Lecture 18: Spherical trigonometry and navigation

1 April, 2019

Overview

Last time.

- A model for spherical geometry
- Euclid Book I revisited.
  - What happens when we change the parallel axiom?
- The angle sum of a triangle in spherical geometry

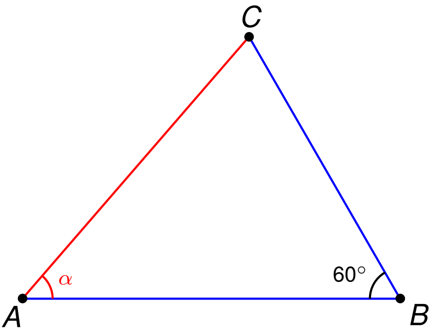
Today.

- Spherical trigonometry (the sine and cosine rule on the sphere)
- Solving triangles on the sphere
- Applications to navigation.

Exercise

Exercise. Consider the triangle ΔABC in the diagram below. We have $|AB| = 5$, $|BC| = 4$ and $\angle ABC = 60^\circ$.

Compute $|AC|$ and $\angle BAC$.



Solution

Solution.

Solution. Recall the cosine rule

$$|AC|^2 = |AB|^2 + |BC|^2 - 2|AB| \cdot |BC| \cos \angle ABC$$

Substituting in the given data for $|AB|$, $|AC|$ and $\angle ABC$ gives us

$$|AC|^2 = 5^2 + 4^2 - 2 \times 5 \times 4 \times \cos 60 \quad \Leftrightarrow \quad |AC| = \sqrt{21}$$

Now recall the sine rule

$$\frac{\sin \angle ABC}{|AC|} = \frac{\sin \angle BCA}{|AB|} = \frac{\sin \angle CAB}{|BC|}$$

Therefore we have

$$\sin \angle BAC = |BC| \times \frac{\sin \angle ABC}{|AC|} \quad \Leftrightarrow \quad \sin \angle BAC = \frac{4\sqrt{3}}{2\sqrt{21}} \approx 0.756$$

and so $\angle BAC \approx 49.11^\circ$.

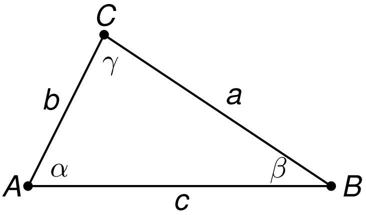
Trigonometry on the sphere

An important problem in geometry (with applications to navigation) is to understand the relationship between the sidelengths and angles of a triangle.

In Euclidean geometry we have the well-known formulas

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \text{(cosine rule)}$$

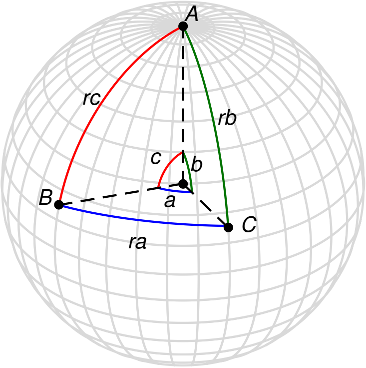
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \quad \text{(sine rule)}$$



Trigonometry on the sphere

Recall from last time that “straight lines” on the sphere are arcs of a great circle (the intersection of the sphere with a plane through the origin).

The length of an arc is the radius of the sphere multiplied by the angle (in radians) subtended by the arc at the origin.



In the diagram on the left, the blue, green and red arcs subtend angles of a, b and c radians at the origin.

The lengths of the arcs on the sphere are then ra, rb and rc.

Trigonometry on the sphere

In spherical geometry, the cosine rule and the sine rule are different due to the curvature of the sphere.

Theorem. (Spherical cosine rule and sine rule)
Consider a triangle ΔABC on the unit sphere (radius $r = 1$) and suppose that the lengths of the sides a, b, c are each less than $\frac{\pi}{2}$. Then

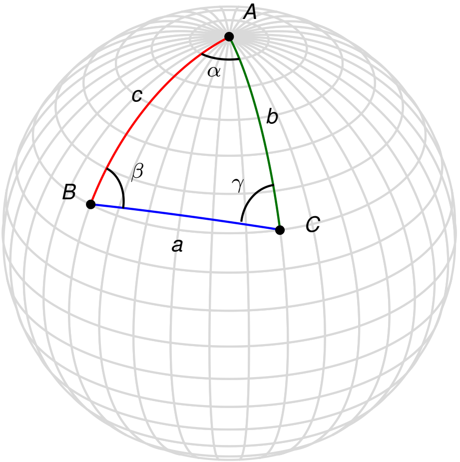
$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha \qquad \text{(spherical cosine rule)}$$

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma} \qquad \text{(spherical sine rule)}$$

To keep things simple, we consider the unit sphere where the length of an arc is equal to the angle in radians subtended by that arc at the origin.

In a real-life problem (such as navigation on the surface of the earth) to compute the length of an arc, we have to remember to multiply the angle by the radius of the sphere.

Trigonometry on the sphere



$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha \qquad \text{(spherical cosine rule)}$$

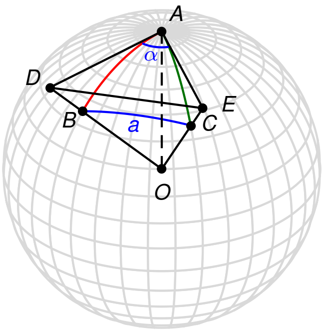
$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma} \qquad \text{(spherical sine rule)}$$

Proof. (Spherical cosine rule)
At the vertex A, draw the tangents to the lines AB and AC. Since the sidelengths b and c are both less than $\frac{\pi}{2}$ then these tangents will meet the lines OB and OC at two points D and E.

Since AD and AE are tangents to the sphere, then $\angle OAD = \frac{\pi}{2} = \angle OAE$.

Therefore (by Pythagoras) we have $|OD|^2 = |OA|^2 + |AD|^2$ and also $|OE|^2 = |OA|^2 + |AE|^2$.

Note that $\angle DOE = a$, since it subtends an arclength of a on the unit sphere. The Euclidean cosine rule applied to the triangles ΔADE and ΔODE gives us



$$|DE|^2 = |AD|^2 + |AE|^2 - 2|AD| \cdot |AE| \cos \alpha$$
$$|DE|^2 = |OD|^2 + |OE|^2 - 2|OD| \cdot |OE| \cos a$$

Trigonometry on the sphere

Proof. (Spherical cosine rule cont.)
Subtracting the two equations gives us

$$0 = 2|OA|^2 + 2|AD| \cdot |AE| \cos \alpha - 2|OD| \cdot |OE| \cos a$$

and so

$$\cos a = \frac{|OA|}{|OE|} \cdot \frac{|OA|}{|OD|} + \frac{|AE|}{|OE|} \cdot \frac{|AD|}{|OD|} \cos \alpha$$

Note that $\angle OAD = 90^\circ$ and so $\frac{|OA|}{|OD|} = \cos c$ (since $\angle AOD$ subtends the arclength c on the unit sphere).
Similarly, we get $\frac{|OA|}{|OE|} = \cos b$, $\frac{|AE|}{|OE|} = \sin b$ and $\frac{|AD|}{|OD|} = \sin c$.
Therefore

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha$$

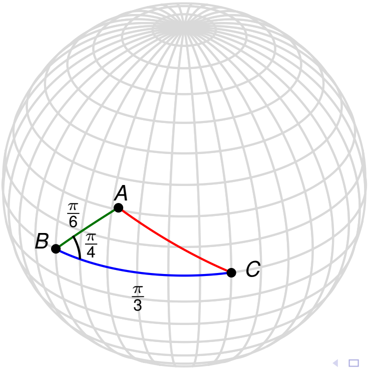
We will prove the spherical sine rule in **Tutorial 10**.

Exercise

Exercise. Consider a triangle ΔABC on the unit sphere, with $|AB| = \frac{\pi}{6}$ radians, $|BC| = \frac{\pi}{3}$ radians and $\angle ABC = \frac{\pi}{4}$ radians.

(a) Use the spherical sine and cosine rule to compute $|AC|$, $\angle CAB$ and $\angle BCA$.

(b) Use the angle excess formula from Lecture 17 to compute the area of ΔABC.



Solution

Solution.

Solution. First use the spherical cosine rule to compute $|AC|$.

$$\cos|AC| = \cos\frac{\pi}{6}\cos\frac{\pi}{3} + \sin\frac{\pi}{6}\sin\frac{\pi}{3}\cos\frac{\pi}{4}$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} \approx 0.739$$

Therefore $|AC| \approx 0.739$ radians. Now we can use the sine rule to compute $\angle CAB$ and $\angle BCA$.

$$\frac{\sin\angle CAB}{\sin|BC|} = \frac{\sin\angle BCA}{\sin|AB|} = \frac{\sin\angle ABC}{\sin|AC|} = \frac{\sin\frac{\pi}{4}}{\sin 0.739} \approx 1.05$$

Therefore $\sin\angle BCA \approx 1.05\sin\frac{\pi}{6} \approx 0.525$, so $\angle BCA \approx 0.55$ radians.

Similarly, we have $\sin\angle CAB \approx 1.05\sin\frac{\pi}{3} \approx 0.91$ and so either $\angle CAB \approx 1.14 < \frac{\pi}{2}$ or $\angle CAB \approx 2 > \frac{\pi}{2}$. If $\angle CAB = 1.14$ radians, then $\angle ABC + \angle BCA + \angle CAB = 2.48 < \pi$, which cannot happen since the angle sum is greater than π.

Therefore we must have $\angle CAB = 2$.

Solution (cont.)

Solution (cont.) Now we can compute the area using the angle excess formula on the unit sphere.

$$\text{Area}(\Delta ABC) = \angle ABC + \angle BCA + \angle CAB - \pi$$

Substituting in the values from the previous computation gives us

$$\text{Area}(\Delta ABC) = \frac{\pi}{4} + 0.55 + 2 - \pi = 0.19$$

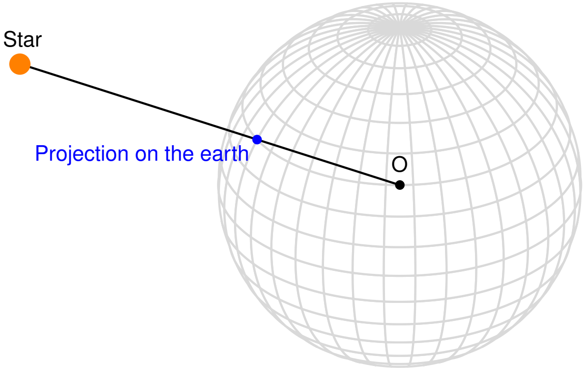
Recall that the area of the unit sphere is 4π.

Celestial Navigation using Spherical Geometry

Now suppose you are sailing on the ocean and you want to find your position on the earth. Unfortunately your GPS system is not working, but you can use your knowledge of spherical geometry together with some information about the position of the stars in the sky to locate your position on the earth.

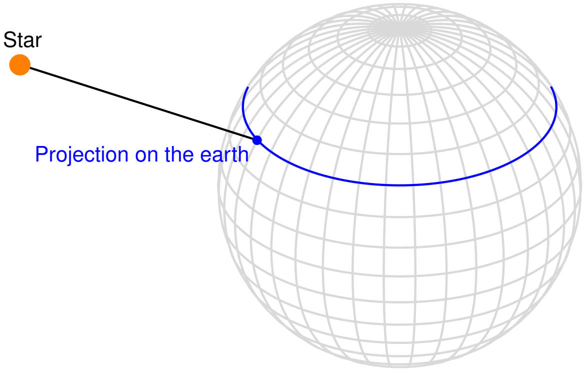
Even if your GPS system does work, do you know **how** it works? (More on this later today)

Each star in the sky corresponds to a ray from the centre of the earth. The intersection of this ray with the surface of the earth is projection of the star on the earth.



Celestial Navigation

Knowing the position of the projection of each star is essential for locating our position, however the position keeps changing as the earth rotates about its axis, and also as the earth rotates about the sun.

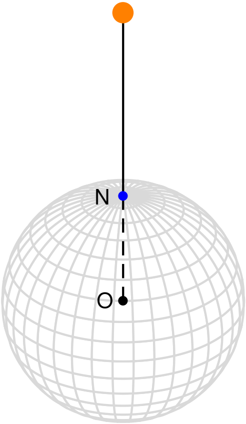


Information about the position of stars at a given time and date is available in a Nautical Almanac.

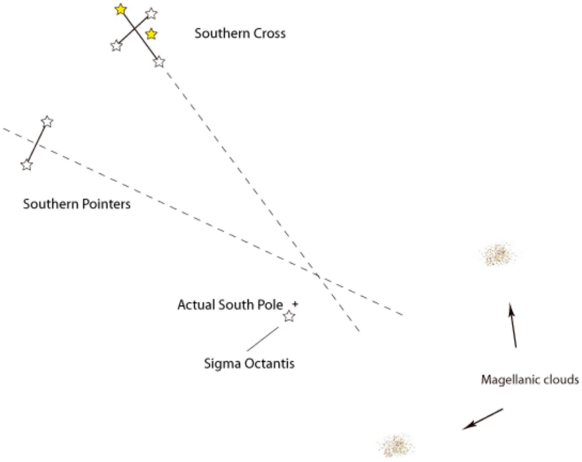
The simplest situation is when a star lies directly over one of the poles of the earth, so that the projection is independent of the rotation of the earth.

The star Polaris is a bright star (almost) directly above the north pole.

Even as the earth rotates, the projection of Polaris on the earth stays in the same position, and therefore we always know its position without needing to consult an almanac.



Finding a point above the south pole

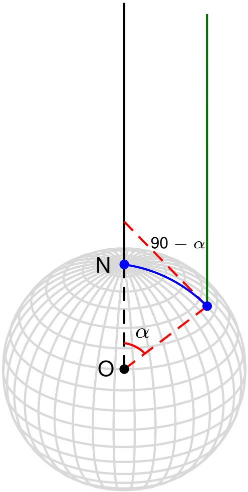


There is no bright star above the south pole, however you can locate a point above the south pole using the Southern Cross. (See the explanation here.)

Once you find the position of a star above one of the poles, you can determine your latitude by measuring the angle between that star and the horizon.

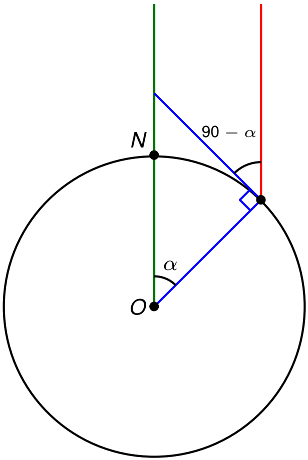
One of the most common instruments for doing this is the Sextant.

Remember, the star is very far away, so a line from you to the star will be (almost) parallel to a line from the centre of the earth to the north pole.



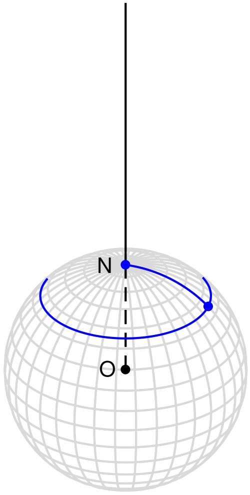
Determining your latitude

Here is a cross-sectional picture.



Therefore, by measuring the angle of the north star (Polaris) above the horizon, you know your latitude on the sphere, but not the longitude.

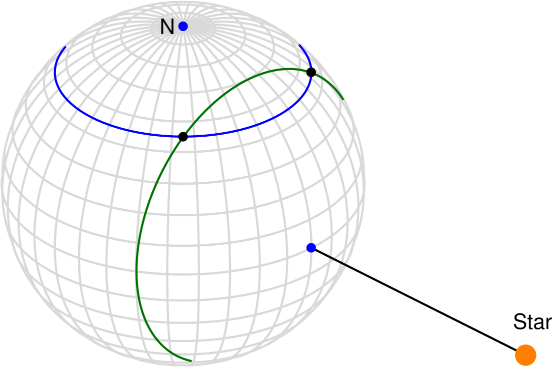
In geometric terms, you know that your position is on a circle centred on the axis through the north pole.



Given another star (for which you know the position of the projection on the earth), once again you can measure the angle that the star makes with the horizon.

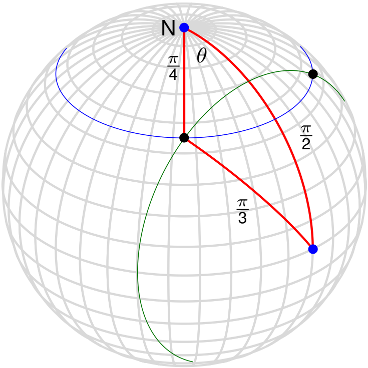
This tells you that your location is on a circle centred at the projection of the star on the earth.

Therefore, your position is at one of the points where the two circles intersect.



Question. What is an efficient way to find your position?

Exercise. Suppose you are on latitude 45°, and you measure your distance from a point of latitude 0° and longitude 90° as 60°. What is your latitude and longitude?



Solution

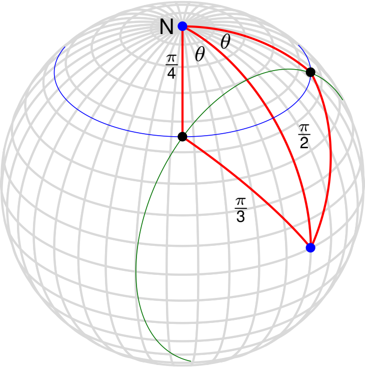
Solution.

Solution

Solution. Use the cosine rule to compute θ. We have

$$\cos\frac{\pi}{3} = \cos\frac{\pi}{2}\cos\frac{\pi}{4} + \sin\frac{\pi}{2}\sin\frac{\pi}{4}\cos\theta$$

and so $\frac{1}{2} = 0 + \frac{\sqrt{2}}{2}\cos\theta$. Therefore $\theta = \frac{\pi}{4}$ radians, or $\theta = 45°$.



The circles intersect at two points, corresponding to the two choices of orientation of the triangle.

How to choose between the two solutions?

In our previous example, we had two solutions for our possible position, due to the fact that two circles generally intersect at two points.

In a real navigation problem, you could often exclude one of the solutions based on common sense (e.g. if you are in the Pacific Ocean, then you can exclude a solution that says you are in the Atlantic Ocean).

For a more robust method, you can measure your position relative to a third star, which will then give you three circles. If all three circles intersect at a point then the intersection point is unique.

Of course, there are always small errors in your measurements, and so the three circles may not all intersect at a single point. Therefore your solution is only an approximation.

Measuring with respect to three or more stars will reduce the error in the approximation.

How does GPS work?

The GPS system in your phone uses similar calculations to those in the previous exercise.

The receiver in your phone can measure the distance from a number of satellites orbiting the earth.

This places your position on a sphere in three-dimensional space.

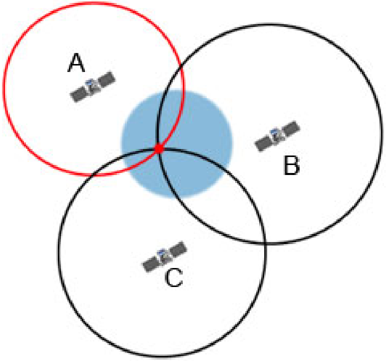
Since you know that you are on the surface of the earth, then the receiver can place your position on the intersection of this sphere with the earth, which is a circle on the surface of the earth.

Therefore, the receiver can compute your position in terms of the intersection of these spheres.

Connecting to three satellites will give you a precise position on the surface of the earth, while connecting to four satellites will also give you your altitude.

How does GPS work?

Here is a picture of the spheres from three satellites intersecting at a point on the surface of the earth.



You can read more about it here and here.

Next time

In Friday's class we will focus on construction problems related to spherical geometry. You can also ask questions about the constructions from the rest of the course.

Next week we will introduce hyperbolic geometry. Many of the ideas are analogous to what we have seen this week.

- The hyperboloid model for hyperbolic geometry
- The Poincaré disk model for hyperbolic geometry
- Failure of the parallel axiom
- Applications of hyperbolic geometry
- Relationship between angle sum and area of a triangle
- Hyperbolic trigonometry