Bootstrapping and confidence intervals

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Prologue:

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Feedback and exercises

- XX of you filled out the feedback survey. Main take-aways:
  - TBA
- What were the main problems with the exercises?

Learning Goals

- Understand the concept of bootstrapping and for what it is useful
- Understand the concept of a confidence interval and learn how to compute and interpret them
- Understand how we can become more confident in our estimation of population parameters

Motivation

Why bootstrapping?

- Usually we cannot study the populations of interest directly → **statistical inference** via **random samples**
- To interpret estimates from random samples, we must know about the properties of our samples
  - Especially: their **sampling distribution** → determines confidence in estimates
- In the previous session we learned how to study the sampling distributions of point estimates using MCS
- But this assumed that we can draw many samples, or even know certain properties of the population
  - In practice we can draw only one sample and know little about the population → **bootstrapping**

Why bootstrapping

- The idea of bootstrapping is to study the sampling distribution of our point estimate by **re-sampling** our sample

- This means: we draw many sub-samples from our sample and study them via MCS

- It turns out that this does **not** help us to improve our estimates...

- ...but gives accurate information about the sampling distribution of our estimate → **quantify uncertainty** due to random sampling

Re-sampling somehow allows us to "pull ourselfes up by your bootstraps"

An introductory example

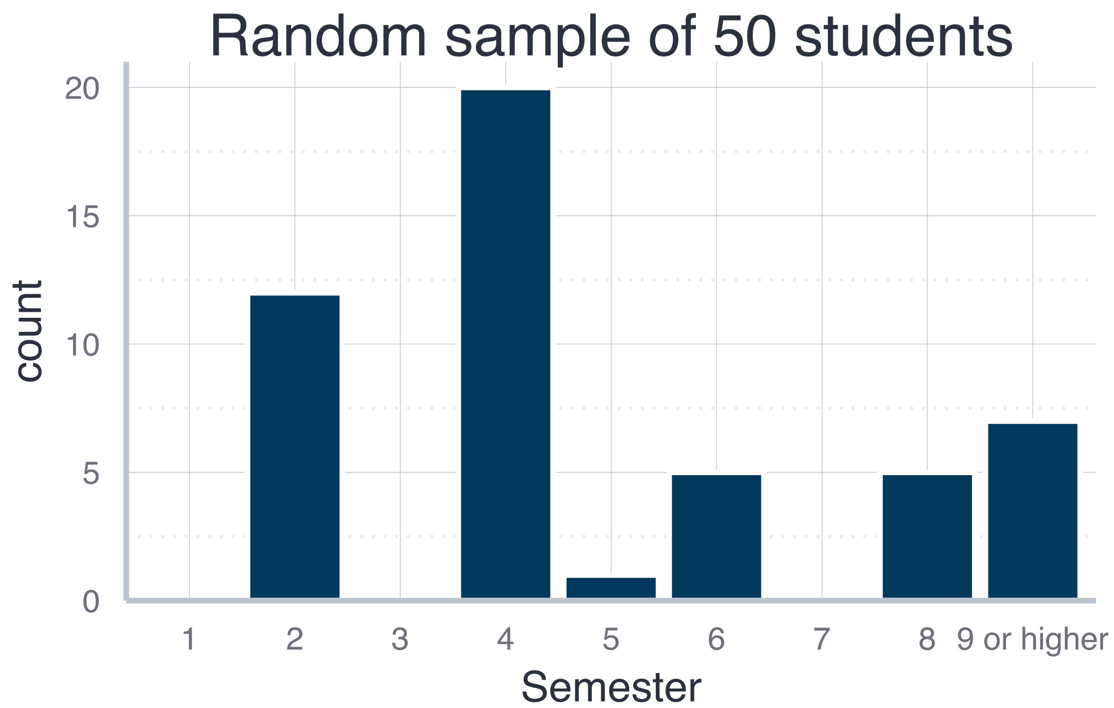
- Suppose we want to know the average study semester of EUF students
  - **Population parameter** of interest: μ (average study semester)
  - We could make a census and ask all students but this is to much work
- We might therefore ask a **random sample of 100 students** about their study semester and infer the population parameter of interest
  - **Sample statistic**: sample mean $\bar{x}$ (or $\hat{\mu}$) of the study semester

An introductory example

- Suppose that the distribution of study semesters in our sample is as such:

- The average study semester of our sample is given as $\bar{x} = 4.84$

- If our sample was drawn randomly, this would be a good guess for the average study semester in the population

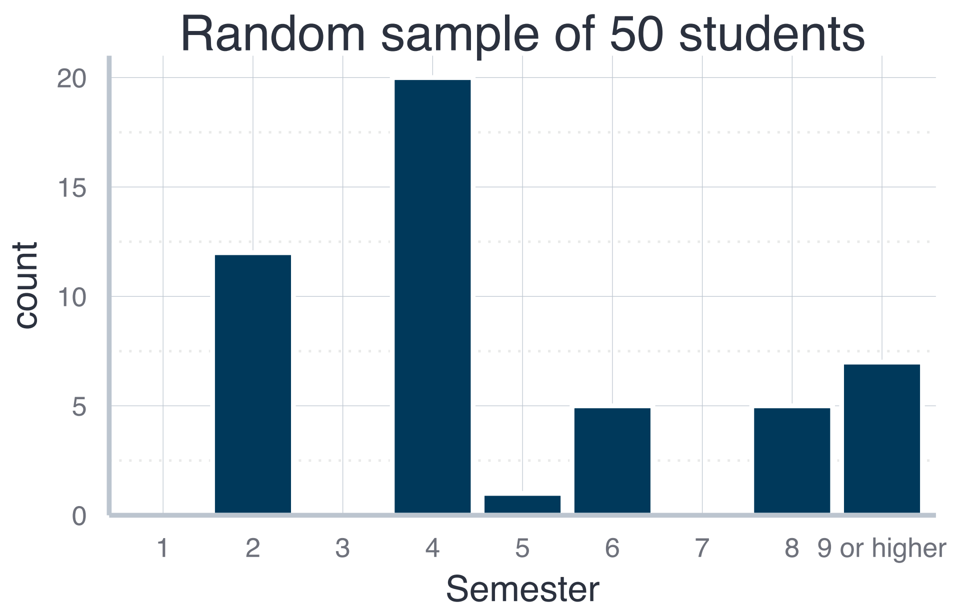


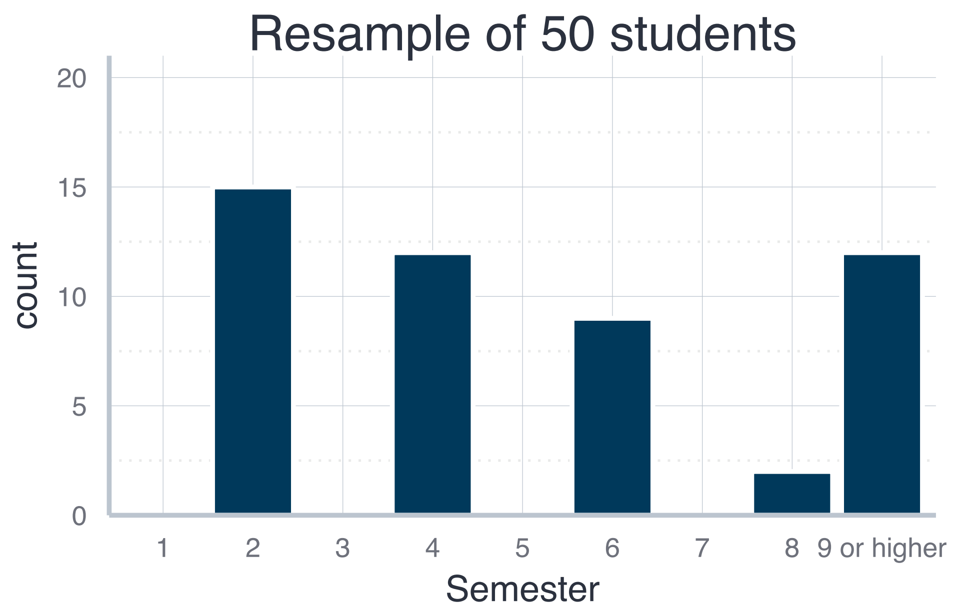
- However, if we asked another 50 students, we would most likely get a different sample statistic → sample variation

- Thus, it might be good and honest to quantify the uncertainty around our guess of `4.84` → requires knowledge about sampling distribution of $\bar{x}$

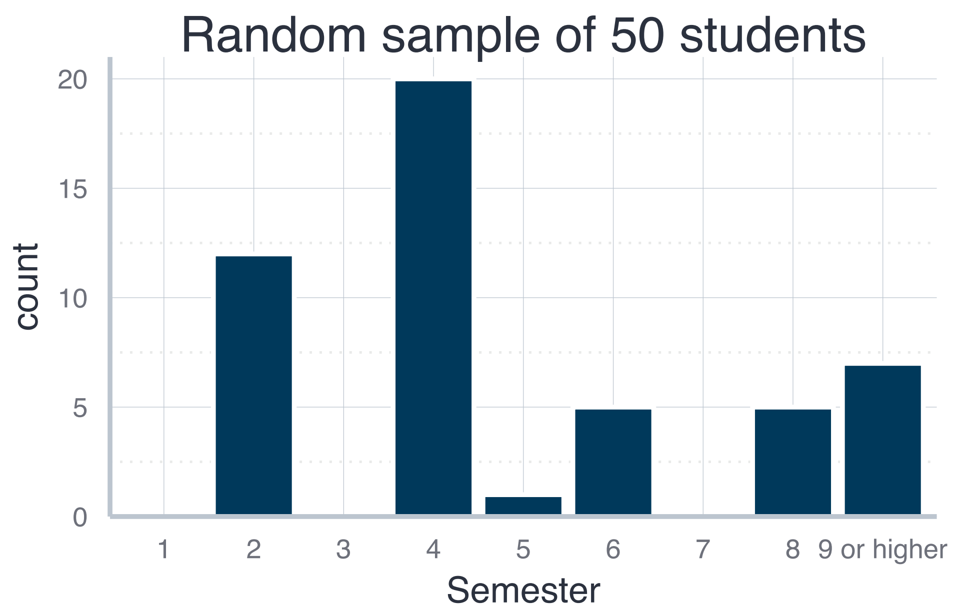
An introductory example

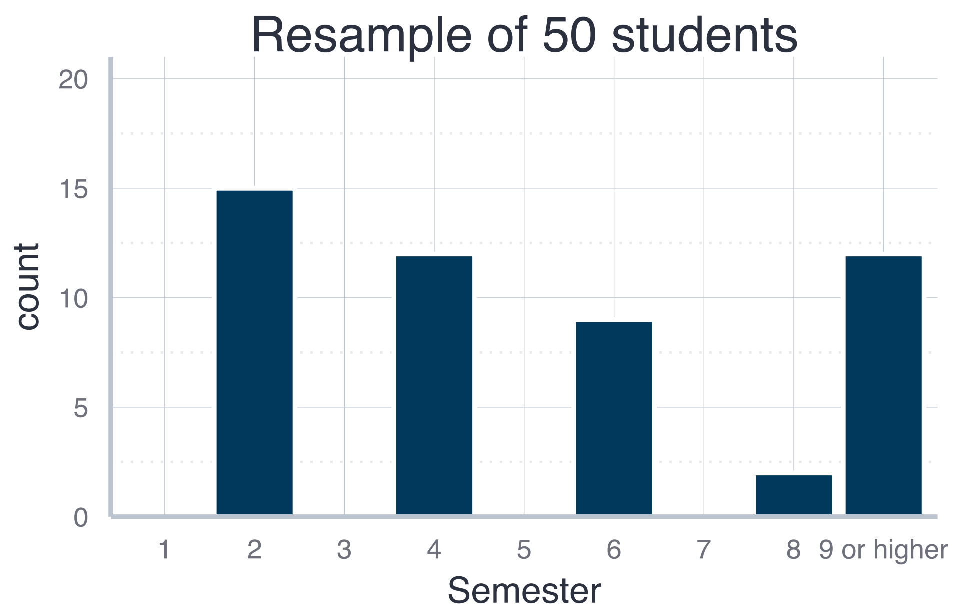
- But how can we get information about the sampling distribution without repeating the process of drawing a sample many, many times?
- The answer: re-sampling, i.e. drawing a sample from our sample
  - Draw 100 elements from our sample but with replacement (since $n = 100$)
  - What we are doing is called **re-sampling with replacement**
- Here is the result of a single re-sampling activity:





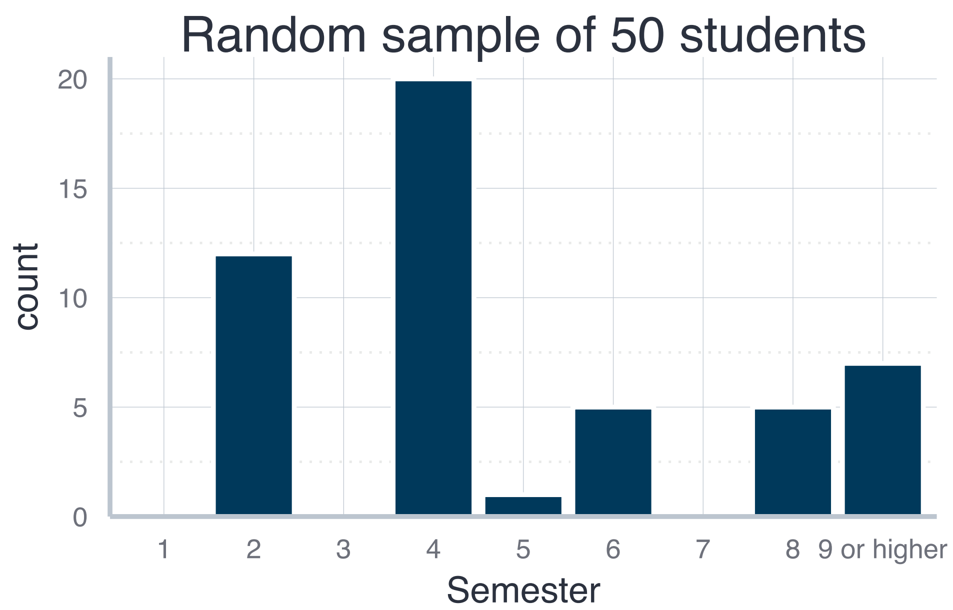
An introductory example

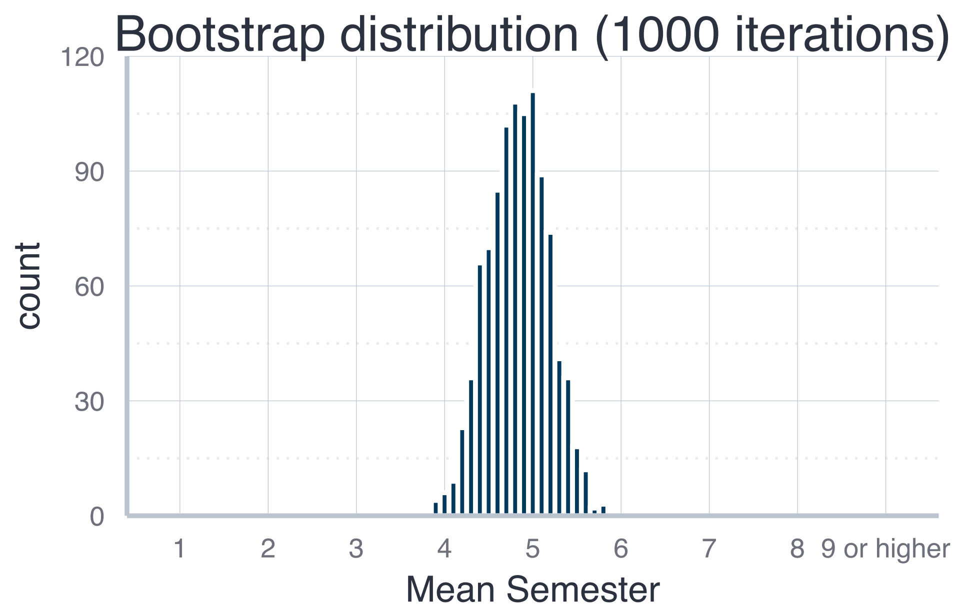




- The re-sample has some similarities to the original sample, but is not identical → sampling variation due to re-sampling
- But does this re-sample help us understand the sampling distribution of our guess $\bar{x}$ for the population parameter μ?
- No, since it is only a single re-sample!
  - Lets repeat this process 100 times and compute $\bar{x}$ for each re-sample!

An introductory example





- The figure on the right side is called the bootstrap distribution
  - Result of re-sampling the original sample many times and compute the sample statistic of interest for each iteration
  - Note that this distribution looks approximately like a normal distribution
- The bootstrap distribution is an **approximation to the sampling distribution** of our sample statistic $\bar{x}$

Central take-aways and implementation

- We were interested in the population parameter μ, i.e. the average study semester of EUF students
- We drew a **single random sample** and computed the sample mean $\bar{x}$
- In principle, this is not a bad guess for μ, but we were aware that the sample mean is subject to **sample variation**
- To get information about the sample distribution of $\bar{x}$ we did **re-sampling with replacement** and produced a **bootstrap distribution**
  - This is the distribution of sample means for 1000 re-samples from our original sample
- The bootstrap distribution **approximates the sample distribution** of $\bar{x}$

Central take-aways and implementation

- To draw a sample or repeated samples you may use the convenience function `moderndive::rep_sample_n()`:

```
moderndive::rep_sample_n(
    tbl = pop_data,
    size = 50,
    replace = TRUE,
    reps = 1000)
```

The population from which samples should be drawn

The size of the samples

Should re-samples be drawn with or without replacement?

Ho many samples should be drawn?

- The code above draws 1000 samples of size 50 from the tibble `pop_data`; each sample is drawn with replacement

- The function always produces tibbles of the following form:

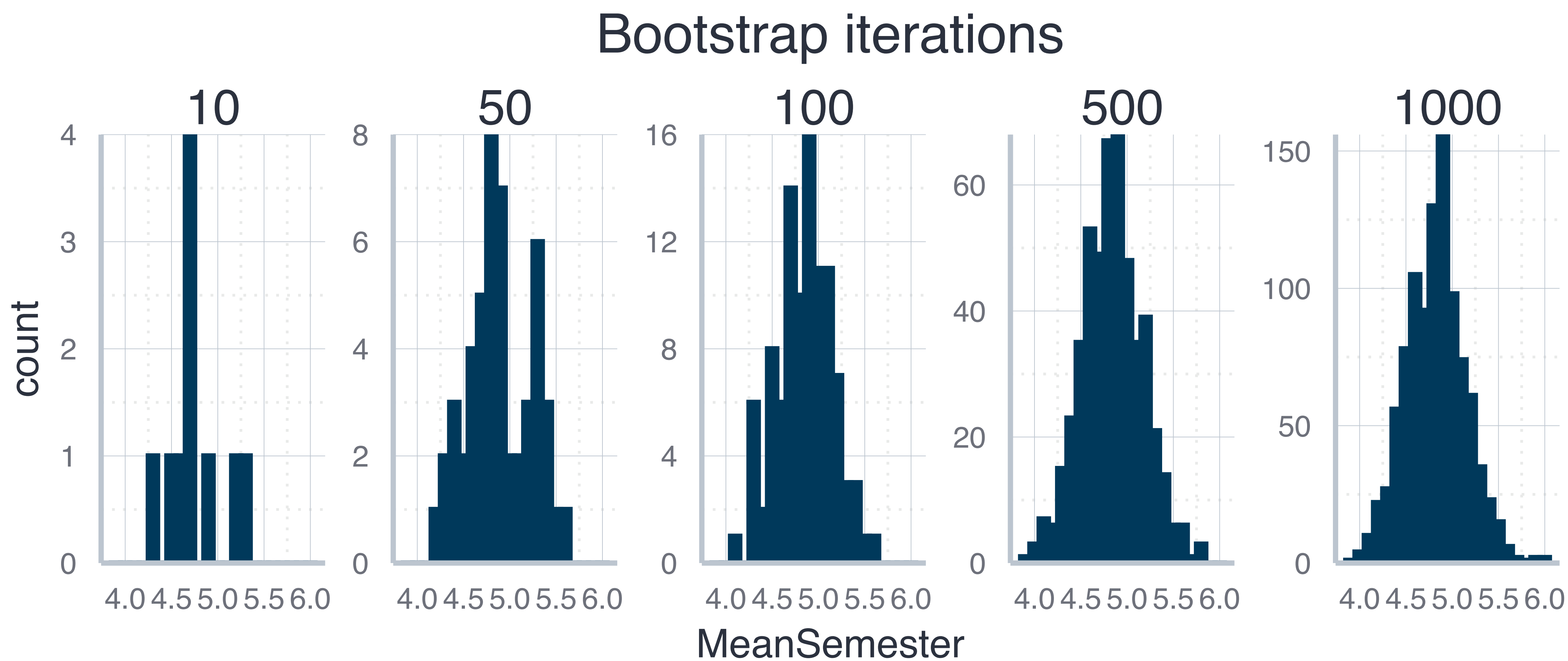
```
# A tibble: 50 × 3
# Groups:   replicate [1]
  replicate Semester    ID
      <int> <fct>    <int>
1         1 2         4237
2         1 4         4818
3         1 8         3937
4         1 4         4089
```

Exercise 1: Constructing a bootstrap distribution

- Consider the data set `T15-SemesterSample.csv` from the course page
  - Contains a sample of EUF students and their study semester
- Compute the bootstrap distribution for the sample mean as discussed above to answer the following two questions:

1. What is the effect of the number of iterations during the bootstrap resampling process?
   - Look at the resulting distributions for 10, 50, 100, 500, and 1000 replications! What do you observe?
2. How could you use the bootstrap distribution to quantify your uncertainty about the sample mean and its usefulness to estimate μ?

Exercise 1: Constructing a bootstrap distribution



- For higher number of iterations, the values look more and more normally distributed
- Example solution is available online

Confidence intervals

Motivation for confidence intervals

- We used the bootstrap to get an approximation of the sampling distribution of our point estimate $\bar{x}$
  - But we would we want this information in the first place?
- Since point estimates are different for each sample drawn, it would be nice quantify our confidence in the particular estimate
  - Are we sure that the estimate is very close to the true parameter μ?
  - Or might the point estimate be rather far away due to sampling variation?
- If the sampling variation is very high, a single point estimate is more likely to be misleading than if the sampling variation is low
- Thus, a better (or at least: more honest) alternative to a point estimate is a **confidence interval**: an interval for which are are pretty sure it contains μ

Motivation for confidence intervals

- As alway, Ismay & Kim (2022) have a nice analogy:

Point estimate	Confidence interval
	

- The bootstrap distribution makes it easy to construct intervals for which we can be confident they contain μ

Image source:
Ismay & Kim (2022)

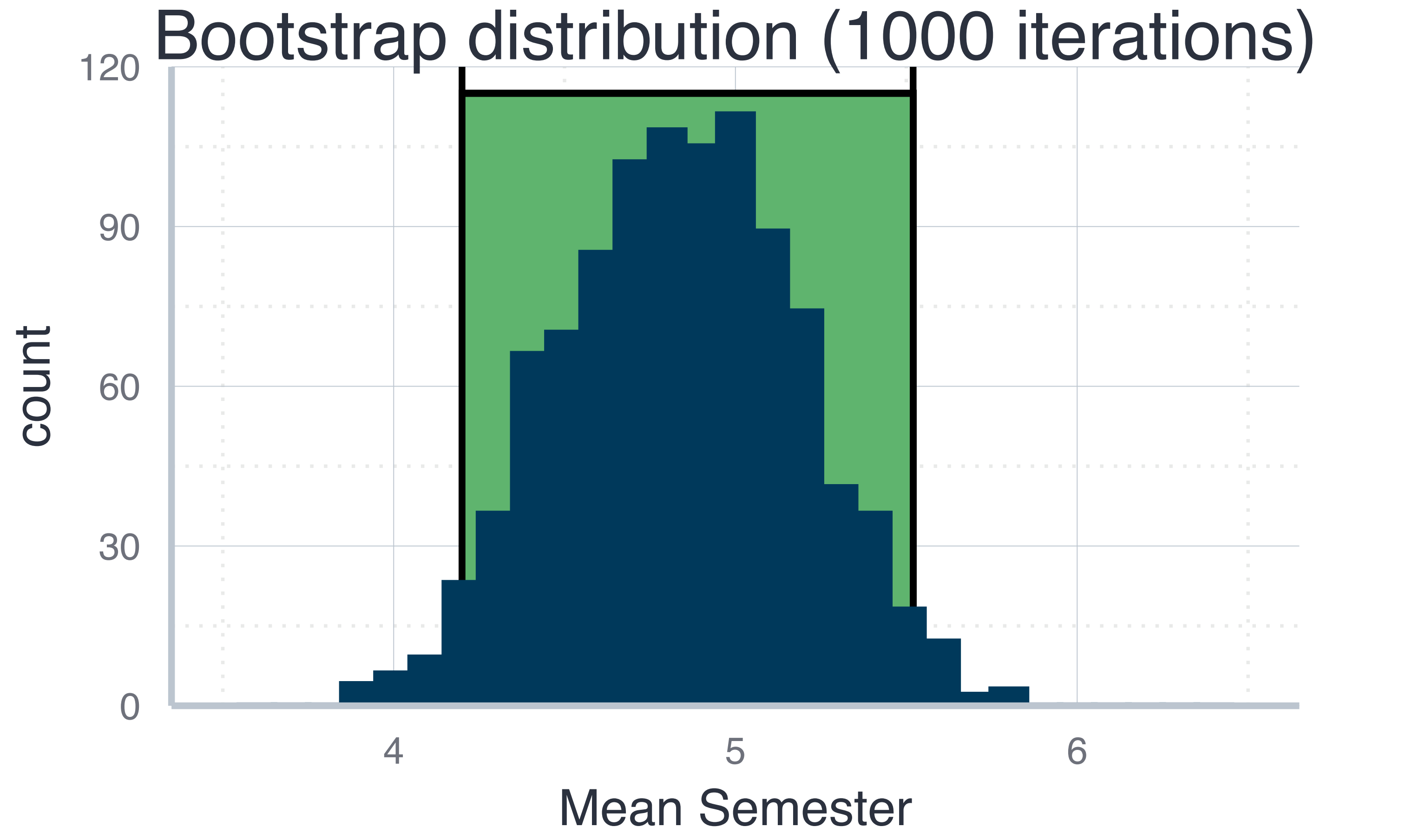
Constructing a confidence interval

- To construct a confidence interval, the following steps are necessary:
  1. Choose the desired level of confidence
  2. Do the bootstrapping
  3. Choose the method to compute the confidence interval
  4. Compute the confidence interval
  5. If desired, visualise the results
- We now learn how this can be done using the package `infer`
- We focus on the **percentile method** to compute confidence intervals
  - An alternative method is described in the mandatory readings

Percentile method: intuition

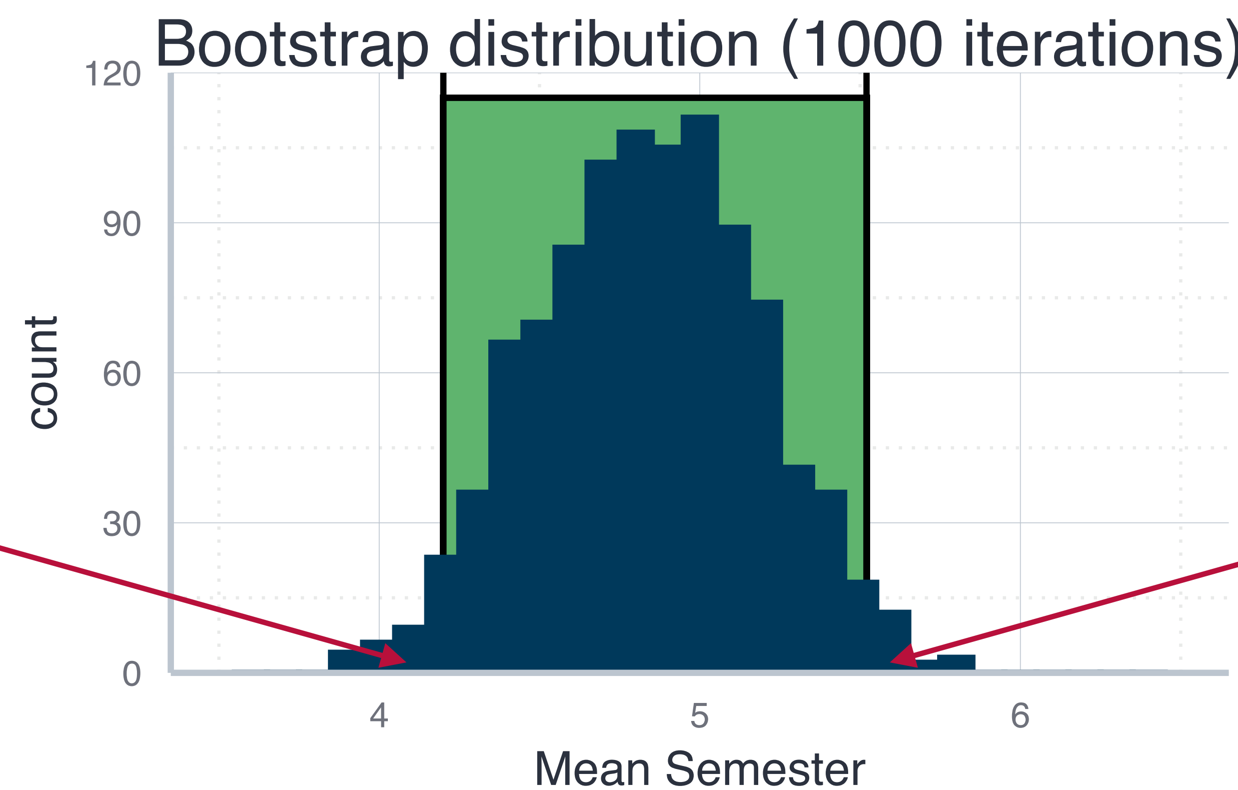
- We first need to specify the desired level of confidence
  - Confidence is measure in percent, typical values are 90%, 95% or 99%
  - The higher the confidence the larger the confidence interval

- Assume we want to construct a 95%-confidence interval
  - We just pick the middle 95% of the bootstrap distribution:



Percentile method: intuition

- Assume we want to construct a 95%-confidence interval
  - We just pick the middle 95% of the bootstrap distribution
  - To get the quantitative thresholds, compute the 2.5 and 97.5th percentile:



$P_{2.5} = 4.2$

```
quantile(
    x = sample_vals,
    probs = 0.025
)
```

$P_{97.5} = 5.52$

```
quantile(
    x = sample_vals,
    probs = 0.975
)
```

$$CI_{95\%} \left[4.2;\ 5.52\right]$$

Confidence interval: the whole workflow

1. Specify the variable that is of main interest using `infer::specify()`
2. Generate basis for the bootstrap distribution using `infer::generate()`
3. Generate the actual bootstrap distribution using `infer::calculate()`
4. Process the bootstrap distribution further, e.g. to create visualisations or to compute the actual CI

Confidence interval: the whole workflow

1. Specify the variable that is of main interest using `infer::specify()`

  - In the present case: $\bar{x}$ as estimate for the the population parameter μ:

```
data_used %>%
    infer::specify(formula = MeanSemester ~ NULL)
```

  - The `~ NULL` part is because we are only interested in the sampling distribution of $\bar{x}$ → later we will adjust this notation to, e.g., the regression context

2. Generate basis for the bootstrap distribution using `infer::generate()`

```
data_used %>%
    infer::specify(formula = MeanSemester ~ NULL)
    infer::generate(reps = 1000, type = "bootstrap")
```

  - `reps` controls the number iterations, `type` should always be `"bootstrap"`

Confidence interval: the whole workflow

3. Generate the actual bootstrap distribution using `infer::calculate()`

```
bootstrap_dist <- data_used %>%
    infer::specify(formula = MeanSemester ~ NULL)
    infer::generate(reps = 1000, type = "bootstrap") %>%
    infer::calculate(stat = "mean")
```

- In our case the statistic of interest is the `mean`, so we set `stat` to `"mean"`

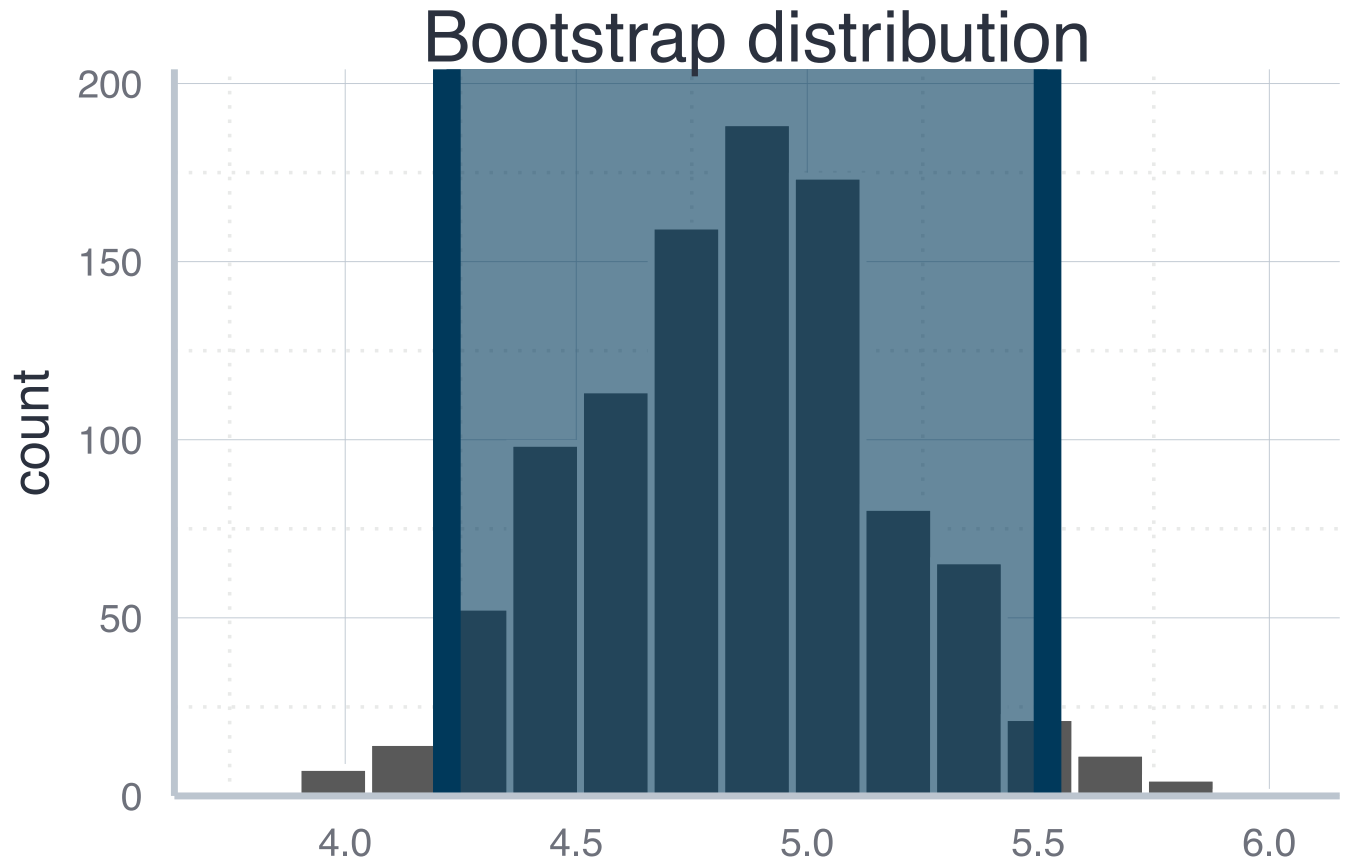
- The code above produces the bootstrap distribution that forms the basis for all further analysis steps
  - For instance, to compute the confidence intervals we use `infer::get_ci()`:

```
conf_ints <- bootstrap_dist %>%
    infer::get_ci(
        level = 0.95,
        type = "percentile")
```

Confidence interval: the whole workflow

- We can also take a more visual approach to create a plot with the confidence intervals directly using `infer::visualize()`:

```
visualize(student_boot_dist) +
    infer::shade_confidence_interval(
        endpoints = conf_ints)
```



Interpreting confidence intervals

- The interpretation of confidence intervals is not straightforward 😵‍💫
- Its main purpose is of the CI is to provide a corridor for which we are confident that it contains the population parameter of interest
- The problem: we do not know the true value for μ so we will never know whether our CI actually contains μ or not 🙄
- But what we can do is to consider an artificial situation in which we *know* μ and this way study the effectiveness of constructing CI using the method

➡ Lets run an example!

Exercise 2: how well do confidence intervals work?

- Consider the data set `DataScienceExercises::EUFstudents`
- Contains a census for all EUF students and their height → we know that $\mu = 166.5808$
  - This allows us to test whether our method to construct CI actually works

- To do so, we will conduct a MCS. To prepare it, do the following once:
  - Draw a random sample from the population
  - Compute the 95% percent confidence interval
  - Check whether the confidence interval contains the true average height
- To test whether an interval the true value you may use `ifelse()`!

The role of confidence

- We have the code to test whether a CI contains a true value once, now we iterate this process 100 times to draw more general conclusions
    - It turns out that about 95% if the CI contain the true value
- This is where the 95% come from:
    - We expect 95% of the CIs so constructed contain the true value
    - But in reality we only draw one sample and we can only construct the CI once
    - Nevertheless, this gives us a quantitative measure for the confidence in our statement
- What it we computed an 80% confidence interval?
    - Right, we expect 80% of the CIs so constructed contain the true value

How to interpret confidence intervals

- But what is the correct way to interpret a confidence interval?
- Assume we have $CI_{X\%} = \left[a; b\right]$, then the correct interpretation is:

If we repeated our sampling procedure a large number of times, we expect about X% of the resulting confidence intervals to capture the value of the population parameter.

- An informal variant frequently use is:

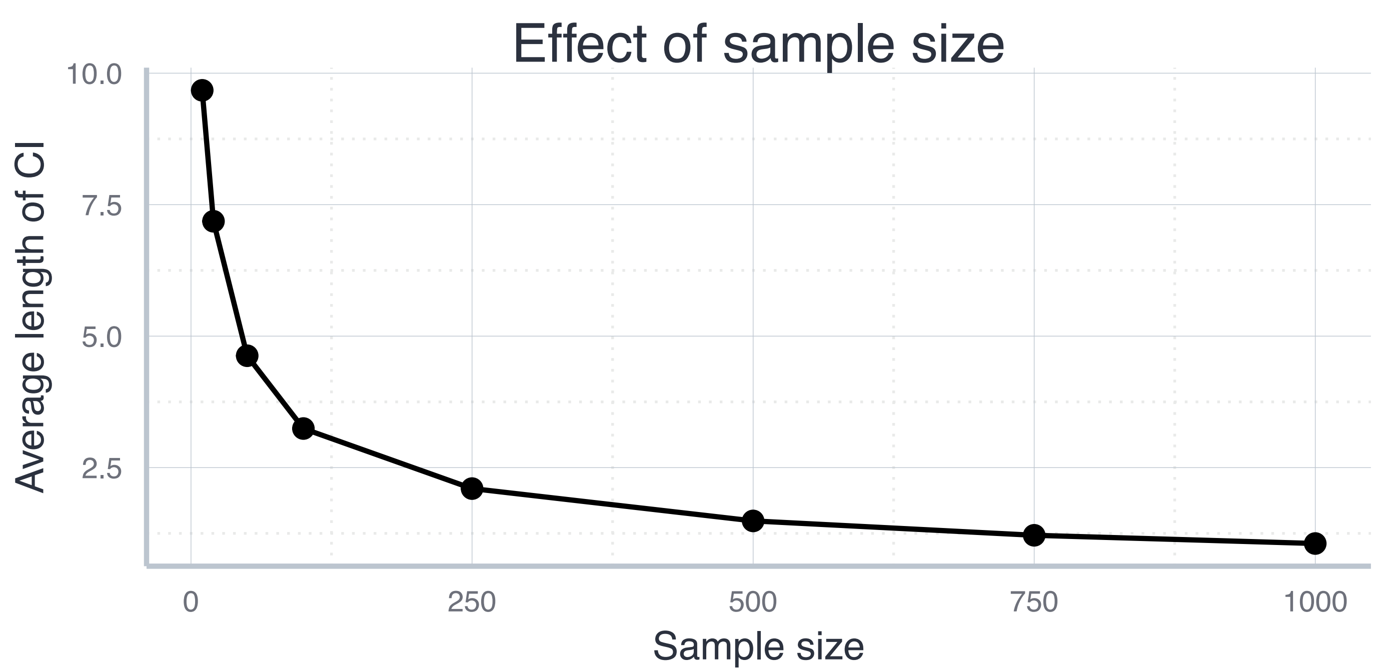
We are X% "confident" that $CI_{X\%} = \left[a; b\right]$ captures the value of the population parameter.

- A wrong interpretation is:

There is an X% probability that $CI_{X\%} = \left[a; b\right]$ contains μ.

Final remarks

- Note that there are two main determinants for the size of a CI:
  - First, the larger the confidence level, the larger the confidence interval
  - Second, the larger our sample, the smaller the confidence interval
- This indicates how important sample size is in practice
- Larger samples allow for more confident point estimates
- If we have a fixed sample size, we can increase confidence only by making less informative guesses



Summary & outlook

Summary

- In reality we can only use a single random sample to make an inference about an unknown population parameter
- The sample must be random in order to allow for effective inference
- But since it is random we also have random variation
- To quantify our confidence in our estimate we would like to take into account the effect of this random variation → get information about sampling distribution of the estimate
- A good approximation for the (unknown) sampling distribution is the bootstrap distribution
- The bootstrap distribution is obtained by doing **re-sampling with replacement** on our sample

Summary

- Bootstrap distributions **cannot improve the point estimates** as such
  - Their sample statistics differ from the population parameter of interest
- The standard error of the bootstrap distribution is good **approximation** for standard error of the (unknown) **sampling distribution**
- There are two main determinants for the size of a confidence interval:
  - The higher the confidence, the larger the interval
  - The larger the sample size, the smaller the interval
- This illustrates how important the sample size is in practice

Outlook

- In practice, confidence intervals are less frequently used than p-values, which we encounter in the next session
- But CI represent a more intuitive and transparent measure for the uncertainty associated with point estimates
- Next session we will apply confidence intervals to the case of regressions analysis and complement it with other tools to assess our model quality

Tasks until next week:

1. Fill in the **quick feedback survey** on Moodle
2. Read the **tutorials** and **lecture notes** posted on the course page
3. Do the **exercises** provided on the course page and **discuss problems** and difficulties via the Moodle forum