

$$\int_0^1 \sqrt{x + \sqrt{x}} \, dx の計算$$

$$x = t^4 \text{ で置換積分すると、} \int_0^1 \sqrt{x + \sqrt{x}} \, dx = \int_0^1 4t^4 \sqrt{1+t^2} \, dt$$

$t = \sinh \theta$ で置換積分すると、 $\operatorname{arcsinh}(1) = \log(1 + \sqrt{2}) = \alpha$ として

$$\begin{aligned} \int_0^1 4t^4 \sqrt{1+t^2} \, dt &= 4 \int_0^\alpha \sinh^4 \theta \cosh^2 \theta d\theta = 4 \int_0^\alpha \sinh^6 \theta + \sinh^4 \theta \, d\theta \\ \sinh^6 \theta &= \frac{1}{2^6} (e^\theta - e^{-\theta})^6 = \frac{1}{2^5} (\cosh 6\theta - 6 \cosh 4\theta + 15 \cosh 2\theta - 10) \\ \sinh^4 \theta &= \frac{1}{2^4} (e^\theta - e^{-\theta})^4 = \frac{1}{2^3} (\cosh 4\theta - 4 \cosh 2\theta + 3) \text{ より、} \\ 4 \int_0^\alpha \sinh^6 \theta + \sinh^4 \theta \, d\theta &= 4 \int_0^\alpha \frac{1}{2^5} \cosh 6\theta - \frac{1}{2^4} \cosh 4\theta - \frac{1}{2^5} \cosh 2\theta + \frac{1}{2^4} \, d\theta \\ &= \frac{1}{48} (\sinh 6\alpha - 3 \sinh 4\alpha - 3 \sinh 2\alpha + 12\alpha) \end{aligned}$$

$$\begin{aligned} \sinh \alpha &= 1 & \cosh \alpha &= \sqrt{2} & \sinh 2\alpha &= 2\sqrt{2} & \cosh 2\alpha &= 3 & \sinh 4\alpha &= 12\sqrt{2} \\ \cosh 4\alpha &= 17 & \sinh 6\alpha &= \sinh 4\alpha \cosh 2\alpha + \sinh 2\alpha \cosh 4\alpha &= 70\sqrt{2} \text{ より、} \end{aligned}$$

$$\frac{1}{48} (\sinh 6\alpha - 3 \sinh 4\alpha - 3 \sinh 2\alpha + 12\alpha) = \frac{7}{12} \sqrt{2} + \frac{1}{4} \alpha = \boxed{\frac{7}{12} \sqrt{2} + \frac{1}{4} \log(1 + \sqrt{2})}$$