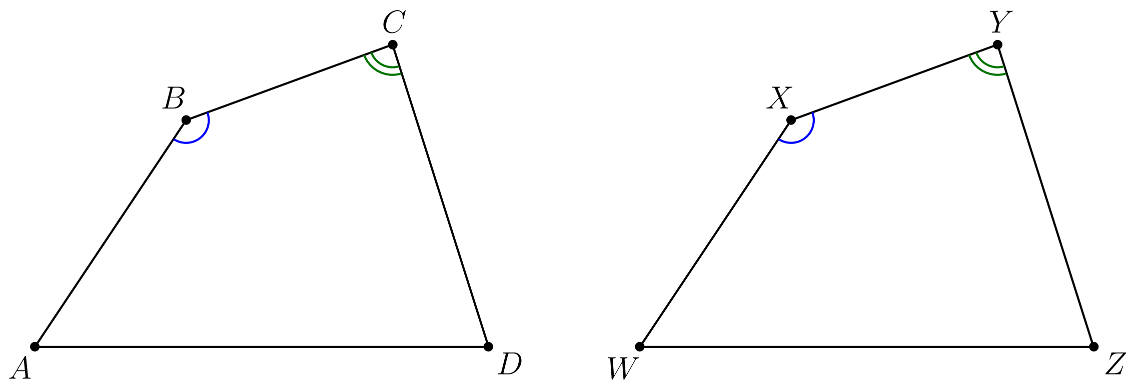
MA2219 Introduction to Geometry - Homework 1
Due in class on Friday 8 February

1. Let $ABCD$ and $WXYZ$ be quadrilaterals, and suppose that $|AB| = |WX|$, $|BC| = |XY|$, $|CD| = |YZ|$, $\angle ABC = \angle WXY$ and $\angle BCD = \angle XYZ$. Prove that $ABCD$ and $WXYZ$ are congruent (i.e. all of the corresponding sides and angles are equal).

   Remark. This result is used in Leonardo da Vinci's proof of Pythagoras' theorem from Tutorial 3.



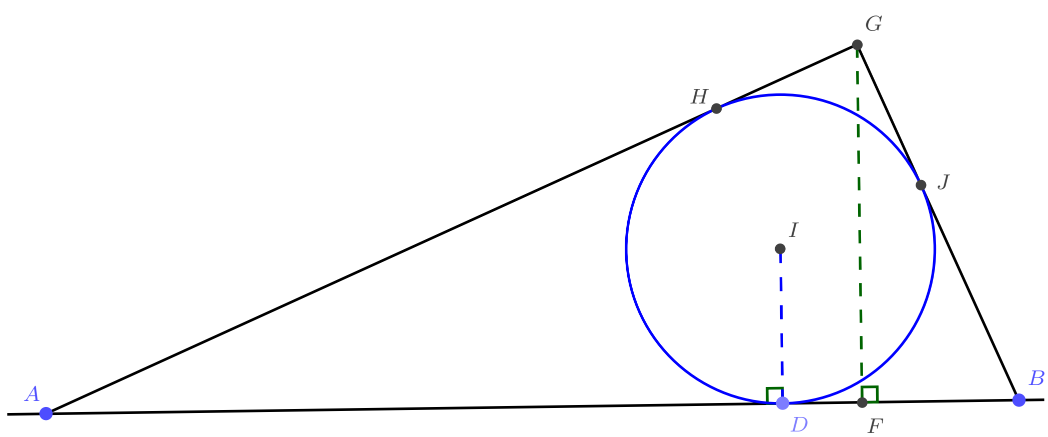
2. Consider the diagram below of a right-angled triangle with an inscribed circle centred at I. Let $x = |AD|$, $y = |DB|$ and $h = |FG|$. Recall that there are three ways to write the area of this triangle.

- As half base times height $\text{Area}(\Delta ABG) = \frac{1}{2}(x+y)h$,
- after rotating the triangle so that BG is the base, as half base times height $\text{Area}(\Delta ABG) = \frac{1}{2}|BG| \cdot |AG|$, and
- using the inradius formula from Lecture 5

$$\text{Area}(\Delta ABG) = |DI| \cdot \frac{1}{2}(|AB| + |BG| + |GA|).$$

(a) Use the second and the third method to prove that the area of the triangle is xy.

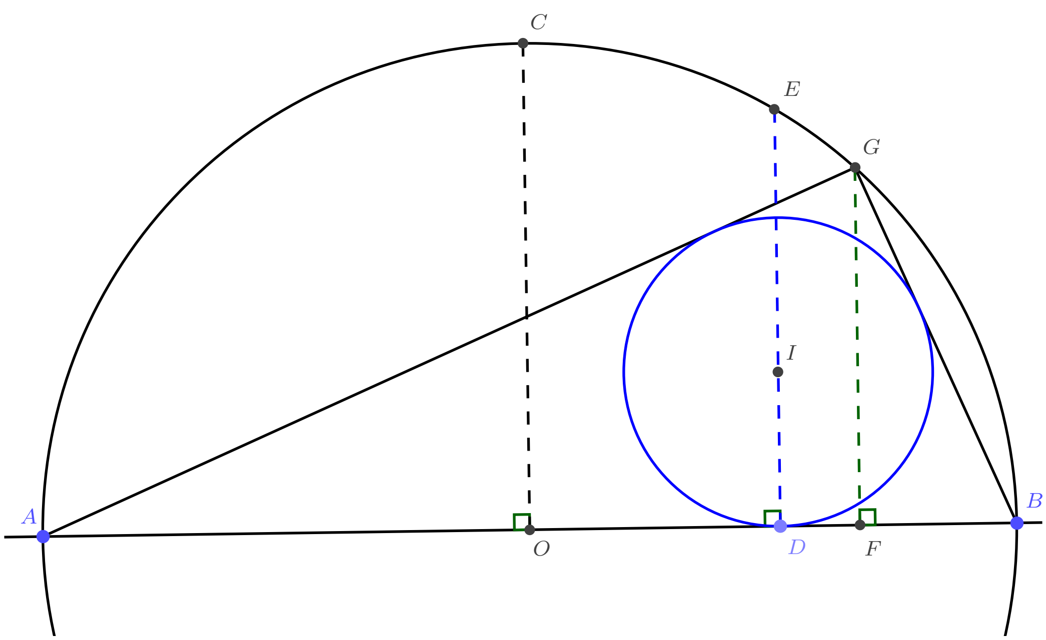
(b) Now use the first method to prove that $h = \frac{2xy}{x+y}$.



3. Given two numbers x and y, Euclid considered three ways of computing their mean.

   - The *arithmetic mean* is the average $M_a = \frac{1}{2}(x + y)$.
   - The *geometric mean* is $M_g = \sqrt{xy}$.
   - The *harmonic mean* is $M_h = \frac{2xy}{x+y} = \frac{2}{\frac{1}{x}+\frac{1}{y}}$.

   In the following diagram, let $x = |AD|$ and $y = |DB|$. Prove that $M_a = |OC|$, $M_g = |DE|$ and $M_h = |FG|$, and then prove that $M_a \geq M_g \geq M_h$.



4. Given a triangle ΔABC, choose a point X_2 on side BC. Choose points P and Q on the line AX_2, and define Y_1 and Y_2 on side CA as the intersection with the extension of BQ and BP respectively. Similarly, define Z_2 and Z_3 on side AB as the intersection with the extension of CP and CQ respectively.

   Define S as the intersection of BY_2 with CZ_3 and U as the intersection of BY_1 with CZ_2. Define X_1 as the intersection of BC with the extension of AS and define X_3 as the intersection of BC with the extension of AU. Define R to be the intersection of AX_1 with CZ_2 and define T as the intersection of BY_2 with AX_3.

   Now define Z_1 as the intersection of AB with the extension of CT and Y_3 as the intersection of CA with the extension of BR.

   Use Ceva's theorem to prove that the cevians AX_2, BY_3 and CZ_1 are concurrent (at the point V in the diagram below).

   (In the next homework we will give a different proof using Desargues' theorem.)