Learning Report(5.4)

Article : Modeling and Vibration Control of the Flapping-Wing Robotic Aircraft with Output Constraint

1 Dynamics Modeling

The kinetic energy of the flexible wings $E_k(t)$ is

$$E_kt) = \frac{1}{2}m\int_0^L [\dot{y}(x,t)]^2 dx + \frac{1}{2}I_p\int_0^L [\dot{\theta}(x,t)]^2 dx \tag{1}$$

The potential energy of the flexible wing $E_p(t)$ is

$$E_p(t) = \frac{1}{2}EI_b\int_0^L [y''(x,t)]^2 dx + \frac{1}{2}GJ\int_0^L [\theta'(x,t)]^2 dx \tag{2}$$

The virtual work done by the bending and torsion rigidity coupling can be expressed as

$$\delta W_c(t) = mx_ec\int_0^L \ddot{y}(x,t)\delta\theta(x,t)dx + mx_ec\int_0^L \ddot{\theta}(x,t)\delta y(x,t)dx \tag{3}$$

The virtual work done by the Kelvin-Voigt damping on the wing is

$$\delta W_d(t) = -\eta EI_b\int_0^L \dot{y}''(x,t)\delta\theta(x,t)dx - \eta GJ\int_0^L \dot{\theta}'(x,t)\delta\theta'(x,t)dx \tag{4}$$

The virtual work done by distributed disturbance can be obtained as follows

$$\delta W_f(t) = \int_0^L [F_b(x,t)\delta y(x,t) - x_acF_b(x,t)\delta\theta(x,t)]dx \tag{5}$$

The virtual work done by the control force can be represented as

$$\delta W_u(t) = U_1(t)\delta y(L,t) + U_2(t)\delta\theta(L,t) \tag{6}$$

The total virtual work can be expressed as

$$\delta W(t) = \delta[W_c(t) + W_d(t) + W_f(t) + W_u(t)] \tag{7}$$

Lemma 1 *Calculus of variations:*

$$\delta y = \frac{\partial y}{\partial x}\delta x \tag{8}$$

Lemma 2 *Partial integration of variational method:*

$$\int_a^b \frac{\partial F}{\partial y'}\delta y' dx = -\int_a^b \frac{d}{dx}(\frac{\partial F}{\partial y'})\partial y dx + \frac{\partial F}{\partial y'}\delta y \mid_a^b \tag{9}$$

$$\int_a^b \frac{\partial F}{\partial y''}\delta y'' dx = \int_a^b \frac{d^2}{dx^2}(\frac{\partial F}{\partial y''})\partial y dx - \frac{d}{dx}\frac{\partial F}{\partial y''}\delta y \mid_a^b + \frac{\partial F}{\partial y''}\delta y' \mid_a^b \tag{10}$$

Substituting above lemmas into the Hamilton's principle $\int_{t_1}^{t_2} \delta[E_k(t) - E_p(t) + W(t)]dt = 0$, where $\delta y(x,t) = 0$, $\delta\theta(t) = 0$ at $t = t_1, t_2$, yields

$$\begin{aligned}
&\textstyle\int_{t_1}^{t_2}\int_0^L \{-m\ddot{y}(x,t)\delta y(x,t) - I_p\ddot{\theta}(x,t)\delta\theta(x,t) - EI_b y''''(x,t)\delta y(x,t) + GJ\theta''(x,t)\delta\theta(x,t) \\
&+ mx_e c\ddot{y}(x,t)\delta\theta(x,t) + mx_e\ddot{\theta}(x,t)\delta y(x,t) - \eta EI_b\dot{y}''''(x,t)\delta y(x,t) + \eta GJ\dot{\theta}''(x,t)\delta\theta(x,t) \\
&+ F_b(x,t)\delta y(x,t) - x_a c F_b(x,t)\delta\theta(x,t)\}dxdt + \textstyle\int_{t_1}^{t_2}\{-EI_b y''(x,t)\delta y'(x,t)|_0^L \\
&+ EI_b y'''(x,t)\delta y(x,t)|_0^L - GJ\theta'(x,t)\delta\theta(x,t)|_0^L - \eta EI_b\dot{y}''(x,t)\delta y'(x,t)|_0^L \\
&+ \eta EI_b\dot{y}'''(x,t)\delta y(x,t)|_0^L - \eta GJ\dot{\theta}'(x,t)\delta\theta(x,t)|_0^L + U_1(t)\delta y(L,t) + U_2(t)\delta\theta(L,t)\}dt = 0
\end{aligned} \tag{11}$$

Therefore we can get the governing equations as:

$$m\ddot{y}(x,t) + EI_b y''''(x,t) - mx_e c\ddot{\theta}(x,t) + \eta EI_b\dot{y}''''(x,t) = F_b(x,t) \tag{12}$$

$$I_p\ddot{\theta}(x,t) - GJ\theta''(x,t) - mx_e c\ddot{y}(x,t) - \eta GJ\dot{\theta}''(x,t) = -x_a c F_b(x,t) \tag{13}$$

and the boundary conditions are:

$$\theta(0,t) = y(0,t) = y'(0,t) = y''(L,t) = 0 \tag{14}$$

$$EI_b y'''(L,t) + \eta EI_b\dot{y}'''(L,t) = -U_1(t) \tag{15}$$

$$GJ\theta'(L,t) + \eta GJ\dot{\theta}(L,t) = U_2(t) \tag{16}$$

2 Boundary Control Design

The dynamic model of the system is expressed by partial differntial governing equations and ODE boundary conditions. Using the proposed IBLF-based method, the model-based boundary controls are designed as:

$$U_1(t) = -\frac{[k1y(L,t) + k_2\dot{y}(L,t)]}{D^2 - y^2(L,t)} \tag{17}$$

$$U_2(t) = -\frac{[k_3\theta(L,t) + k_4\dot{\theta}(L,t)]}{\phi^2 - \theta^2(L,t)} \tag{18}$$

The barrier Lyapunov function candidate(LFC) is proposed as follows

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{19}$$

where each term is respectively defined as

$$\begin{aligned} V_1(t) =& \frac{am}{2}\int_0^L [\dot{y}(x,t)]^2 dx + \frac{aEI_b}{2}\int_0^L [y''(x,t)]^2 dx \\ &+ \frac{aI_p}{2}\int_0^L [\dot{\theta}(x,t)]^2 dx + \frac{aGJ}{2}\int_0^L [\theta'(x,t)]^2 dx \end{aligned} \tag{20}$$

$$V_2(t) = \frac{ak_1 + bk_2}{2} ln\frac{D^2}{D^2 - y^2(L,t)} + \frac{ak_3 + bk_4}{2} ln\frac{\phi^2}{\phi^2 - \theta^2(L,t)} \tag{21}$$

$$\begin{aligned} V_3(t) =& bm\int_0^L \dot{y}(x,t)y(x,t)dx + bI_p\int_0^L \dot{\theta}(x,t)\theta(x,t)dx \\ &- bmx_ec\int_0^L [y(x,t)\dot{\theta}(x,t) + \theta(x,t)\dot{y}(x,t)]dx \\ &- amx_ec\int_0^L \dot{\theta}(x,t)\dot{y}(x,t)dx \end{aligned} \tag{22}$$

Lemma 3 *Let $\phi_1(x,t), \phi_2(x,t) \in \mathbb{R}$ with $x \in [0,L]$, there is the inequality:*

$$|\phi_1\phi_2| = |(\frac{1}{\sqrt{\delta}}\phi_1)(\sqrt{\delta}\phi_2)| \le \frac{1}{\delta}\phi_1^2 + \delta\phi_2^2 \tag{23}$$

$\forall\phi_1, \phi_2 \in \mathbb{R}$ *and* $\delta > 0$.

Lemma 4 *Let $\phi(x,t) \in \mathbb{R}$ be an integrable function defined on $x \in [0,L]$ and $t \in [0,\infty)$, satisfying the boundary condition $\phi(0,t) = 0$, then the following inequality holds:*

$$\phi^2(x,t) \le L\int_0^L [\phi'(x,t)]^2 dx \tag{24}$$

if the function $\phi(x,t)$ satisfies the boundary condition $\phi'(0,t)=0$ further, then the following inequalities also holds:

$$\int_0^L \phi^2(x,t)dx \le L^2 \int_0^L [\phi'(x,t)]^2 dx \tag{25}$$

$$\phi'^2(x,t) \le L \int_0^L [\phi''(x,t)]^2 dx \tag{26}$$

Theorem 1 . *The barrier LFC given by (24) can be upper and lower bounded as:*

$$0 \le \xi_1[\upsilon(t)+V_2(t)] \le V(t) \le \xi_2[\upsilon(t)+V_2(t)] \tag{27}$$

where ξ_1 and ξ_2 are two positive numbers.

Proof: We define a new auxiliary function

$$\xi(t) = \int_0^L \{[\dot{\theta}(x,t)]^2 dx + [\dot{y}(x,t)]^2 dx + [\theta'(x,t)]^2 dx + [y''(x,t)]^2 dx\} \tag{28}$$

We can obtain that:

$$\mu_1 \upsilon(t) \le V_1(t) \le \mu_2 \upsilon(t) \tag{29}$$

where $\mu_1 = \frac{a}{2} min(n, I_p, EI_b, GJ) > 0$, and $\mu_2 = \frac{a}{2} max(n, I_p, EI_b, GJ) > 0$.

Applying **Lemma 7** and **8** to $V_3(t)$, we obtain

$$\begin{aligned} |V_3(t)| \le (bm + bmx_e c + amx_e c) \int_0^L [\dot{y}(x,t)]^2 dx + (bI_p + bmx_e c + amx_e c) \int_0^L [\dot{\theta}(x,t)]^2 dx \\ +(bm + bmx_e c)L^4 \int_0^L [y''(x,t)]^2 dx + (bI_p + bmx_e c)L^2 \int_0^L [\theta'(x,t)]^2 dx \le \mu_3 \upsilon(t) \end{aligned} \tag{30}$$

where

$$\mu_3 = max\{bm, bmx_e c + amx_e c, bI_p + bmx_e c + amx_e c, (bm + bmx_e c)L^4, (bI_p + bmx_e c)L^2\} \ge 0$$

We have

$$0 \le \beta_1 \upsilon(t) \le V_1(t) + V_3(t) \le \beta_2 \upsilon(t) \tag{31}$$

and $\beta_1 = \mu_1 - \mu_3, \beta_2 = \mu_2 + \mu_3$, and we chose appropriate values of a and b to guarantee $\beta_1 \ge 0$.

We can prove that the barrier LFC has upper and lower bounds as follows

$$0 \leq \xi_1[\upsilon(t) + V_2(t)] \leq V(t) \leq \xi_2[\upsilon(t) + V_2(t)] \tag{32}$$

where $\xi_1 = min(\beta_1, 1)$ and $\xi_2 = max(\beta_2, 1)$ are positive constants.

Theorem 2 . *Using the proposed control laws, time derivative of the barrier LFC is negative definite, and its upper bound as follow*

$$\dot{V}(t) \leq -\xi V(t) + \varepsilon \tag{33}$$

where ξ is a positive constant.

Proof: Differentiating (24) with respect to time leads to

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \tag{34}$$

In detail,

$$\begin{aligned}\dot{V}_1(t) =& am \int_0^L \dot{y}(x,t)\ddot{y}(x,t)dx + aI_p \int_0^L \ddot{\theta}(x,t)\dot{\theta}(x,t)dx \\ &+ aGJ \int_0^L \dot{\theta}'(x,t)\theta'(x,t)dx + aEI_b \int_0^L \dot{y}''(x,t)y''(x,t)dx\end{aligned} \tag{35}$$

Similarly, $\dot{V}_2(t)$ is written as

$$\dot{V}_2(t) = \frac{(ak_1 + bk_2)y(L,t)\dot{y}(L,t)}{D^2 - y^2(L,t)} + \frac{(ak_3 + bk_4)\theta(L,t)\dot{\theta}(L,t)}{\phi^2 - \theta^2(L,t)} \tag{36}$$

and

$$\begin{aligned}\dot{V}_3(t) =& bm \int_0^L \ddot{y}(x,t)y(x,t)dx + bm \int_0^L [\dot{y}(x,t)]^2dx + bI_p \int_0^L \ddot{\theta}(x,t)\theta(x,t)dx + bI_p \int_0^L [\dot{\theta}(x,t)]^2dx \\ &- bmx_ec \int_0^L [y(x,t)\ddot{\theta}(x,t) + 2\dot{y}(x,t)\dot{\theta}(x,t) + \ddot{y}(x,t)\theta(x,t)]dx \\ &- amx_ec \int_0^L [\ddot{y}(x,t)\dot{\theta}(x,t) + \dot{y}(x,t)\ddot{\theta}(x,t)]dx\end{aligned} \tag{37}$$

With the system governing equations (refgovern1) and (refgovern2), and from the boundary conditions (refbound1)-(refbound3), applying the **Lemma 7** and **8**, we enlarge $\dot{V}_1(t)$ and $\dot{V}_3(t)$ as

$$\begin{aligned}
\dot{V}_1(t) \leq & -(\frac{a\eta EI_b}{2L^4} - \sigma_1 a)\int_0^L [\dot{y}(x,t)]^2 dx - (\frac{a\eta GJ}{2L^2} - \sigma_2 a x_a c)\int_0^L [\dot{\theta}(x,t)]^2 dx - \frac{a\eta EI_b}{2}\int_0^L [\dot{y}''(x,t)]^2 dx \\
& - \frac{a\eta GJ}{2}\int_0^L [\dot{\theta}'(x,t)]^2 dx + amx_e c\int_0^L [\dot{\theta}(x,t)\ddot{y}(x,t) + \dot{y}(x,t)\ddot{\theta}(x,t)]dx - aEI_b\dot{y}(x,t)y'''(L,t) \\
& - a\eta EI_b\dot{y}(L,t) + aGJ\dot{\theta}(L,t)\theta'(L,t) + a\eta GJ\dot{\theta}(L,t)\dot{\theta}'(L,t) + (\frac{a}{\sigma_1} + \frac{ax_a c}{\sigma_2})LF_{b\max}^2 \qquad (38)
\end{aligned}$$

where $\sigma_1 > 0$ and $\sigma_2 > 0$.

$$\begin{aligned}
\dot{V}_3(t) \leq & -bEI_b y(L,t)y'''(L,t) - bEI_b\int_0^L [y''(x,t)]^2 dx - b\eta EI_b y(L,t)\dot{y}'''(L,t) + \frac{b\eta EI_b}{\sigma_3}\int_0^L [y''(x,t)]^2 dx \\
& + b\eta EI_b\sigma_3\int_0^L [\dot{y}''(x,t)]^2 dx + bGJ\theta'(L,t) - bGJ\int_0^L [\theta'(x,t)]^2 dx + b\eta GJ\theta(L,t)\dot{\theta}'(L,t) \\
& + \frac{b\eta GJ}{\sigma_4}\int_0^L [\theta'(x,t)]^2 dx + \sigma_4 b\eta GJ\int_0^L [\dot{\theta}'(x,t)]^2 dx + bm\int_0^L [\dot{y}(x,t)]^2 dx + bI_p\int_0^L [\dot{\theta}(x,t)]^2 dx \\
& - amx_e c\int_0^L [\dot{y}(x,t)\ddot{\theta}(x,t) + \ddot{y}(x,t)\dot{\theta}(x,t)]dx + 2bmx_e c\sigma_5\int_0^L [\dot{y}(x,t)]^2 dx + \sigma_6 bL^4\int_0^L [y''(x,t)]^2 dx \\
& + \frac{2bmx_e c}{\sigma_5}\int_0^L [\dot{\theta}(x,t)]^2 dx + \sigma_7 bL^2 x_a c\int_0^L [\theta'(x,t)]^2 dx + (\frac{b}{\sigma_6} + \frac{bx_a c}{\sigma_7})LF_{b\max}^2 \qquad (39)
\end{aligned}$$

where $\sigma_3 - \sigma_7$ are positive constants.

Substituting (36),(38) and (39), we have

$$\begin{aligned}
\dot{V}(t) \leq & -(\frac{a\eta EI_b}{2L^4} - \sigma_1 a - bm - 2bmx_e c\sigma_5)\int_0^L [\dot{y}(x,t)]^2 dx - (bEI_b - \frac{b\eta EI_b}{\sigma_3} - \sigma_6 bL^4)\int_0^L [y''(x,t)]^2 dx \\
& -(\frac{a\eta GJ}{2L^2} - \sigma_2 a x_a c - bI_p - \frac{2bmx_e c}{\sigma_5})\int_0^L [\dot{\theta}(x,t)]^2 dx - (bGJ - \frac{b\eta GJ}{\sigma_4} - \sigma_7 bL^4 x_a c)\int_0^L [\theta'(x,t)]^2 dx \\
& - bk_1 ln\frac{D^2}{D^2 - y^2(L,t)} - bk_3 ln\frac{\phi^2}{\phi^2 - \theta^2(L,t)} - (\frac{a\eta EI_b}{2} - b\eta EI_b\sigma_3)\int_0^L [\dot{y}''(x,t)]^2 dx \\
& -(\frac{a\eta GJ}{2} - \sigma_4 b\eta GJ)\int_0^L [\dot{\theta}'(x,t)]^2 dx - \frac{ak_2[\dot{y}(L,t)]^2}{D^2 - y^2(L,t)} - \frac{ak_4[\dot{\theta}(L,t)]^2}{\phi^2 - \theta^2(L,t)} \\
& + (\frac{a}{\sigma_1} + \frac{ax_a c}{\sigma_2} + \frac{b}{\sigma_6} + \frac{bx_a c}{\sigma_7})LF_{b\max}^2 \\
\leq & -\xi_3[\upsilon(t) + V_2(t)] + \varepsilon \qquad (40)
\end{aligned}$$

and

$$
\begin{aligned}
\xi_3 =& min(\frac{a\eta EI_b}{2L^4} - \sigma_1 a - bm - 2bmx_e c\sigma_5, bEI_b - \frac{b\eta EI_b}{\sigma_3} - \sigma_6 bL^4, \frac{a\eta GJ}{2L^2} - \sigma_2 ax_a c - bI_p - \frac{2bmx_e c}{\sigma_5}, \\
& bGJ - \frac{b\eta GJ}{\sigma_4} - \sigma_7 bL^4 x_a c, \frac{2bk_1}{ak_1+bk_2}, \frac{2bk_3}{ak_3+bk_4}) \\
\varepsilon =& (\frac{a}{\sigma_1} + \frac{ax_a c}{\sigma_2} + \frac{b}{\sigma_6} + \frac{bx_a c}{\sigma_7})LF_{\mathrm{bmax}}^2
\end{aligned}
$$

At the same time, positive constants a, b and $\sigma_1 - \sigma_7$ are chosen to satisfy the following inequalities:

$$\frac{a\eta EI_b}{2L^4} - \sigma_1 a - bm - 2bmx_e c\sigma_5 \geq 0 \tag{41}$$

$$bEI_b - \frac{b\eta EI_b}{\sigma_3} - \sigma_6 bL^4 \geq 0 \tag{42}$$

$$\frac{a\eta GJ}{2L^2} - \sigma_2 ax_a c - bI_p - \frac{2bmx_e c}{\sigma_5} \geq 0 \tag{43}$$

$$bGJ - \frac{b\eta GJ}{\sigma_4} - \sigma_7 bL^4 x_a c \geq 0 \tag{44}$$

$$\frac{a\eta EI_b}{2} - b\eta EI_b\sigma_3 \geq 0 \tag{45}$$

$$\frac{a\eta GJ}{2} - \sigma_4 b\eta GJ \geq 0 \tag{46}$$

Combining **Theorem 1** and **2**, we obtain:

$$\dot{V}(t) \leq -\xi V(t) + \varepsilon \tag{47}$$

where $\xi = \frac{\xi_3}{\xi_2}$.

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$$E_p(t) = \frac{1}{2}EI_b\int_0^L [y''(x,t)]^2 dx + \frac{1}{2}GJ\int_0^L [\theta'(x,t)]^2 dx \tag{2}$$

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$$\delta W_d(t) = -\eta EI_b\int_0^L \dot{y}''(x,t)\delta\theta(x,t)dx - \eta GJ\int_0^L \dot{\theta}'(x,t)\delta\theta'(x,t)dx \tag{4}$$

The virtual work done by distributed disturbance can be obtained as follows

$$\delta W_f(t) = \int_0^L [F_b(x,t)\delta y(x,t) - x_acF_b(x,t)\delta\theta(x,t)]dx \tag{5}$$

The virtual work done by the control force can be represented as

$$\delta W_u(t) = U_1(t)\delta y(L,t) + U_2(t)\delta\theta(L,t) \tag{6}$$

The total virtual work can be expressed as

$$\delta W(t) = \delta[W_c(t) + W_d(t) + W_f(t) + W_u(t)] \tag{7}$$

Lemma 5 *Calculus of variations:*

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$$\int_a^b \frac{\partial F}{\partial y'}\delta y' dx = -\int_a^b \frac{d}{dx}(\frac{\partial F}{\partial y'})\partial y dx + \frac{\partial F}{\partial y'}\delta y \mid_a^b \tag{9}$$

$$\int_a^b \frac{\partial F}{\partial y''}\delta y'' dx = \int_a^b \frac{d^2}{dx^2}(\frac{\partial F}{\partial y''})\partial y dx - \frac{d}{dx}\frac{\partial F}{\partial y''}\delta y \mid_a^b + \frac{\partial F}{\partial y''}\delta y' \mid_a^b \tag{10}$$

Integrating $E_k(t)$ with respect to t yield

$$\begin{aligned}\int_{t_1}^{t_2} \delta E_k dt =& m\int_{t_1}^{t_2}\int_0^L \dot{y}(x,t)\delta\dot{y}(x,t)dxdt + I_p\int_{t_1}^{t_2}\int_0^L \dot{\theta}(x,t)\delta\dot{\theta}(x,t)dxdt \\ =& m\int_0^L [\dot{y}(x,t)\delta y(x,t)]|_{t_1}^{t_2} dx - m\int_{t_1}^{t_2}\int_0^L [\ddot{y}(x,t)\delta y(x,t)]dxdt \\ &+ I_p\int_0^L [\dot{\theta}(x,t)\delta\theta(x,t)]|_{t_1}^{t_2} dx - I_p\int_{t_1}^{t_2}\int_0^L [\ddot{\theta}(x,t)\delta\theta(x,t)]dxdt \\ =& -m\int_{t_1}^{t_2}\int_0^L [\ddot{y}(x,t)\delta y(x,t)]dxdt - I_p\int_{t_1}^{t_2}\int_0^L [\ddot{\theta}(x,t)\delta\theta(x,t)]dxdt\end{aligned} \tag{11}$$

Similarly, we can obtain variation about the potential energy

$$\begin{aligned}\int_{t_1}^{t_2} \delta E_p dt =& EI_b\int_{t_1}^{t_2}\int_0^L y''(x,t)\delta y''(x,t)dxdt + GJ\int_{t_1}^{t_2}\int_0^L \theta'(x,t)\delta\theta'(x,t)dxdt \\ =& EI_b\int_{t_1}^{t_2} [y''(x,t)\delta y'(x,t)]|_0^L dt - EI_b\int_{t_1}^{t_2} [y'''(x,t)\delta y(x,t)]|_0^L dt \\ &+ EI_b\int_{t_1}^{t_2}\int_0^L y''''(x,t)\delta y(x,t)dxdt + GJ\int_{t_1}^{t_2} [\theta'(x,t)\delta\theta(x,t)]|_0^L dt \\ &- GJ\int_{t_1}^{t_2}\int_0^L \theta''(x,t)\delta\theta(x,t)dxdt\end{aligned} \tag{12}$$

The variation about stiffness coupling can be expressed

$$\int_{t_1}^{t_2} \delta W_c(t)dt = mx_e c\int_{t_1}^{t_2}\int_0^L \ddot{y}(x,t)\delta\theta(x,t)dxdt + mx_e c\int_{t_1}^{t_2}\int_0^L \ddot{\theta}(x,t) \tag{13}$$

The variation of Kelvin-Voigt damping on he right wing as follow

$$\begin{aligned}\int_{t_1}^{t_2} \delta W_d(t)dt =& -\eta EI_b \int_{t_1}^{t_2} [\dot{y}''(x,t)\delta y'(x,t)]|_0^L dt + \eta EI_b \int_{t_1}^{t_2} [\dot{y}'''(x,t)\delta y(x,t)]|_0^L dt \\ & -\eta EI_b \int_{t_1}^{t_2} \int_0^L \dot{y}''''(x,t)\delta y(x,t)dxdt - \eta GJ \int_{t_1}^{t_2} [\dot{\theta}'(x,t)\delta\theta(x,t)]|_0^L dt \\ & + \eta GJ \int_{t_1}^{t_2} \int_0^L \dot{\theta}''(x,t)\delta\theta(x,t)dxdt \end{aligned} \tag{14}$$

The variation about distributed disturbance and control force can be represented

$$\int_{t_1}^{t_2} \delta W_f(t)dt = \int_{t_1}^{t_2} \int_0^L [F_b(x,t)\delta y(x,t) - x_a c F_b(x,t)\delta\theta(x,t)]dxdt \tag{15}$$

$$\int_{t_1}^{t_2} \delta W_u(t)dt = \int_{t_1}^{t_2} [U_1(t)\delta y(L,t) - U_2(t)\delta\theta(L,t)]dt \tag{16}$$

Substituting above equations into the Hamilton's principle $\int_{t_1}^{t_2} \delta[E_k(t) - E_p(t) + W(t)]dt = 0$, where $\delta y(x,t) = 0$, $\delta\theta(t) = 0$ at $t = t_1, t_2$, yields

$$\begin{aligned} & \textstyle\int_{t_1}^{t_2} \int_0^L \{-m\ddot{y}(x,t)\delta y(x,t) - I_p\ddot{\theta}(x,t)\delta\theta(x,t) - EI_b y''''(x,t)\delta y(x,t) + GJ\theta''(x,t)\delta\theta(x,t) \\ & + m x_e c\ddot{y}(x,t)\delta\theta(x,t) + m x_e\ddot{\theta}(x,t)\delta y(x,t) - \eta EI_b\dot{y}''''(x,t)\delta y(x,t) + \eta GJ\dot{\theta}''(x,t)\delta\theta(x,t) \\ & + F_b(x,t)\delta y(x,t) - x_a c F_b(x,t)\delta\theta(x,t)\}dxdt + \textstyle\int_{t_1}^{t_2}\{-EI_b y''(x,t)\delta y'(x,t)|_0^L \\ & + EI_b y'''(x,t)\delta y(x,t)|_0^L - GJ\theta'(x,t)\delta\theta(x,t)|_0^L - \eta EI_b\dot{y}''(x,t)\delta y'(x,t)|_0^L \\ & + \eta EI_b\dot{y}'''(x,t)\delta y(x,t)|_0^L - \eta GJ\dot{\theta}'(x,t)\delta\theta(x,t)|_0^L + U_1(t)\delta y(L,t) + U_2(t)\delta\theta(L,t)\}dt = 0 \end{aligned} \tag{17}$$

Therefore we can get the governing equations as:

$$m\ddot{y}(x,t) + EI_b y''''(x,t) - m x_e c\ddot{\theta}(x,t) + \eta EI_b\dot{y}''''(x,t) = F_b(x,t) \tag{18}$$

$$I_p\ddot{\theta}(x,t) - GJ\theta''(x,t) - m x_e c\ddot{y}(x,t) - \eta GJ\dot{\theta}''(x,t) = -x_a c F_b(x,t) \tag{19}$$

and the boundary conditions are:

$$\theta(0,t) = y(0,t) = y'(0,t) = y''(L,t) = 0 \tag{20}$$

$$EI_b y'''(L,t) + \eta EI_b\dot{y}'''(L,t) = -U_1(t) \tag{21}$$

$$GJ\theta'(L,t) + \eta GJ\dot{\theta}(L,t) = U_2(t) \tag{22}$$

2 Boundary Control Design

The dynamic model of the system is expressed by partial differential governing equations and ODE boundary conditions. Using the proposed IBLF-based method, the model-based boundary controls are designed as:

$$U_1(t) = -\frac{[k_1 y(L,t) + k_2 \dot{y}(L,t)]}{D^2 - y^2(L,t)} \tag{23}$$

$$U_2(t) = -\frac{[k_3 \theta(L,t) + k_4 \dot{\theta}(L,t)]}{\phi^2 - \theta^2(L,t)} \tag{24}$$

The barrier Lyapunov function candidate(LFC) is proposed as follows

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{25}$$

where each term is respectively defined as

$$\begin{aligned} V_1(t) =& \frac{am}{2}\int_0^L [\dot{y}(x,t)]^2 dx + \frac{aEI_b}{2}\int_0^L [y''(x,t)]^2 dx \\ &+ \frac{aI_p}{2}\int_0^L [\dot{\theta}(x,t)]^2 dx + \frac{aGJ}{2}\int_0^L [\theta'(x,t)]^2 dx \qquad (26) \\ V_2(t) =& \frac{ak_1 + bk_2}{2} ln \frac{D^2}{D^2 - y^2(L,t)} + \frac{ak_3 + bk_4}{2} ln \frac{\phi^2}{\phi^2 - \theta^2(L,t)} \qquad (27) \\ V_3(t) =& bm \int_0^L \dot{y}(x,t) y(x,t) dx + bI_p \int_0^L \dot{\theta}(x,t)\theta(x,t) dx \\ &- bmx_e c \int_0^L [y(x,t)\dot{\theta}(x,t) + \theta(x,t)\dot{y}(x,t)] dx \\ &- amx_e c \int_0^L \dot{\theta}(x,t)\dot{y}(x,t) dx \qquad (28) \end{aligned}$$

Lemma 7 *Let $\phi_1(x,t), \phi_2(x,t) \in \mathbb{R}$ with $x \in [0, L]$, there is the inequality:*

$$|\phi_1 \phi_2| = |(\frac{1}{\sqrt{\delta}}\phi_1)(\sqrt{\delta}\phi_2)| \le \frac{1}{\delta}\phi_1^2 + \delta\phi_2^2 \tag{29}$$

$\forall \phi_1, \phi_2 \in \mathbb{R}$ *and* $\delta > 0$.

Lemma 8 *Let $\phi(x,t) \in \mathbb{R}$ be an integrable function defined on $x \in [0, L]$ and $t \in [0, \infty)$, satisfying the boundary condition $\phi(0,t) = 0$, then the following inequality holds:*

$$\phi^2(x,t) \le L \int_0^L [\phi'(x,t)]^2 dx \tag{30}$$

if the function $\phi(x,t)$ satisfies the boundary condition $\phi'(0,t)=0$ further, then the following inequalities also holds:

$$\int_0^L \phi^2(x,t)dx \leq L^2 \int_0^L [\phi'(x,t)]^2 dx \tag{31}$$

$$\phi'^2(x,t) \leq L \int_0^L [\phi''(x,t)]^2 dx \tag{32}$$

Theorem 1 . *The barrier LFC given by (24) can be upper and lower bounded as:*

$$0 \leq \xi_1[\upsilon(t)+V_2(t)] \leq V(t) \leq \xi_2[\upsilon(t)+V_2(t)] \tag{33}$$

where ξ_1 and ξ_2 are two positive numbers.

Proof: We define a new auxiliary function

$$\xi(t) = \int_0^L \{[\dot{\theta}(x,t)]^2 dx + [\dot{y}(x,t)]^2 dx + [\theta'(x,t)]^2 dx + [y''(x,t)]^2 dx\} \tag{34}$$

We can obtain that:

$$\mu_1 \upsilon(t) \leq V_1(t) \leq \mu_2 \upsilon(t) \tag{35}$$

where $\mu_1 = \frac{a}{2} min(n, I_p, EI_b, GJ) > 0$, and $\mu_2 = \frac{a}{2} max(n, I_p, EI_b, GJ) > 0$.

Applying **Lemma 7** and **8** to $V_3(t)$, we obtain

$$\begin{aligned} |V_3(t)| \leq (bm + bmx_e c + amx_e c) \int_0^L [\dot{y}(x,t)]^2 dx + (bI_p + bmx_e c + amx_e c) \int_0^L [\dot{\theta}(x,t)]^2 dx \\ +(bm + bmx_e c)L^4 \int_0^L [y''(x,t)]^2 dx + (bI_p + bmx_e c)L^2 \int_0^L [\theta'(x,t)]^2 dx \leq \mu_3 \upsilon(t) \end{aligned} \tag{36}$$

where

$$\mu_3 = max\{bm, bmx_e c + amx_e c, bI_p + bmx_e c + amx_e c, (bm + bmx_e c)L^4, (bI_p + bmx_e c)L^2\} \geq 0$$

We have

$$0 \leq \beta_1 \upsilon(t) \leq V_1(t) + V_3(t) \leq \beta_2 \upsilon(t) \tag{37}$$

and $\beta_1 = \mu_1 - \mu_3, \beta_2 = \mu_2 + \mu_3$, and we chose appropriate values of a and b to guarantee $\beta_1 \geq 0$.

We can prove that the barrier LFC has upper and lower bounds as follows

$$0 \leq \xi_1[\upsilon(t) + V_2(t)] \leq V(t) \leq \xi_2[\upsilon(t) + V_2(t)] \tag{38}$$

where $\xi_1 = min(\beta_1, 1)$ and $\xi_2 = max(\beta_2, 1)$ are positive constants.

Theorem 2 . *Using the proposed control laws, time derivative of the barrier LFC is negative definite, and its upper bound as follow*

$$\dot{V}(t) \leq -\xi V(t) + \varepsilon \tag{39}$$

where ξ is a positive constant.

Proof: Differentiating (24) with respect to time leads to

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \tag{40}$$

In detail,

$$\begin{aligned}\dot{V}_1(t) =& am \int_0^L \dot{y}(x,t)\ddot{y}(x,t)dx + aI_p \int_0^L \ddot{\theta}(x,t)\dot{\theta}(x,t)dx \\ &+ aGJ \int_0^L \dot{\theta}'(x,t)\theta'(x,t)dx + aEI_b \int_0^L \dot{y}''(x,t)y''(x,t)dx\end{aligned} \tag{41}$$

Similarly, $\dot{V}_2(t)$ is written as

$$\dot{V}_2(t) = \frac{(ak_1 + bk_2)y(L,t)\dot{y}(L,t)}{D^2 - y^2(L,t)} + \frac{(ak_3 + bk_4)\theta(L,t)\dot{\theta}(L,t)}{\phi^2 - \theta^2(L,t)} \tag{42}$$

and

$$\begin{aligned}\dot{V}_3(t) =& bm \int_0^L \ddot{y}(x,t)y(x,t)dx + bm \int_0^L [\dot{y}(x,t)]^2 dx + bI_p \int_0^L \ddot{\theta}(x,t)\theta(x,t)dx + bI_p \int_0^L [\dot{\theta}(x,t)]^2 dx \\ &- bmx_e c \int_0^L [y(x,t)\ddot{\theta}(x,t) + 2\dot{y}(x,t)\dot{\theta}(x,t) + \ddot{y}(x,t)\theta(x,t)]dx \\ &- amx_e c \int_0^L [\ddot{y}(x,t)\dot{\theta}(x,t) + \dot{y}(x,t)\ddot{\theta}(x,t)]dx\end{aligned} \tag{43}$$

With the system governing equations (8) and (9), and from the boundary conditions (12)-(22), applying the **Lemma 7** and **8**, we enlarge $\dot{V}_1(t)$ and $\dot{V}_3(t)$ as

$$\begin{aligned}\dot{V}_1(t) \leq &-(\frac{a\eta EI_b}{2L^4}-\sigma_1 a)\int_0^L[\dot{y}(x,t)]^2dx-(\frac{a\eta GJ}{2L^2}-\sigma_2 a x_a c)\int_0^L[\dot{\theta}(x,t)]^2dx-\frac{a\eta EI_b}{2}\int_0^L[\dot{y}''(x,t)]^2dx\\ &-\frac{a\eta GJ}{2}\int_0^L[\dot{\theta}'(x,t)]^2dx+amx_ec\int_0^L[\dot{\theta}(x,t)\ddot{y}(x,t)+\dot{y}(x,t)\ddot{\theta}(x,t)]dx-aEI_b\dot{y}(x,t)y'''(L,t)\\ &-a\eta EI_b\dot{y}(L,t)+aGJ\dot{\theta}(L,t)\theta'(L,t)+a\eta GJ\dot{\theta}(L,t)\dot{\theta}'(L,t)+(\frac{a}{\sigma_1}+\frac{ax_ac}{\sigma_2})LF_{b\max}^2 \end{aligned} \tag{44}$$

where $\sigma_1 > 0$ and $\sigma_2 > 0$.

$$\begin{aligned}\dot{V}_3(t) \leq &-bEI_by(L,t)y'''(L,t)-bEI_b\int_0^L[y''(x,t)]^2dx-b\eta EI_by(L,t)\dot{y}'''(L,t)+\frac{b\eta EI_b}{\sigma_3}\int_0^L[y''(x,t)]^2dx\\ &+b\eta EI_b\sigma_3\int_0^L[\dot{y}''(x,t)]^2dx+bGJ\theta'(L,t)-bGJ\int_0^L[\theta'(x,t)]^2dx+b\eta GJ\theta(L,t)\dot{\theta}'(L,t)\\ &+\frac{b\eta GJ}{\sigma_4}\int_0^L[\theta'(x,t)]^2dx+\sigma_4 b\eta GJ\int_0^L[\dot{\theta}'(x,t)]^2dx+bm\int_0^L[\dot{y}(x,t)]^2dx+bI_p\int_0^L[\dot{\theta}(x,t)]^2dx\\ &-amx_ec\int_0^L[\dot{y}(x,t)\ddot{\theta}(x,t)+\ddot{y}(x,t)\dot{\theta}(x,t)]dx+2bmx_ec\sigma_5\int_0^L[\dot{y}(x,t)]^2dx+\sigma_6bL^4\int_0^L[y''(x,t)]^2dx\\ &+\frac{2bmx_ec}{\sigma_5}\int_0^L[\dot{\theta}(x,t)]^2dx+\sigma_7bL^2x_ac\int_0^L[\theta'(x,t)]^2dx+(\frac{b}{\sigma_6}+\frac{bx_ac}{\sigma_7})LF_{b\max}^2 \end{aligned} \tag{45}$$

where $\sigma_3 - \sigma_7$ are positive constants.

Substituting (44),(42) and (45), we have

$$\begin{aligned}\dot{V}(t) \leq &-(\frac{a\eta EI_b}{2L^4}-\sigma_1a-bm-2bmx_ec\sigma_5)\int_0^L[\dot{y}(x,t)]^2dx-(bEI_b-\frac{b\eta EI_b}{\sigma_3}-\sigma_6bL^4)\int_0^L[y''(x,t)]^2dx\\ &-(\frac{a\eta GJ}{2L^2}-\sigma_2ax_ac-bI_p-\frac{2bmx_ec}{\sigma_5})\int_0^L[\dot{\theta}(x,t)]^2dx-(bGJ-\frac{b\eta GJ}{\sigma_4}-\sigma_7bL^4x_ac)\int_0^L[\theta'(x,t)]^2dx\\ &-bk_1ln\frac{D^2}{D^2-y^2(L,t)}-bk_3ln\frac{\phi^2}{\phi^2-\theta^2(L,t)}-(\frac{a\eta EI_b}{2}-b\eta EI_b\sigma_3)\int_0^L[\dot{y}''(x,t)]^2dx\\ &-(\frac{a\eta GJ}{2}-\sigma_4b\eta GJ)\int_0^L[\dot{\theta}'(x,t)]^2dx-\frac{ak_2[\dot{y}(L,t)]^2}{D^2-y^2(L,t)}-\frac{ak_4[\dot{\theta}(L,t)]^2}{\phi^2-\theta^2(L,t)}\\ &+(\frac{a}{\sigma_1}+\frac{ax_ac}{\sigma_2}+\frac{b}{\sigma_6}+\frac{bx_ac}{\sigma_7})LF_{b\max}^2\\ \leq &-\xi_3[\upsilon(t)+V_2(t)]+\varepsilon \end{aligned} \tag{46}$$

and

$$
\begin{aligned}
\xi_3 =& min(\frac{a\eta EI_b}{2L^4} - \sigma_1 a - bm - 2bmx_e c\sigma_5, bEI_b - \frac{b\eta EI_b}{\sigma_3} - \sigma_6 bL^4, \frac{a\eta GJ}{2L^2} - \sigma_2 ax_a c - bI_p - \frac{2bmx_e c}{\sigma_5}, \\
& bGJ - \frac{b\eta GJ}{\sigma_4} - \sigma_7 bL^4 x_a c, \frac{2bk_1}{ak_1 + bk_2}, \frac{2bk_3}{ak_3 + bk_4}) \\
\varepsilon =& (\frac{a}{\sigma_1} + \frac{ax_a c}{\sigma_2} + \frac{b}{\sigma_6} + \frac{bx_a c}{\sigma_7})LF_{\mathrm{bmax}}^2
\end{aligned}
$$

At the same time, positive constants a, b and $\sigma_1 - \sigma_7$ are chosen to satisfy the following inequalities:

$$
\frac{a\eta EI_b}{2L^4} - \sigma_1 a - bm - 2bmx_e c\sigma_5 \geq 0 \tag{47}
$$

$$
bEI_b - \frac{b\eta EI_b}{\sigma_3} - \sigma_6 bL^4 \geq 0 \tag{48}
$$

$$
\frac{a\eta GJ}{2L^2} - \sigma_2 ax_a c - bI_p - \frac{2bmx_e c}{\sigma_5} \geq 0 \tag{49}
$$

$$
bGJ - \frac{b\eta GJ}{\sigma_4} - \sigma_7 bL^4 x_a c \geq 0 \tag{50}
$$

$$
\frac{a\eta EI_b}{2} - b\eta EI_b\sigma_3 \geq 0 \tag{51}
$$

$$
\frac{a\eta GJ}{2} - \sigma_4 b\eta GJ \geq 0 \tag{52}
$$

Combining **Theorem 1** and **2**, we obtain:

$$
\dot{V}(t) \leq -\xi V(t) + \varepsilon \tag{53}
$$

where $\xi = \frac{\xi_3}{\xi_2}$.

Theorem 3 . *For the dynamical system of the flexible wing, which is described by governing movement equations (8) and (9) as well as boundary conditions (12)-(22), under the proposed boundary control (23),(24), if the initial conditions are bounded, then the closed-loop system is uniformly bounded.*

Proof: Integrating of the inequality (53) by multiplying $e^{-\xi t}$

$$
\begin{aligned}
V(t) \leq & \left(V(0) - \frac{\varepsilon}{\xi}\right) e^{-\xi t} + \frac{\varepsilon}{\xi} \\
\leq & V(0)e^{-\xi t} + \frac{\varepsilon}{\xi} \in \mathcal{L}_\infty
\end{aligned} \tag{54}
$$

which implies that $V(t)$ is bounded. Utilizing Lemma 1 and 2, we have

$$\begin{aligned}\frac{1}{L^3}y^2(x,t) \leq& \frac{1}{L^2}\int_0^L [y'(x,t)]^2 dx \leq \int_0^L [y''(x,t)]^2 dx \\ \leq& \upsilon(t) \leq \frac{1}{\xi_2}V(t) \in \mathcal{L}_\infty \end{aligned} \tag{55}$$

$$\frac{1}{L^2}\theta^2(x,t) \leq \int_0^L [\theta'(x,t)]^2 dx \leq \upsilon(t) \leq \frac{1}{\xi_2}V(t) \in \mathcal{L}_\infty \tag{56}$$

Appropriately rearranging the terms of (54)-(56), we obtain that $y(x,t)$ and $\theta(x,t)$ are uniformly bounded as follows:

$$|y(x,t)| \leq \sqrt{\frac{L^3}{\xi_2}\left(V(0)e^{-\xi t} + \frac{\varepsilon}{\xi}\right)} \tag{57}$$

$$|\theta(x,t)| \leq \sqrt{\frac{L}{\xi_2}\left(V(0)e^{-\xi t} + \frac{\varepsilon}{\xi}\right)} \tag{58}$$

Learning Report(5.26)

Article : Boundary Control of Flexible Aircraft Wings for Vibration Suppression

1 Dynamics Modeling

The kinetic energy of the flexible wings $E_k(t)$ is

$$\begin{aligned} E_k t) =& \frac{1}{2}\rho \int_{-L}^{0} [\dot{y_L}(x,t)]^2 dx + \frac{1}{2} I_p \int_{-L}^{0} [\dot{\theta_L}(x,t)]^2 dx + \frac{1}{2}\rho \int_{0}^{L} [\dot{y_R}(x,t)]^2 dx \\ &+ \frac{1}{2} I_p \int_{0}^{L} [\dot{\theta_R}(x,t)]^2 dx + \frac{m_b}{2}[\dot{y}(0,t)]^2 + \frac{I_p}{2}[\dot{\theta}(0,t)]^2 \end{aligned} \tag{1}$$

The potential energy of the flexible wing $E_p(t)$ is

$$E_p(t) = \frac{1}{2}\int_{-L}^{0} EI[y_L''(x,t)]^2 + GJ[\theta_L'(x,t)]^2 dx + \frac{1}{2}\int_{0}^{L} EI[y_R''(x,t)]^2 + GJ[\theta_R'(x,t)]^2 dx \tag{2}$$

The virtual work done by the bending and torsion rigidity coupling can be expressed as

$$\begin{aligned} \delta W_c(t) =& \rho x_e c \int_{-L}^{0} [\ddot{y_L}(x,t)\delta\theta_L(x,t) + \ddot{\theta_L}(x,t)\delta y_L(x,t)] dx \\ &+ \rho x_e c \int_{0}^{L} [\ddot{y_R}(x,t)\delta\theta_R(x,t) + \ddot{\theta_R}(x,t)\delta y_R(x,t)] dx \end{aligned} \tag{3}$$

The virtual work done by the Kelvin-Voigt damping on the wing is

$$\begin{aligned} \delta W_d(t) =& -\eta EI \int_{-L}^{0} \dot{y_L}''(x,t)\delta\theta_L(x,t) dx - \eta GJ \int_{0}^{L} \dot{\theta_L}'(x,t)\delta\theta_L'(x,t) dx \\ &- \eta EI \int_{0}^{L} \dot{y_R}''(x,t)\delta\theta_R(x,t) dx - \eta GJ \int_{0}^{L} \dot{\theta_R}'(x,t)\delta\theta_R'(x,t) dx \end{aligned} \tag{4}$$

The virtual work done by distributed disturbance can be obtained as follows

$$\begin{aligned} \delta W_f(t) =& \int_{-L}^{0} [F_{bL}(x,t)\delta y_L(x,t) - x_a c F_{bL}(x,t)\delta\theta_L(x,t)] dx \\ &+ \int_{0}^{L} [F_{bR}(x,t)\delta y_R(x,t) - x_a c F_{bR}(x,t)\delta\theta_R(x,t)] dx \end{aligned} \tag{5}$$

The virtual work of boundary control force $U_1(t)$ and moment $U_2(t)$ as below

$$\delta W_u(t) = U_1(t)\delta y(0,t) + U_2(t)\delta\theta(0,t) \tag{6}$$

The total virtual work can be expressed as

$$\delta W(t) = \delta[W_c(t) + W_d(t) + W_f(t) + W_u(t)] \tag{7}$$

Substituting above equations into the Hamilton's principle $\int_{t_1}^{t_2} \delta[E_k(t) - E_p(t) + W(t)]dt = 0$, we get the dynamics equations of flexible wings and the body as follows

$$\rho \ddot{y}_L(x,t) + EIy_L''''(x,t) - \rho x_e c\ddot{\theta}_L(x,t) + \eta EI\dot{y}_L''''(x,t) = F_{bL}(x,t) \tag{8}$$

$$I_p\ddot{\theta}_L(x,t) - GJ\theta_L''(x,t) - \rho x_e c\ddot{y}_L(x,t) - \eta GJ\dot{\theta}_L''(x,t) = -x_a cF_{bL}(x,t) \tag{9}$$

$$\rho \ddot{y}_R(x,t) + EIy_R''''(x,t) - \rho x_e c\ddot{\theta}_R(x,t) + \eta EI\dot{y}_R''''(x,t) = F_{bR}(x,t) \tag{10}$$

$$I_p\ddot{\theta}_R(x,t) - GJ\theta_R''(x,t) - \rho x_e c\ddot{y}_R(x,t) - \eta GJ\dot{\theta}_R''(x,t) = -x_a cF_{bR}(x,t) \tag{11}$$

and the boundary conditions are:

$$y_L(0,t) = y_R(0,t) = y(0,t) \tag{12}$$

$$\theta_L(0,t) = \theta_R(0,t) = \theta(0,t) \tag{13}$$

$$y_L'(0,t) = y_R'(0,t) = y''(L,t) = 0 \tag{14}$$

$$y_L''(-L,t) = y_R''(L,t) = 0 \tag{15}$$

$$y_L'''(-L,t) + \eta\dot{y}_L'''(-L,t) = 0 \tag{16}$$

$$y_R'''(L,t) + \eta\dot{y}_R'''(L,t) = 0 \tag{17}$$

$$\theta_L'''(-L,t) + \eta\dot{\theta}_L'''(-L,t) = 0 \tag{18}$$

$$\theta_R'''(L,t) + \eta\dot{\theta}_R'''(L,t) = 0 \tag{19}$$

$$m_b\ddot{y}(0,t) - EIy_L'''(0,t) - \eta EI\dot{y}_L'''(0,t) + EIy_R'''(0,t) + \eta EI\dot{y}_R'''(0,t) = U_1(t) \tag{20}$$

$$I_p\ddot{\theta}(0,t) + GJ\theta_L'(0,t) + \eta GJ\dot{\theta}_L'(0,t) - GJ\theta_R'(0,t) - \eta GJ\dot{\theta}_R'(0,t) = U_2(t) \tag{21}$$

Assumption 1 *The spatiotemporal-varying distributed air-loads $F_{bL}(x,t)$ and $F_{bR}(x,t)$ are assumed to be bounded. That is, constants $\bar{F}_{bL} \in \mathbb{R}^+$ and $\bar{F}_{bR} \in \mathbb{R}^+$ make $|F_{bL}(x,t)| \leq \bar{F}_{bL}(x,t), \forall(x,t) \in [-L,0] \times [0,\infty)$ and $|F_{bR}(x,t)| \leq \bar{F}_{bR}(x,t), \forall(x,t) \in [0,L] \times [0,\infty)$. This is a reasonable assumption,* *since the distributed air loads have finite energy.*

2 Boundary Control Design

The target of our controller is to suppress the bending and torsional deformations of flexible wings in time via exerting boundary control force $U_1(t)$ and moment $U_2(t)$ at the wing root. In addition, as $t \to \infty$,the uniform ultimately bounded (UUB) of the closed-loop system states $y_{L(R)}(x,t), \theta_{L(R)}(x,t)$ and so on are guaranteed.

Under the condition that system parameters are known,we design boundary control laws as

$$U_1(t) = -k_1 \dot{y}(0,t) \tag{22}$$

$$U_2(t) = -k_2 \dot{\theta}(0,t) \tag{23}$$

where k_1 and k_2 are two positive control gains.

Consider the Lyapunov function candidate(LFC) is proposed as follows

$$V(t) = V_1(t) + V_2(t) + \Delta(t) \tag{24}$$

where each term is respectively defined as

$$\begin{aligned} V_1(t) =& \frac{a\rho}{2} \int_{-L}^{0} [\dot{y}_L(x,t)]^2 dx + \frac{aEI}{2} \int_{-L}^{0} [y''_L(x,t)]^2 dx \\ &+ \frac{aI_p}{2} \int_{-L}^{0} [\dot{\theta}_L(x,t)]^2 dx + \frac{aGJ}{2} \int_{-L}^{0} [\theta'_L(x,t)]^2 dx \\ & \frac{a\rho}{2} \int_{0}^{L} [\dot{y}_R(x,t)]^2 dx + \frac{aEI}{2} \int_{0}^{L} [y''_R(x,t)]^2 dx \\ &+ \frac{aI_p}{2} \int_{0}^{L} [\dot{\theta}_R(x,t)]^2 dx + \frac{aGJ}{2} \int_{0}^{L} [\theta'_R(x,t)]^2 dx \end{aligned} \tag{25}$$

$$V_2(t) = \frac{a}{2} m_b [\dot{y}(0,t)]^2 + \frac{a}{2} I_p [\dot{\theta}(0,t)]^2 \tag{26}$$

$$\Delta(t) = -a\rho x_e c \int_{-L}^{0} \dot{y}_L(x,t)\dot{\theta}_L(x,t)dx - a\rho x_e c \int_{0}^{L} \dot{y}_R(x,t)\dot{\theta}_R(x,t)dx \tag{27}$$

Theorem 1 *The Lyapunov candidate function $V(t)$ has upper and lower bounds*

$$0 \le \xi_1[V_1(t) + V_2(t)] \le V(t) \le \xi_2[V_1(t) + V_2(t)] \tag{28}$$

where $\xi_1 \geq 0$ and $\xi_2 \geq 0$ are two positive numbers.

Proof: Defining an auxiliary function $\nu(t)$

$$\begin{aligned}\nu(t) = &\int_{-L}^{0}[\dot{y}_L(x,t)]^2dx + \int_{-L}^{0}[\dot{\theta}_L(x,t)]^2dx + \int_{-L}^{0}[y''_L(x,t)]^2dx + \int_{-L}^{0}[\theta'_L(x,t)]^2dx \\ &+ \int_{0}^{L}[\dot{y}_R(x,t)]^2dx + \int_{0}^{L}[\dot{\theta}_R(x,t)]^2dx + \int_{0}^{L}[y''_R(x,t)]^2dx + \int_{0}^{L}[\theta'_R(x,t)]^2dx \end{aligned} \tag{29}$$

From the definition of $V_1(t)$, its lower bound can be expressed with $\nu(t)$ as

$$\gamma_1\nu(t) \leq V_1(t) \tag{30}$$

where $\gamma_1 = \frac{a}{2}min\{\rho, I_p, EI, GJ\}$.

Furthermore, we have

$$\begin{aligned}|\Delta(t)| &\leq a\rho x_e c\left[\int_{-L}^{0}(\dot{y}_L^2(x,t)+\dot{\theta}_L^2(x,t))dx + \int_{0}^{L}(\dot{y}_R^2(x,t)+\dot{\theta}_R^2(x,t))dx\right] \\ &\leq a\rho x_e c\nu(t) \leq \gamma_2 V_1(t)\end{aligned} \tag{31}$$

where $\nu_2 = \frac{a\rho x_e c}{\nu_1}$, that is

$$-\nu_2 V_1(t) \leq \delta(t) \leq \nu_2 V_1(t) \tag{32}$$

Then, we add $V_1(t)$ to the both sides of

$$(1-\nu_2)V_1(t) \leq V_1(t) + \Delta(t) \leq (1+\nu_2)V_1(t) \tag{33}$$

We can chose proper parameters to make sure that $0 < \nu_2 < 1$. Further more, we have

$$\begin{aligned}0 &\leq \xi_1[V_1(t)+V_2(t)] \leq V_1(t)+V_2(t)+\Delta(t) \\ &\leq \xi_2[V_1(t)+V_2(t)]\end{aligned} \tag{34}$$

where $\xi_1 = min\{1-\gamma_2, 1\} = 1-\gamma_2, \xi_2 = max\{1+\gamma_2, 1\} = 1+\gamma_2$.

Theorem 2 *If the boundary control laws are utilized, the time derivative of $V(t)$ has upper bound*

$$\dot{V}(t) \leq -\xi V(t) + \varepsilon \tag{35}$$

where $\xi > 0$ is a constant.

Proof: The derivative of the Lyapunov candidate (24) with respect to time as follows

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{\delta}(t) \tag{36}$$

Further more, we have

$$\begin{aligned} \dot{V}_1(t) &= \int_{-L}^{0} [a\rho\dot{y}_L(x,t)\ddot{y}_L(x,t) + aEIy_L''(x,t)\dot{y}_L''(x,t) \\ &\quad + aI_p\dot{\theta}_L(x,t)\ddot{\theta}_L(x,t) + aGJ\theta_L'(x,t)\dot{\theta}_L'(x,t)]dx \\ &\quad + \int_0^L [a\rho\dot{y}_R(x,t)\ddot{y}_R(x,t) + aEIy_R''(x,t)\dot{y}_R''(x,t) \\ &\quad + aI_p\dot{\theta}_R(x,t)\ddot{\theta}_R(x,t) + aGJ\theta_R'(x,t)\dot{\theta}_R'(x,t)]dx \end{aligned} \tag{37}$$

Substituting the governing equations (8)-(11) into the above equation, integrating by parts and using boundary conditions (12)-(21). Furthermore, the above formula can be:

$$\begin{aligned} \dot{V}(t) &= a\int_{-L}^{0} \dot{y}_L(x,t)F_{bL}(x,t)dx - ax_ac\int_{-L}^{0} \dot{\theta}_L(x,t)F_{bL}(x,t)dx \\ &\quad + a\rho x_ec\int_{-L}^{0} [\dot{y}_L(x,t)\ddot{\theta}_L(x,t) + \dot{\theta}_L(x,t)\ddot{y}_L(x,t)]dx \\ &\quad - a\eta EI\int -L^0(\dot{y}_L''(x,t))^2dx - a\eta GJ\int_{-L}^{0}[\dot{\theta}_L'(x,t)]^2dx + a\int_0^L \dot{y}_R(x,t)F_{bL}(x,t)dx \\ &\quad - ax_ac\int_0^L \dot{\theta}_R(x,t)F_{bL}(x,t)dx + a\rho x_ec\int_0^L [\dot{y}_R(x,t)\ddot{\theta}_R(x,t) + \dot{\theta}_R(x,t)\ddot{y}_R(x,t)]dx \\ &\quad - a\eta EI\int -L^0(\dot{y}_R''(x,t))^2dx - a\eta GJ\int_0^L [\dot{\theta}_R'(x,t)]^2dx \\ &\quad - aEI\dot{y}(0,t)[y_L'''(0,t) - y_R'''(0,t)] - a\eta EI\dot{y}(0,t)[\dot{y}_L'''(0,t) - \dot{y}_R'''(0,t)] \\ &\quad + aGJ\dot{\theta}(0,t)[\theta_L(0,t) - \theta_R'(0,t)] + a\eta GJ\dot{\theta}(0,t)[\dot{\theta}_L'(0,t) - \dot{\theta}_R'(0,t)] \end{aligned} \tag{38}$$

Differentiating $V_2(t)$ respect to time t, $\dot{V}_2(t)$ is calculated as

$$\dot{V}_2(t) = am_b\dot{y}(0,t)\ddot{y}(0,t) + aI_p\dot{\theta}(0,t)\ddot{\theta}(0,t) \tag{39}$$

Substituting the boundary conditions (20) and (21) into it

$$\begin{aligned}\dot{V}_2(t) &= a\dot{y}(0,t)[U_1(t) + EIy_L'''(0,t) + \eta EI\dot{y}_L'''(0,t) - EIy_R'''(0,t) - \eta EI\dot{y}_R'''(0,t)] \\ &\quad + a\dot{\theta}(0,t)[U_2(t) - GJ\theta_L'(0,t) - \eta GJ\dot{\theta}_L'(0,t) + GJ\theta_R'(0,t) + \eta GJ\dot{\theta}_R'(0,t)]\end{aligned} \tag{40}$$

Then, differentiating $\Delta(t)$ with respect to t, it yields

$$\begin{aligned}\dot{\Delta}(t) &= -a\rho x_e c\int_{-L}^{0}[\ddot{y}_L(x,t)\dot{\theta}_L(x,t) + \dot{y}_L(x,t)\ddot{\theta}_L(x,t)]dx \\ &\quad -a\rho x_e c\int_{0}^{L}[\ddot{y}_R(x,t)\dot{\theta}_R(x,t) + \dot{y}_R(x,t)\ddot{\theta}_R(x,t)]dx\end{aligned} \tag{41}$$

From (38),(40) and (41), substituting the proposed control laws (22) and (23), we can rewrite the time derivative of Lyapunov function(36) as:

$$\begin{aligned}\dot{V}(t) &= -ak_1[\dot{y}(0,t)]^2 - ak_2[\dot{\theta}(0,t)]^2 - a\eta EI\int_{-L}^{0}(\dot{y}_L''(x,t))^2dx - a\eta GJ\int_{-L}^{0}(\dot{\theta}(x,t))^2dx \\ &\quad + a\int_{-L}^{0}\dot{y}_L(x,t)F_{bL}(x,t)dx - ax_ac\int_{-L}^{0}\dot{\theta}_L(x,t)F_{bL}(x,t)dx - a\eta EI\int_{0}^{L}(\dot{y}_R''(x,t))^2dx \\ &\quad - a\eta GJ\int_{0}^{L}(\theta_R'(x,t))^2dx + a\int_{0}^{L}\dot{y}_R(x,t)F_{bR}(x,t)dx - ax_ac\int_{0}^{L}\dot{\theta}_R(x,t)F_{bR}(x,t)dx\end{aligned} \tag{42}$$

Lemma 9 *Let $\phi_1(x,t), \phi_2(x,t) \in \mathbb{R}$ with $x \in [0,L]$, there is the inequality:*

$$|\phi_1\phi_2| = |(\frac{1}{\sqrt{\delta}}\phi_1)(\sqrt{\delta}\phi_2)| \leq \frac{1}{\delta}\phi_1^2 + \delta\phi_2^2 \tag{43}$$

$\forall\phi_1, \phi_2 \in \mathbb{R}$ and $\delta > 0$.

Lemma 10 *Let $\phi(x,t) \in \mathbb{R}$ be an integrable function defined on $x \in [0,L]$ and $t \in [0,\infty)$, satisfying the boundary condition $\phi(0,t) = 0$, then the following inequality holds:*

$$\phi^2(x,t) \leq L\int_{0}^{L}[\phi'(x,t)]^2dx \tag{44}$$

if the function $\phi(x,t)$ satisfies the boundary condition $\phi'(0,t) = 0$ further, then the following inequal-

ities also holds:

$$\int_0^L \phi^2(x,t)dx \leq L^2 \int_0^L [\phi'(x,t)]^2 dx \tag{45}$$

$$\phi'^2(x,t) \leq L \int_0^L [\phi''(x,t)]^2 dx \tag{46}$$

Substituting control laws (22) and (23), with Lemma 1 and Lemma 2, and Assumption 1, we obtain that

$$\begin{aligned}
\dot{V}(t) \leq & -ak_1[\dot{y}(0,t)]^2 - ak_2[\dot{\theta}(0,t)]^2 - (\frac{a\eta EI}{L^4} - a\delta_1)\int_{-L}^{0}[\dot{y}_L(x,t)]^2 dx - (\frac{a\eta EI}{L^4} - a\delta_3)\int_0^L [\dot{y}_R(x.t)]^2 dx \\
& -(\frac{a\eta GJ}{L^2} - ax_a c\delta_2)\int_{-L}^{0}[\dot{\theta}_L(x,t)]^2 - (\frac{a\eta GJ}{L^2} - ax_a c\delta_4)\int_0^L [\dot{\theta}_R(x,t)]^2 dx \\
& +(\frac{a}{\delta_1} + \frac{ax_a c}{\delta_2})\int_{-L}^{0}[F_{bL}(x,t)]^2 dx + (\frac{a}{\delta_3} + \frac{ax_a c}{\delta_4})\int_0^L [F_{bR}(x,t)]^2 dx \\
\leq & -\xi_3[V_1(t) + V_2(t)] + \varepsilon
\end{aligned} \tag{47}$$

The positive constants a and $\delta_1 - \delta_4$ are chosen to satisfy following condition

$$\sigma_1 = (\frac{a\eta EI}{L^4} - a\delta_1) \geq 0 \tag{48}$$

$$\sigma_2 = (\frac{a\eta EI}{L^4} - a\delta_3) \geq 0 \tag{49}$$

$$\sigma_3 = (\frac{a\eta GJ}{L^2} - ax_a c\delta_2) \geq 0 \tag{50}$$

$$\sigma_4 = (\frac{a\eta GJ}{L^2} - ax_a c\delta_4) \geq 0 \tag{51}$$

Meanwhile

$$\varepsilon = (\frac{a}{\delta_1} + \frac{ax_a c}{\delta_2})\bar{F}_{bL}^2 + (\frac{a}{\delta_3} + \frac{ax_a c}{\delta_4})\bar{F}_{bR}^2 \tag{52}$$

$$\xi_3 = min\frac{2}{a}\{\frac{\sigma_1}{\rho}, \frac{\sigma_2}{\rho}, \frac{\sigma_3}{I_p}, \frac{\sigma_4}{I_p}\} \tag{53}$$

Combining Theorem 1 and (43), we can prove that

$$\dot{V}(t) \leq -\xi V(t) + \varepsilon \tag{54}$$

where $\xi = \frac{\xi_3}{\xi_2} > 0$.

Theorem 3 . *For the dynamical system of the flexible wing, which is described by governing movement equations (8)-(11) as well as boundary conditions (12)-(21), under the proposed boundary control (22) and (23), we can obtain that: All the system signals, especially boundary outputs* $y_L(-L,t), y_R(L,t)$ *and* $\theta_L(-L,t), \theta_R(L,t)$ *can realize uniformly ultimate bounded (UUB) as* t *tends to infinity, they will eventually converge to small compact sets.*

Proof: Integrating of the inequality (54) by multiplying $e^{-\xi t}$

$$\begin{aligned} V(t) \leq & \left(V(0) - \frac{\varepsilon}{\xi}\right) e^{-\xi t} + \frac{\varepsilon}{\xi} \\ \leq & V(0)e^{-\xi t} + \frac{\varepsilon}{\xi} \in \mathcal{L}_\infty \end{aligned} \tag{55}$$

which implies that $V(t)$ is bounded. Utilizing Lemma 1 and 2, we have

$$\begin{aligned} \frac{1}{L^3} y_L^2(x,t) \leq & \frac{1}{L^2} \int_{-L}^{0} [y_L'(x,t)]^2 dx \leq \int_{-L}^{0} [y_L''(x,t)]^2 dx \\ \leq & \nu(t) \leq \frac{1}{\gamma_1}[V_1(t) + V_2(t)] \leq \frac{1}{\gamma_1 \xi_1} V(t) \in \mathcal{L}_\infty \end{aligned} \tag{56}$$

$$\frac{1}{L}\theta_L^2(x,t) \leq \int_{-L}^{0} [\theta_L'(x,t)]^2 dx \leq \nu(t) \leq \frac{1}{\gamma_1}[V_1(t) + V_2(t)] \leq \frac{1}{\gamma_1 \xi_1} \in \mathcal{L}_\infty \tag{57}$$

where γ_1 and ξ_1 are dimensionless positive constants.

Deformations of the right wing side can be obtained in the same way. Appropriately rearranging the formulae (55)-(57), two DOF deformations of the flexible wings $y_{L(R)}(x,t)$ and $\theta_{L(R)}(x,t)$ have upper bounds as follows

$$|y_{L(R)}(x,t)| \leq \sqrt{\frac{L^3}{\gamma_1 \xi_1}\left(V(0)e^{-\xi t} + \frac{\varepsilon}{\xi}\right)} \tag{58}$$

$$|\theta_{L(R)}(x,t)| \leq \sqrt{\frac{L}{\gamma_1 \xi_1}\left(V(0)e^{-\xi t} + \frac{\varepsilon}{\xi}\right)} \tag{59}$$

Based on above two formulae (58) and (59), we can state that when t intends to infinity, $y(L,t)$ and $\theta(L,t)$ will remain in the small domains, that are

$$|y_{L(R)}(x,t)| \leq \sqrt{\frac{L^3 \varepsilon}{\gamma_1 \xi_1 \xi}}, \forall x \in [-L, L] \tag{60}$$

$$|\theta_{L(R)}(x,t)| \leq \sqrt{\frac{L \varepsilon}{\gamma_1 \xi_1 \xi}}, \forall x \in [-L, L] \tag{61}$$

Learning Report(6.18)

Article : Lyapunov-Based Control of Mechanical Systems (Chapter 6)

1 System Model

The model for the cantilevered Euler-Bernoulli beam system shown in Figure 6.1 is described by a PDE of the form:

$$\rho u_{tt}(x,t) + EIu_{xxxx}(x,t) = 0 \tag{1}$$

with the following boundary conditions:

$$u(0,t) = u_x(0,t) = u_{xx}(L,t) = 0 \tag{2}$$

and

$$mu_{tt}(L,t) - EIu_{xxx}(L,t) = f(t) \tag{3}$$

where x and t represent the independent spatial and time variables,respectively.

Property 1 *If the potential energy of the system given by (1) through (3), which is given by*

$$E_p = \frac{1}{2}EI\int_0^L u_{\sigma\sigma}^2(\sigma,t)d\sigma \tag{4}$$

is bounded $\forall t \in [0,\infty)$ *and* $\forall x \in [0,L]$, *then* $\frac{\partial^n}{\partial x^n}u(x,t)$ *is bounded for* $n = 2,3,4$, $\forall t \in [0,\infty)$, *and* $\forall x \in [0,L]$.

Property 2 *If the kinetic energy of the system of (1) through (3) which is given by*

$$E_k = \frac{1}{2}\rho\int_0^L u_t^2(\sigma,t)d\sigma + \frac{1}{2}mu_t^2(L,t) \tag{5}$$

is bounded $\forall t \in [0,\infty)$, *then* $\frac{\partial^n}{\partial x^n}u_t(x,t)$ *is bounded for* $n = 0,1,2,3$, $\forall t \in [0,\infty)$, *and* $\forall x \in [0,L]$.

2 Problem Statement

The primary control objective is to design the boundary control force $f(t)$ such that the beam displacement $u(x,t)$ is driven to zero $forall x \in [0,L]$ as $t \to \infty$. We first define an auxiliary signal, denoted by $\eta(t)$, as follow:

$$\eta(t) = u_t(L,t) - u_{xxx}(L,t) \tag{6}$$

We differentiate (6) with respect to time, multiply the resulting expression by m, and then utilize (3) to substitute for $mu_{tt}(L,t)$ to produce

$$m\dot{\eta}(t) = -mu_{xxxt}(L,t) + EIu_{xxx}(L,t) + f(t) \tag{7}$$

The above open-loop equation will form the basis for the design of the model-based and adaptive control laws.

3 Model-Based Control Law

Given the structure of the open-loop dynamics of (7), the control force is designed as follow:

$$f(t) = mu_{xxxt}(L,t) - EIu_{xxx}(L,t) - k_s\eta(t) \tag{8}$$

where k_s is a positive control gain. After substituting (8) into (7), we obtain the following closed-loop dynamics for $\eta(t)$:

$$m\dot{\eta}(t) = -k_s\eta(t) \tag{9}$$

Theorem 1 *The model-based boundary control law given by (8) ensures that the beam displacement is exponentially regulated in the following sense:*

$$|u(x,t)| \le \sqrt{\frac{2\lambda_2 L^3}{\lambda_1 EI}\kappa_o exp\left(-\frac{\lambda_3}{\lambda_2}t\right)} \quad \forall x \in [0,L] \tag{10}$$

provided the control gain k_s is selected to satisfy the following inequality:

$$k_s > \frac{EI}{2} \tag{11}$$

where $\lambda_1, \lambda_2,$ and λ_3 are some positive bounding constants, and the positive constant κ_o is given by

$$\kappa_o = \frac{1}{2}\rho \int_0^L u_t^2(\sigma,0)d\sigma + \frac{1}{2}EI \int_0^L u_{\sigma\sigma}^2(\sigma,0)d\sigma + (u_t(L,0) - u_{xxx}(L,0))^2 \tag{12}$$

Proof: To prove the result given by (10), we begin by defining the following function:

$$V(t) = E_b(t) + \frac{1}{2}m\eta^2(t) + E_c(t) \tag{13}$$

where the beam's energy-related term $e_b(t)$ and the "cross" term $E_c(t)$ are defined as

$$E_b(t) = \frac{1}{2}\rho \int_0^L u_t^2(\sigma,t)d\sigma + \frac{1}{2}EI \int_0^L u_{\sigma\sigma}^2(\sigma,t)d\sigma \tag{14}$$

and

$$E_c(t) = 2\beta\rho \int_0^L \sigma u_t(\sigma,t)u_\sigma(\sigma,t)d\sigma \tag{15}$$

with β being a positive weighting constant, which can be made sufficiently small to ensure $V(t)$ is always non-negative.$E_c(t)$ is bounded as follow:

$$E_c = 2\beta\rho \int_0^L \sigma u_t u_\sigma d\sigma \leq 2\beta\rho L \int_0^L (u_t^2 + u_\sigma^2)d\sigma \tag{16}$$

$$\leq 2\beta\rho L max\{1, L^2\} \int_0^L (u_t^2 + u_{\sigma\sigma}^2)d\sigma \tag{17}$$

$$\leq 4\beta\rho L \frac{max\{1, L^2\}}{min\{\rho, EI\}} \frac{1}{2} \int_0^L (\rho u_t^2 + EI u_{\sigma\sigma}^2) \tag{18}$$

Then we can get the following inequalities:

$$-4\beta\rho L \frac{max\{1, L^2\}}{min\{\rho, EI\}} E_b \leq E_c \leq 4\beta\rho L \frac{max\{1, L^2\}}{min\{\rho, EI\}} E_b \tag{19}$$

If β is selected according to

$$\beta < \frac{min\{\rho, EI\}}{4\rho L max\{1, L^2\}} \tag{20}$$

we can use (18) to state that

$$0 \leq \xi_1 E_b \leq E_b + E_c \leq \xi_2 E_b \tag{21}$$

for some positive constants ξ_1 and ξ_2. Given the definition of (13) and the inequality given by (21), we can formulate the following bounds on $V(t)$:

$$\lambda_1(E_b(t) + \eta^2(t)) \leq V(t) \leq \lambda_2(E_b(t) + \eta^2(t)) \tag{22}$$

where λ_1 and λ_2 are defined as follows:

$$\lambda_1 = min\{1 - 4\beta\rho L\frac{max\{1, L^2\}}{min\{\rho, EI\}}, \frac{1}{2}m\} > 0 \tag{23}$$

$$\lambda_2 = min\{1 + 4\beta\rho L\frac{max\{1, L^2\}}{min\{\rho, EI\}}, \frac{1}{2}m\} > 0 \tag{24}$$

We differentiate (13) with respect to time, we have

$$\dot{V}(t) = \dot{E}_b(t) + \dot{E}_c(t) - k_s\eta^2(t) \tag{25}$$

where (9) has been utilized. We differentiate (14) with respect to time to obtain

$$\dot{E}_b = -EI\int_0^L u_t u_{\sigma\sigma\sigma\sigma} d\sigma + EI\int_0^L u_{\sigma\sigma}u_{\sigma\sigma t} d\sigma \tag{26}$$

where (1) has been utilized. If we integrate by, part twice, the first term on he right-hand side of (25), we obtain

$$\dot{E}_b = EIu_t(L)u_{xxx}(L) \tag{27}$$

where the boundary conditions given in (2) have been applied. finally, upon the application of (6),(26) can be rewritten as

$$\dot{E}_b = -\frac{EI}{2}(u_t^2(L) + u_{xxx}^2(L)) + \frac{EI}{2}\eta^2 \tag{28}$$

We differentiate (15) with respect to time and then apply (1) to produce

$$\dot{E}_c = A_1 + A_2 \tag{29}$$

where

$$A_1 = 2\beta\rho\int_0^L \sigma u_t u_{\sigma t} d\sigma \qquad A_2 = -2\beta EI\int_0^L \sigma u_\sigma u_{\sigma\sigma\sigma\sigma} d\sigma \tag{30}$$

After integrating, by parts, the expression for A_1 given in (29), we obtain

$$A_1 = 2\beta\rho\left(Lu_t^2(L) - \int_0^L u_t^2 d\sigma\right) - 2\beta\rho\int_0^L \sigma u_t u_{\sigma t} d\sigma \tag{31}$$

where (2) has been used. We can rearrange (31) to yield

$$A_1 = \beta\left(L\rho u_t^2(L) - \rho\int_0^L u_t^2 d\sigma\right) \tag{32}$$

$$A_2 = -2\beta EI\left(Lu_x(L)u_{xxx}(L) - \int_0^L u_\sigma u_{\sigma\sigma\sigma} d\sigma - \int_0^L \sigma u_{\sigma\sigma} u_{\sigma\sigma\sigma} d\sigma\right) \tag{33}$$

upon application of (2). After integrating, by parts, the last integral on the right-hand side of (33), we obtain

$$A_2 = -2\beta EI\left(Lu_x(L)u_{xxx}(L) + 2\int_0^L u_{\sigma\sigma}^2 d\sigma + \int_0^L \sigma u_{\sigma\sigma} u_{\sigma\sigma\sigma} d\sigma\right) \tag{34}$$

where (2) has been again used. After adding the expressions given by (33) and (34), we obtain

$$A_2 = -\beta EI\left(2Lu_x(L)u_{xxx}(L) + 3\int_0^L u_{\sigma\sigma}^2 d\sigma\right) \tag{35}$$

We can now substitute (32) and (35) into (29), and then substitute the resulting expression along with (28) into (25) to produce

$$\begin{aligned}\dot{V} = &-\left(\frac{EI}{2} - \beta L\rho\right)u_t^2(L) - \frac{EI}{2}u_{xxx}^2(L) - \left(k_s - \frac{EI}{2}\right)\eta^2 - 2\beta E_b \\ &- 2\beta EI\int_0^L u_{\sigma\sigma}^2 d\sigma - 2\beta EILu_x(L)u_{xxx}(L)\end{aligned} \tag{36}$$

where (14) has been utilized. Then we can obtain the following upper bound for $\dot{V}(t)$:

$$\begin{aligned}\dot{V} \leq &-\left(\frac{EI}{2} - \beta L\rho\right)u_t^2(L) - \left(k_s - \frac{EI}{2}\right)\eta^2 - 2\beta E_b \\ &- 2EIL\beta\left(\frac{1}{L^2} - \delta\right)u_x^2(L) - EI\left(\frac{1}{2} - \frac{2L\beta}{\delta}\right)u_{xxx}^2(L)\end{aligned} \tag{37}$$

From (37), it is not difficult to see that if the control gain k_s and the constants δ, β are selected to satisfy the following conditions:

$$k_s > \frac{EI}{2}, \qquad \delta < \frac{1}{L^2}, \qquad \beta < min\{\frac{EI}{2\rho L}, \frac{\delta}{4L}\} \tag{38}$$

then $\dot{V}(t)$ can be upper bounded by a non-positive scalar function as shown below:

$$\dot{V}(t) \leq -\lambda_3(E_b(t) + \eta^2(t)) \tag{39}$$

where λ_3 is defined as follow:

$$\lambda_3 = min\{k_s - \frac{EI}{2}, 2\beta\} > 0 \tag{40}$$

From (22) and (39), we can obtain the following upper bound for the time derivative of $V(t)$:

$$\dot{V}(t) \leq -\frac{\lambda_3}{\lambda_2} V(t) \tag{41}$$

Upon application of Lemma A.4 to (41), we have

$$V(t) \leq V(0)exp(-\frac{\lambda_3}{\lambda_2} t) \tag{42}$$

From (21) and (14), we have that

$$v(0) \leq \lambda_2(E_b(0) + \eta^2(0)) \tag{43}$$

and

$$\frac{1}{2L^3} EIu^2(x,t) \leq \frac{1}{2} EI \int_0^L u^2_{\sigma\sigma}(\sigma,t)d\sigma \leq E_b(t) \leq \frac{1}{\lambda_1} V(t) \tag{44}$$

The result given by (10) and (12) now directly follows by combining (42),(43) and (44), and the using the definitions given by (6) and (14).□