# Lecture 9: Collinearity

11 February, 2018

# Overview

**Last time.**

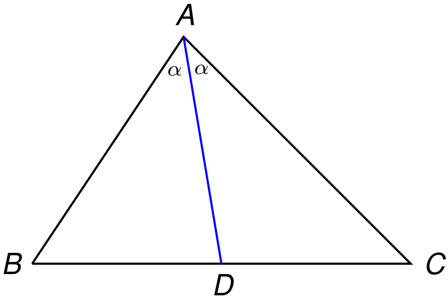
- Brahmagupta's formula for the area of a cyclic quadrilateral.
- Properties of non-cyclic quadrilaterals
- More constructions in Euclidean geometry

**Today.**

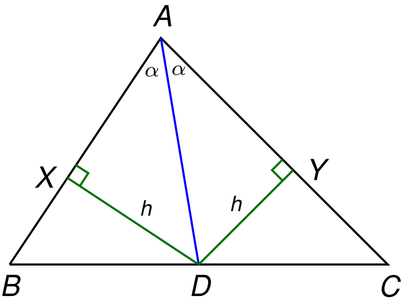
- Menelaus' theorem on collinearity
- Desargues' theorem
- Pappus' theorem
- Pascal's hexagon theorem

# Exercise

**Exercise.** Let $\Delta ABC$ be a triangle and let $D$ be the intersection of the angle bisector of $\angle BAC$ with $BC$. Show that $\frac{|BD|}{|DC|} = \frac{|AB|}{|AC|}$.



# Solution (hint)




$$\text{Area}(\Delta ABD) = \frac{1}{2}h|AB|, \quad \text{Area}(\Delta ACD) = \frac{1}{2}h|AC|$$

**Solution.**

**Solution.** First note that the triangles $\Delta ABD$ and $\Delta ADC$ are between the same parallels, and so Euclid Prop. VI.1 shows that the ratio of the areas is equal to the ratio of the bases

$$\frac{\text{Area}(\Delta ABD)}{\text{Area}(\Delta ADC)} = \frac{|BD|}{|DC|}$$

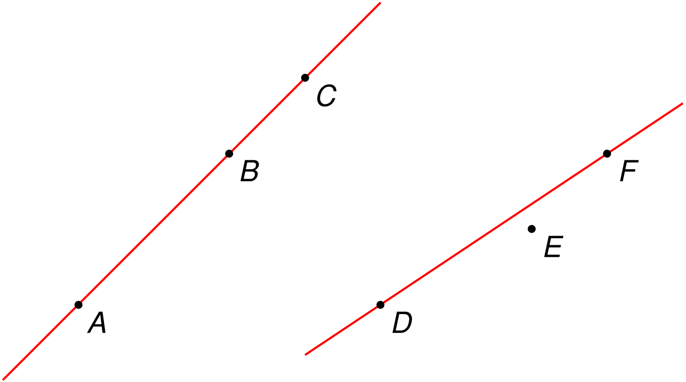
Now drop altitudes from the point $D$ to the sides $AB$ and $AC$. Let $X$ and $Y$ denote the respective points on $AB$ and $AC$.

Note that $\text{Area}(\Delta ABD) = \frac{1}{2}|DX| \cdot |AB|$ and $\text{Area}(\Delta ACD) = \frac{1}{2}|DY| \cdot |AC|$.

Since $\angle XAD = \alpha = \angle YAD$, the side $AD$ is common, and $\angle DXA = 90^\circ = \angle DYA$ then $\Delta AXD \cong \Delta AYD$ by AAS. Therefore $|DX| = |DY|$, and so

$$\frac{|BD|}{|DC|} = \frac{\text{Area}(\Delta ABD)}{\text{Area}(\Delta ADC)} = \frac{\frac{1}{2}|AB| \cdot |DX|}{\frac{1}{2}|AC| \cdot |DY|} = \frac{|AB|}{|AC|}$$

**Definition.** Three points $A$, $B$, $C$ are collinear if and only if the line through any two points also contains the third.



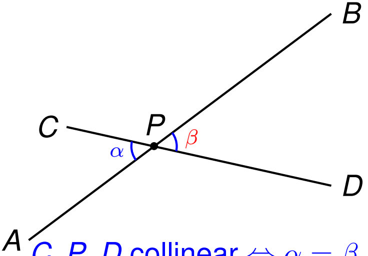

$A$, $B$, $C$ collinear

$D$, $E$, $F$ not collinear

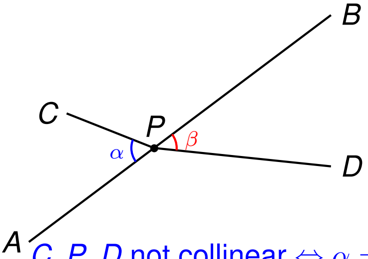
# Methods for proving collinearity

We have already encountered some proofs where we had to prove that three points are collinear.

For example, in Lecture 7 we used the fact that $C$, $P$, $D$ are collinear if and only if they make opposite angles equal with a straight line $AB$ passing through $P$.




$C$, $P$, $D$ collinear $\Leftrightarrow \alpha = \beta$




$C$, $P$, $D$ not collinear $\Leftrightarrow \alpha \neq \beta$

In Lecture 6 we used a different method to prove that the orthocentre, centroid and circumcentre of a triangle are all collinear. (Constructing the Euler line)

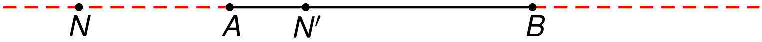
# Menelaus' theorem

Are there any more general theorems that we can use?

**Theorem.** (Menelaus) Let $\Delta ABC$ be a triangle and let $L, M, N$ be points on the extensions of the three sides $BC$, $AC$ and $AB$ respectively. Then $L, M, N$ are collinear if and only if

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = -1$$

**Remark.** In the statement above we take the lengths of the *directed segments*. This means that $\frac{AN}{NB} > 0$ if $N$ lies between $A$ and $B$, and $\frac{AN}{NB} < 0$ if $N$ does not lie between $A$ and $B$.

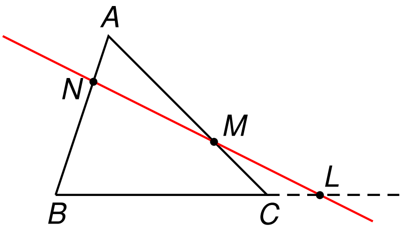


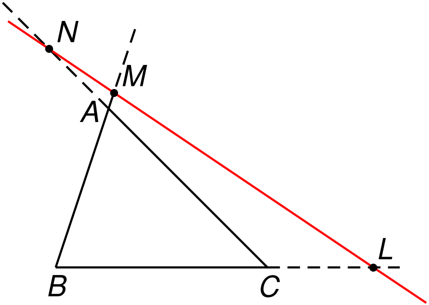

In the diagram above, $\frac{AN}{NB} < 0$ and $\frac{AN'}{N'B} > 0$.

# Methods for proving collinearity

Note that it is only possible for the three points to be collinear if either

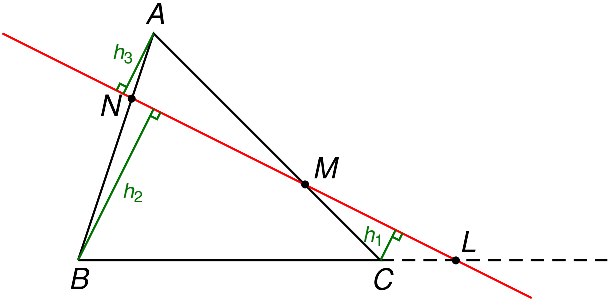
- two points lie inside the edges of the triangle and one point lies outside, or
- all three points lie outside the edges of the triangle.



# Proof of Menelaus' theorem ($\Rightarrow$)

**Proof.** $\Rightarrow$ Suppose first that $L, M, N$ are collinear. Then we can drop perpendiculars from each vertex to the line $LMN$ as in the picture below.




There are two similar right-angled triangles formed by these perpendiculars, the line $LMN$ and the line $BC$. Therefore

$$\frac{|LB|}{|LC|} = \frac{h_2}{h_1}$$

# Proof of Menelaus' theorem ($\Rightarrow$)

$$\frac{|LB|}{|LC|} = \frac{h_2}{h_1}$$

The same idea applied to the sides $AC$ and $AB$ gives us

$$\frac{|MC|}{|MA|} = \frac{h_1}{h_3}, \quad \text{and} \quad \frac{|NA|}{|NB|} = \frac{h_3}{h_2}$$

Multiplying the three equations together gives us

$$\frac{|LB|}{|LC|} \cdot \frac{|MC|}{|MA|} \cdot \frac{|NA|}{|NB|} = \frac{h_2}{h_1} \cdot \frac{h_1}{h_3} \cdot \frac{h_3}{h_2} = 1$$

If we take the length of the directed segments instead, then we get either: (a) one minus sign (if one point is outside the triangle and two are inside) or (b) three minus signs (if all three points are outside the triangle). Therefore we have

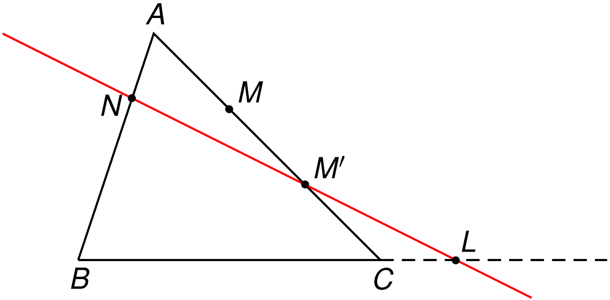
$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = -1$$

# Proof of Menelaus' theorem ($\Leftarrow$)

**Proof of $\Leftarrow$.** Now suppose that $\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = -1$. Suppose (for contradiction) that $L, M, N$ are not collinear. Let $M'$ be the point of intersection of $LN$ with $AC$. Then $L, M', N$ are collinear and so (by the proof above)

$$\frac{BL}{LC} \cdot \frac{CM'}{M'A} \cdot \frac{AN}{NB} = -1$$

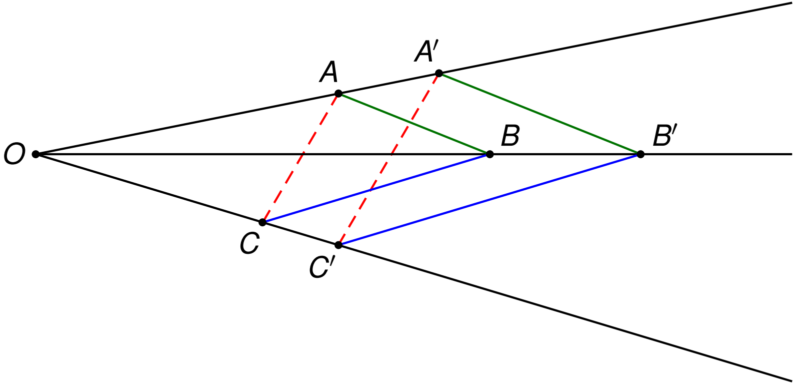
Therefore $\frac{CM'}{M'A} = \frac{CM}{MA}$ and so the points $M$ and $M'$ coincide. ∎



# Desargues' theorem

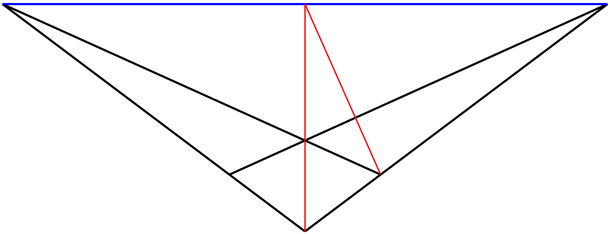
We proved a special case of Desargues' theorem in Lecture 4.

**Theorem.** Suppose that $A$ and $A'$ are collinear, $B$ and $B'$ are collinear, and $C$ and $C'$ are collinear as in the diagram below. If $AB$ is parallel to $A'B'$ and $BC$ is parallel to $B'C'$ then $AC$ is parallel to $A'C'$.



When we study projective geometry later in the course, we will see that parallel lines meet at infinity. (For example, when drawing in perspective, parallel lines meet at the horizon.)

Another way of interpreting this special case of Desargues' theorem is to say that the intersection point of each pair of lines is on the horizon line at infinity. Therefore the intersections are all collinear.



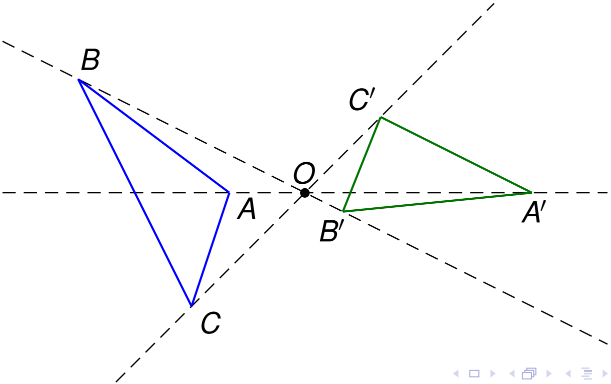
We'll talk more about the line at infinity when we study projective geometry.

# Perspective from a point

The most general way to state Desargues' theorem is in terms of perspective.

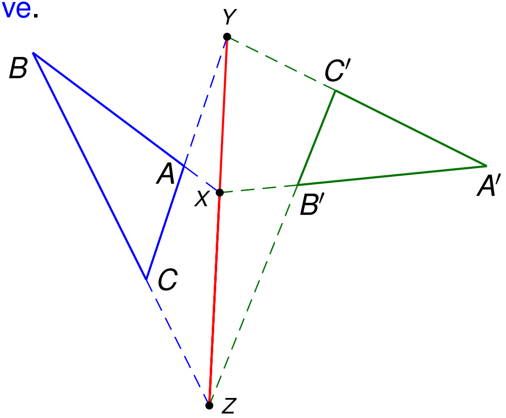
**Definition.** Two polygons are in perspective from a point $O$ if and only if the lines joining corresponding points are concurrent at $O$.

The point $O$ is called the *centre of perspective*.



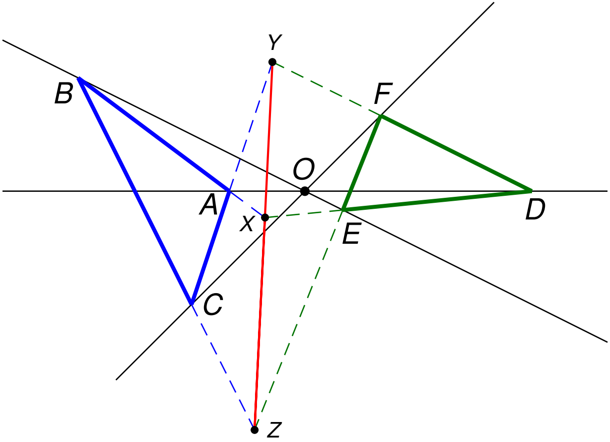
# Perspective from a line

**Definition.** Two triangles are in perspective from a line if and only if the points of intersection of the corresponding lines are collinear.
The line containing the points of intersection is called the axis of perspective.



# Desargues' theorem

**Theorem.** (Desargues) Let $\Delta ABC$ and $\Delta DEF$ be two triangles in the plane. Then $\Delta ABC$ and $\Delta DEF$ are in perspective from a point if and only if they are in perspective from a line.




Click the link for an interactive picture of Desargues' theorem.

# Proof of Desargues' theorem ($\Rightarrow$)

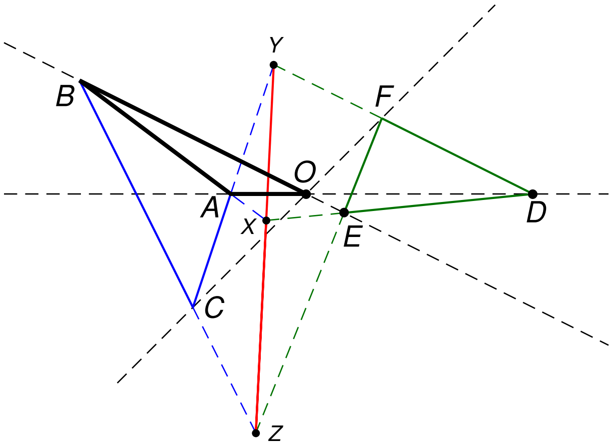
**Proof.** $\Rightarrow$ First suppose that $\Delta ABC$ and $\Delta DEF$ are in perspective from a point $O$. Let $X$ be the intersection of $AB$ and $DE$, $Y$ be the intersection of $AC$ and $DF$ and $Z$ be the intersection of $BC$ and $EF$.

Since $X$ is on the line $DE$, then $D, E, X$ are all collinear. Moreover, we also know that

- $X$ is also on $AB$ (since it is the intersection of $AB$ and $DE$),
- $D$ is on the line $OA$ (since $\Delta DEF$ and $\Delta ABC$ are in perspective from $O$), and
- $E$ is on the line $OB$ (same reason).

Therefore, the points $D, E, X$ are collinear and they lie on the sides of the triangle $\Delta OAB$, and so we can apply Menelaus' theorem in this situation.

# Proof of Desargues' theorem ($\Rightarrow$)




Consider the triangle $\Delta OAB$. The point $D$ lies on $OA$, the point $E$ lies on $OB$ and the point $X$ lies on $AB$.

# Proof of Desargues' theorem ($\Rightarrow$)

Menelaus' theorem shows us that

$$\frac{AX}{XB} \cdot \frac{OD}{DA} \cdot \frac{BE}{EO} = -1$$

The same idea applied to the collinear points $D, F, Y$ and the triangle $\Delta OAC$ shows that

$$\frac{CY}{YA} \cdot \frac{AD}{DO} \cdot \frac{OF}{FC} = -1$$

Finally, $E, F, Z$ are collinear and lie on the sides of $\Delta OBC$, so

$$\frac{BZ}{ZC} \cdot \frac{OE}{EB} \cdot \frac{CF}{FO} = -1$$

Combining these equations and cancelling gives us

$$\frac{AX}{XB} \cdot \frac{CY}{YA} \cdot \frac{BZ}{ZC} = -1$$

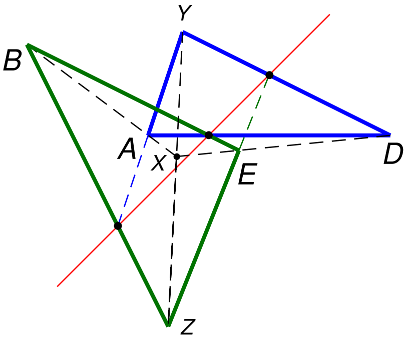
The converse of Menelaus' theorem then shows that $X, Y, Z$ are collinear.

# Proof of Desargues' theorem ($\Leftarrow$)

Now suppose that $X, Y, Z$ are collinear. We want to show that $AD$, $BE$ and $CF$ intersect at a point $O$.

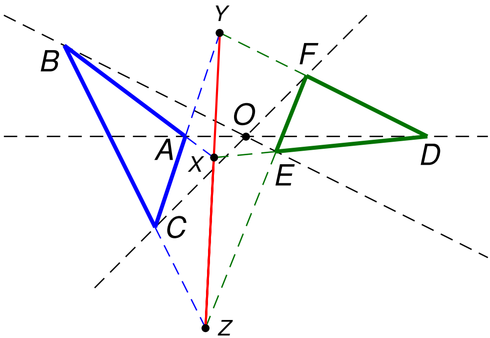
Since $X$ lies on the three lines $AB$, $DE$ and $YZ$, then the triangles $\Delta ADY$ and $\Delta BEZ$ are in perspective from the point $X$.

Therefore, Desargues theorem shows that they are in perspective from a line (drawn in red in the diagram below).



# Proof of Desargues' theorem ($\Leftarrow$)

We claim that this line must be $CF$. To see this, note that $D, F, Y$ and $E, F, Z$ are all collinear. Therefore $F$ lies on both $DY$ and $EZ$, and so the intersection of $DY$ and $EZ$ must be at $F$. Similarly, the point $C$ lies on both $AY$ and $BZ$. Therefore the intersection of $AD$ and $BE$ must occur on the line $CF$. Therefore $AD$, $BE$ and $CF$ are all concurrent and so the two triangles $\Delta ABC$ and $\Delta DEF$ are in perspective from a point. ∎



# Triangles in perspective from more than one point

What if two triangles are in perspective from two points? In other words, there are two points $P$ and $Q$ such that $\Delta ABC$ and $\Delta DEF$ are in perspective from $P$ and $\Delta ABC$ and $\Delta FDE$ are in perspective from $Q$?

Note that the order of the vertices is important; we are asserting that $AD$, $BE$ and $CF$ are concurrent at $P$ and that $AF$, $BD$ and $CE$ are concurrent at $Q$.

In this case we can also use Menelaus' and Desargues' theorem to show that $\Delta ABC$ and $\Delta EFD$ are in perspective from another point $O$.

If two triangles are in perspective from two points, then they are in perspective from a third point.

The proof uses the symmetry between all of the conditions resulting from Menelaus' theorem.

# Sketch of proof

The proof is quite long and so I will only sketch the idea here. We will see how it works by doing an example of the construction during Lecture 10.

The idea is that there are nine different intersection points for the two triangles (call them $X_1, X_2, X_3, Y_1, \ldots, Z_3$).

Three of them lie on $DE$, three on $EF$ and three on $DF$.

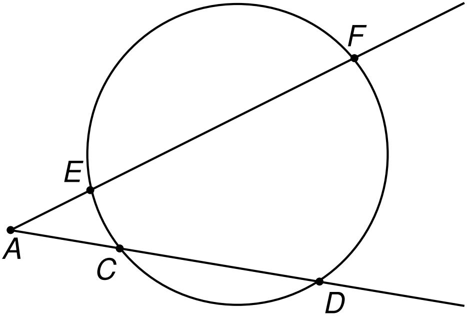
These three collinearity conditions give us a relationship between the side lengths by Menelaus' theorem.

Since the triangles are in perspective from two points, then Desargues' theorem shows that they are in perspective from two different axes. This gives us two more collinearity conditions from Menelaus' theorem.

After substituting these into the previous equation then we can apply Menelaus' theorem to the resulting equation. This gives us a third collinearity condition.

See the end of today's slides for more details.

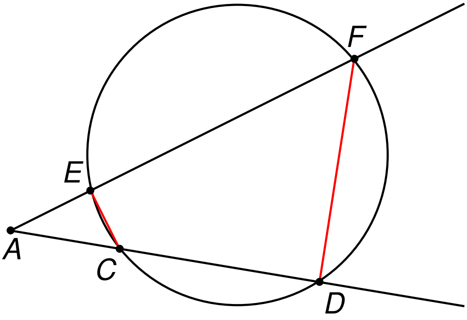
**Exercise.** Let $A$ be a point outside a circle, let $C$ and $E$ be points on the circle such that the lines $AC$ and $AE$ cut the circle at distinct points $D$ and $F$ as in the diagram below. Prove that $|AC| \cdot |AD| = |AE| \cdot |AF|$.



**Solution.**

**Solution.** Since $CDEF$ is cyclic then $\angle ECD + \angle EFD = 180$. Therefore $\angle ACE = 180 - \angle ECD = \angle EFD = \angle AFD$ and so the triangles $\Delta AEC$ and $\Delta ADF$ are similar. Therefore the ratios of the corresponding sides are equal and we have

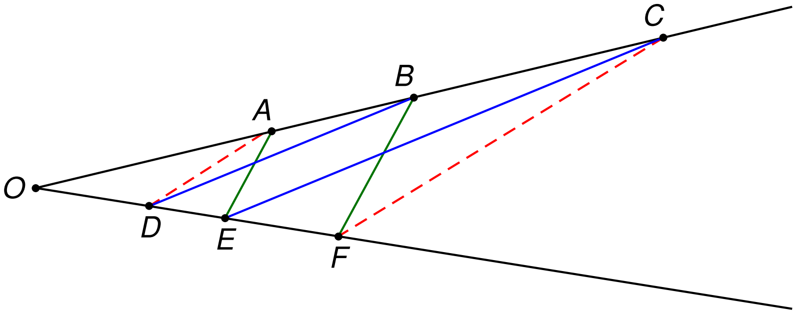
$$\frac{|CA|}{|AE|} = \frac{|FA|}{|AD|} \Rightarrow |AC| \cdot |AD| = |AE| \cdot |AF|.$$



# Pappus' theorem

We proved a special case of Pappus' theorem in Lecture 4.

**Theorem.** Suppose that $A$, $B$, $C$ lie on the same line (collinear), and $D$, $E$, $F$ lie on another line as in the diagram below. If $AE$ is parallel to $BF$ and $DB$ is parallel to $EC$, then $AD$ is parallel to $CF$.

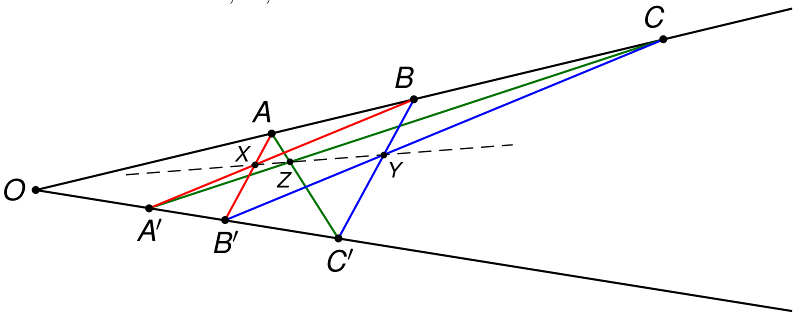



Another interpretation of this is that the intersection points of the three pairs of lines all lie on the line at infinity.

# Pappus' theorem

Just as we did for Desargues' theorem, we can generalise Pappus' theorem by asserting that all the intersection points have to be collinear.

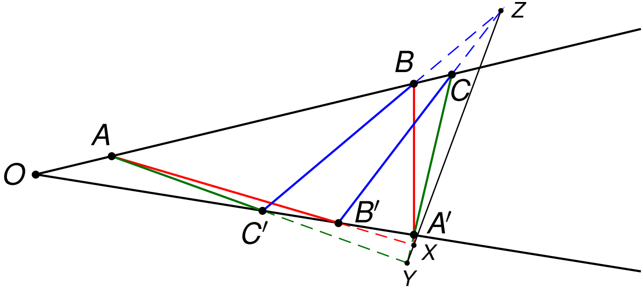
**Theorem.** (Pappus) Let $A, B, C$ be collinear and $A', B', C'$ be collinear as well. Let $X$ be the intersection of $AB'$ and $A'B$, $Y$ the intersection of $B'C$ and $BC'$ and $Z$ the intersection of $AC'$ and $A'C$. Then $X, Y, Z$ are collinear.




Click the link for an interactive picture of Pappus' theorem.

# Pappus' theorem (another picture)

Below is another picture of Pappus' theorem. Note that the intersection points $X, Y, Z$ are collinear.




The previous picture is neater, but less useful for the proof, since we want to construct points such as the intersection of $AB'$ and $BC'$. In the previous picture these lines are parallel, and so the intersection is at infinity (which is difficult to draw!).

This picture is not as neat, but it has the advantage that the intersection points used in the proof all lie on the page.

# Pappus' theorem

**Another interpretation.** We can think of the hexagon formed by $AB'CA'BC'$. Then if we extend the opposite sides, their intersection points are at $X, Y, Z$. Pappus' theorem says that these intersections are collinear if $A, B, C$ and $A', B', C'$ are collinear.

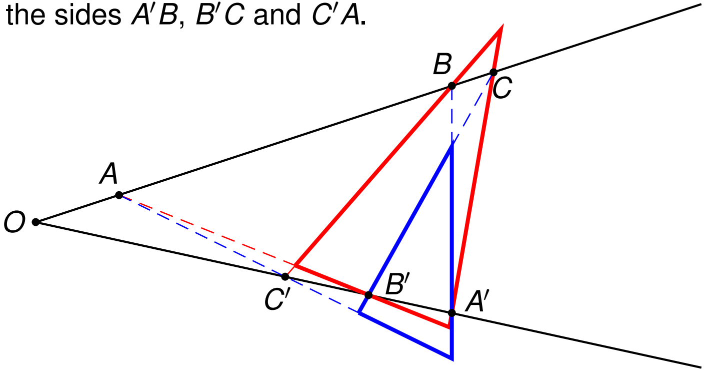
The intersection points of the opposite sides of a hexagon are collinear if the vertices alternate between two straight lines.

**How can we prove Pappus' theorem?**

Last year's lecture notes contain a proof using Menelaus' theorem. (See Theorem 6.8 on pp55-56)

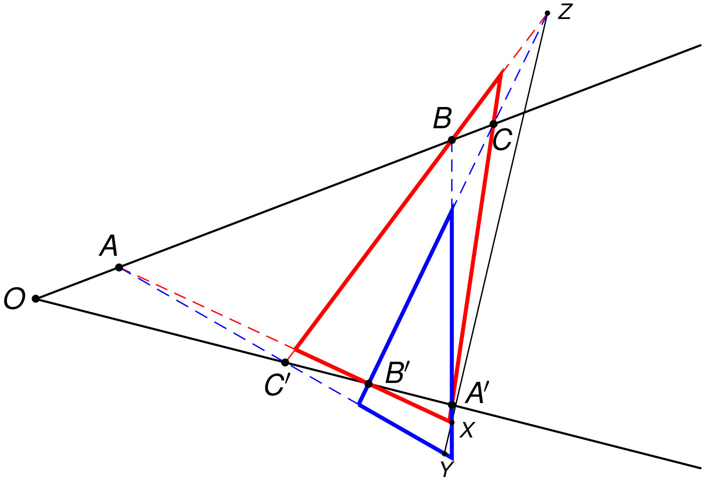
Here we give an alternative proof using our previous result on triangles in perspective from two points.

**Proof.** Consider the triangle formed by extending the sides $AB'$, $BC'$ and $CA'$. We also have a triangle formed by extending the sides $A'B$, $B'C$ and $C'A$.




These triangles are in perspective from the lines $ABC$ and $A'B'C'$. Therefore they are in perspective from a third line as well. This turns out to be the required collinearity condition. ∎

# Pappus' theorem (final picture)




The two triangles are also in perspective from a third line, the line through $X, Y, Z$.

# Pascal's hexagon theorem

Another application of Desargues' theorem is *Pascal's hexagon theorem*.

**Theorem.** (Pascal) If a hexagon has a circumscribed circle, then the intersections of the opposite sides are collinear.

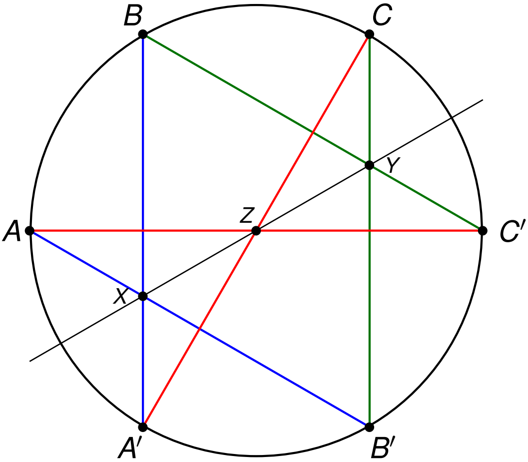
More precisely, let $A, B', C, A', B, C'$ be six points on a circle, let $X$ be the intersection of $AB'$ and $A'B$, let $Y$ be the intersection of $B'C$ and $BC'$ and let $Z$ be the intersection of $A'C$ and $AC'$. Then $X, Y, Z$ are collinear.

**Remark.** This statement is very similar to Pappus' theorem, except we now have a circle instead of two lines. Do the two theorems follow from the same principle?

There is a more general theorem which says that for any hexagon inscribed in a conic section (e.g. ellipse, hyperbola, parabola), the intersection points of the opposite sides are collinear. The circle is an example of a conic, and a pair of intersecting lines is also an example of a (degenerate) conic.

# Pascal's hexagon theorem (picture)

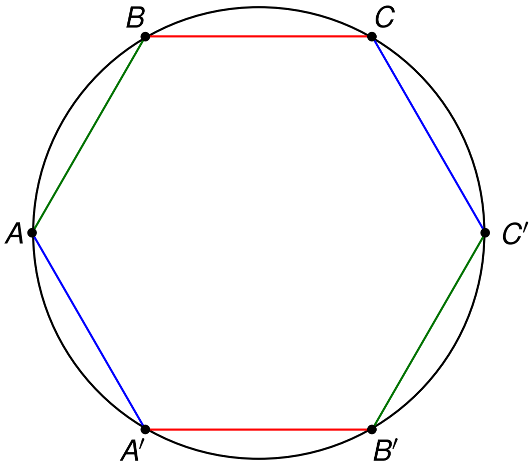
A picture of Pascal's theorem for evenly spaced points $A, B, C, A', B', C'$ is given below. The hexagon is $AB'CA'BC'$.



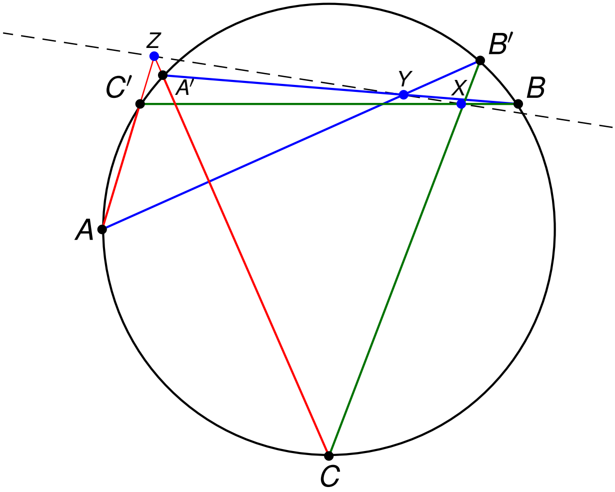

Click the link for an interactive picture of Pascal's theorem.

# Pascal's hexagon theorem (regular hexagon picture)

For a regular hexagon $ABCC'B'A'$, the opposite sides are all parallel. In this case we say that the opposite sides all intersect on the line at infinity.



# Pascal's hexagon theorem (another picture)



**Sketch of Proof.** Let $\Delta PQR$ be the triangle formed by extending the lines $AB'$, $BC'$ and $CA'$.

Let $\Delta P'Q'R'$ be the triangle formed by extending the lines $A'B$, $B'C$ and $C'A$.

Also let $X, Y, Z$ be the three points of intersection.

We want to show that these two triangles are in perspective, since then the axis of perspective will be the line $XYZ$.

Now note that $AB'BC'$ is a cyclic quadrilateral with a pair of opposite sides intersecting at $P$. Using the result of the exercise, we see that

$$\frac{AP}{PB} = \frac{C'P}{PB'}$$

If we continue this for the other sides of the triangle $\Delta PQR$ then we have

$$\frac{BQ}{QC} = \frac{A'Q}{QC'}, \quad \text{and} \quad \frac{CR}{RA} = \frac{B'R}{RA'}$$

# Pascal's hexagon theorem (sketch of proof)

We combine these equations to obtain

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RA} = \frac{C'P}{PB'} \cdot \frac{A'Q}{QC'} \cdot \frac{B'R}{RA'}$$

Then to finish the proof, we use the condition from the "Background details" notes (see the end of today's slides) to show that the triangles $\Delta PQR$ and $\Delta P'Q'R'$ are in perspective from the line formed by the remaining three intersection points, which are $X, Y, Z$. ∎

# Next time

In the next lecture we will study properties of circles.

- Recall properties of circles that we have already proved.
- Define the power of a point with respect to a circle
- Define the radical axis of a circle and study its properties.

We will also continue with the ruler and compass constructions related to this week's topics.

# Background details for the proofs (not tested)

Let $\Delta ABC$ and $\Delta A'B'C'$ be two triangles. Define the intersection points by

$$X_1 = BC \cap B'C',\ Y_1 = BC \cap A'C',\ Z_1 = BC \cap A'B'$$
$$X_2 = AB \cap A'B',\ Y_2 = AB \cap B'C',\ Z_2 = BC \cap A'C'$$
$$X_3 = AC \cap A'C',\ Y_3 = AC \cap A'B',\ Z_3 = AC \cap B'C'$$

Then $X_1$, $Y_2$ and $Z_3$ are all on the line $B'C'$. Therefore, Menelaus' theorem applied to the triangle $\Delta ABC$ implies that

$$\frac{BX_1}{X_1C} \cdot \frac{AY_2}{Y_2B} \cdot \frac{CZ_3}{Z_3A} = -1$$

Similarly, $X_2$, $Y_3$ and $Z_1$ are all on the line $A'B'$ and $X_3$, $Y_1$ and $Z_2$ are all on the line $A'C'$. Therefore

$$\frac{AX_2}{X_2B} \cdot \frac{CY_3}{Y_3A} \cdot \frac{BZ_1}{Z_1C} = -1, \text{ and } \frac{CX_3}{X_3A} \cdot \frac{BY_1}{Y_1C} \cdot \frac{AZ_2}{Z_2B} = -1$$

# Background details for the proofs (not tested)

If we multiply all of these equations together and rearrange, then we get

$$\frac{BX_1}{X_1C}\cdot\frac{AX_2}{X_2B}\cdot\frac{CX_3}{X_3A}\cdot\frac{BY_1}{Y_1C}\cdot\frac{AY_2}{Y_2B}\cdot\frac{CY_3}{Y_3A}\cdot\frac{BZ_1}{Z_1C}\cdot\frac{AZ_2}{Z_2B}\cdot\frac{CZ_3}{Z_3A}=-1$$

Now suppose that

$$\frac{BX_1}{X_1C}\cdot\frac{AX_2}{X_2B}\cdot\frac{CX_3}{X_3A}\cdot\frac{BY_1}{Y_1C}\cdot\frac{AY_2}{Y_2B}\cdot\frac{CY_3}{Y_3A}=1$$

Then $\frac{BZ_1}{Z_1C}\cdot\frac{AZ_2}{Z_2B}\cdot\frac{CZ_3}{Z_3A}=-1$ and so $Z_1, Z_2$ and $Z_3$ are collinear.
Therefore the triangles are in perspective from the line $Z_1Z_2Z_3$.
As a special case, we see that if $X_1, X_2$ and $X_3$ are collinear and $Y_1$, $Y_2$, $Y_3$ are collinear then $Z_1, Z_2, Z_3$ are collinear.
If triangles $\Delta ABC$ and $\Delta A'B'C'$ are in perspective from two axes then they are in perspective from a third axis.