Lecture 10: Circles

15 February, 2019

Overview

Last time.

- Menelaus' theorem (can use this to prove collinearity of three points)
- Desargues' theorem
- Pappus' theorem
- Pascal's hexagon theorem

Today.

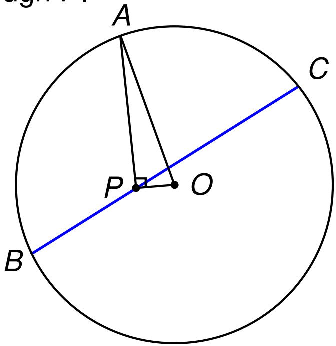
- Recall basic properties of circles
- The power of a point with respect to a circle
- The radical axis of two circles
- Pencils of circles

Exercise

Exercise. Consider a circle with centre O, let P be a point inside the circle and let B and C be points on the circle such that B, P, C are collinear.

Show that $|AP|^2 = |BP| \cdot |CP|$ in the diagram below.

Hint. First show that the quantity $|BP| \cdot |CP|$ is independent of the choice of line through P.

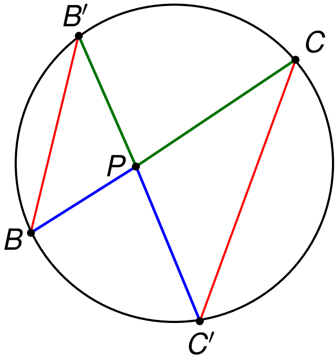


Solution

Solution.

Solution. Consider another pair of points B', C' on the circle such that B', C', P are collinear.

The arc BC' subtends equal angles at C and B'. The angles $\angle BPB'$ and $\angle CPC'$ are opposite and therefore equal. Therefore $\Delta BB'P \sim \Delta C'CP$ and so $|BP| \cdot |CP| = |B'P| \cdot |C'P|$

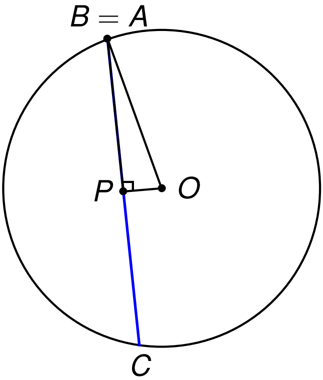


Solution (cont.)

Since $|BP| \cdot |CP|$ is independent of the line through P then we can choose $B = A$.

Then OP is perpendicular to BC and so it is the perpendicular bisector (since O is the centre of the circle).

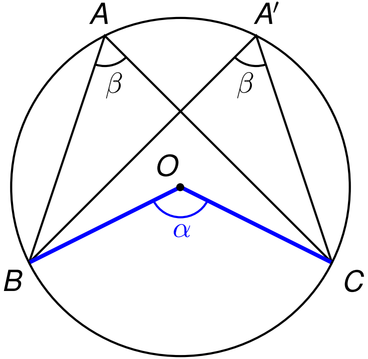
Therefore $|AP| = |BP| = |CP|$ and so $|AP|^2 = |BP| \cdot |CP|$.



Useful properties of circles (already proven)

Two properties of circles that we will use throughout today's lecture are

1. Equal arcs subtend equal angles in the same segment (Euclid Prop. III.21).

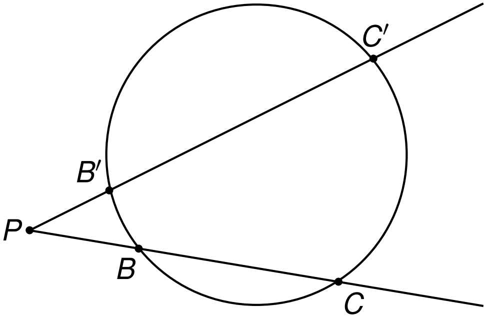


$$\angle BAC = \angle BA'C$$

Useful properties of circles (already proven)

2. If P is a point outside a circle, then for any line from P intersecting the circle in two points B and C, the quantity $|PB| \cdot |PC|$ is invariant.

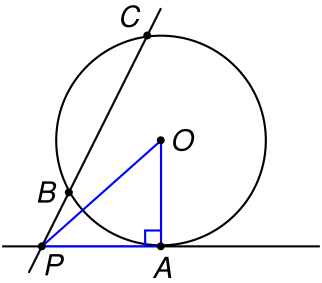
In the diagram below, we have $|PB| \cdot |PC| = |PB'| \cdot |PC'|$.
(recall the exercise from Lecture 9)



Definition. Consider a circle C with centre O and radius r. Let P be a point not on the circle. The power of P with respect to the circle C is defined to be

$$\mathcal{P}_C(P) = |OP|^2 - r^2$$

Question. What is the geometric meaning of the power?



Suppose that P is outside the circle. Let PA be a tangent to the circle and PBC be a line passing through the circle.
In Tutorial 4 we will prove that

$$|PA|^2 = |PB| \cdot |PC|$$

Power of a point with respect to a circle

Pythagoras' theorem also shows us that $|PA|^2 + |AO|^2 = |OP|^2$. Therefore, since $|AO| = r$

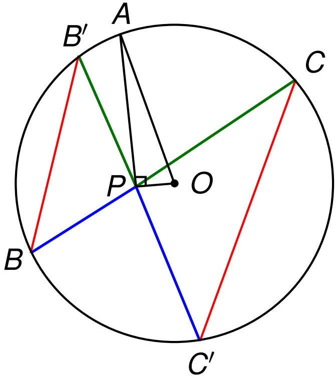
$$\mathcal{P}_C = |OP|^2 - r^2 = |OP|^2 - |AO|^2 = |PA|^2 = |PB| \cdot |PC|$$

Therefore then we have three equivalent definitions of the power of a point lying outside the circle:

- the square of the distance from P to a tangent point A,
- the product of the distances from P to collinear points B and C on the circle, or
- the original definition $\mathcal{P}_C(P) = |OP|^2 - |OA|^2 = |OP|^2 - r^2$.

Power of a point with respect to a circle

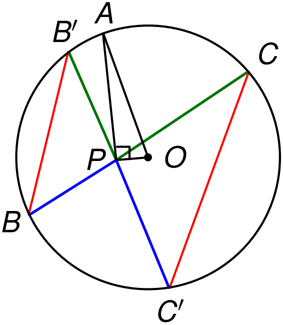
Question. What if P lies inside the circle?



The arc BC' subtends equal angles at C and B'. The angles $\angle BPB'$ and $\angle CPC'$ are opposite and therefore equal. Therefore $\Delta BB'P \sim \Delta C'CP$ and so $\frac{|BP|}{|B'P|} = \frac{|C'P|}{|CP|}$. Therefore

$$|BP| \cdot |CP| = |B'P| \cdot |C'P|$$

Power of a point with respect to a circle



We also know by Pythagoras' theorem that

$$\begin{aligned} |AP|^2 &= |OA|^2 - |OP|^2 \\ &= r^2 - |OP|^2 \end{aligned}$$

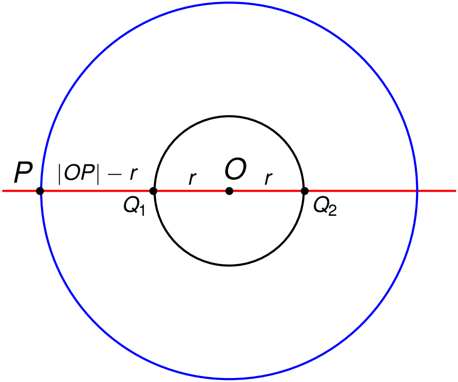
We showed in the exercise that $|PA|^2 = |PB| \cdot |PC|$.

Therefore we have three equivalent definitions of the power of a point P *inside* a circle $\mathcal{C}$.

- the difference $\mathcal{P}_{\mathcal{C}}(P) = |OP|^2 - |OA|^2$ (where $|OA|$ is the radius of the circle), or
- the square $\mathcal{P}_{\mathcal{C}}(P) = -|AP|^2$, where A is a point on the circle such that $AP \perp OP$.
- As a product $\mathcal{P}_{\mathcal{C}}(P) = -|PB| \cdot |PC|$.

The locus of points with the same power

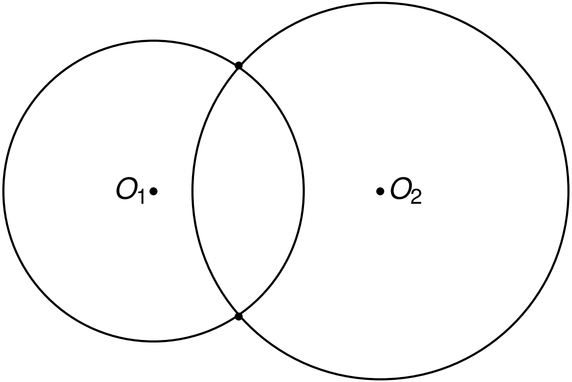
Given a circle with centre O and radius r, fix the value of the power $x = |OP|^2 - r^2$. What is the locus of points with power x? Since $|OP| = \sqrt{x + r^2}$ is fixed then the locus will be a circle of radius $\sqrt{x + r^2}$. We have $x = (|OP| + r)(|OP| - r)$.



Given any point P, the line from P through the centre O will intersect the circle at two points Q_1 and Q_2 whose distance from P is $|OP| + r$ and $|OP| - r$.

Exercise. Consider two circles in the plane with distinct centres O_1 and O_2. Describe the set of points that have equal power with respect to both circles.

Hint. First try the case where the circles intersect

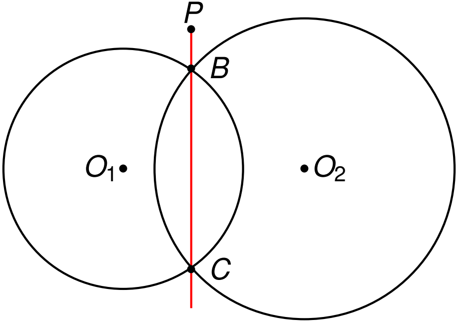


Solution

Solution.

Solution. Consider first the case where the circles intersect. Let C_1 be the circle with centre O_1 and C_2 the circle with centre O_2. Let B and C be the intersection points. Then if P lies on the line BC, we have (using *directed lengths* PB and PC)

$$\mathcal{P}_{C_1}(P) = PB \cdot PC \quad \text{and} \quad \mathcal{P}_{C_2}(P) = PB \cdot PC$$



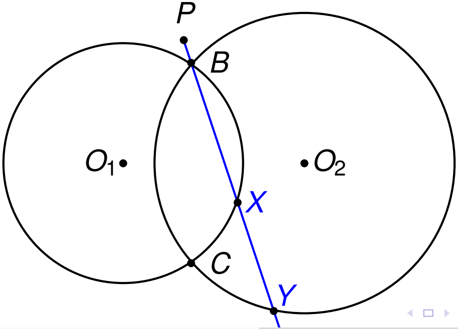
Therefore all of the points on the line BC have equal power with respect to both circles and $BC \perp O_1O_2$.

What about points not on this line? Is it possible that they have equal power?

Given a point P, construct the line PB (which intersects both circles). If P is not on the line BC then there are distinct points X and Y such that

$$\mathcal{P}_{C_1}(P) = PB \cdot PX, \quad \mathcal{P}_{C_2}(P) = PB \cdot PY$$

Since $PX \neq PY$ then $\mathcal{P}_{C_1}(P) \neq \mathcal{P}_{C_2}(P)$.



What if the circles do not intersect?

On the line O_1O_2, construct a point X such that $|O_1X|^2 - r_1^2 = |O_2X|^2 - r_2^2$ (i.e. X has equal power with respect to both circles).

If PX is perpendicular to O_1O_2, Pythagoras' theorem shows

$$|O_1P|^2 - r_1^2 = |O_1X|^2 - r_1^2 + |XP|^2$$

and

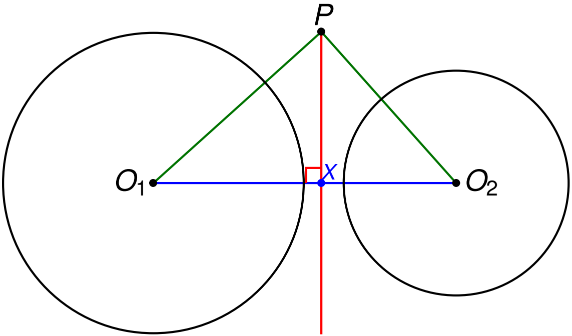
$$|O_2P|^2 - r_2^2 = |O_2X|^2 - r_2^2 + |XP|^2$$

Since $|O_1X|^2 - r_1^2 = |O_2X|^2 - r_2^2$ then $\mathcal{P}_{C_1}(P) = \mathcal{P}_{C_2}(P)$.

If PX is not perpendicular to O_1O_2 then we can use a similar idea to show that $\mathcal{P}_{C_1}(P) \neq \mathcal{P}_{C_2}(P)$.

Therefore the set of all points that have equal power to both circles is a line through X perpendicular to O_1O_2.

Solution (cont.)



The radical axis of two circles

We showed in the previous exercise that the locus of points which have equal power with respect to two circles is a line.

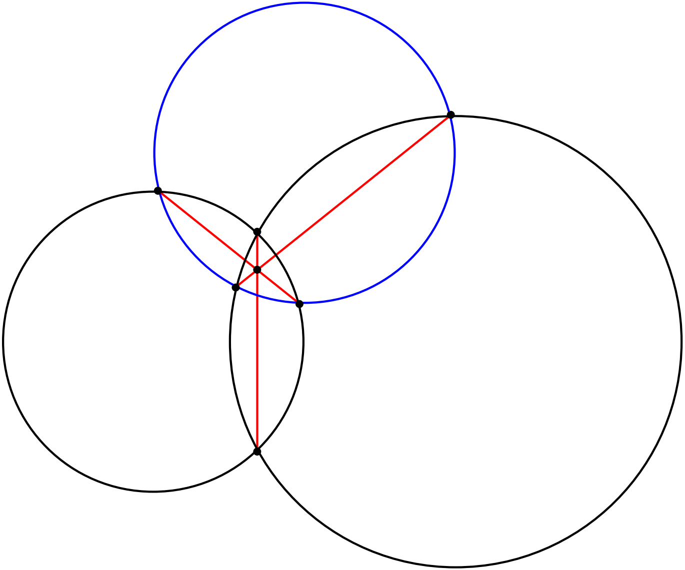
Moreover, this line is always perpendicular to the line joining the two centres of the circles.

Definition. Given two circles, the radical axis is the locus of points having equal power with respect to both circles.

Question. Suppose we are given a line in the plane. What are the pairs of circles that have this line as the radical axis?

In order to answer this, we will consider what happens when we add a third circle and look at the radical axes for the three pairs of circles.

The radical centre



The radical centre

Proposition. Given three circles in the plane, either the three radical axes are concurrent, or they are all parallel (concurrent at infinity). The point of concurrency is called the radical centre.

Proof. The radical axis of C_1 and C_2 consists of points that have the same power, $\mathcal{P}_{C_1}(P) = \mathcal{P}_{C_2}(P)$.

The radical axis of C_2 and C_3 consists of points that have the same power with respect to C_2 and C_3, $\mathcal{P}_{C_2}(P) = \mathcal{P}_{C_3}(P)$.

If these lines are not parallel, then they will intersect at a point Q. This point satisfies

$$\mathcal{P}_{C_1}(Q) = \mathcal{P}_{C_2}(Q) = \mathcal{P}_{C_3}(Q)$$

therefore it also lies on the radical axis of C_1 and C_3.

If the two radical axes are parallel, then they cannot intersect the radical axis of C_1 and C_3 at any points, since then the argument above shows that they must all be concurrent. Therefore, in this case, all three radical axes are parallel. ∎

Coaxial circles

Definition. Given two circles C_1 and C_2, a third circle C_3 is said to be coaxial with C_1 and C_2 if and only if the radical axis of C_1 and C_2 is the same as the radical axis of C_2 and C_3 (and hence the same as that of C_1 and C_3).

Definition. The collection of all circles coaxial with a given pair of circles is called a pencil of circles.

Lemma. All circles in a pencil have centres lying on the same line.

Proof. Fix one circle C_1 in the pencil. The line connecting the centre of C_1 with the centre of any other circle in the pencil must be perpendicular to the radical axis.

Therefore all of the centres lie on the line perpendicular to the radical axis which passes through C_1. ■

Classification of pencils

Proposition.

(a) If the original pair of circles intersect at two points, then a circle is in the pencil if and only if it contains those two points.
(In this case the pencil is called elliptic.)

(b) If the original pair of circles intersect at one point of tangency, then a circle is in the pencil if and only if it contains this point and it is tangent to the common tangent line (which is the radical axis).
(In this case the pencil is called parabolic.)

(c) If the original pair of circles do not intersect, then none of the circles in the pencil intersect.
(In this case the pencil is called hyperbolic.)

Proof of (a). Let C_1 and C_2 be circles intersecting at two points X, Y. Then the previous exercise shows that the radical axis of C_1 and C_2 is the line through X and Y.

If C_3 is any other circle in the pencil, then the pair C_3 and C_1 have the same radical axis as C_1 and C_2, which is the line through X, Y. Since X, Y are both on C_1 then they have zero power with respect to C_1, and therefore they must also have zero power with respect to C_3 (since the radical axis consists of points with the same power with respect to both circles).

Therefore X, Y are onthe circle C_3. Since this is true for any circle in the pencil then X, Y are on all the circles in the pencil.

Conversely, suppose that a circle C_3 passes through X, Y. Then the radical axis of C_3 and C_1 is the line through X, Y and so C_3 is in the pencil.

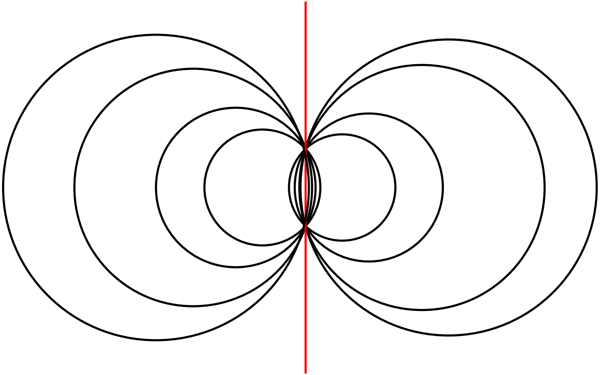
Proof of (b). Let C_1 and C_2 be circles intersecting at one point X. Then X has zero power with respect to both circles, and so it lies on the radical axis. Any other point P on the tangent line through X has power $|PX|^2$ with respect to both circles (using the tangent line definition of power), and therefore the radical axis must be the tangent line.

If C_3 is any other circle in the pencil, then X must also have zero power with respect to C_3, and so C_3 contains X. The same argument as above shows that the radical axis must be tangent to C_3.

Conversely, if C_3 is any circle which contains X and is tangent to the radical axis then C_3 and C_1 have the same radical axis.

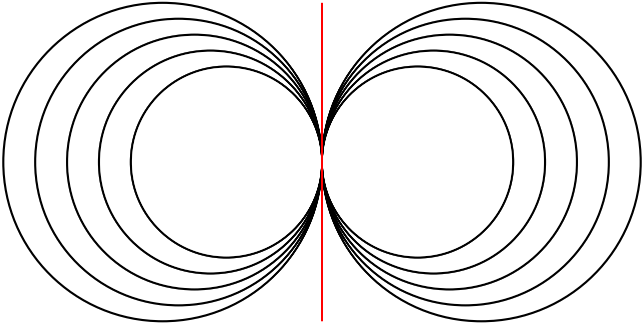
Proof of (c). Suppose (for contradiction) that a pair of circles in the pencil intersect. Then the pencil must be elliptic and all the circles pass through the points of intersection, which contradicts our assumption that the original pair of circles do not intersect.

An elliptic pencil



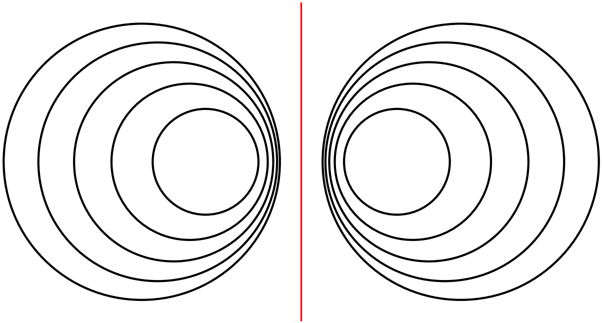
A pencil of intersecting coaxial circles

A parabolic pencil



A pencil of tangent coaxial circles

A hyperbolic pencil



A pencil of non-intersecting coaxial circles

Next time

We will continue to study pencils of circles and their properties.

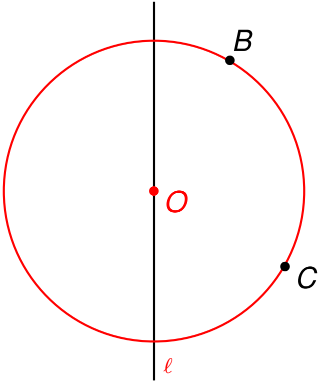
- Orthogonal pairs of pencils
- More properties of pencils
- Poncelet's theorem
- Apollonius' problem

Many properties of circles and pencils of circles also apply to general conic sections. We will also begin a preview of the chapter on conic sections.

Constructions

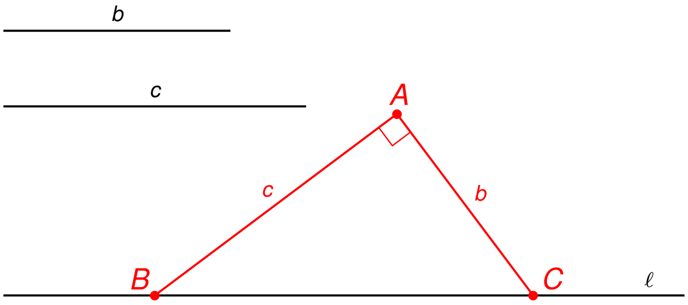
Let B, C be two points in the plane and let ℓ be a line which is not perpendicular to BC. Construct a circle containing the points B, C and which has centre O on ℓ.

What happens if ℓ is perpendicular to BC?



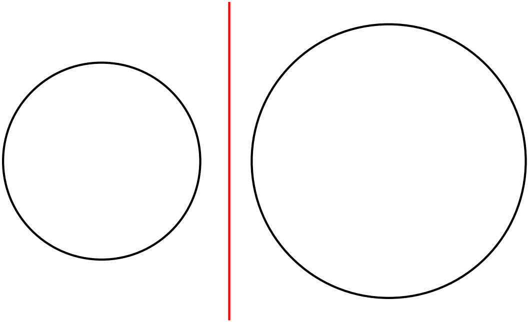
Constructions

Given a line ℓ and two lengths b, c, construct a right-angled triangle ΔABC such that the hypotenuse lies on ℓ and the two legs have length b, c.



Constructions

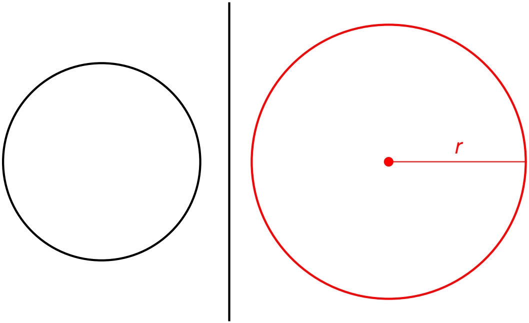
Given two non-intersecting circles, construct the radical axis.



Constructions

Given a circle and a line ℓ, construct another circle such that ℓ is the radical axis of the two circles.

Now do the construction so that the second circle has a given radius r.



Constructions

Given equally sized squares as in the figure below, construct the midpoints of AB and AC *without using a compass* (i.e. you are only allowed to use your ruler).