1. Courant Sigma Model

1.1. Courant algebroids.

Definition 1.1. A *Courant algebroid* on a manifold X is a tuple $(E, \langle -,- \rangle, a, [\![-,-]\!])$ where

(i) $E \to X$ is a (finite rank) vector bundle, whose sheaf of smooth sections we denote $\mathcal{E}$;

(ii) $\langle -,- \rangle : \mathcal{E} \times \mathcal{E} \to C^\infty_X$ is a nondegenerate, symmetric, bilinear pairing;

(iii) $a : \mathcal{E} \to \mathcal{T}_X$ is a C^∞_X-linear map, called the anchor;

(iv) $[\![-,-]\!] : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a bilinear operator.

This data is required to satisfy the following equations. We denote $x, y, z \in \mathcal{E}$, $f \in C^\infty_X$.

(1) BW: list

Every Courant algebroid defines a Lie algebroid in the obvious way. The conditions above imply that the tuple $(\mathcal{E}, a, [\![-,-]\!])$ has the structure of a Lie algebroid on X.

The collection of Courant algebroids on X form a 1-groupoid that we denote $\mathcal{CA}(X)$. The objects are simply Courant algebroids, and the morphisms are bundle isomorphisms preserving the pairing, anchor, and bracket. Note that given any $U \subset X$ one has a natural map of groupoids $\mathcal{CA}(X) \to \mathcal{CA}(U)$ given by restriction. The 1-groupoid of Courant algebroids satisfies a nice gluing law with respect to affine charts on X. Namely, there is an equivalence of groupoids

$$\mathcal{CA}(X) \simeq \lim_{U \subset X} \mathcal{CA}(U)$$

where the limit is taken over the category of affine subsets of X.

Theorem 1.2 ([?Roytenberg])**.** *Let L be a Lie algebroid. There is a one-to-one correspondence between 2-shifted (Roytenberg) symplectic structures on the dg manifold $[X/L] = (X, \mathbf{C}^*(L))$ and Courant algebroid structures on L. BW: say as equivalence of groupoids*

If we relax the symplectic definition to include homotopy coherent 2-shifted symplectic structures one finds the notion of a *twisted* Courant algebroid.

There is a particularly well-behaved class of Courant algebroids that are important for us. First, note that the linear dual of the anchor map determines a map

$$a^* : \mathcal{T}_X^\vee = \Omega^1_X \to \mathcal{E}^\vee \cong \mathcal{E}.$$

In the isomorphism, we have identified $\mathcal{E}$ with its dual via the pairing $\langle -,- \rangle$. A Courant algebroid is *exact* if the resulting sequence of locally free sheaves

$$0 \to \Omega^1_X \xrightarrow{a^\vee} \mathcal{E} \xrightarrow{a} \mathcal{T}_X \to 0$$

is exact. The above exact sequence determines a class in $H^1(X, \Omega^2_{cl})$.

Theorem 1.3. [?Severa, ?SafPym, ?...] *The stack of exact Courant algebroids on X is equivalent to the stack of of 1-shifted closed two-forms $\Omega^2_{cl}(X)[1]$. In particular, every exact Courant algebroid is completely determined, up to isomorphism, by the class $[H] \in H^1(X, \Omega^2_{cl})$, called its "Ševera" class.*

1.2. **The Courant σ-model.** Every Courant algebroid defines the following dg Lie algebroid

$$\left(\mathcal{T}_X^\vee[1] \xrightarrow{a^\vee} \mathcal{E}\right) \xrightarrow{a} \mathcal{T}_X,$$

that we denote by $\mathcal{L}_{\mathcal{C}}$. Here, the parentheses indicate the differential on the Lie algebroid, and a is the anchor. By [???] we know $(X, \mathrm{enh}(\mathcal{L}_{\mathcal{C}}))$ is an L_∞-space equipped with a 2-shifted symplectic structure that we denote by $\omega_{\mathcal{L}}$.

Definition 1.4. Let $\mathcal{E}$ be a Courant algebroid and M a three-manifold. The perturbative Courant σ-model of maps from M to $\mathcal{E}$ near the smooth map $f : M \to X$ has underlying space of fields

$$\Omega^*(M, f^*\mathrm{enh}(\mathcal{L}_{\mathcal{C}}))[1].$$

The (-1)-shifted pairing is defined by:

$$\langle \alpha, \alpha' \rangle = \int \omega_{\mathcal{L}}(\alpha \wedge \alpha').$$

The action functional is encoded by the local L_∞-structure on $\Omega^*(M, f^*\mathrm{enh}(\mathcal{L}_{\mathcal{C}}))$.

To begin, let's suppose that f is a constant map. Then, as a graded $\Omega^\#_X$-module, the space of fields has the form

$$\Omega^\#(M) \otimes \Omega^\#(X, T_X^\vee[2] \oplus E[1] \oplus T_X).$$

RG: Exact Courant algebroids are classified by their Severa class $H \in \Omega^3_{cl}(X)$. Twisting the canonical topological boundary condition via this class may give a description of the A-model with H-flux H. See R. Szabo's work or the JHEP article of Bonechi–Cattaneo–Iraso (RG: they get it as a certain gauged fixing for the Poisson sigma model where the Poisson structure is the inverse of a Kahler form).

1.3. **Dirac Structures.** Suppose $\mathcal{E}$ is a Courant algebroid (may or may not be exact) on X.

A Dirac structure is a subbundle $L \subset E$ such that

(i) L is Largrangian with respect to the pairing $\langle -, - \rangle$;
(ii) L is involutive with respect to the bracket $[-, -]$.

There is a slightly modified version of a Dirac structure that is relative to a closed submanifold $i : Y \hookrightarrow X$. A Dirac structure on the pair $(\mathcal{E}, Y)$ is a subbundle $L \subset f^*E$ such that

(i) L is Largrangian with respect to the pairing $\langle -, - \rangle$;
(ii) L is compatible with the anchor, in the sense that $a(L) \subset TY \subset f^*TX$.
(ii) L is involutive with respect to the bracket $[-, -]$.

BW: Pavel classifies Lagrangian structures on morphisms $[Y/\mathcal{M}] \to [X/\mathcal{E}]$ where $\mathcal{E}$ is a Courant algebroid so $[X/\mathcal{E}]$ is 2-symplectic. In the case $Y = X$ you get the first type of Dirac structures. In the general case, you get the second type.

In the exact case.

Proposition 1.5. *Suppose we use the Dirac structure for the standard exact Courant algebroid $\mathcal{E} = T_X \oplus T_X^\vee$ defined by a Poisson structure (X, Π). The corresponding boundary theory for the Courant σ-model of maps from $\Sigma \times \mathbb{R}_{\geq 0}$ is equivalent to the Poisson σ-model on Σ with target (X, Π).*

1.4. **Link Invariants.** Let E be a Courant algebroid and R a representation up to homotopy of the associated 2-symplectic Lie algebroid. Further, assume that R is equipped with an invariant trace. Wilson loop observables determine invariants for links in a 3-manifold source manifold (anomalies?).

BW: I claim there are no anomalies for any Courant σ-model. This should probably wait till a later paper, but perhaps it's good for us to study the local deformation complex still...