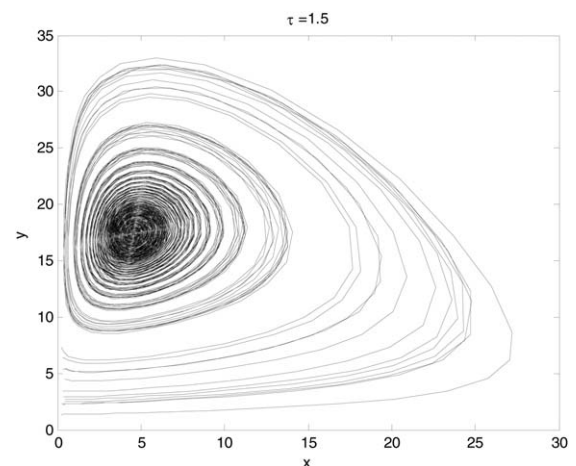
Variation of prey and predator biomass with the increasing time when  $\tau=1.5 < \tau^*$ .

Now, we can easily evaluate the numerical value of net feedback gain from Theorem 4. The numerical solution provides  $u > \max(4.26146, 23.873, 12.653)$ .

Let us consider,  $u=25$ . Interior equilibrium solution of system (8) is  $(3.901, 3.326, 3.901, 2.5)$  when  $s=0$ . However, we find that, when  $s=0.1$ , the interior equilibrium solution of system (8) becomes  $(3.574, 3.119, 3.574, 2.525)$ .

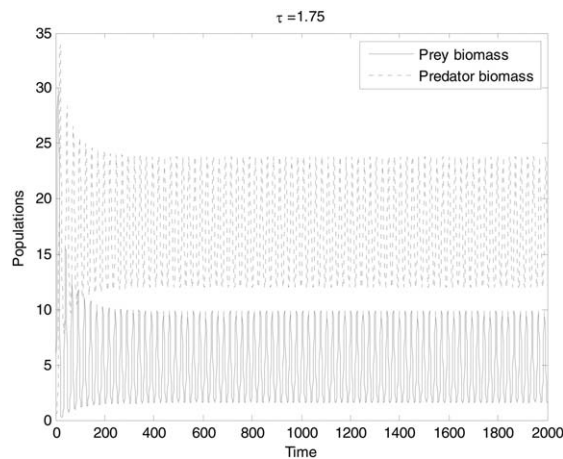
Figures 1 and 2 depict that the differential algebraic model system (10) is stable when  $s$  increases through

**FIGURE 10**



Phase space trajectories of prey and predator biomass beginning with different initial levels when  $\tau=1.5 < \tau^*$ .

**FIGURE 11**



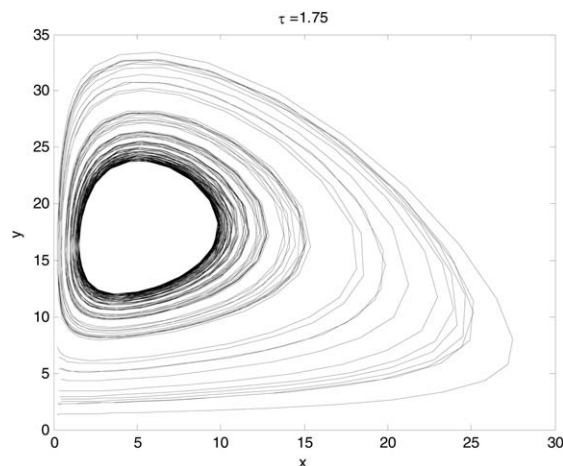
Variation of prey and predator biomass with the increasing time when  $\tau = 1.75 = \tau^*$ .

zero. Therefore, it is clear that singularity induced bifurcation phenomenon can be eliminated from differential algebraic system (8) at the interior equilibrium point when net economic profit of the system increases through zero.

#### 4.2. Hopf Bifurcation

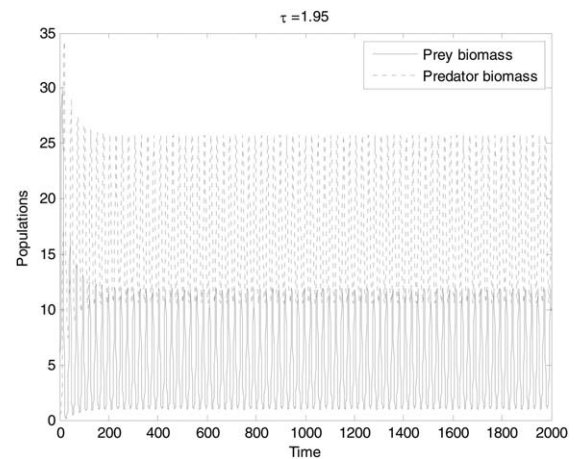
To numerically validate the existence of Hopf bifurcation at the interior equilibrium of system (8) let us consider the following set of parameters:

**FIGURE 12**



Phase space trajectories of prey and predator biomass beginning with different initial levels when  $\tau = 1.75 = \tau^*$ .

**FIGURE 13**

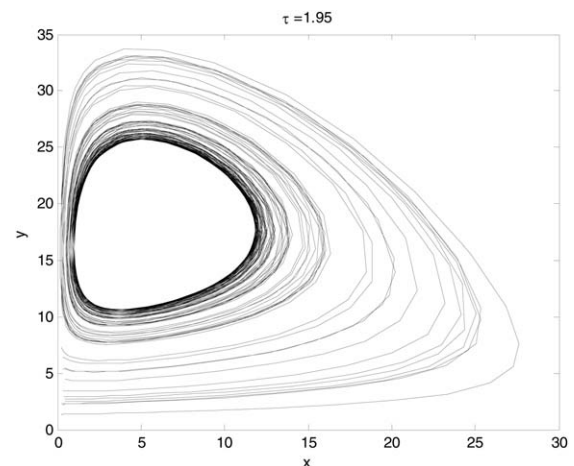


Variation of prey and predator biomass with the increasing time when  $\tau = 1.95 > \tau^*$ .

$r=1.2, K=100, \alpha=0.75, a=5, q=0.2, d=0.001, \beta=0.6, \gamma=0.025, p=12, c=0.5$ .

Let us consider the value of  $\tau=0.42$  then it is observed from the Figures 3 and 4 that  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  is locally asymptotically stable and the populations  $x$  and  $y$  converge to their steady states in finite time. But, if we gradually increase the value of  $\tau$ , keeping other parameters fixed, then, by Theorem3, it is possible to get the critical value

**FIGURE 14**



Phase space trajectories of prey and predator biomass beginning with different initial levels when  $\tau = 1.95 > \tau^*$ .

of  $\tau^*=0.52$  such that  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  loses its stability as  $\tau$  passes through  $\tau^*$ . Figures 5 and 6 clearly show the result. It is also noted that if we consider the value of  $\tau=0.62$ , then it is evident from Figures 7 and 8 that the positive equilibrium  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  is unstable and there is a periodic orbit near  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ .

Hopf bifurcation phenomenon can also be illustrated using the following set of numerical parameters of the system (8).

$$r=2.5, K=100, \alpha=0.8, a=15, q=0.5, d=0.001, \beta=0.75, \\ \gamma=0.01, p=12, c=0.5$$

It may be noted that the critical value of the bifurcation parameter is  $\tau^*=1.75$ . It is clearly observed from Figures 9 and 10 that  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  is locally asymptotically stable and the populations  $x$  and  $y$  converge to their steady states in finite time when  $\tau=1.5$ . However, a simple Hopf bifurcation occurs at  $\tau=\tau^*$ , depicted by Figures 11 and 12, and consequently for  $\tau > \tau^*$  the interior equilibrium point,  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$  approaches to a periodic solution and there exists a periodic orbit near  $P(\hat{x}, \hat{y}, \hat{z}, \hat{E})$ . Figures 13 and 14 clearly show the result.

## 5. CONCLUDING REMARKS

The article analyzes the dynamical behavior of a prey predator system with Holling type III functional response function. Differential-algebraic systems theory is used to investigate the dynamic behavior of the system. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations thus we have studied the effects of continuous time delay on the dynamics of prey predator system. We have studied the existence of singularity induced bifurcation phenomenon. At singularity induced bifurcation, one eigenvalue of the system Jacobian  $M$  at the equilibrium point  $P_3$  becomes unbounded and the other keeps stable. From a biological point of view, singularity induced bifurcation shows that economic profit may cause impulsive phenomenon, that is, rapid expansion of biological population, which may result in ecosystem unbalance and even biological disaster. Furthermore, we have designed state feedback controller to stabilize the system at positive economic rent. Numerical simulations are carried out to prove that state feedback controller can be designed to resume the stability of the system at positive economic interest.

It is observed that intraspecific competition of predators plays an important role to the dynamics of the sys-

tem. In this regard, it may be noted that the system is stable without time delay, but it is possible to derive a critical value of total economic profit such that the system becomes unstable when total economic profit passes through the critical value and system enters into Hopf type small amplitude periodic solution. Conversely, continuous time delay can also cause of a stable equilibrium to become unstable. We have shown the occurrence of Hopf bifurcation when the time delay passes through its critical value.

We have already provided several real examples in the introduction part where prey population is subjected to harvest as we have considered in our model. It is also obvious that the density of prey population would be decreased with the increasing time and harvesting effort. Again, Holling type III functional response is originally proposed as a model to represent a predator switching its preference from one focal prey to an alternative prey at times when the focal prey is at low densities or depleted. Therefore, in this regard, the entire analysis presented in this article seems to be more realistic and extended compare to the analysis reported by other researchers.

The functional form of harvest, in the present model, is considered using the phrase CPUE hypothesis, but this functional form gives us some unrealistic features like, unbounded linear increase of  $h$  with  $x$  for fixed  $E$ , and so forth. These hypothetical features can be removed using the functional form,  $h=qEx/(uE+vx)$ . To avoid the complexity of the system, I did not consider this modified functional form of harvest in the present model, but this concern would be addressed in my future research problems. In the present modelling approach, I have considered biological as well as economic factors into account. But the natural extension of the model is to introduce more general profit function. This would make the model more interesting and fruitful in a real-world fisheries management. To achieve the commercial purpose of the fishery, it is possible to determine optimal strategies using game theory which may be considered another important area to explore.

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