Topology of the moduli space of Higgs bundles over a compact Riemann surface

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Goal

The moduli space of Higgs bundles was defined by Hitchin in 1987. In Hitchin's words "...the moduli space of all solutions turns out to be a manifold with an extremely rich geometric structure".

In this talk I will describe some of the features of this moduli space.

- Definition of Higgs bundles and the moduli space of Higgs bundles
- Properties of the moduli space
- Survey of results on the cohomology of the moduli space (Hitchin, Gothen, Hausel, Hausel-Rodriguez-Villegas, Garcia-Prada-Heinloth-Schmitt)
- Morse theory for the Yang-Mills-Higgs functional and new results about the equivariant cohomology of the space of semistable Higgs bundles (joint with G. Daskalopoulos and R. Wentworth).

Holomorphic bundles

Let X be a compact Riemann surface and let $p : E \to X$ be a smooth complex vector bundle.

Smooth classification of bundles. E is determined up to C^∞ isomorphism by its rank n and first Chern class $c_1(E) \in H^2(X, \mathbb{Z})$.

E is a *holomorphic bundle* if all of the transition functions are holomorphic $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C})$

The classification of holomorphic bundles up to isomorphism is more complicated and the moduli space of holomorphic bundles is an interesting object in gauge theory and algebraic geometry.

Gauge-theoretic description of holomorphic bundles

Each trivialisation $p^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}^n$ has a natural $\bar{\partial}$-operator

$$\bar{\partial} : \Omega^0(U_\alpha \times \mathbb{C}^n) \to \Omega^{0,1}(U_\alpha \times \mathbb{C}^n)$$

If the transition functions are all holomorphic then this operator is well-defined on overlapping trivialisations

$$\bar{\partial}(g_{\alpha\beta}(z)s(z)) = g_{\alpha\beta}(z)\,\bar{\partial}(s(z))$$

and so it extends to a $\mathbb{C}$-linear operator $\bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$.

This operator satisfies the Leibniz rule

$$\bar{\partial}_E(fs) = (\bar{\partial}f)s + f\bar{\partial}_E s, \quad f \in \Omega^0(X), s \in \Omega^0(E)$$

and we can extend it to a $\mathbb{C}$-linear operator $\Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$.

We automatically have $\bar{\partial}_E^2 = 0$ since $\Omega^{0,2}(E) = \{0\}$ when the base manifold has complex dimension one.

Therefore, $\{\text{holomorphic bundles}\} \longrightarrow \{\bar{\partial} - \textit{operators}\}$.

Gauge-theoretic description of holomorphic bundles (cont.)

What about the converse? Do $\bar{\partial}$-operators determine holomorphic bundles?

Theorem. [Newlander-Nirenberg] Let $E \to X$ be a complex vector bundle over a complex manifold X. An operator $\bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$ satisfying

- the Leibniz rule, and
- $\bar{\partial}_E^2 = 0$

defines a holomorphic structure on E.

Therefore, $\{\text{holomorphic bundles}\} \overset{\textit{bijective}}{\longleftrightarrow} \{\bar{\partial} - \textit{operators}\}$.

When X is a compact Riemann surface (and so the condition $\bar{\partial}_E^2 = 0$ is trivial) then the space of all such $\bar{\partial}$ operators is an affine space locally modelled on $\Omega^{0,1}(\operatorname{End}(E))$.

$$\mathcal{A}^{0,1} := \bar{\partial}_{E_0} + \Omega^{0,1}(\operatorname{End}(E))$$

Gauge-theoretic description of holomorphic bundles (cont.)

Now fix a Hermitian metric H on the fibres of E. Given a holomorphic structure $\bar{\partial}_E$ on E, there is a unique *Chern connection* compatible with the metric

$$D_{H,E} : \Omega^0(E) \to \Omega^1(E), \quad \text{such that the } (0,1) \text{ part is } \bar{\partial}_E$$

We write

$$D_{H,E} = \bar{\partial}_E + \partial_{H,E} \quad \text{where } \partial_{H,E} : \Omega^0(E) \to \Omega^{1,0}(E)$$

Let $\mathcal{A} := d_{E_0} + \Omega^1(\mathrm{ad}(E))$ denote the space of all connections compatible with the metric.

For each Hermitian metric, the Chern connection construction defines a homeomorphism

$$\mathcal{A}^{0,1} \xrightarrow{\cong} \mathcal{A}$$

Gauge-theoretic description of holomorphic bundles (cont.)

Using the Chern connection construction, we can define the curvature of a holomorphic structure $F(\bar{\partial}_E, H) = D_{H,E}^2$ and hence the *Yang-Mills functional*

$$\mathrm{YM}(\bar{\partial}_E, H) = \| F(\bar{\partial}_E, H) \|^2$$

The *Yang-Mills equations* are the Euler-Lagrange equations for the critical points of this functional

$$D_{H,E}^* F(\bar{\partial}_E, H) = 0$$

Remark. The Yang-Mills functional and the Yang-Mills equations depend on the metric H. Some natural questions are:

- Given a holomorphic structure $\bar{\partial}_E$, do there exist "good" metrics for which $\bar{\partial}_E$ is a solution to the Yang-Mills equations?
- Is $\bar{\partial}_E$ is a minimiser for the Yang-Mills functional?

How to classify holomorphic bundles up to equivalence?

When E is a line bundle, the classification of holomorphic structures up to isomorphism is a classical result due to Abel and Jacobi.

The holomorphic line bundles on a compact Riemann surface are classified up to isomorphism by the points in the *Jacobian*

$$\mathrm{Jac}(X) := \mathbb{C}^g / \Lambda$$

where $\Lambda \cong \mathbb{Z}^{2g}$ is the period lattice of X.

In the gauge-theoretic setting of connections, this theorem says that

$$\mathrm{Jac}(X) \cong \mathcal{A}^{0,1} / \mathcal{G}^{\mathbb{C}} \overset{(\textit{metric})}{\cong} \mathcal{A} / \mathcal{G}^{\mathbb{C}}$$

When $\mathit{rank}(E) > 1$ then the problem is more complicated. The isomorphism classes do not form a "good" moduli space (it is not even Hausdorff).

How to classify holomorphic bundles up to equivalence?

Mumford's Geometric Invariant Theory shows how to construct a coarse moduli space by restricting to the subset of stable or semistable bundles.

Definition. A holomorphic bundle $E \to X$ is *stable* (resp. *semistable*) if and only if for every proper non-zero holomorphic sub-bundle $F \subset E$ we have

$$\frac{\deg(F)}{\operatorname{rank}(F)} < \frac{\deg(E)}{\operatorname{rank}(E)} \quad \left(\text{resp. } \frac{\deg(F)}{\operatorname{rank}(F)} \leq \frac{\deg(E)}{\operatorname{rank}(E)}\right)$$

A bundle is *polystable* iff it is the direct sum of stable bundles of the same slope (degree-rank ratio).

Now we can define the *moduli space of stable/semistable bundles*

$$\mathcal{M}_{st}(E) := \mathcal{A}^{0,1}_{st} / \mathcal{G}^{\mathbb{C}}, \quad \mathcal{M}_{ss}(E) := \mathcal{A}^{0,1}_{ss} // \mathcal{G}^{\mathbb{C}}$$

where the double quotient $//$ indicates that we identify orbits whose closures intersect.

Hitchin-Kobayashi correspondence

The Hitchin-Kobayashi correspondence is a general principle which asserts that the moduli space should have an interpretation as a symplectic quotient.

Given a fixed Hermitian metric, the minimum of the Yang-Mills functional is achieved by connections D with curvature $F_D = c \cdot \mathrm{id}$, where c is a topological invariant of E.

Theorem. [Narasimhan-Seshadri, Donaldson, Uhlenbeck-Yau]

$$\mathcal{M}_{ss}(E) := \mathcal{A}^{0,1}_{polyst}/\mathcal{G}^{\mathbb{C}} \cong \{F_D = c \cdot \mathrm{id}\}/\mathcal{G}$$

where $\mathcal{G}$ is the group of gauge transformations that preserve the Hermitian metric on E.

Equivalently, a bundle is polystable iff there exists a unique Hermitian metric such that the bundle minimises the Yang-Mills functional.

Higgs bundles

Higgs bundles are a generalisation of holomorphic bundles ("holomorphic bundles with extra structure").

The moduli space has many interesting features as a consequence of this extra structure.
(Nonabelian Hodge theory, Mirror symmetry, Langlands program)

Definition. [Hitchin (1987), Simpson (1988)] A *Higgs bundle* is a pair $(\bar{\partial}_E, \phi)$ consisting of a holomorphic bundle $\bar{\partial}_E$ and a section $\phi \in \Omega^{1,0}(\mathrm{End}(E))$ such that $\bar{\partial}_E \phi = 0$ and $\phi \wedge \phi = 0$.

(Note that $\phi \wedge \phi = 0$ is automatic when $\dim_{\mathbb{C}} X = 1$.)

Let $\mathcal{B}$ denote the space of all Higgs bundles with underlying smooth bundle $E \to X$.

Unlike the space of holomorphic bundles (which is affine), the space of Higgs bundles is singular (more later).

Constructing the moduli space of Higgs bundles

Definition. A Higgs bundle $(\bar{\partial}_E, \phi)$ is *stable* (resp. *semistable*) if and only if for every proper non-zero Higgs sub-bundle $F \subset E$ we have

$$\frac{\deg(F)}{\operatorname{rank}(F)} < \frac{\deg(E)}{\operatorname{rank}(E)} \quad \left(\text{resp. } \frac{\deg(F)}{\operatorname{rank}(F)} \leq \frac{\deg(E)}{\operatorname{rank}(E)} \right)$$

A Higgs bundle is *polystable* iff it is the direct sum of stable Higgs bundles with the same slope.

Now we can define

$$\mathcal{M}_{ss}^{Higgs}(E) := \mathcal{B}_{ss} // \mathcal{G}^{\mathbb{C}} = \mathcal{B}_{polyst} / \mathcal{G}^{\mathbb{C}}, \quad \mathcal{M}_{st}^{Higgs}(E) := \mathcal{B}_{st} / \mathcal{G}^{\mathbb{C}}$$

where the double quotient // indicates that we identify orbits whose closures intersect.

Hitchin-Kobayashi correspondence

Given a fixed Hermitian metric on E, the *Yang-Mills-Higgs functional* is

$$\mathrm{YMH}(\bar{\partial}_E, \phi) := \| F(\bar{\partial}_E) + [\phi, \phi^*] \|^2$$

Let YMH_{min} denote the space of Higgs pairs minimising YMH.

Hitchin (1987) ($\dim_{\mathbb{C}} X = 1$) and Simpson (1988) ($\dim_{\mathbb{C}} X \geq 1$) proved a Hitchin-Kobayashi correspondence for Higgs pairs.

Theorem. [Hitchin, Simpson]
A Higgs bundle $(\bar{\partial}_E, \phi)$ is polystable iff there exists a unique Hermitian metric such that $(\bar{\partial}_E, \phi) \in \mathrm{YMH}_{min}$.
Equivalently

$$\mathcal{M}_{ss}^{Higgs}(E) := \mathcal{B}_{polyst} / \mathcal{G}^{\mathbb{C}} \cong \mathrm{YMH}_{min} / \mathcal{G}$$

What does the space of Higgs bundles look like?

For a complex vector bundle E over a compact Riemann surface X, the space of all holomorphic structures is an affine space locally modelled on $\Omega^{0,1}(\mathrm{End}(E))$.

The situation is more complicated for Higgs bundles, since the condition $\bar{\partial}_E \phi = 0$ is not of constant rank.

There is a map from the space of Higgs bundles to the affine space of holomorphic bundles

$$\begin{aligned} \mathcal{B} &\to \mathcal{A}^{0,1} \\ (\bar{\partial}_E, \phi) &\mapsto \bar{\partial}_E \end{aligned}$$

For each holomorphic structure $\bar{\partial}_E$, the fibre of this map is $H^{1,0}(\mathrm{End}(E)) = \{\phi \mid \bar{\partial}_E \phi = 0\} \cong H^0(\mathrm{End}(E) \otimes K)$.

The dimension of the fibre depends on $\bar{\partial}_E$. In particular, we see that $\mathcal{B}$ has singularities for $\mathrm{rank}(E) \geq 2$.

Structure of the moduli space of Higgs bundles

When X is a compact Riemann surface, the moduli space of *stable* Higgs bundles is a smooth manifold.
This is not necessarily true for semistable or polystable Higgs bundles, since singularities may occur at $E = F_1 \oplus F_2$ (bundles such as this have larger isotropy group).

When the degree and rank of E are coprime then $(\bar{\partial}_E, \phi)$ is stable iff it is semistable (and hence $\mathcal{M}_{ss}^{Higgs}$ is smooth in this case).

Theorem. [Hitchin ($\dim_{\mathbb{C}} X = 1$), Fujiki ($\dim_{\mathbb{C}} X \geq 1$)]
$\mathcal{M}_{ss}^{Higgs}$ is a hyperkähler quotient.

There is an action of $\mathbb{C}^*$ on $\mathcal{M}_{ss}^{Higgs}$

$$t \cdot [(\bar{\partial}_E, \phi)] = [(\bar{\partial}_E, t\phi)]$$

The fixed points of this action are called *variations of Hodge structure* (due to Simpson). In low rank these can be classified via the theory of holomorphic chains.

Structure of the moduli space of Higgs bundles (cont.)

The action of $\mathrm{U}(1) \subset \mathbb{C}^*$ is Hamiltonian. The associated moment map is $\mu[(\bar{\partial}_E, \phi)] = \sqrt{-1}\|\phi\|_{L^2}^2$.

Theorem. [Hitchin, 1987]
When the degree and rank are coprime then the function $f = |\mu|$ is proper and Morse-Bott.

Hitchin (1987, rank 2) and Gothen (1995, rank 3), (2002, $G = \mathrm{SU}(2,1)$ or $\mathrm{U}(2,1)$) classified the critical sets and computed the Morse index and the cohomology of each critical set.

As a consequence, when the degree and rank are coprime, they can compute the Betti numbers of the moduli space via Morse-Bott theory.

Hitchin's calculation for $\operatorname{rank}(E) = 2$, $\det E = L$

At the non-minimal critical sets of $f = \|\phi\|^2_{L^2}$ the bundle splits $E = L_1 \oplus L_2$ and the Higgs field simplifies to $\phi \in H^0(L_1^* L_2 \otimes K)$. The result is a special case of a *holomorphic chain*

$$L_1 \xrightarrow{\phi} L_2$$

where $\deg L_2 < \deg L_1 \leq \deg L_2 + 2g - 2$.

(Note that $L_1 \oplus L_2$ is unstable as a *holomorphic bundle*, but the pair $(L_1 \oplus L_2, \phi)$ is stable as a *Higgs bundle*.)

These critical sets are classified by $\ell = \deg L_1$. If we fix the determinant of $E = L_1 \oplus L_2$, $d = \deg(E)$ then the critical set is isomorphic to $S^{d-2\ell+2g-2}X$, the symmetric product of X.

Hitchin's calculation for rank$(E) = 2$, det $E = L$ (cont.)

The minimal critical set for $f = \|\phi\|^2$ is at $\phi = 0$, which gives us $\mathcal{M}_{ss}(E)$, the moduli space of stable *holomorphic* bundles.

Hitchin proved that when rank$(E) = 2$, deg$(E) = 1$ then

$$P_t(\mathcal{M}_{ss}^{Higgs}(E)) = P_t(\mathcal{M}_{ss}(E)) + \sum_{\ell=1}^{g-1} t^{\lambda_\ell} P_t(S^{1-2\ell+2g-2}X)$$

Gothen obtained similar formulas for rank$(E) = 3$, deg$(E) = 1$ and also when the structure group is U$(2,1)$ or SU$(2,1)$.

The idea of the calculation is the same, but classifying the critical sets and computing the Morse index is more complicated.

Recently, Garcia-Prada, Heinloth and Schmitt computed the Poincaré polynomial for rank$(E) = 4$ and coprime degree.

Counting rational points (Hausel)

Another approach to computing cohomology of a projective variety is to count the rational points on a variety defined over a finite field $\mathbb{F}_q$.

Harder and Narasimhan (1975) did this for moduli spaces of stable bundles with coprime rank and degree.

One can then compute the cohomology using the Weil conjectures (proved by Deligne).

For (quasiprojective) moduli spaces of Higgs bundles, this program was carried out by Hausel (2006). He developed a new theory "arithmetic harmonic analysis" to compute the number of rational points on a holomorphic symplectic quotient.

This leads to conjectures on the Betti numbers and, more generally, on the mixed Hodge polynomials for moduli spaces of Higgs bundles. These conjectures have been verified in low rank (Hausel-Villegas-Rodriguez, 2008).

Back to holomorphic bundles (Atiyah and Bott)

A famous result of Atiyah and Bott shows that we can inductively compute the (equivariant) cohomology of the moduli space of semistable holomorphic bundles using the Morse theory of the Yang-Mills functional.

Idea. Fix a Hermitian metric on E. The Yang-Mills functional $f(\bar{\partial}_E) = \mathrm{YM}(\bar{\partial}_E, H)$ is an energy function on the space of holomorphic bundles.

The Hitchin-Kobayashi correspondence shows that (modulo gauge) the minimum can be identified with the moduli space of semistable bundles.

Atiyah and Bott (1983) classified the non-minimal critical sets and showed that there is an algebraic stratification by Harder-Narasimhan type with "good" properties.

Daskalopoulos (1992) and Rade (1992) later showed that this stratification coincides with the Morse stratification by the gradient flow of the Yang-Mills functional.

Atiyah and Bott's calculation

Basic idea. (All calculations in $\mathcal{G}$-equivariant cohomology)

$$P_t^{\mathcal{G}}(\mathcal{A}_{ss}^{0,1}) = \text{cohomology of total space}$$
$$- \text{contributions from critical sets}$$

More precisely.

$$P_t^{\mathcal{G}}(\mathcal{A}_{ss}^{0,1}(E)) = P_t(B\mathcal{G}) - \sum_{\text{crit. sets } C_k} t^{\lambda_k} P_t^{\mathcal{G}}(C_k)$$

The same methods also show that the inclusion $A_{ss}^{0,1}(E) \hookrightarrow \mathcal{A}^{0,1}(E)$ induces a surjection in equivariant cohomology

$$H_{\mathcal{G}}^*(A^{0,1}(E)) \twoheadrightarrow H_{\mathcal{G}}^*(A_{ss}^{0,1}(E))$$

This last fact is known as *Kirwan surjectivity* and gives a description of the generators of $H_{\mathcal{G}}^*(A_{ss}^{0,1}(E))$.

Later work by Jeffrey-Kirwan (1998) and Earl-Kirwan (2004) gives a complete description of the intersection pairings and the generators/relations for the cohomology ring.

Atiyah and Bott for Higgs bundles

Can we carry out this program for Higgs bundles?

Atiyah and Bott's calculation does not require that the moduli space is smooth, since the Morse theory only sees the moduli space as the lowest critical set.

Therefore, a calculation for Higgs bundles in the spirit of Atiyah and Bott would give us

- new results on the Betti numbers when the degree and rank are not coprime
- insight into the question of Kirwan surjectivity for Higgs bundles.

Problem. As we saw before, the space of Higgs bundles is singular. How to do Morse theory on this space?

We are trading the difficulties caused by the singularities in the moduli space for a new set of difficulties caused by the singularities in the space of Higgs bundles.

Stratification of the space of Higgs bundles

Question. Fix a Hermitian metric on E. Is there a Morse stratification of the space of Higgs bundles given by the Yang-Mills-Higgs functional?

$$\mathrm{YMH}(\bar{\partial}_E, \phi) = \| F(\bar{\partial}_E) + [\phi, \phi^*] \|^2$$

Theorem.[W. (2008)]
For any initial condition $(\bar{\partial}_0, \phi_0) \in \mathcal{B}$, the gradient flow of the Yang-Mills-Higgs functional converges to a critical point of YMH and therefore there is a Morse stratification of $\mathcal{B}$.

Moreover, the Morse stratification coincides with the Harder-Narasimhan stratification from algebraic geometry.

The limit of the flow is isomorphic to the graded object of the Harder-Narasimhan-Seshadri double filtration of $(\bar{\partial}_0, \phi_0)$.

Jiayu Li and Xi Zhang have recently generalised this result to compact Kähler manifolds of any dimension.

Results on the topology of the moduli space

We choose an ordering on the strata and denote the Morse stratification by

$$\mathcal{B} = \bigcup_{k=0}^{\infty} \mathcal{B}_k$$

Theorem. [Daskalopoulos, Weitsman, Wentworth, W. (2011)] Let $E \to X$ be a complex vector bundle of rank 2 and degree either even or odd. Then

$$P_t^{\mathcal{G}}(\mathcal{B}_{ss}) = P_t^{\mathcal{G}}(\mathcal{B}) - \sum_{k=0}^{\infty} t^{\lambda_k} P_t^{\mathcal{G}}(\mathcal{B}_k) + \sum_{\ell=1}^{g-1} t^{\lambda_\ell} P_t(S^{\deg(E)-2\ell+2g-2}X)$$

Moreover, the Kirwan map $H_{\mathcal{G}}^*(\mathcal{B}) \to H_{\mathcal{G}}^*(\mathcal{B}_{ss})$ is surjective for structure group $G = \mathrm{GL}(2, \mathbb{C})$.

The Kirwan map is *not* surjective for structure group $G = \mathrm{SL}(2, \mathbb{C})$.

Results on the topology of the moduli space (cont.)

The idea of the proof is to first use the gradient flow to reduce the calculation to the critical sets.

We then interpret the singularities at each critical set via the theory of holomorphic chains.

When $\mathrm{rank}(E) = 2$ then the holomorphic chain is just a symmetric product $S^{d_k}X$ (fixed determinant) or $S^{d_k}X \times \mathrm{Jac}(X)$ (non-fixed determinant).

Basic idea.

$$\begin{aligned} P_t^{\mathcal{G}}(\mathcal{B}_{ss}) = \text{cohomology of total space} \\ - \text{contributions from critical sets} \\ + \text{correction terms from singularities} \end{aligned}$$

The correction terms arise due to a failure of YMH to be equivariantly perfect on the singular space $\mathcal{B}$.

Results on the topology of the moduli space (cont.)

The next theorem interprets the failure of Kirwan surjectivity in terms of the action of the mapping class group on the moduli space.

Theorem. [Hitchin, Donaldson (1987)] Let $E \to X$ have rank 2, degree zero and fixed determinant. Then

$$\mathcal{M}_{ss}^{Higgs}(E) \cong \mathrm{Hom}(\pi_1(X), \mathrm{SL}(2, \mathbb{C}))/\!/ \mathrm{SL}(2, \mathbb{C})$$

Therefore there is an induced action of the mapping class group $\mathrm{Mod}(X)$ on $\mathcal{M}_{ss}^{Higgs}(E)$.

Theorem. [Daskalopoulos, Wentworth, W. (2010)] When E has rank 2, degree zero and fixed determinant then the Torelli group acts non-trivially on the cohomology of $\mathcal{M}_{ss}^{Higgs}(E)$ and hence the same is true for $\mathrm{Hom}(\pi_1(X), \mathrm{SL}(2, \mathbb{C}))/\!/ \mathrm{SL}(2, \mathbb{C})$.

The proof involves showing that (a) the Torelli group acts trivially on the image of the Kirwan map, and (b) the Torelli group acts non-trivially on each correction term in the Morse theory

Results on the topology of the moduli space (cont.)

More recently, we have extended these results to U$(2,1)$ and SU$(2,1)$ bundles, generalising Gothen's results. The singularities are more complicated and we require some new ideas to overcome this, but the basic idea of computing correction terms still holds.

Theorem. [Wentworth, W. (2013)]
Let $\mathcal{B}$ denote the space of U$(2,1)$, PU$(2,1)$ or SU$(2,1)$ Higgs bundles of any degree. Then

- we can compute $P_t^{\mathcal{G}}(\mathcal{B}_{ss})$ via the Morse theory of the Yang-Mills-Higgs functional,
- the Kirwan map is surjective for U$(2,1)$ and PU$(2,1)$ Higgs bundles, and
- the Kirwan map is not surjective for SU$(2,1)$ Higgs bundles with Toledo invariant $\tau \leq \frac{4}{3}(g-1)$. Moreover, the Torelli group acts non-trivially on the equivariant cohomology of these spaces iff $\tau < \frac{4}{3}(g-1)$.

Interpretation via Morse complex

Morse-Smale-Witten theory obtains more information about the cohomology ring structure using gradient flow lines for the Morse function.

The *Morse complex* consists of the cohomology of the critical sets together with differentials determined by the gradient flow lines. The Morse-Smale-Witten theory shows that the cohomology of the complex $H^\ell = \ker d_\ell / \operatorname{im} d_{\ell-1}$ is the cohomology of the manifold.

For Higgs bundles we can write down a conjectural Morse complex with extra terms corresponding to the correction terms from our previous calculation.

The differentials from (approximate) flow lines then correspond to cancellation of the correction terms. The Morse complex then computes the cohomology of the total space $\mathcal{B}$.

Relationship to Nakajima quiver varieties

There are many analogies between Higgs bundles and Nakajima quiver varieties. These were originally developed to construct moduli of instantons on ALE hyerkähler 4-manifolds (Kronheimer-Nakajima (1990)).

Later, Nakajima (2001) constructs representations of quantum affine algebras on spaces of topological invariants of quiver varieties.

For certain special cases of low rank, the same idea of correction terms gives the correct Betti numbers for the moduli space (previously computed by Nakajima (2004) and Hausel (2006)).

Making these calculations rigorous would give a proof of Kirwan surjectivity for these moduli spaces.

A more recent result (W. 2013) shows that for quivers there is a relationship between approximate flow lines and Nakajima's Hecke correspondence.

Relationship to Nakajima quiver varieties (cont.)

Questions.

- Is it possible to carry out the Morse theory for Nakajima quiver varieties? Other quivers with relations?
- Can we construct representations of quantum algebras on a Morse complex? What about other algebras?
- What about Higgs bundles of higher rank?