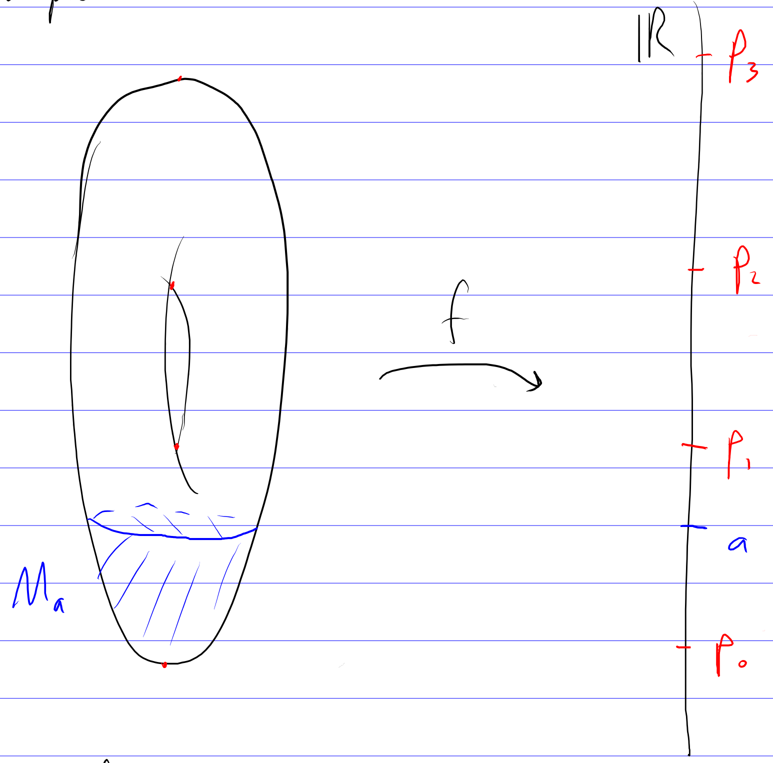
Algebraic and Geometric classification of Yang-Mills-Higgs flow lines

1. Motivation (Morse Theory)

Morse theory aims to relate the topology of a manifold M to analytic properties of $\underbrace{\text{"well-behaved"}}_{\text{"Morse", "Morse-Bott"...}}$ smooth functions $f: M \to \mathbb{R}$.

Basic Example:



This is enough information to determine Morse inequalities

$$b_k = \dim_{\mathbb{C}} H^k(M, \mathbb{C}), \quad c_k = \# \text{ crit. points of index } k$$

$$P_t(M) = \sum_{k=0}^{n} b_k t^k \qquad M_t(f) = \sum_{k=0}^{n} c_k t^k$$

(Poincaré polynomial) (Morse polynomial)

Then

$$M_t(f) - P_t(M) = (1+t) R(t)$$

polynomial with non-negative coefficients

In particular $c_k \geq b_k$ for all k (Weak Morse inequalities)

We would like to use analysis to determine more information about the topology of M, in particular

(a) $P_t(M)$
(b) cup product structure on $H^*(M)$.

We can do this if we know more information about spaces of gradient flow lines connecting critical points

requires a Riemannian metric on M.

For the Yang-Mills flow and Yang-Mills-Higgs flow, these spaces have a very interesting algebraic and geometric structure.

2. Holomorphic and Higgs bundles over Riemann surfaces.

Let X be a compact Riemann surface.
$E \to X$ a smooth $\mathbb{C}$-vector bundle (rank r, degree d)

Holomorphic structure on E $\Leftrightarrow$ Holomorphic transition functions

$\Leftrightarrow$ Well-defined operator $\bar\partial_A : \Omega^0(E) \to \Omega^{0,1}(E)$ such that
(i) $\bar\partial_A^{\,2} = 0$
(ii) $\bar\partial_A(fs) = f(\bar\partial_A s) + (\bar\partial f)s$
$f \in \Omega^0(X)$, $s \in \Omega^0(E)$

The $\Leftarrow$ implication is a non-trivial theorem (Newlander–Nirenberg)

$H^0(E) = \ker(\bar\partial_A) \subset \Omega^0(E)$ = space of holomorphic sections

For any bundle associated to E (e.g. $\mathrm{End}\,E$ = bundle of endomorphisms of E $= E \otimes E^*$) there is an associated $\bar\partial$-operator.

$$\bar\partial_{E\otimes E^*} = \bar\partial_E \otimes \mathrm{id} + \mathrm{id} \otimes \bar\partial_{E^*}$$

Gauge group: $\mathcal{G}^{\mathbb{C}} = \mathrm{Aut}(E)$ = smooth sections of the frame bundle (locally $GL(r,\mathbb{C})$-valued)

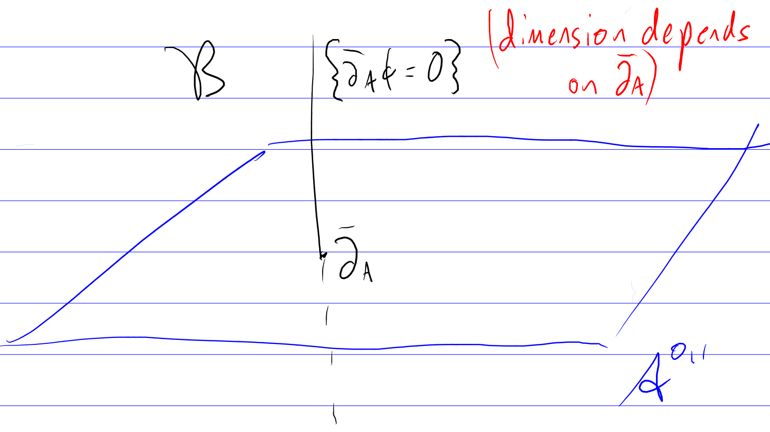
Gauge action $g \cdot (\bar\partial_A s) = (g \bar\partial_A g^{-1})(gs)$

The space of all such holomorphic structures is an affine space
$$\boxed{\mathcal{A}^{0,1} = \bar\partial_{A_0} + \Omega^{0,1}(\mathrm{End}\,E)}$$

A Higgs bundle (or Higgs structure on E) is a pair $(\bar\partial_A, \phi)$ where $\phi \in H^0(\mathrm{End}\,E \otimes K)$ (K = canonical bundle of X)

The space of all Higgs bundles is
$$\mathcal{B} = \{(\bar\partial_A, \phi) \mid \bar\partial_A \phi = 0\} \subset \mathcal{A}^{0,1} \times \Omega^0(\mathrm{End}\,E \otimes K)$$

Forgetful map $\mathcal{B} \to \mathcal{A}^{0,1}$
$(\bar\partial_A, \phi) \mapsto \bar\partial_A$



Fibre over $\bar\partial_A$ is $H^0(\mathrm{End}\,E \otimes K)$

dimension depends on $\bar\partial_E$

We would like to construct a moduli space of holomorphic/Higgs bundles up to gauge equivalence.

$\mathcal{B}/\mathcal{G}^{\mathbb{C}}$ does not have good properties (e.g. it is non-Hausdorff)

Instead we restrict to an open subset of stable/semistable bundles.

$(\bar\partial_A, \phi)$ is

stable iff $\boxed{\dfrac{\deg F}{\operatorname{rank} F} < \dfrac{\deg E}{\operatorname{rank} E}}$

semistable iff $\boxed{\dfrac{\deg F}{\operatorname{rank} F} \leq \dfrac{\deg E}{\operatorname{rank} E}}$

for every proper non-zero ϕ-invariant holomorphic subbundle $F \subset E$.

Then define the moduli space of stable (resp. semistable) Higgs bundles by
$$\mathcal{M}^{st}(r,d) = \mathcal{B}^{st}/\mathcal{G}^{\mathbb{C}}, \quad \mathcal{M}^{ss}(r,d) = \mathcal{B}^{ss}/\!/\mathcal{G}^{\mathbb{C}}$$

All the objects defined above are holomorphic.

In order to define curvature, Yang-Mills-Higgs energy, etc. we need to fix a Hermitian metric on E.

3. Yang-Mills / Yang-Mills-Higgs energy

Fix a Hermitian metric h on E

Then each holomorphic structure $\bar\partial$ has an associated Chern connection $d_{A,h}$ compatible with the metric, such that the following diagram commutes.

$$\begin{tikzcd} \Omega^0(E) \arrow[r, "d_{A,h}"] \arrow[dr, "\bar\partial_A"'] & \Omega^1(E) \arrow[d, "\text{Project onto } (0,1) \text{ part}"] \\ & \Omega^{0,1}(E) \end{tikzcd}$$

Let $F_{A,h}$ be the curvature of $d_{A,h}$

$\mathcal{G} = \{ g \in \mathcal{G}^{\mathbb{C}} \mid g^* g = id \}$ Unitary gauge group

Now define $YM_h : \mathcal{A}^{0,1} \to \mathbb{R}$ and $YMH_h : \mathcal{B} \to \mathbb{R}$ by

$$YM_h(\bar\partial_A) = \| F_{A,h} \|^2_{L^2} = \int_X |F_{A,h}|^2 \, dvol_X \qquad \text{Yang-Mills energy}$$

$$\text{and} \quad YMH_h(\bar\partial_A, \phi) = \| F_{A,h} + [\phi, \phi^*] \|^2_{L^2} \qquad \text{Yang-Mills-Higgs energy}$$

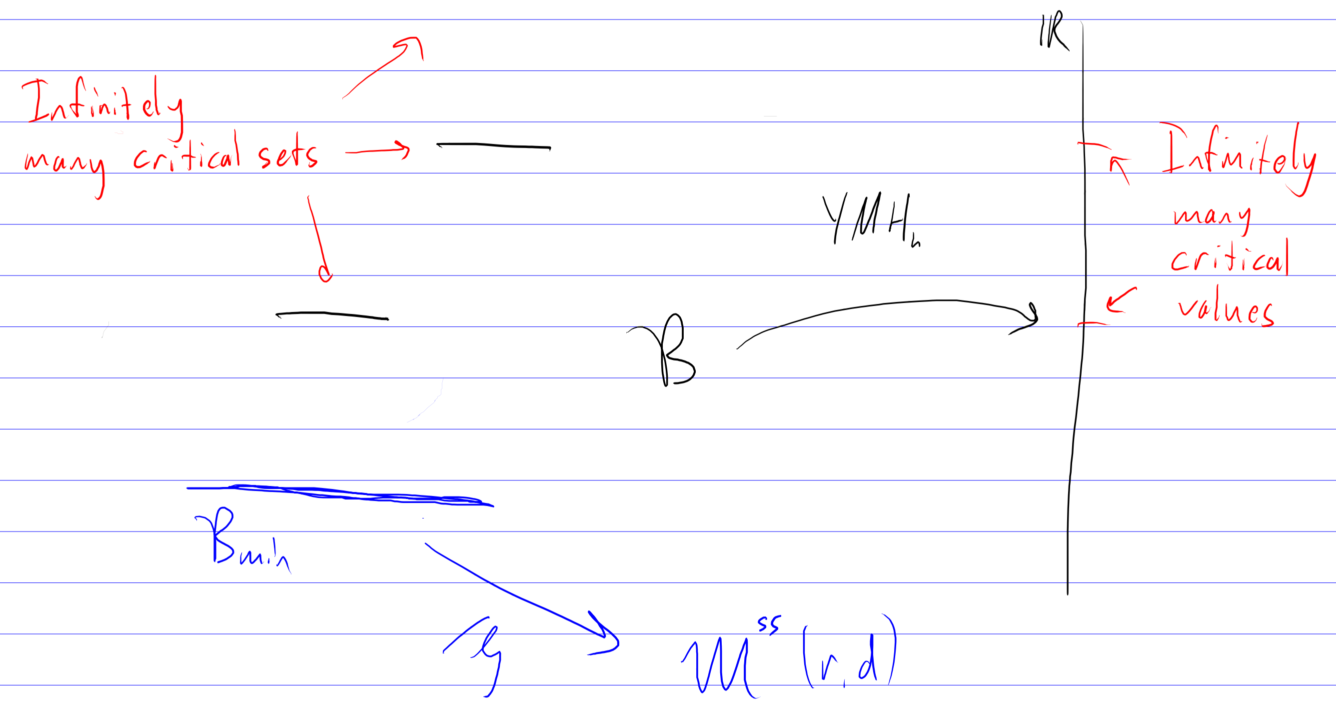
The energy minimising connections satisfy $*(F_{A,h} + [\phi, \phi^*]) = \lambda \cdot id$ λ is a constant depending on the degree and rank of E

(Hermitian-Einstein connections)

The critical points are direct sums of Hermitian-Einstein connections.

$$*(F_{A,h} + [\phi, \phi^*]) = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

We can think of these energy functions as height functions on the spaces $\mathcal{A}^{0,1}$ and $\mathcal{B}$ (analogous to the picture of the torus from above)



Thm (Narasimhan & Seshadri '65, Donaldson '83, Hitchin '87, Simpson '88)

$$\mathcal{B}_{min} / \mathcal{G} \cong \mathcal{M}^{ss}(r,d)$$

It is natural to study the downwards gradient flow of YMH_h on $\mathcal{B}$.

$$\boxed{\begin{aligned} \frac{\partial A}{\partial t} &= i \bar\partial_{A_t} * (F_{A_t} + [\phi_t, \phi_t^*]) \\ \frac{\partial \phi}{\partial t} &= i [\phi_t, *(F_{A_t} + [\phi_t, \phi_t^*])] \end{aligned}}$$

If X is Kähler, then a solution (for $t \geq 0$) is generated by $\mathcal{G}^{\mathbb{C}}$.

$$(\bar\partial_{A_t}, \phi_t) = g_t \cdot (\bar\partial_{A_0}, \phi_0)$$

$$\text{where} \quad \frac{\partial g_t}{\partial t} g_t^{-1} = -i * (F_{A_t} + [\phi_t, \phi_t^*]), \quad g_0 = id$$

We can also think of the flow in terms of changing the metric.

Define $h_t = g_t^* g_t$. Then

$$\frac{\partial h_t}{\partial t} + \Delta_{A_0} h_t = (\text{linear in } h_t) + (\text{nonlinear in } h_t)$$

So the downwards flow resembles a nonlinear heat equation.

This is great when $t \geq 0$!

The flow exists for all $t \geq 0$
It converges to a critical point as $t \to \infty$

This is a huge problem when $t < 0$.

For some initial conditions, the flow may not exist for any $t < 0$.

We need to work around this if we are to classify flow lines between critical sets.

4. Results about the downwards flow.

The Yang-Mills / Yang-Mills-Higgs heat flow is well-studied for $t \geq 0$.

(i) Long-time existence (Donaldson '85, Simpson '88)

Key idea: Distance-decreasing property.

Two initial conditions $(\bar\partial_{A_0}, \phi_0)$ and $g_0 \cdot (\bar\partial_{A_0}, \phi_0)$ $\quad g_0 \in \mathcal{G}^{\mathbb{C}}$

The two solutions of the flow are $(\bar\partial_{A_t}, \phi_t)$ and $g_t \cdot (\bar\partial_{A_t}, \phi_t)$

Let $h_t = g_t^* g_t$ and define $\underbrace{\sigma(h_t) = tr(h_t) + tr(h_t^{-1}) - 2\,\mathrm{rank}\,E}_{\text{"distance" in space of metrics}}$

Then $\left(\frac{\partial}{\partial t} + \Delta\right) \sigma(h_t) \leq 0$

(ii) Convergence (Råde '92, W. '08)

$\lim_{t \to \infty} (\bar\partial_{A_t}, \phi_t)$ exists and is a critical point.

(iii) Identification of the limit (Daskalopoulos '92, W. '08)

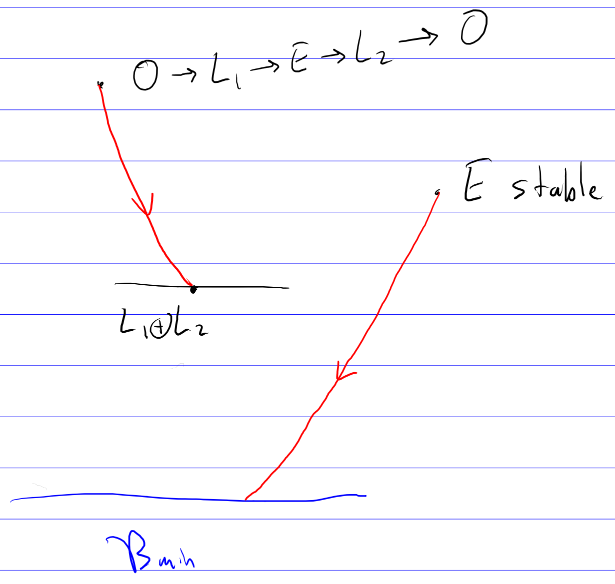
Every holomorphic / Higgs bundle admits a "Harder-Narasimhan-Seshadri" double filtration which measures the failure of the bundle to be stable.

Theorem (Daskalopoulos '92, W. '08)

The limit is isomorphic to the graded object of the Harder-Narasimhan-Seshadri filtration of the initial condition.

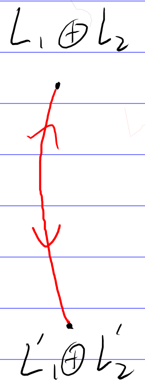
Example: $\mathrm{rank}\, E = 2$. $L_1 \subset E$ is the line subbundle of maximal degree.

$\Rightarrow$ H-N-S filtration is $0 \subset L_1 \subset E$



What about flow lines?

Example:



(i) What are the conditions on L_1, L_2, L_1', L_2' for a flow line to connect the critical points $L_1 \oplus L_2$ and $L_1' \oplus L_2'$?

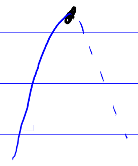
(ii) What does the space of flow lines look like?

(iii) What about broken/unbroken flow lines?

5. The reverse flow near a critical point.

To simplify the exposition I will focus on the case of rank 2 Yang-Mills flow

$L_1 \oplus L_2 \qquad \deg L_1 > \deg L_2$



$$\bar\partial_A = \begin{pmatrix} \bar\partial_{A_1} & 0 \\ 0 & \bar\partial_{A_2} \end{pmatrix}$$

Extensions

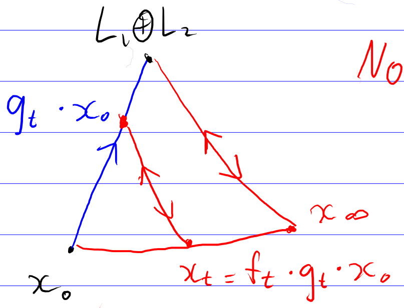
$0 \to L_2 \to E \to L_1 \to 0$ "Linearised unstable manifold"

Holomorphic structure $\begin{pmatrix} \bar\partial_{A_1} & 0 \\ a & \bar\partial_{A_2} \end{pmatrix}$, $a \in H^{0,1}(L_1^* L_2) \cong \underbrace{H^0(L_2^* L_1 \otimes K)}_{\text{dimension computable by Riemann-Roch.}}$

Act by the gauge transformation

$$g_t = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \qquad \lambda_2 > \lambda_1$$

$$g_t \begin{pmatrix} \bar\partial_{A_1} & 0 \\ 0 & \bar\partial_{A_2} \end{pmatrix} g_t^{-1} = \begin{pmatrix} \bar\partial_{A_1} & 0 \\ e^{(\lambda_2 - \lambda_1)t} a & \bar\partial_{A_2} \end{pmatrix} \longrightarrow \begin{pmatrix} \bar\partial_{A_1} & 0 \\ 0 & \bar\partial_{A_2} \end{pmatrix} \quad \text{as } t \to -\infty$$



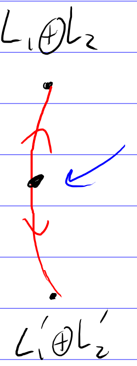
Now flow down using the nonlinear heat flow

Theorem (W.) (i) This process converges as $t \to \infty$.

(ii) $x_\infty := \lim_{t \to \infty} f_t \cdot g_t \cdot x \in \mathcal{G}^{\mathbb{C}} \cdot x$

(iii) One can construct a solution to the reverse heat flow from x_∞ up to the critical point $L_1 \oplus L_2$.

(iv) Conversely, every solution to the reverse heat flow that converges to $L_1 \oplus L_2$ can be constructed in this way.

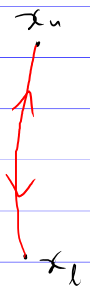


admits a double filtration

$\deg L_1 > \deg L_1' > \deg L_2' > \deg L_2$

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & L_2 \arrow[d] & & \\ 0 \arrow[r] & L_1' \arrow[r] & E \arrow[r] \arrow[d] & L_2' \arrow[r] & 0 \\ & & L_1 \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

More generally (arbitrary rank, also works for Higgs bundles) a point on a flow line between two critical points x_ℓ and x_u must admit two filtrations

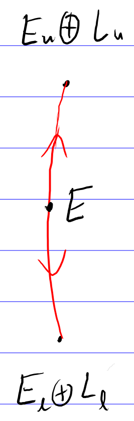


(i) H-N-S filtration whose graded object is isomorphic to x_ℓ

(ii) A filtration whose graded object is a direct sum of stable bundles with increasing slope $= \dfrac{\deg}{\text{rank}}$

6. Consequences of the main analytic theorem

(a) Hecke correspondences via spaces of flow lines



L_u, L_ℓ line bundles

E_ℓ, E_u arbitrary rank, E_ℓ stable, E_u polystable

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & L_u & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & E_\ell & \to & E & \to & L_\ell & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & E_u & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

Then E_ℓ is a subsheaf of E_u

$\Rightarrow$ they are related by a Hecke modification

$$0 \to \mathcal{O}_{E_\ell} \to \mathcal{O}_{E_u} \to \bigoplus_{i=1}^{n} \mathbb{C}_{p_i} \to 0$$

$\mathcal{F}_\ell^u$ = points that flow up to C_u and down to C_ℓ

$$\begin{array}{ccccc} & & \mathcal{F}_\ell^u & & \\ & \swarrow & \downarrow & \searrow & \\ \text{lower crit. set} = C_\ell & & \mathcal{P}_\ell^u & & C_u = \text{upper crit. set} \\ \downarrow & \swarrow & & \searrow & \downarrow \\ \mathcal{M}^{ss}(E_\ell) & & & & \mathcal{M}^{ss}(E_u) \end{array}$$

pairs of critical points connected by a flow line modulo gauge

Theorem (W.) $\mathcal{P}_\ell^u$ is the Hecke correspondence.

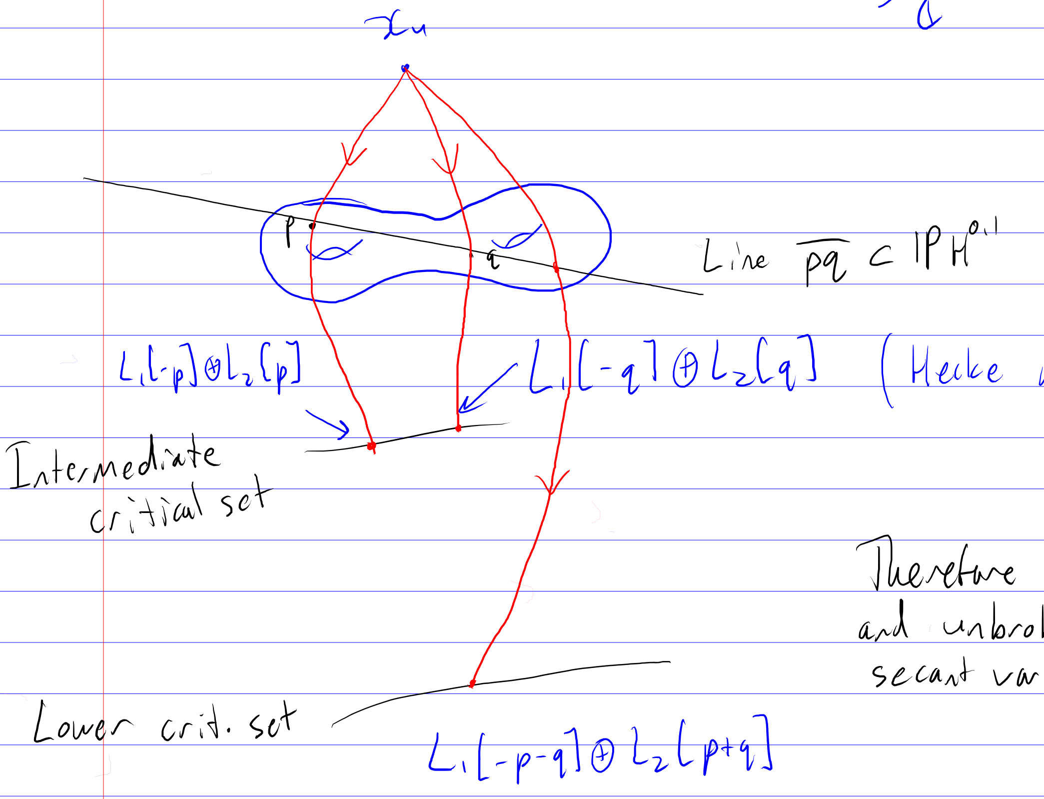
(b) Geometric classification of broken/unbroken flow lines

Fix an upper critical point $x_u = L_1 \oplus L_2$.

Projective embedding $X \hookrightarrow \mathbb{P}H^{0,1}(L_1^* L_2)$ ← space of extensions from before

$$\downarrow$$

$W_{x_u} / \mathbb{C}^*$ ← "unstable manifold of x_u"



Therefore we can distinguish broken and unbroken flow lines using secant varieties of $X \subset \mathbb{P}H^{0,1}(L_1^* L_2)$

Higher rank Yang-Mills: Use secant varieties of $\mathbb{P}E \subset \mathbb{P}H^{0,1}$

Higgs bundles: Need to take the Higgs field into account.