OLG model: tax-transfer

lump-sum

① Different but not that different!

Note: $\sigma > 0$

$$U(c) = \ln(c^\sigma)$$
$$= \sigma \ln(c)$$

"NEW"

c.f. in lecture/tutorial $U(c) = \ln(c)$

② Different but not that different!

Production:

$$Y_t = K_t^\alpha (A_t N_t)^{1-\alpha}$$

(A_t grows at constant rate g.)

Government

Tax (lump-sum) the young/workers some amount T_t.

Funds benefits to old, b_t.

Part 1

Accounting: government budget constraint.

At every date t,

$$\underbrace{\underbrace{N_{t-1}}_{\text{\# old at } t}\, b_t}_{\text{total benefit to all old agents}} = \underbrace{N_t T_t}_{\text{total tax revenue raised from young agents}}$$

Since we assumed $N_t = (1+n) N_{t-1}$,

then we can also write:

$$b_t = (1+n) T_t$$

Part 2

Consider a date-t young agent:

$$\max_{s_t,\, c_{y,t},\, c_{o,t+1}} \left\{ u(c_{y,t}) + \beta u(c_{o,t+1}) : \begin{array}{l} c_{y,t} = w_t - s_t - T_t \\ c_{o,t+1} = R^e_{t+1} s_t + \underbrace{(1+n) T^e_{t+1}}_{b^e_{t+1}} \end{array} \right\}$$

expected future

FONC:

$$s_t: \quad -u'(w_t - s_t - T_t) + \beta u'\left[R^e_{t+1} s_t + (1+n) T^e_{t+1}\right] R^e_{t+1} = 0 \quad (1)$$

Given assumption $u(c) = \ln(c^\sigma) = \sigma \times \ln(c)$, then (1) becomes

$$\frac{-\cancel{\sigma}\,1}{w_t - s_t - T_t} + \frac{\beta \cancel{\sigma}}{R^e_{t+1} s_t + (1+n) T^e_{t+1}} \cdot R^e_{t+1} = 0.$$

Simplify ...

$$\boxed{s_t^* = \frac{\beta}{1+\beta}(w_t - T_t) - \frac{1}{1+\beta}\frac{(1+n) T^e_{t+1}}{R^e_{t+1}}}$$

Firm's profit maximization ...

FONC:

k_t : $$\underbrace{\alpha A_t^{1-\alpha} k_t^{\alpha-1}}_{\text{? MPK}} = \underbrace{R_t}_{\substack{\text{gross rental} \\ \text{cost of capital}}} \qquad (R_t = 1 + r_t - \delta)$$

N_t : $$\underbrace{(1-\alpha) A_t^{1-\alpha} k_t^{\alpha}}_{\text{MPN}} = \underbrace{w_t}_{\text{wage rate.}}$$

Market clearing

Sufficient to check that capital market clears...

$$\underbrace{K_{t+1}}_{\substack{\text{total capital} \\ \text{demanded}}} = \underbrace{s_t^* N_t}_{\substack{\text{total supply} \\ \text{of saving from} \\ \text{young at } t}}$$

$$\Rightarrow \quad \underbrace{\frac{K_{t+1}}{N_{t+1}}}_{k_{t+1}} \left(\frac{N_{t+1}}{N_t}\right) = s_t^* \quad \Rightarrow \quad k_{t+1} = \frac{s_t^*}{1+n}$$

<u>DEFINITION</u> A <u>recursive competitive equilibrium</u> (RCE) with lump-sum taxes/transfers $\{T_t\}_{t=0}^{\infty}$, given initial states k_0 and A_0, given $\{A_t\}_{t=0}^{\infty}$ at each date t:

(a) Households save optimally:

$$s_t^* = \frac{\beta}{1+\beta}(w_t - T_t) - \frac{1}{1+\beta}\frac{(1+n)T_{t+1}^e}{R_{t+1}^e}.$$

(b) Firms max. profit:

$$\alpha A_t^{1-\alpha} k_t^{\alpha-1} = R_t$$

$$(1-\alpha) A_t^{1-\alpha} k_t^{\alpha} = w_t$$

(c) Markets clear:

$$(1+n) k_{t+1} = s_t^*$$

(d) Government budget balances:

$$b_t = (1+n) T_t$$

(e) Agents have perfect foresight:

$$R_{t+1}^e = R_{t+1} \text{ and } T_{t+1}^e = T_{t+1}.$$

Simplifying... RCE – TT can be summarized by a scalar difference equation in k_t:

$$k_{t+1} = \frac{\beta\left((1-\alpha)A_t^{1-\alpha}h_t^{\alpha} - T_t\right) - \dfrac{(1+n)\,T_{t+1}}{\alpha(h_{t+1}/A_{t+1})^{\alpha-1}}}{(1+n)(1+\beta)} \qquad (2)$$

Recall, the question asks you to define a stationary transformation:

$$\tilde{k}_t = \frac{k_t}{A_t}$$

Divide (2) by A_t on both sides:

$$\vdots$$

$$\boxed{\tilde{k}_{t+1} = \frac{\beta\left((1-\alpha)\tilde{k}_t^{\alpha} - \tilde{T}_t\right) - \frac{(1+n)\tilde{T}_{t+1}}{\alpha \tilde{k}_{t+1}^{\alpha-1}}}{(1+g)(1+n)(1+\beta)} \qquad (3)}$$

RCE-TT in terms of stationary variables.

Part 3 $T_t = \tau w_t$

We can show (see tutorial)

for positive savings:

$\vdots$

$$\boxed{-\frac{\alpha(1+\beta)}{1-\alpha} < \tau < 1} \quad (4)$$

But we also need to check that lifetime income of agents is positive ...

$$\underbrace{(\tilde{w}_t - \tilde{T}_t)}_{\boxed{A}\text{ after tax income of young}} + \underbrace{\frac{(1+n)\tilde{T}_{t+1}}{\tilde{R}_{t+1}}}_{\boxed{B}\text{ P.V. of benefit when old}} > 0$$

This implies that

$$\underbrace{(1-\alpha)(1-\tau)\tilde{k}_t^{\alpha}}_{\boxed{A}} + \underbrace{\frac{(1+n)\,\tau(1-\alpha)\tilde{k}_{t+1}^{\alpha}}{\alpha\tilde{k}_{t+1}^{\alpha-1}}}_{\boxed{B}} > 0$$

Use (3), we can simplify last line to

$$\boxed{\tau > -\frac{\alpha}{1-\alpha}} \qquad (5)$$

Note that (5) implies first inequality in (4).

$$\therefore \boxed{\tau_{min} = \frac{\alpha}{1-\alpha}}$$

Part 4

$$\boxed{\text{Show that} \ldots \quad \frac{\partial \tilde{k}}{\partial \tau} \gtreqless 0.}$$

From RCE-TT description in (3), impose steady state: $\boxed{\tilde{k} = \tilde{k}_{t+1} = \tilde{k}_t}$ → (3) you get:

$$\tilde{k} = \left[\underbrace{\frac{\alpha\beta(1-\alpha)}{(1+g)(1+n)}}_{(+)\text{ monotonic}} \underbrace{\left[\alpha(1+\beta) + (1-\alpha)\tau\right](1-\tau)}_{\equiv R(\tau)} \right]^{\frac{1}{1-\alpha}}$$

sign$\left(\frac{\partial \tilde{k}}{\partial \tau}\right)$ depends on this!

monotonic

Checks:

$$\text{sign}\left(\frac{\partial \tilde{k}}{\partial \tau}\right) ?$$

Equivalently, check:

$$\frac{\partial K(\tau)}{\partial \tau} = \left[(1-\alpha) - \alpha(1+\beta)\right] - 2(1-\alpha)\tau$$

$\text{sign}\left(\frac{\partial K(\tau)}{\partial \tau}\right)$ is ambiguous.

Note:

- If $\tau < \frac{(1-\alpha) - \alpha(1+\beta)}{2(1-\alpha)}$ then $\frac{\partial K(\tau)}{\partial \tau} > 0$

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