ACTION OF THE MAPPING CLASS GROUP ON CHARACTER VARIETIES AND HIGGS BUNDLES

OSCAR GARCÍA-PRADA AND GRAEME WILKIN

Abstract. We consider the action of the orientation preserving mapping class group $\mathrm{Mod}^+(S)$ of an oriented compact surface S of genus $g \geqslant 2$ on the moduli space $\mathcal{R}(S,G)$ of representations of $\pi_1(S)$ in a connected semisimple real Lie group G. We identify the fixed points of the action of a finite subgroup $\Gamma \subset \mathrm{Mod}^+(S)$ on $\mathcal{R}(S,G)$, in terms of G-Higgs bundles equipped with a Γ-pseudoequivariant structure on a Riemann surface $X = (S,J)$, where J is an element in the Teichmüller space of S for which $\Gamma \subset \mathrm{Aut}(X)$, whose existence is guaranteed by Kerckhoff's solution of the Nielsen realization problem. The Γ-equivariant G-Higgs bundles are in turn in correspondence with parabolic Higgs bundles on $Y = X/\Gamma$, where the weights on the parabolic points are determined by the Γ-equivariant structure. This generalizes work of Nasatyr & Steer for $G = \mathrm{SL}(2,\mathbb{R})$ and Boden, Andersen & Grove and Furuta & Steer for $G = \mathrm{SU}(n)$.

1. Introduction

Let S be a compact oriented surface of genus greater than one, and G be a real reductive Lie group. Consider the moduli space of representations or character variety $\mathcal{R}(S,G)$ defined as the space of reductive representations of the fundamental group of S in G modulo conjugation by elements of G. These are very important varieties that play a central role in geometry, topology, Teichmüller theory and theoretical physics. A fundamental problem is that of understanding the action of the mapping class group or modular group of the surface $\mathrm{Mod}(S)$ in $\mathcal{R}(S,G)$. In this paper, we consider the action of a finite subgroup Γ of the orientation preserving subgroup $\mathrm{Mod}^+(S) \subset \mathrm{Mod}(S)$ and give a description of the fixed-point subvariety.

A crucial step in our study is provided by a theorem of Kerckhoff solving the Nielsen realization problem [20]. This theorem proves the existence of a complex structure J on S, such that, if $X := (S,J)$ is the corresponding Riemann surface, Γ is a subgroup of the group of holomorphic automorphisms of X. We can then use holomorphic methods, and in particular the theory of G-Higgs bundles over X. To define a G-Higgs bundle, we consider a maximal compact subgroup $H \subset G$, and a Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. A G-Higgs bundle is a pair (E,φ) consisting of a $H^{\mathbb{C}}$-bundle E, where $H^{\mathbb{C}}$ is the complexification of H, and a holomorphic section φ of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the bundle associated to the complexification of the isotropy representation of H in $\mathfrak{m}$ and K is the canonical line bundle of X. The non-abelian Hodge theory correspondence establishes a homeomorphism

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between $\mathcal{R}(S,G)$ and the moduli space of polystable G-bundles over $X = (S,J)$ for any complex structure J on S. Now, if J is the complex structure given by Kerckhoff's theorem, the action of Γ on $\mathcal{R}(S,G)$ coincides with the natural action of $\Gamma \subset \mathrm{Aut}(X)$ on $\mathcal{M}(X,G)$ via pull-back, as one can prove tracing the non-abelian Hodge theory correspondence. Our problem becomes then that of analysing the fixed points $\mathcal{M}(X,G)^\Gamma$.

The fixed-point subvariety $\mathcal{M}(X,G)^\Gamma$ is described in terms of G-Higgs bundles equipped with a Γ-pseudoequivariant sructure defined by a group cocycle $c \in Z^2(\Gamma, Z')$, where Z' is a subgroup of the the centre of $H^{\mathbb{C}}$ — here Z' acts trivially on Γ. When Z' is contained in the kernel of the isotropy representation, these are lifts of true Γ-equivariant structures on the associated G/Z'-Higgs bundles. A (Γ, c)-pseudoequivariant structure defines isomorphism classes of pseudorepresentations (or projective representations in another terminology) of the isotropy subgroups $\Gamma_x \subset \Gamma$. It is well-known that there is only a finite number of points $x \in X$ for which $\Gamma_x \neq \{1\}$, and Γ_x is a cyclic group. Fixing the cocycle c and the pseudorepresentation classes σ at the points with $\Gamma_x \neq \{1\}$, we define a moduli space of (Γ, c)-pseudoequivariant G-Higgs bundles with fixed σ. Our main result is Theorem 4.6.

Generalising a well-known result for vector bundles [22, 13, 24, 5, 2, 1], and principal bundles [33, 3], we establish in Theorem 5.1 a correspondence between Γ-equivariant G-Higgs bundles over X and parabolic G-Higgs bundles over $Y := X/\Gamma$. The weights of the parabolic structure are determined by the representation classes σ defined by the equivariant structure. In particular, if Z' is contained in the kernel of the isotropy representation there is a map from the moduli space of G-Higgs bundles over X to the moduli space of G/Z'-Higgs bundles and hence a map from the moduli space of (Γ, c)-pseudoequivariant G-Higgs bundles over X to the moduli space of parabolic G/Z'-Higgs bundles over Y. It would be very interesting to find a parabolic description of the general Γ-pseudoequivariant objects. This may involve a certain twisting of the construction given in the equivariant case. We leave this for future work.

In the process of writing up this paper, we came across the work of Schaffhauser [28] were cocycles are also used to study equivariant structures on vector bundles.

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2. Moduli space of representations and the mapping class group

In this section S is an oriented smooth compact surface of genus $g \geqslant 2$ and G is a connected real reductive Lie group.

2.1. Moduli space of representations.

By a **representation** of $\pi_1(S)$ in G we mean a homomorphism $\rho\colon \pi_1(S) \to G$. The set of all such homomorphisms, denoted $\mathrm{Hom}(\pi_1(S), G)$, is an analytic variety, which is algebraic if G is algebraic. The group G acts on $\mathrm{Hom}(\pi_1(S), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$$

for $g \in G$, $\rho \in \mathrm{Hom}(\pi_1(S), G)$ and $\gamma \in \pi_1(S)$. If we restrict the action to the subspace $\mathrm{Hom}^+(\pi_1(S), g)$ consisting of reductive representations, the orbit space is Hausdorff. By a **reductive representation** we mean one that, composed with the adjoint representation in the Lie algebra of G, decomposes as a sum of irreducible representations. If G is algebraic this is equivalent to the Zariski closure of the image of $\pi_1(S)$ in G being a reductive group. (When G is compact every representation is reductive). The **moduli space of representations** or **character variety** of $\pi_1(S)$ in G is defined to be the orbit space

$$\mathcal{R}(S,G) = \mathrm{Hom}^+(\pi_1(S),G)/G.$$

It has the structure of an analytic variety (see e.g. [15]) which is algebraic if G is algebraic and is real if G is real or complex if G is complex. If G is complex then $\mathcal{R}(S,G)$ is the GIT quotient

$$\mathcal{R}(S,G) = \mathrm{Hom}(\pi_1(S),G) /\!\!/ G.$$

Let $\rho : \pi_1(S) \to G$ be a representation of $\pi_1(S)$ in G. Let $Z_G(\rho)$ be the centralizer in G of $\rho(\pi_1(S))$. We say that ρ is **irreducible** if and only if it is reductive and $Z_G(\rho) = Z(G)$, where $Z(G)$ is the centre of G.

2.2. **The mapping class group.** The **mapping class group** or **modular group** of S is defined as

$$\mathrm{Mod}(S) = \pi_0 \,\mathrm{Diff}(S),$$

where $\mathrm{Diff}(S)$ is the group of diffeomorphisms of S. We also consider the subgroup

$$\mathrm{Mod}^+(S) = \pi_0 \,\mathrm{Diff}^+(S),$$

where $\mathrm{Diff}^+(S)$ is the subgroup of $\mathrm{Diff}(S)$ consisting of orientation-preserving diffeomorphisms. We have an exact sequence

$$1 \to \mathrm{Mod}^+(S) \to \mathrm{Mod}(S) \to \mathbb{Z}/2 \to 1. \tag{2.1}$$

By the Dehn–Nielsen–Baer theorem, $\mathrm{Mod}(S)$ is isomorphic to $\mathrm{Out}(\pi_1(S))$, the group of outer automorphisms of $\pi_1(S)$, and hence acts in the obvious way on $\mathcal{R}(S,G)$.

Let $\Gamma \subset \mathrm{Mod}(S)$ be a finite subgroup. The main goal of this paper is to investigate the fixed points $\mathcal{R}(S,G)^\Gamma$. A crucial step to do this is provided by Kerckhoff's solution of the **Nielsen realization problem** [20]:

Theorem 2.1. *Let $\Gamma \subset \mathrm{Mod}(S)$ be a finite subgroup. There exists an element J in the Teichmüller space of S such that, if $X = (S,J)$ and $\widetilde{\mathrm{Aut}}(X)$ is the group of automorphisms of X which are either holomorphic or antiholomorphic, $\Gamma \subset \widetilde{\mathrm{Aut}}(X)$. In particular, if $\Gamma \subset \mathrm{Mod}^+(S)$, one has $\Gamma \subset \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the group of holomorphic automorphisms of X.*

Remark 2.2. This had been proved by Nielsen [25] for cyclic groups and by Fenchel [12] for solvable groups. Thanks to Theorem 2.1 the problem of studying the action of Γ on $\mathcal{R}(S,G)$ can be reduced to studying the action of Γ on the moduli space of G-Higgs bundles on X.

Remark 2.3. In fact, if X is not hyperelliptic, $\Gamma = \mathrm{Aut}(X)$ if $\Gamma \subset \mathrm{Mod}^+(S)$, and $\Gamma = \widetilde{\mathrm{Aut}}(X)$ if Γ is not contained in $\mathrm{Mod}^+(S)$.

2.3. **Moduli space of G-Higgs bundles.** Here X is a compact Riemann surface and G is a real reductive Lie group. We fix a maximal compact subgroup H of G. The Lie algebra $\mathfrak{g}$ of G is equipped with an involution θ that gives the Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H. We fix a metric B in $\mathfrak{g}$ with respect to which the Cartan decomposition is orthogonal. This metric is positive definite on $\mathfrak{m}$ and negative definite on $\mathfrak{h}$. We have $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{h}$. From the isotropy representation $H \to \mathrm{Aut}(\mathfrak{m})$, we obtain the representation $\iota : H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$. When G is semisimple we take B to be the Killing form. In this case B and a choice of a maximal compact subgroup H determine a Cartan decomposition (see [21] for details).

A G-Higgs bundle on X consists of a holomorphic principal $H^{\mathbb{C}}$-bundle E together with a holomorphic section $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the associated vector bundle with fibre $\mathfrak{m}^{\mathbb{C}}$ via the complexified isotropy representation, and K is the canonical line bundle of X.

If G is compact, $H = G$ and $\mathfrak{m} = 0$. A G-Higgs bundle is hence simply a holomorphic principal $G^{\mathbb{C}}$-bundle. If $G = H^{\mathbb{C}}$, where now H is a compact Lie group, H is a maximal compact subgroup of G, and $\mathfrak{m} = i\mathfrak{h}$. In this case, a G-Higgs bundle is a principal $H^{\mathbb{C}}$-bundle together with a section $\varphi \in H^0(X, E(\mathfrak{h}^{\mathbb{C}}) \otimes K) = H^0(X, E(\mathfrak{g}) \otimes K)$, where $E(\mathfrak{g})$ is the adjoint bundle. This is the original definition for complex Lie groups given by Hitchin in [18].

There is a notion of stability for G-Higgs bundles (see [14]). To explain this we consider the parabolic subgroups of $H^{\mathbb{C}}$ defined for $s \in i\mathfrak{h}$ as

$$P_s = \{g \in H^{\mathbb{C}} \ : \ e^{ts} g e^{-ts} \text{ is bounded as } t \to \infty\}. \tag{2.2}$$

A Levi subgroup of P_s is given by $L_s = \{g \in H^{\mathbb{C}} \ : \ \mathrm{Ad}(g)(s) = s\},$. Their Lie algebras are given by

$$\mathfrak{p}_s = \{Y \in \mathfrak{h}^{\mathbb{C}} \ : \ \mathrm{Ad}(e^{ts})Y \text{ is bounded as } t \to \infty\},$$
$$\mathfrak{l}_s = \{Y \in \mathfrak{h}^{\mathbb{C}} \ : \ \mathrm{ad}(Y)(s) = [Y, s] = 0\}.$$

We consider the subspaces

$$\mathfrak{m}_s = \{Y \in \mathfrak{m}^{\mathbb{C}} \ : \ \iota(e^{ts})Y \text{ is bounded as } \ t \to \infty\}$$
$$\mathfrak{m}_s^0 = \{Y \in \mathfrak{m}^{\mathbb{C}} \ : \ \iota(e^{ts})Y = Y \ \text{ for every } \ t\}.$$

One has that $\mathfrak{m}_s$ is invariant under the action of P_s and $\mathfrak{m}_s^0$ is invariant under the action of L_s.

An element $s \in i\mathfrak{h}$ defines a character χ_s of $\mathfrak{p}_s$ since $\langle s, [\mathfrak{p}_s, \mathfrak{p}_s]\rangle = 0$. Conversely, by the isomorphism $(\mathfrak{p}_s/[\mathfrak{p}_s, \mathfrak{p}_s])^* \cong \mathfrak{z}_{L_s}^*$, where $\mathfrak{z}_{L_s}$ is the centre of the Levi subalgebra $\mathfrak{l}_s$, a character χ of $\mathfrak{p}_s$ is given by an element in $\mathfrak{z}_{L_s}^*$, which gives, via the invariant metric, an element $s_\chi \in \mathfrak{z}_{L_s} \subset i\mathfrak{h}$. When $\mathfrak{p}_s \subset \mathfrak{p}_{s_\chi}$, we say that χ is an antidominant character of $\mathfrak{p}$. When $\mathfrak{p}_s = \mathfrak{p}_{s_\chi}$ we say that χ is a strictly antidominant character. Note that for $s \in i\mathfrak{h}$, χ_s is a strictly antidominant character of $\mathfrak{p}_s$.

Let now (E, φ) be a G-Higgs bundle over X, and let $s \in i\mathfrak{h}$. Let P_s be defined as above. For $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ a reduction of the structure group of E from $H^{\mathbb{C}}$ to P_s, we define the degree relative to σ and s, or equivalently to σ and χ_s in terms of the curvature of connections using Chern–Weil theory. For this, define $H_s = H \cap L_s$ and $\mathfrak{h}_s = \mathfrak{h} \cap \mathfrak{l}_s$.

Then H_s is a maximal compact subgroup of L_s, so the inclusion $H_s \subset L_s$ is a homotopy equivalence. Since the inclusion $L_s \subset P_s$ is also a homotopy equivalence, given a reduction σ of the structure group of E to P_s one can further restrict the structure group of E to H_s in a unique way up to homotopy. Denote by E'_σ the resulting H_s principal bundle. Consider now a connection A on E'_σ and let $F_A \in \Omega^2(X, E'_\sigma(\mathfrak{h}_s))$ be its curvature. Then $\chi_s(F_A)$ is a 2-form on X with values in $i\mathbb{R}$, and

$$\deg(E)(\sigma, s) := \frac{i}{2\pi}\int_X \chi_s(F_A). \tag{2.3}$$

We define the subalgebra $\mathfrak{h}_{\mathrm{ad}}$ as follows. Consider the decomposition $\mathfrak{h} = \mathfrak{z} + [\mathfrak{h}, \mathfrak{h}]$, where $\mathfrak{z}$ is the centre of $\mathfrak{h}$, and the isotropy representation $\mathrm{ad} = \mathrm{ad} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$. Let $\mathfrak{z}' = \ker(\mathrm{ad}_{|\mathfrak{z}})$ and take $\mathfrak{z}''$ such that $\mathfrak{z} = \mathfrak{z}' + \mathfrak{z}''$. Define the subalgebra $\mathfrak{h}_{\mathrm{ad}} := \mathfrak{z}'' + [\mathfrak{h}, \mathfrak{h}]$. The subindex ad denotes that we have taken away the part of the centre $\mathfrak{z}$ acting trivially via the isotropy representation ad.

Definition 2.4. *We say that a G-Higgs bundle (E, φ) is:*

semistable *if for any $s \in i\mathfrak{h}$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$, we have that* $\deg(E)(\sigma, s) \geqslant 0$*;*

stable *if for any $s \in i\mathfrak{h}_{\mathrm{ad}}$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$, we have that* $\deg(E)(\sigma, s) > 0$*;*

polystable *if it is semistable and for any $s \in i\mathfrak{h}_{\mathrm{ad}}$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$ and* $\deg(E)(\sigma, s) = 0$*, there is a holomorphic reduction of the structure group $\sigma_L \in \Gamma(E_\sigma(P_s/L_s))$ to a Levi subgroup L_s such that $\varphi \in H^0(X, E_{\sigma_L}(\mathfrak{m}_s^0) \otimes K) \subset H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$.*

We define the **moduli space of polystable G-Higgs bundles** $\mathcal{M}(X, G)$ as the set of isomorphism classes of polystable G-Higgs bundles on X.

The notion of stability emerges from the study of the Hitchin equations. The equivalence between the existence of solutions to these equations and the polystability of Higgs bundles is known as the **Hitchin-Kobayashi correspondence**, which we state below.

Theorem 2.5. *Let (E, φ) be a G-Higgs bundle over a Riemann surface X. Then (E, φ) is polystable if and only if there exists a reduction h of the structure group of E from $H^{\mathbb{C}}$ to H, such that*

$$F_h - [\varphi, \tau_h(\varphi)] = 0 \tag{2.4}$$

where $\tau_h : \Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{m}^{\mathbb{C}}))$ is the combination of the anti-holomorphic involution in $E(\mathfrak{m}^{\mathbb{C}})$ defined by the compact real form at each point determined by h and the conjugation of 1*-forms, and F_h is the curvature of the unique H-connection compatible with the holomorphic structure of E.*

A G-Higgs bundle (E, φ) is said to be **simple** if $\mathrm{Aut}(E, \varphi) = Z(H^{\mathbb{C}}) \cap \ker(\iota)$ where $Z(H^{\mathbb{C}})$ the centre of $H^{\mathbb{C}}$. A G-Higgs bundle (E, φ) is said to be **infinitesimally simple** if the infinitesimal automorphism space $\mathrm{aut}(E, \varphi)$ is isomorphic to $H^0(X, E(\ker d\iota \cap Z(\mathfrak{h}^{\mathbb{C}}))$ where $Z(\mathfrak{h}^{\mathbb{C}})$ denotes the Lie algebra of $Z(H^{\mathbb{C}})$.

Thus a G-Higgs bundle is (infinitesimally) simple if its (infinitesimal) automorphism group is as small as possible. It is clear that a simple G-Higgs bundle is infinitesimally simple. If

G is complex then ι is the adjoint representation and (E,φ) is simple (resp. infinitesimally simple) if $\mathrm{Aut}(E,\varphi) = Z(G)$ (resp. $\mathrm{aut}(E,\varphi) = Z(\mathfrak{g})$).

The basic link between representations of $\pi_1(S)$ and Higgs bundles is given by the following (see [14] and references there).

Theorem 2.6. *Let S be a compact surface and $X = (S, J)$ be the Riemann surface defined by any complex structure on S. Let G be a real reductive Lie group. There is a homeomorphism $\mathcal{R}(S,G) \xrightarrow{\cong} \mathcal{M}(X,G)$, where the image of the irreducible representations is the subspace of stable and simple G-Higgs bundles.*

From Theorems 2.1 and 2.6 we conclude the following.

Proposition 2.7. *Let $\Gamma \subset \mathrm{Mod}^+(S)$ be a finite subgroup. Let J be a complex structure given by Kerckhoff's theorem and $X = (S, J)$ be the corresponding Riemann surface. Then, under the correspondence of $\mathcal{R}(S,G)$ with the moduli space of G-Higgs bundles $\mathcal{M}(X,G)$ given by Theorem 2.6, the action of Γ on $\mathcal{R}(S,G)$ coincides with the holomorphic action of $\Gamma \subset \mathrm{Aut}(X)$ given by $\gamma \cdot (E,\varphi) = (\gamma^* E, \gamma^*\varphi)$ for $\gamma \in \Gamma$ and $(E,\varphi) \in \mathcal{M}(X,G)$. We thus have that $\mathcal{R}(S,G)^\Gamma$ and $\mathcal{M}(X,G)^\Gamma$ are in bijective correspondence.*

Proof. Given any $\gamma \in \Gamma \subset \mathrm{Mod}(S)$, Kerckhoff's theorem [20, Thm. 5] guarantees a unique diffeomorphism f in the isotopy class of γ such that $f^*J = J$. The action of γ on $\mathcal{R}(S,G)$ is defined by $\gamma\cdot[\rho] = [f^*\rho] = [\rho \circ f_*]$, which induces an action $\gamma \cdot [\nabla] = [f^*\nabla]$ on the space of equivalence classes of flat connections. The induced action of γ on $\mathcal{M}(X,G)$ via Theorem 2.6 (which is well-defined since $f^*J = J$) is then defined by $\gamma \cdot [(E,\varphi)] = [(\gamma^*E, \gamma^*\varphi)]$. Therefore the fixed point sets $\mathcal{R}(S,G)^\Gamma$ and $\mathcal{M}(X,G)^\Gamma$ are in bijective correspondence. $\square$

3. Pseudoequivariant structures on principal bundles and associated vector bundles

In this section X is a compact Riemann surface of genus bigger than one, $\Gamma \subset \mathrm{Aut}(X)$ is a subgroup of the group of holomorphic automorphisms of X, and G is a connected complex reductive Lie group.

3.1. Γ-pseudoequivariant structure on a principal bundle.

Let $Z := Z(G)$ be the centre of G and let $Z' \subset Z$ be a subgroup. We consider a 2-**cocycle** $c \in Z^2(\Gamma, Z')$, where the action of Γ on Z' is trivial. This is a map $c : \Gamma\times\Gamma \to Z'$ satisfying the cocycle condition

$$c(\gamma\gamma', \gamma'')c(\gamma,\gamma') = c(\gamma, \gamma'\gamma'')c(\gamma', \gamma'').$$

These objects emerge in the study of "lifts" to G of representations of Γ in G/Z', that is maps $\sigma : \Gamma \to G$ such that $\sigma(\gamma\gamma') = c(\gamma,\gamma')\sigma(\gamma)\sigma(\gamma')$. We will refer to such a map as a **pseudorepresentation or pseudohomomorphism** of Γ in G with cocycle c.

Let E be a holomorphic G-bundle over X. Let $c \in Z^2(\Gamma, Z')$. A (Γ, c)**-pseudoequivariant structure on** E consists of a collection of bundle maps $\widetilde{\gamma} : E \to E$ covering $\gamma : X \to X$ for every $\gamma \in \Gamma$, satisfying

$$\widetilde{\gamma\gamma'} = c(\gamma,\gamma')\widetilde{\gamma}\widetilde{\gamma}',$$

and $\widetilde{\mathrm{Id}_X} = \mathrm{Id}_E$. This imposes the condition $c(\gamma, 1) = 1$ for every $\gamma \in \Gamma$. Let $\mathrm{Aut}_\Gamma(E)$ (resp. $\mathrm{Aut}(E)$) the group of holomorphic bundle automorphisms of E covering Γ (resp. covering the identity of X). There is an exact sequence

$$1 \to \mathrm{Aut}(E) \to \mathrm{Aut}_\Gamma(E) \to \Gamma.$$

A (Γ, c)-pseudoequivariant structure on E is simply a pseudorepresentation $\Gamma \to \mathrm{Aut}_\Gamma(E)$ with cocycle c. This is clear since, if E' is the G/Z'-principal bundle associated to E via the projection $G \to G/Z'$, and $\mathrm{Aut}_\Gamma(E')$ is the group of bundle automorphisms of E' covering Γ, we have an exact sequence

$$1 \to Z' \to \mathrm{Aut}_\Gamma(E) \to \mathrm{Aut}_\Gamma(E') \to 1.$$

Of course a (Γ, c)-pseudoequivariant structure on E defines a genuine Γ-equivariant structure on E'. Moreover two Γ-pseudoequivariant structures on E for two cocycles c and c' define the same Γ-equivariant structure on E' if and only if there exists a function $f : G \to Z'$ such that the corresponding pseudorepresentations σ and σ' in $\Gamma \to \mathrm{Aut}_\Gamma(E)$ are related by $\sigma' = f\sigma$, and

$$c'(\gamma, \gamma') = f(\gamma\gamma')f(\gamma)^{-1}f(\gamma')^{-1}c(\gamma, \gamma'). \tag{3.1}$$

This defines a natural equivalence relation in the set of (Γ, c)-pseudoequivariant structures on E, whose equivalence classes are parametrised by the cohomology group $H^2(\Gamma, Z')$.

Remark 3.1. Of course if $Z' = Z$, $G/Z' = \mathrm{Ad}(G)$ and $E' = P(E) := E/Z$.

There is an alternative way of thinking of a (Γ, c)-pseudoequivariant structure as a genuine equivariant structure on E for the action of a larger group. Namely, a 2-cocycle c defines an extension of groups

$$1 \to Z' \to \Gamma_c \to \Gamma \to 1.$$

Two cocycles are cohomologous if and only if the corresponding extensions are equivalent, i.e. equivalence classes of extensions of Γ by Z' with trivial action of Γ on Z' are parametrised by $H^2(\Gamma, Z')$.

We have the following.

Proposition 3.2. *(Γ, c)-pseudoequivariant structures on E are in bijection with central Γ_c-equivariant structures on E, where Γ_c acts on X via the projection $\Gamma_c \to \Gamma$, and by central we mean that the action of Z' in the kernel of the extension above is the natural action of Z' on E.*

Proof. It follows from group representation theory (see [26] for example) that a pseudorepresentation $\Gamma \to \mathrm{Aut}_\Gamma(E)$ with cocycle c is equivalent to a representation $\rho : \Gamma_c \to \mathrm{Aut}_{\Gamma_c}(E)$ fitting in the following commutative diagram, where $\widetilde{\rho}$ is the induced representation

$$\begin{tikzcd}
0 \arrow[r] & Z' \arrow[r] \arrow[d, "\mathrm{Id}"'] & \Gamma_c \arrow[r] \arrow[d, "\rho"'] & \Gamma \arrow[r] \arrow[d, "\tilde{\rho}"'] & 1 \\
1 \arrow[r] & Z' \arrow[r] & \mathrm{Aut}_{\Gamma_c}(E) \arrow[r] & \mathrm{Aut}_\Gamma(E') \arrow[r] & 1.
\end{tikzcd}$$

This completes the proof. ∎

Recall that a G-bundle E is said to be **simple** if $\mathrm{Aut}(E) \cong Z$. We have the following.

Proposition 3.3. *Let E be a simple G-bundle over X such that $E \cong \gamma^* E$ for every $\gamma \in \Gamma$. Then E admits a (Γ, c)-pseudoequivariant structure with $c \in Z^2(\Gamma, Z)$.*

Proof. The condition $E \cong \gamma^* E$ for every $\gamma \in \Gamma$ implies that we have an exact sequence

$$1 \to \mathrm{Aut}(E) \to \mathrm{Aut}_\Gamma(E) \to \Gamma \to 1.$$

Now, since E is simple $\mathrm{Aut}(E) \cong Z$ and hence we have an extension

$$1 \to Z \to \mathrm{Aut}_\Gamma(E) \to \Gamma \to 1.$$

This extension is determined by a cocycle $c \in Z^2(\Gamma, Z)$, which is precisely the obstruction to having a proper Γ-equivariant structure on E, i.e. a homomorphism $\Gamma \to \mathrm{Aut}_\Gamma(E)$ splitting the exact sequence. However we have a pseudohomomorphism of Γ in $\mathrm{Aut}_\Gamma(E)$ with cocycle c, that is a (Γ, c)-pseudoequivariant structure. □

3.2. **Isotropy subgroups associated to a (Γ, c)-pseudoequivariant structure.** Let $x \in X$, and

$$\Gamma_x := \{\gamma \in \Gamma \ \ : \ \ \gamma(x) = x\}$$

be the corresponding **isotropy subgroup**. Let $\mathscr{P} = \{x \in X \ : \ \Gamma_x \neq \{1\}\}$.

Proposition 3.4. *(1) $\mathscr{P}$ consists of a finite number of points $\{x_1, \ldots, x_r\} \subset X$.*

(2) For each $x_i \in \mathscr{P}$ the isotropy subgroup $\Gamma_{x_i} \subset \Gamma$ is cyclic.

A (Γ, c)-pseudoequivariant structure on a G-bundle $\pi : E \to X$ determines the following. For each $x \in \mathscr{P}$ and $e \in E$ such that $\pi(e) = x$, there is a pseudohomomorphism

$$\sigma_e : \Gamma_x \to G$$

defined by

$$\widetilde{\gamma}(e) = e\sigma_e(\gamma). \tag{3.2}$$

One can easily check the following

Proposition 3.5. *(1) The 2-cocycle of σ_e is the restriction of $c \in Z^2(\Gamma, Z')$ to Γ_x, and hence an element of $Z^2(\Gamma_x, Z')$.*

(2) Let $e' \in \pi^{-1}(x)$, with $e' = eg$ for $g \in G$. Then $\sigma_{e'}(\gamma) = g^{-1}\sigma_e(\gamma)g$.

(3) Let Γ_x be of order n and γ be a generator of Γ_x. Then $\sigma_e(\gamma^n) = \zeta(\gamma) \in Z'$, with

$$\zeta(\gamma) = \prod_{i=1}^{n-1} c(\gamma, \gamma^i).$$

The composition of σ_e with the projection $G \to G/Z'$, defines a homomorphism $\rho_e : \Gamma_x \to G/Z'$. Of course if c is trivial, i.e., if we have a genuine Γ-equivariant structure on E, then σ_e itself is a homomorphism.

Let $c \in Z^2(\Gamma_x, Z')$ define the set

$$R_c(\Gamma_x, G) := \{\text{pseudohomomorphisms } \ \Gamma_x \to G \ \text{ with cococycle } c\}/G,$$

where G is acting by conjugation.

From Proposition 3.5 we have the following.

Proposition 3.6. *A (Γ, c)-pseudoequivariant structure on E defines for every $x \in \mathscr{P}$ an element $\sigma_x \in R_c(\Gamma_x, G)$, where we are considering for every x, the restriction of the cocycle $c \in Z^2(\Gamma, Z')$ to Γ_x.*

The following is clear.

Proposition 3.7. *Let c and c' be 2-cocycles in $Z^2(\Gamma, Z')$. Let $\sigma_x \in R_c(\Gamma_x, G)$ and $\sigma'_x \in R_{c'}(\Gamma_x, G)$ the corresponding classes. Then the projections of σ_x and σ'_x in $R(\Gamma_x, G/Z') := \mathrm{Hom}(\Gamma_x, G/Z')/(G/Z')$ coincide.*

The next result shows that the Γ action defines a bijection between spaces of pseudorepresentations of isotropy groups over points in X related by a deck transformation.

Proposition 3.8. *The action of Γ on X induces an action on $\mathscr{P}$. Let $\mathscr{Q} = \mathscr{P}/\Gamma$. If x and x' are in the same class in $\mathscr{Q}$ there is a bijection between $R_c(\Gamma_x, G)$ and $R_c(\Gamma_{x'}, G)$ under which σ_x and $\sigma_{x'}$ are in correspondence. This bijection induces a canonical bijection $R(\Gamma_x, G/Z') \to R(\Gamma_{x'}, G/Z')$.*

Proof. There exists $\gamma_0 \in \Gamma$ such that $x' = \gamma_0 \cdot x$ and so $\Gamma_{x'} = \gamma_0 \Gamma_x \gamma_0^{-1}$. Let $\tilde{\gamma}_0$ denote the lift of γ_0 to $\mathrm{Aut}_\Gamma(E)$. Given any e_x in the fibre E_x, let $e_{x'} := \tilde{\gamma}_0(e_x)$. For any $\gamma \in \Gamma_x$, let $\gamma' = \gamma_0 \gamma \gamma_0^{-1}$ be the corresponding element of $\Gamma_{x'}$ and let $\tilde{\gamma}$ and $\tilde{\gamma}' = \tilde{\gamma}_0 \tilde{\gamma} \tilde{\gamma}_0^{-1}$ denote the respective lifts to $\mathrm{Aut}_\Gamma(E)$. Using (3.2) we have

$$\tilde{\gamma}(e_x) = e_x \sigma_{e_x}(\gamma) \quad \text{and} \quad \tilde{\gamma}'(e_{x'}) = e_{x'} \sigma_{e_{x'}}(\gamma').$$

Therefore

$$e_{x'} \sigma_{e_{x'}}(\gamma') = \tilde{\gamma}'(e_{x'}) = \tilde{\gamma}_0 \tilde{\gamma} \tilde{\gamma}_0^{-1}(e_{x'}) = \tilde{\gamma}_0 \tilde{\gamma}(e_x) = \tilde{\gamma}_0 e_x \sigma_{e_x}(\gamma) = e_{x'} \sigma_{e_x}(\gamma)$$

and so $\sigma_{e_{x'}}(\gamma') = \sigma_{e_{x'}}(\gamma_0 \gamma \gamma_0^{-1}) = \sigma_{e_x}(\gamma)$. Therefore we see that σ_{e_x} determines $\sigma_{e_{x'}}$ and vice versa, and so the same is true for σ_x and $\sigma_{x'}$.

Therefore, a choice of γ_0 such that $x' = \gamma_0 \cdot x$ determines a bijection $R_c(\Gamma_x, G) \to R_c(\Gamma_{x'}, G)$ mapping pseudohomomorphisms $\sigma \mapsto \sigma'$ with $\sigma'(\gamma') := \sigma(\gamma_0^{-1} \gamma' \gamma_0)$, and this bijection maps σ_x to $\sigma_{x'}$.

A pseudohomomorphism $\sigma : \Gamma_x \to G$ descends to a homomorphism $\bar{\sigma} : \Gamma_x \to G/Z'$. The bijection $\sigma \mapsto \sigma'$ defined above induces a map $\bar{\sigma} \mapsto \bar{\sigma}'$ defined by

$$\bar{\sigma}'(\gamma') := \bar{\sigma}(\gamma_0^{-1} \gamma' \gamma_0).$$

Given any other choice γ_1 such that $\Gamma_{x'} = \gamma_1 \Gamma_x \gamma_1^{-1}$, we have $\gamma_1 \gamma_0^{-1} \in \Gamma_{x'}$ and so (since $\bar{\sigma}$ is a homomorphism) for any $\gamma' \in \Gamma_{x'}$ we have

$$\bar{\sigma}(\gamma_1^{-1} \gamma' \gamma_1) = \bar{\sigma}(\gamma_1^{-1} \gamma_0) \bar{\sigma}(\gamma_0^{-1} \gamma' \gamma_0) \bar{\sigma}(\gamma_1^{-1} \gamma_0)^{-1}.$$

Therefore the conjugacy class of $\bar{\sigma}'$ in $R(\Gamma_{x'}, G/Z')$ is well-defined and independent of the choice of γ_0 such that $\Gamma_{x'} = \gamma_0 \Gamma_x \gamma_0^{-1}$. $\square$

3.3. **Γ-pseudoequivariant structures on associated vector bundles.** Let now V be a rank n holomorphic complex vector bundle over X. Similarly as above, one can define a (Γ, c)-pseudoequivariant structure on V by considering a cocycle $c \in Z^2(\Gamma, \mathbb{C}^*)$, where we identify $\mathbb{C}^*$ with the centre of $\mathrm{GL}(n, \mathbb{C})$, the structure group of V. This is simply a pseudorepresentation of Γ in $\mathrm{Aut}_\Gamma(V)$, the group of holomorphic automorphisms of V covering Γ, with cocycle c.

Let E be a principal G-bundle and $\rho : G \to GL(\mathbb{V})$ a representation of G in a complex vector space $\mathbb{V}$. Consider the associated vector bundle $V := E(\mathbb{V})$. Let $c \in Z^2(\Gamma, Z')$ and $c_\rho \in Z^2(\Gamma, \mathbb{C}^*)$ be the cocycle induced by $\rho|_{Z'} : Z' \to \mathbb{C}^* \cong Z(\mathrm{GL}(\mathbb{V}))$. Since there is a homomorphism $\mathrm{Aut}_\Gamma(E) \to \mathrm{Aut}_\Gamma(V)$, it is clear that a (Γ, c)-pseudoequivariant structure on E defines a (Γ, c_ρ)-pseudoequivariant structure on V. In particular if $Z' \subset \ker \rho$, then c_ρ is trivial and hence we have a genuine Γ-equivariant structure on V.

4. Pseudoequivariant structures on Higgs bundles and fixed points

In this section X is a compact Riemann surface of genus bigger than one, $\Gamma \subset \mathrm{Aut}(X)$ is a subgroup of the group of holomorphic automorphisms of X, and G is a connected real reductive Lie group. As in Section 2.3, we fix a maximal compact subgroup H of G. The Lie algebra $\mathfrak{g}$ of G is equipped with an involution θ that gives the Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H. Consider the action of Γ on the moduli space of G-Higgs bundles $\mathcal{M}(X, G)$ given by the rule:

$$\gamma \cdot (E, \varphi) = (\gamma^* E, \gamma^* \varphi),$$

for $(E, \varphi) \in \mathcal{M}(X, G)$ and $\gamma \in \Gamma$. Our task in this section is to identify the fixed points $\mathcal{M}(X, G)^\Gamma$ for this action.

4.1. Γ-pseudoequivariant structures on G-Higgs bundles.

Let (E, φ) be a G-Higgs bundles over X. We will define now pseudoequivariant structures on (E, φ). To do this, let $Z = Z(H^{\mathbb{C}})$, and let $Z' \subset Z$. Choose a 2-cocycle $c \in Z^2(\Gamma, Z')$. Recall from Section 3.3, that this defines a 2-cocycle $c_\iota \in Z^2(\Gamma, \mathbb{C}^*)$, via the isotropy representation $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$.

A (Γ, c)**-pseudoequivariant structure on** (E, φ) is a (Γ, c)-pseudoequivariant structure on E, such that for every $\gamma \in \Gamma$ the following diagram commutes:

$$\begin{array}{ccc} E(\mathfrak{m}^{\mathbb{C}}) \otimes K & \xrightarrow{\widetilde{\gamma}} & E(\mathfrak{m}^{\mathbb{C}}) \otimes K \\ \Big\uparrow \varphi & & \Big\uparrow \varphi \\ X & \xrightarrow{\gamma} & X \end{array} ,$$

where $\widetilde{\gamma}$ is the collection of maps defining the (Γ, c_ι)-pseudoequivariant structure on $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$ induced by the (Γ, c)-pseudoequivariant structure of E (see Section 3.3), and the natural Γ-equivariant structure on K defined by the action of Γ on X.

Remark 4.1. If $\chi : \Gamma \to \mathbb{C}^*$ is a character of Γ, we can consider a more general Γ-pseudoequivariant structure on (E, φ) by replacing the above condition on the Higgs field by the commutativity of the following diagram

$$\begin{array}{ccc} E(\mathfrak{m}^{\mathbb{C}}) \otimes K & \xrightarrow{\widetilde{\gamma}} & E(\mathfrak{m}^{\mathbb{C}}) \otimes K \\ \Big\uparrow \varphi & & \Big\uparrow \chi(\gamma) \cdot \varphi, \\ X & \xrightarrow{\gamma} & X \end{array}$$

where by $\chi(\gamma)\cdot\varphi$ we mean the multiplication of φ by $\chi(\gamma)\in\mathbb{C}^*$. Of course if Γ is cyclic of order n we can take $\chi(\gamma)=\exp(\frac{2\pi i}{n})$, identifying Γ with the group of n-th roots of unity. This pseudoequivariant structures will relate to the action of Γ on $\mathcal{M}(G,X)$ given by

$$(E,\varphi)\mapsto(\gamma^*E,\chi(\gamma)\cdot\gamma^*\varphi).$$

The notion of stability for G-Higgs bundles given in Section 2.3 (see [14]) can be extended in a natural way to a G-Higgs bundle equipped with a pseudoequivariant structure. Everything is exactly the same except that the reductions to the parabolic subgroups $P_s\subset H^{\mathbb{C}}$ have to be Γ-equivariant. For this it is important to observe that a pseudoequivariant structure on E defines actually a Γ action on the space of reductions of E to P_s. In fact we have the following more general result.

Lemma 4.2. *Let E be a $H^{\mathbb{C}}$-bundle over X equipped with a (Γ,c)-pseudoequivariant structure, where $c\in Z^2(\Gamma,Z')$, and let $H'\subset H^{\mathbb{C}}$ be a subgroup such that $Z'\subset H'$. Then the pseudoequivariant structure on E induces a a group action of Γ on the space of reductions of E to H'.*

Proof. Recall that a reduction of E to H' is a section of $E(H^{\mathbb{C}}/H')$, the $H^{\mathbb{C}}/H'$-bundle associated to E via the natural action of $H^{\mathbb{C}}$ on $H^{\mathbb{C}}/H'$ on the left. Such a section is equivalent to a map $\psi:E\to H^{\mathbb{C}}/H'$ such that $\psi(eh)=h^{-1}e$, for every $e\in E$ and $h\in H^{\mathbb{C}}$. Now, let $\gamma\in\Gamma$ and define

$$\gamma\cdot\psi(e):=\psi(\widetilde{\gamma}(e)),$$

where $\widetilde{\gamma}$ is given by the (Γ,c)-pseudoequivariant structure on E. We need to check that $(\gamma\gamma')\cdot\psi=\gamma\cdot(\gamma'\cdot\psi)$ for every $\gamma,\gamma'\in\Gamma$. We have

$$((\gamma\gamma')\cdot\psi)(e)=\psi(\widetilde{\gamma\gamma'}(e))=\psi(c(\gamma,\gamma')\widetilde{\gamma}(\widetilde{\gamma}'(e)))=c(\gamma,\gamma')^{-1}\psi(\widetilde{\gamma}(\widetilde{\gamma}'(e)))$$

but since $Z'\subset H'$

$$c(\gamma,\gamma')^{-1}\psi(\widetilde{\gamma}(\widetilde{\gamma}'(e)))=\psi(\widetilde{\gamma}(\widetilde{\gamma}'(e))=(\gamma\cdot(\gamma'\cdot\psi))(e).$$

□

Remark 4.3. An alternative way of proving Lemma 4.2 is to consider the bundle $E'=E/Z'$ with structure group $H^{\mathbb{C}}/Z'$. If H' is a subgroup of $H^{\mathbb{C}}$ with $Z'\subset H'$ then $E(H^{\mathbb{C}}/H')=E'(H^{\mathbb{C}}/H')$. But now since E' has a Γ-equivariant structure we can define an Γ action on $E'(H^{\mathbb{C}}/H')$ and hence on $E(H^{\mathbb{C}}/H')$.

To define the moduli space we fix the cocycle $c\in Z^2(\Gamma,Z')$ and the elements $\sigma_i\in R_c(\Gamma_{x_i},H^{\mathbb{C}})$ for every point $x_i\in\mathscr{P}$ defined by Proposition 3.6. Let $\sigma=(\sigma_1,\cdots,\sigma_r)$. We define $\mathcal{M}(X,G,\Gamma,c,\sigma)$ to be the **moduli space $\mathcal{M}(X,G,\Gamma,c,\sigma)$ of polystable (Γ,c)-pseudoequivariant G-Higgs bundles with fixed classes σ.**

Given a (Γ,c)-pseudoequivariant G-Higgs bundle (E,φ) such that $Z'\subset H$, by Lemma 4.2 we consider the action of Γ on the space of metrics on E, that is on the space of sections of $E(H^{\mathbb{C}}/H)$. The analysis done for the Hitchin–Kobayashi correspondence given in Section 2.3 can be extended to this equivariant situation (see [14] to prove the following).

Theorem 4.4. *Let (E,φ) be a G-Higgs bundle over a Riemann surface X equipped with a (Γ,c)-equivariant structure, with cocycle $c\in Z^2(\Gamma,Z')$ and $Z'\subset H$. Then (E,φ) is* *polystable as a (Γ, c)-equivariant Higgs bundle if and only if there exists a Γ-invariant reduction h of the structure group of E from $H^{\mathbb{C}}$ to H, such that*

$$F_h - [\varphi, \tau_h(\varphi)] = 0. \tag{4.1}$$

From Theorems 4.4 and 2.5 we conclude the following.

Proposition 4.5. *Let $Z' \subset Z \cap H$ and $c \in Z^2(\Gamma, Z')$. Then the forgetful map defines a morphism $\mathcal{M}(X, G, \Gamma, c, \sigma) \to \mathcal{M}(X, G)$.*

4.2. **Fixed points.**

Theorem 4.6. *Let $Z' \subset Z \cap H$ and $\widetilde{\mathcal{M}}(X, G, \Gamma, c, \sigma)$ be the image of the morphism in Proposition 4.5. Then*

(1) If c and c' are cohomologous cocycles in $Z^2(\Gamma, Z')$

$$\widetilde{\mathcal{M}}(X, G, \Gamma, c, \sigma) = \widetilde{\mathcal{M}}(X, G, \Gamma, c', \sigma').$$

(2) For any $Z' \subset Z$ and any cocycle $c \in Z^2(\Gamma, Z')$

$$\widetilde{\mathcal{M}}(X, G, \Gamma, c, \sigma) \subset \mathcal{M}(X, G)^{\Gamma}.$$

(3) Let $\mathcal{M}_*(X, G) \subset \mathcal{M}(X, G)$ the subvariety of G-Higgs bundles which are stable and simple and let $Z' = Z \cap \ker \iota$

$$\mathcal{M}(X, G)^{\Gamma}_* \subset \bigcup_{[c] \in H^2(\Gamma, Z'), [\sigma] \in R(\Gamma_x, H^{\mathbb{C}}/Z')} \widetilde{\mathcal{M}}(X, G, \Gamma, c, \sigma).$$

Proof. To prove (1), we consider the function $f : G \to Z'$ such that c and c' are related by (3.1). This function defines an automorphism of a G-Higgs bundle (E, φ) which sends the pseudoequivariant structure with cocycle c and isotropy σ to a pseudoequivariant structure with cocycle c' and isotropy σ'. The proof of (2) follows immediately from the definition of pseudoequivariant structure. The proof of (3) follows a similar argument to that of Proposition 3.3: The condition $(E, \varphi) \cong (\gamma^* E, \gamma^* \varphi)$ for every $\gamma \in \Gamma$ implies the existence of an exact sequence

$$1 \to \mathrm{Aut}(E, \varphi) \to \mathrm{Aut}_{\Gamma}(E, \varphi) \to \Gamma \to 1,$$

where $\mathrm{Aut}(E, \varphi)$ and $\mathrm{Aut}_{\Gamma}(E, \varphi)$ are the group of automorphisms of (E, φ) covering the identity and Γ respectively. Since we are assuming that (E, φ) is simple $\mathrm{Aut}(E, \varphi) \cong Z' = Z \cap \ker \iota$ and hence we have an extension

$$1 \to Z' \to \mathrm{Aut}_{\Gamma}(E, \varphi) \to \Gamma \to 1.$$

This extension defines a cocycle $c \in Z^2(\Gamma, Z')$, and a pseudohomomorphism $\Gamma \to \mathrm{Aut}_{\Gamma}(E, \varphi)$ with cocycle c, i.e., a (Γ, c)-pseudoequivariant structure on (E, φ). It follows from (1) that the union should run over $[c] \in H^2(\Gamma, Z')$ and $[\sigma] \in R(\Gamma_x, H^{\mathbb{C}}/Z')$. $\square$

5. Equivariant structures and parabolic Higgs bundles

As in the previous section, let X be a compact Riemann surface, let $\Gamma \subset \mathrm{Aut}(X)$ be a finite subgroup, let $Y := X/\Gamma$ and $\pi_Y : X \to Y$ be the associated ramified covering map. The set of points $\mathscr{P} \subset X$ maps by π_Y to a set $\mathscr{S} \subset Y$. In this section we establish a correspondence between Γ-equivariant G-Higgs bundles over X and parabolic G-Higgs bundles over Y with parabolic points $\mathscr{S}$. This extends the well-known correspondences for vector bundles [22, 13, 24, 5, 2, 1], and principal bundles [33, 3]. In particular this implies that if $Z' = Z \cap \ker \iota$ and a G-Higgs bundle (E, φ) is equipped with a (Γ, c)-pseudoequivariant structure with $c \in Z^2(\Gamma, Z')$, then (E', φ) with $E' := E/Z'$ is a $G' = G/Z'$-Higgs bundle with a Γ-equivariant structure and hence is in correspondence with a parabolic G'-Higgs bundle over Y. It would be very interesting to give a parabolic description of the pseudoequivariant structure on (E, φ).

5.1. **Parabolic G-Higgs bundles.**

In this section Y is a compact Riemann surface, and G is a connected real reductive Lie group. We keep the same notation as in the previous sections for a maximal compact subgroup, isotropy representation, etc.

Let $T \subset H$ be a Cartan subgroup, and $\mathfrak{t}$ be its Lie algebra. We consider a Weyl alcove $\mathscr{A} \subset \mathfrak{t}$ (see [4]). Recall that if W is the Weyl group we have

$$\mathscr{A} \cong T/W \cong \mathrm{Conj}(H),$$

where $\mathrm{Conj}(H)$ is the set of conjugacy classes of H. Note that in contrast to the definition of alcove in [4], here $\mathscr{A}$ may contain some walls so that it is a fundamental domain for the action of the affine Weyl group.

Let $\mathscr{S} = \{y_1, \dots, y_s\}$ be a finite set of distinct points of Y and $D = y_1 + \cdots + y_s$ be the corresponding effective divisor.

An element $\alpha \in \sqrt{-1}\mathscr{A}$ defines a parabolic subgroup of $P_\alpha \subset H^{\mathbb{C}}$ given by (2.2). Fix for every point $y_i \in \mathscr{S}$ an element $\alpha_i \in \sqrt{-1}\mathscr{A}$, and denote $\alpha = (\alpha_1, \cdots, \alpha_s)$.

A **parabolic G-Higgs bundle over $(Y, \mathscr{S})$ with weights** α is a pair (E, φ) consisting of a holomorphic $H^{\mathbb{C}}$-bundle E over Y equipped with a reduction of E_{y_i} to P_{α_i} and φ is a holomorphic section of $PE(\mathfrak{m}^{\mathbb{C}}) \otimes K(D))$, where $PE(\mathfrak{m}^{\mathbb{C}})$ is the sheaf of parabolic sections of $E(\mathfrak{m}^{\mathbb{C}})$ (see [4] for details). There is a notion of stability similar to the ones we have already seen in previous sections ([4]).

To define a moduli space one has to fix for every point $y_i \in \mathscr{S}$ the projection $\mathscr{L}_i$ of the residue of φ in $\mathfrak{m}^0_{\alpha_i}/L_{\alpha_i}$, where $\mathfrak{m}^0_{\alpha_i}$ and L_{α_i} are defined as in Section 2.3. Denote $\mathscr{L} = (\mathscr{L}_1, \cdots, \mathscr{L}_s)$. We define $\mathcal{M}(Y, \mathscr{S}, G, \alpha, \mathscr{L})$ to be the **moduli space of parabolic G-Higgs bundles on $(Y, \mathscr{S})$ with weights $\alpha = (\alpha_1, \cdots, \alpha_s)$ and residues $\mathscr{L} = (\mathscr{L}_1, \cdots, \mathscr{L}_s)$**.

5.2. **Γ-equivariant Higgs bundles and parabolic Higgs bundles.**

In this section we describe the correspondence between parabolic G-Higgs bundles on Y and Γ-equivariant G-Higgs bundles on X. For holomorphic vector bundles over a compact Riemann surface, this correspondence originated in [13] and was generalised to higher dimensions in [5]. The extension to Higgs vector bundles was carried out in [24], and for holomorphic principal bundles this correspondence is contained in [33] and [3].

First we begin with the data of a compact Riemann surface X and a finite subgroup $\Gamma \subset \operatorname{Aut}(X)$. Applying the smoothing process of [9, Sec. 2] to the orbifold X/Γ determines a compact Riemann surface Y and a holomorphic map $\pi_Y : X \to Y$ such that Γ is the group of deck transformations of the ramified cover π. Let $\{x_1, \ldots, x_r\}$ denote the ramification points of π and let $D = y_1 + \cdots + y_s$ denote the branch divisor. Each ramification point x_j has a non-trivial isotropy group denoted $\Gamma_{x_j} \subset \Gamma$ which is cyclic of order N_j. Let $N = |\Gamma|$ denote the order of the ramified cover $\pi_Y : X \to Y$.

Let $E \to X$ be a principal $H^{\mathbb{C}}$ bundle, and choose a lift of Γ to the group of C^∞ automorphisms of E. Via this lift, each isotropy group $\Gamma_{x_j} \cong \mathbb{Z}_{N_j}$ acts on the fibre E_{x_j} which determines a representation in $\sigma_j \in R(\Gamma_{x_j}, H^{\mathbb{C}})$ (note that since we are considering equivariant rather than pseudo-equivariant bundles then the cocycle $c \in Z^2(\Gamma, Z')$ is trivial).

Let $\mathscr{C}_{x_j} \in \operatorname{Conj}(H)$ denote the conjugacy class of the generator γ_{x_j} of Γ_{x_j}, which is determined by the representation σ_j. Under the bijection between $\operatorname{Conj}(H)$ and a Weyl alcove $\mathscr{A}$ of H (see [4]) we thus have that each conjugacy class $\mathscr{C}_{x_j}$ corresponds to a weight $\alpha_j \in \sqrt{-1}\mathscr{A}$. Since $|\Gamma_{x_j}| = N_j$ then $e^{2\pi i N_j \alpha_j} = \mathrm{id} \in H^{\mathbb{C}}$. In the following we will always choose the weights α_j in the interior of the Weyl alcove $\sqrt{-1}\mathscr{A}$.

Given a branch point $y \in Y$ and two points $x, x' \in \pi^{-1}(y)$, there is a deck transformation $\gamma \in \Gamma$ such that $x' = \gamma \cdot x$, and the lift of γ to the group of automorphisms of E determines a map on the fibres $\gamma : E_x \to E_{x'}$. Moreover, the isotropy groups are conjugate $\Gamma_{x'} = \gamma \Gamma_x \gamma^{-1}$ and so the conjugacy classes $\mathscr{C}_x$ and $\mathscr{C}_{x'}$ are equal, and hence so are the weights in $\sqrt{-1}\mathscr{A}$ associated to these classes.

Now consider a Γ-equivariant Higgs structure on E, i.e. a holomorphic structure on E together with a Higgs field φ such that (E, φ) is preserved by the action of Γ. For each ramification point x_j, choose a small neighbourhood U_j such that the bundle is trivial $E|_{U_j} \cong U_j \times H^{\mathbb{C}}$ and the Γ-action is trivial

$$e^{\frac{2\pi i}{N_j}} \cdot (z, g) = (e^{\frac{2\pi i}{N_j}} z, e^{2\pi i \alpha_j} \cdot g) \tag{5.1}$$

(as explained in [33], the existence of this trivialisation follows from the equivariant Oka principle of [16]). We now show that after gauging by $z^{-N_j \alpha_j}$ on each trivialisation for $j = 1, \ldots, r$ then the Higgs pair (E, φ) descends to a parabolic Higgs bundle on the quotient $(X \smallsetminus \mathscr{P})/\Gamma$, where the weight at the branch point $\pi(x_j)$ is α_j. This is known for holomorphic vector bundles (cf. [13], [5]) and holomorphic principal bundles (cf. [33], [3]), and so to describe the correspondence for Higgs bundles it only remains to describe the residue of the Higgs field at each branch point in Y, which is a local computation on each neighbourhood U_j. This was worked out for Higgs vector bundles in [24], however this has not appeared in the literature for general G-Higgs bundles and so we include the details below.

Locally, the Higgs field on E has the form $\varphi(z) = f(z) dz$, where $f(z) : U_j \to \mathfrak{m}^{\mathbb{C}}$ is holomorphic. The action of $\operatorname{Ad}_{e^{2\pi i \alpha_j}}$ decomposes $\mathfrak{m}^{\mathbb{C}}$ into eigenspaces

$$\mathfrak{m}^{\mathbb{C}} = \bigoplus_{\beta} \mathfrak{m}^{\mathbb{C}}_\beta$$

where $\mathfrak{m}^{\mathbb{C}}_\beta$ denotes the eigenspace with eigenvalue $e^{2\pi i \beta}$. This decomposition depends on α_j (and hence the ramification point x_j), however to avoid an overload of notation we drop the index j for the ramification point since the meaning will be clear from the context. Note that each $N_j\beta$ is an integer since $e^{2\pi i N_j \alpha_j} = \mathrm{id}$, and since α_j is in the interior of the Weyl alcove then each eigenvalue is strictly less than one. Let $f = \bigoplus_\beta f_\beta$ be the corresponding decomposition of f. Since each f_β is holomorphic then we can write it as a power series

$$f_\beta(z) = \sum_{k=0}^{\infty} a_k^\beta z^k$$

with a_k^β taking values in $\mathfrak{m}_\beta^{\mathbb{C}}$. The induced action of $e^{\frac{2\pi i}{N_j}}$ on φ is given by

$$e^{\frac{2\pi i}{N_j}} \cdot \varphi(z) = \mathrm{Ad}_{e^{2\pi i \alpha_j}} \left(f\left(e^{\frac{2\pi i}{N_j}} z \right) \right) e^{\frac{2\pi i}{N_j}} dz.$$

Therefore, the action on the component $\varphi_\beta = f_\beta dz$ is

$$e^{\frac{2\pi i}{N_j}} \cdot f_\beta(z) dz = e^{2\pi i \beta} \sum_{k=0}^{\infty} a_k^\beta e^{\frac{2\pi i k}{N_j}} z^k \, e^{\frac{2\pi i}{N_j}} dz = \sum_{k=0}^{\infty} a_k^\beta e^{\frac{2\pi i (k+1)}{N_j}} e^{2\pi i \beta} z^k \, dz.$$

If φ is invariant under the action of $\mathbb{Z}_{N_j} \cong \Gamma_{x_j}$ then we see that $a_k^\beta \neq 0$ implies that $k = N_j \ell - N_j \beta - 1$ for some $\ell \in \mathbb{Z}$. Therefore

$$f_\beta(z) dz = \begin{cases} z^{-N_j \beta} \sum_{\ell=0}^{\infty} a^\beta_{N_j \ell - N_j \beta - 1} z^{N_j \ell} \, z^{-1} dz & \text{if } \beta < 0 \\ z^{-N_j \beta} \sum_{\ell=1}^{\infty} a^\beta_{N_j \ell - N_j \beta - 1} z^{N_j \ell} \, z^{-1} dz & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

where the two distinct cases come from the requirement that f_β is holomorphic and hence the power series has non-negative powers of z. To simplify the notation, we will use $b_\ell^\beta = a^\beta_{N_j \ell - N_j \beta - 1}$ in the sequel. On the punctured disk $U_j \setminus \{0\}$, apply the meromorphic gauge transformation $g(z) = z^{N_j \alpha_j} = e^{N_j \alpha_j \log z}$ (note that this is well-defined on the punctured neighbourhood $U_j \setminus \{x_j\}$ since $e^{2\pi i N_j \alpha_j} = \mathrm{id}$). We have $g(z) \cdot \varphi(z) = \sum_\beta g(z) \cdot f_\beta(z) dz$ where

$$g(z) \cdot f_\beta(z) dz = \begin{cases} \sum_{\ell=0}^{\infty} b_\ell^\beta z^{N_j \ell} \, z^{-1} dz & \text{if } \beta < 0 \\ \sum_{\ell=0}^{\infty} b_{\ell+1}^\beta z^{N_j \ell} \, z^{N_j - 1} dz & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

Therefore, after applying the meromorphic gauge transformation $g(z)$, the residue of $g(z) \cdot f_\beta(z)$ is zero if $\beta \geqslant 0$ and equal to b_0^β if $\beta < 0$. Now let $V = \pi(U_j) \subset Y$ and note that (5.1) implies that $\pi : U_j \to V$ is given by $z \mapsto z^{N_j}$. Then $w = z^{N_j}$ satisfies $w^{-1} dw = N_j z^{-1} dz$ and so $g(z) \cdot f_\beta(z)$ can be written as a function of w, i.e. it descends to the quotient $(U_j \setminus \{x_j\})/\Gamma_{x_j}$

$$g(z) \cdot f_\beta(z) = f'_\beta(w) = \begin{cases} \sum_{\ell=0}^{\infty} b_\ell^\beta w^\ell \, \frac{1}{N_j} w^{-1} dw & \text{if } \beta < 0 \\ \sum_{\ell=0}^{\infty} b_{\ell+1}^\beta w^\ell \, \frac{1}{N_j} dw & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

Therefore the Γ-invariant Higgs bundle (E, φ) on X defines a parabolic Higgs bundle (E', φ') on Y with Higgs field $\varphi' \in \Gamma\left(PE'(\mathfrak{m}^{\mathbb{C}}) \otimes K(D) \right)$. In particular, the residue of the Higgs field $\varphi'(w) = f'(w) dw$ is $\bigoplus_{\beta < 0} b_0^\beta$ which is nilpotent and so the projection to $\mathfrak{m}^0_{\alpha_j}/L_{\alpha_j}$ is zero.

Therefore the Γ-equivariant Higgs bundle (E, φ) on X with isotropy representations σ corresponding to weights $\alpha_j \in \sqrt{-1}\mathcal{A}$ in the interior of the Weyl alcove determines a parabolic Higgs bundle (E', φ') on Y with parabolic points $\{y_1, \ldots, y_s\} = \pi(\{x_1, \ldots, x_r\})$, conjugacy classes $\mathscr{C}'_{\pi(x_j)} = \mathscr{C}_{x_j}$ determined by α_j and a parabolic Higgs field with residues in $\bigoplus_{\beta<0} \mathfrak{m}_\beta^\mathbb{C}$, hence the residues are nilpotent. Moreover, gauge equivalent Γ-equivariant Higgs bundles on X descend to parabolic gauge-equivalent parabolic Higgs bundles on Y.

Conversely, given a parabolic G-Higgs bundle (E', φ') on Y with residues in $\bigoplus_{\beta<0} \mathfrak{m}_\beta^\mathbb{C}$ at each parabolic point $y = \pi(x_j)$, in the same way as above let V be a neighbourhood of a branch point y such that the bundle is trivial over $V \smallsetminus \{y\}$ with weight $\alpha_j \in \sqrt{-1}\mathcal{A}$ such that $e^{2\pi i N_j \alpha_j} = \mathrm{id}$. Since the residues are in $\bigoplus_{\beta<0} \mathfrak{m}_\beta^\mathbb{C}$ then the Higgs field $\varphi' \in \Gamma\left(PE'(\mathfrak{m}^\mathbb{C}) \otimes K(D)\right)$ locally has the form

$$f'_\beta(w) = \begin{cases} \sum_{\ell=0}^\infty c_\ell^\beta w^\ell \, w^{-1} dw & \text{if } \beta < 0 \\ \sum_{\ell=0}^\infty c_\ell^\beta w^\ell \, dw & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

After pulling back by the ramified covering map $z \mapsto z^{N_j} = w$, the Higgs field $\varphi(z) = f(z)dz$ upstairs has the form

$$f_\beta(z) = \begin{cases} \sum_{\ell=0}^\infty c_\ell^\beta z^{N_j \ell} \, N_j z^{-1} dz & \text{if } \beta < 0 \\ \sum_{\ell=0}^\infty c_\ell^\beta z^{N_j \ell} \, N_j z^{N_j - 1} dz & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

Applying the gauge transformation $g(z) = z^{-N_j \alpha_j}$ (once again, $e^{2\pi i N_j \alpha_j} = \mathrm{id}$ implies that this is well-defined on the punctured neighbourhood $U_j \smallsetminus \{x_j\}$) gives us

$$g(z) \cdot f_\beta(z) = \begin{cases} z^{-N_j \beta} \sum_{\ell=0}^\infty c_\ell^\beta z^{N_j \ell} \, N_j z^{-1} dz & \text{if } \beta < 0 \\ z^{-N_j \beta} \sum_{\ell=0}^\infty c_\ell^\beta z^{N_j \ell} \, N_j z^{N_j - 1} dz & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

and the same argument as before shows that this is holomorphic and invariant under the action of $\mathbb{Z}_{N_j}$ determined by $\alpha_j \in \sqrt{-1}\mathcal{A}$. Therefore the parabolic Higgs bundle on Y determines a Γ-equivariant Higgs bundle on X.

Now that we have established the correspondence, it only remains to show that the notions of stability, semistability and polystability are also in correspondence. In the case of holomorphic principal bundles, the results of [33, Sec. 2.2] show that, via the correspondence described above, a stable Γ-equivariant bundle upstairs on X corresponds to a stable parabolic bundle on Y. Moreover, the degree of any parabolic reduction of structure group on $E \to X$ is related to the parabolic degree of a parabolic reduction of structure group on $E' \to Y$ by a factor of $\frac{1}{|\Gamma|}$.

For Higgs bundles, the only modification is to restrict to reductions of structure group which are compatible with the Higgs field as described in [4, Sec. 3.2]. For the Higgs bundle (E, φ) over X, given $s \in \sqrt{-1}\mathfrak{h}$ and a Γ-invariant holomorphic reduction $\eta \in \Omega^0(E(H^\mathbb{C}/P_s))$ such that $\varphi \in H^0(X, E_\eta(\mathfrak{m}_s) \otimes K)$, the Γ-invariance of the Higgs field φ implies that the induced reduction of structure group on the parabolic bundle (E', φ') over $Y \smallsetminus \mathscr{S}$ is compatible with the Higgs field, i.e. $\varphi'|_{Y \smallsetminus \mathscr{S}} \in H^0(Y \smallsetminus \mathscr{S}, E'_\eta(\mathfrak{m}_s) \otimes K)$. Conversely, a reduction of structure group on the parabolic bundle (E', φ') over $Y \smallsetminus \mathscr{S}$ which is compatible with the Higgs field φ' lifts to a reduction of (E, φ) over X compatible with φ. Since the degree on X is related to the parabolic degree on Y by a factor of $\frac{1}{|\Gamma|}$ (cf. [33, Sec.

2.3]) then the notion of Γ-equivariant Higgs stability (resp. semistability and polystability) upstairs on X corresponds to the notion of parabolic Higgs stability (resp. semistability and polystability) downstairs on Y.

Therefore we have proved the following bijection of moduli spaces.

Theorem 5.1. *The correspondence described above defines a bijection*

$$\mathcal{M}(X, G, \Gamma, \mathrm{id}, \sigma) \to \mathcal{M}(Y, \mathscr{S}, G, \alpha, 0).$$

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Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Nicolás Cabrera, 13–15, 28049 Madrid, Spain

E-mail address: oscar.garcia-prada@icmat.es

National University of Singapore, Department of Mathematics, Block S17, 10, Lower Kent Ridge Rd, Singapore 119076

E-mail address: graeme@nus.edu.sg