The reverse Yang-Mills-Higgs heat flow in a neighbourhood of a critical point

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New perspectives on Higgs bundles
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The space of Higgs bundles

Let X be a compact Riemann surface and let $E \to X$ be a smooth complex Hermitian vector bundle.

Let $\mathcal{G}^{\mathbb{C}}$ be the complex gauge group and $\mathcal{G} \subset \mathcal{G}^{\mathbb{C}}$ the subgroup preserving the Hermitian structure.

Let $\mathcal{A}^{0,1}$ be the space of holomorphic structures $\bar{\partial}_A : \Omega^0(E) \to \Omega^{0,1}(E)$ (satisfying Leibniz rule and $(\bar{\partial}_A)^2 = 0$).

Define the space of Higgs bundles

$$\mathcal{B} := \{(\bar{\partial}_A, \phi) \in \mathcal{A}^{0,1} \times \Omega^0(\mathrm{End}(E) \otimes K) \mid \bar{\partial}_A \phi = 0, \phi \wedge \phi = 0\}$$

The Yang-Mills-Higgs functional $\mathrm{YMH} : \mathcal{B} \to \mathbb{R}$ is

$$\mathrm{YMH}(\bar{\partial}_A, \phi) = \|F_A + [\phi, \phi^*]\|_{L^2}^2 = \int_X |F_A + [\phi, \phi^*]|^2 \, dvol$$

A pair $(\bar{\partial}_A, \phi)$ minimises YMH iff $i * (F_A + [\phi, \phi^*]) = \lambda \cdot \mathrm{id}$, where $\lambda = \frac{2\pi}{\mathrm{vol}(X)} \cdot \frac{\deg(E)}{\mathrm{rank}(E)} = \frac{2\pi}{\mathrm{vol}(X)} \cdot \mathrm{slope}(E)$.

The moduli space of Higgs bundles

Let $\mathcal{B}_{min} \subset \mathcal{B}$ denote the set of YMH minimisers.

A Higgs bundle $(\bar{\partial}_A, \phi)$ is stable (resp. semistable) if and only if

$$\text{slope}(F) < \text{slope}(E) \quad (\text{resp. } \leq)$$

for every proper, non-zero, ϕ-invariant subbundle $F \subset E$.

$(\bar{\partial}, \phi)$ is polystable $\Leftrightarrow$ direct sum of stable Higgs bundles of the same slope

The moduli space of semistable Higgs bundles is

$$\mathcal{M}_{ss}^{Higgs} := \mathcal{B}_{ss} // \mathcal{G}^{\mathbb{C}} \cong \mathcal{B}_{polyst.} / \mathcal{G}^{\mathbb{C}}.$$

Theorem. (Hitchin '87, Simpson '88)

$$\mathcal{M}_{ss}^{Higgs} \cong \mathcal{B}_{min} / \mathcal{G}$$

Equivalently $(\bar{\partial}_A, \phi)$ is polystable iff there exists $g \in \mathcal{G}^{\mathbb{C}}$ such that $g \cdot (\bar{\partial}_A, \phi) \in \mathcal{B}_{min}$.

Idea of Simpson's proof.

Given an initial condition $(\bar{\partial}_{A_0}, \phi_0) \in \mathcal{B}$, solve for a one-parameter family h_t of metrics satisfying a nonlinear heat equation

$$\frac{dh}{dt} + \Delta_{A_0} h = (\text{nonlinear terms in } h)$$

Equivalently, solve for a 1-parameter family $g_t \in \mathcal{G}^{\mathbb{C}}$ such that

$$\frac{dg}{dt} g_t^{-1} = -i * (F_A + [\phi, \phi^*]), \quad g_0 = \mathrm{id}\,.$$

and set $h_t = g_t^* g_t$. Then the one parameter family $(\bar{\partial}_{A_t}, \phi_t) = g_t \cdot (\bar{\partial}_{A_0}, \phi_0)$ solves the YMH heat flow

$$\frac{\partial A}{\partial t} = i\bar{\partial}_{A_t} * (F_{A_t} + [\phi_t, \phi_t^*])$$
$$\frac{\partial \phi}{\partial t} = i[\phi_t, *(F_{A_t} + [\phi_t, \phi_t^*])]$$

The Yang-Mills-Higgs heat flow (cont.)

Simpson showed that if the initial condition is polystable, then the heat flow converges (on the space of metrics) to a metric minimising YMH.

In general, critical points of YMH correspond to a direct sum decomposition $(E_1, \phi_1) \oplus \cdots \oplus (E_\ell, \phi_\ell)$ of YMH-minimisers.

Theorem. (W., '08)
For any initial condition $(\bar{\partial}_A, \phi) \in \mathcal{B}$

- the YMH heat flow converges in the smooth topology to a unique critical point,
- this critical point is isomorphic to the graded object of the Harder-Narasimhan-Seshadri filtration of $(\bar{\partial}_A, \phi)$, and
- there is a continuous deformation retract from each Harder-Narasimhan stratum to the associated critical set.

A priori the flow depends on the complex structure on X, but the above theorem is valid for any complex structure.

Question. What about flow lines between critical sets?

Yang-Mills-Higgs flow lines

Definition. A YMH flow line is a continuous map $x : \mathbb{R} \to \mathcal{B}$ given by

$$t \mapsto x_t = (\bar{\partial}_{A_t}, \phi_t)$$

which satisfies the YMH heat flow equation for all $t \in \mathbb{R}$.

A flow line connects two critical points x_u and x_ℓ iff

$$x_u = \lim_{t \to -\infty} x_t, \quad x_\ell = \lim_{t \to +\infty} x_t$$

Existing results of Swoboda ('12) and Janner-Swoboda ('15) construct flow lines for a perturbed Yang-Mills flow.

Unfortunately the perturbations destroy the algebraic structure of the flow.

Naito, Kozono, Maeda ('90) constructed the unstable manifold for the Yang-Mills flow using a contraction mapping technique.

This method uses the manifold structure of the space of connections and does not say anything about the isomorphism class of a point in the unstable manifold.

Yang-Mills-Higgs flow lines (cont.)

We would also like to study flow lines on the space of Higgs bundles (which is singular). In addition, we would like to understand the isomorphism classes of critical points connected by flow lines.

Therefore one needs to be careful about perturbing the flow or defining projections using local coordinates.

Goal.

1. Develop a method for constructing flow lines that (a) works on a singular space, and (b) allows us to classify the isomorphism classes.

2. Use this construction to give an algebraic criterion for two critical points to be connected by a YMH flow line.

3. Refine this algebraic criterion further, to give a geometric description of the flow lines between two critical sets in terms of secant varieties of the underlying Riemann surface.

Example: Rank 2 Yang-Mills flow

Example. To illustrate the idea, consider the Yang-Mills flow when $\operatorname{rank}(E) = 2$. Let $x_u = L_1^u \oplus L_2^u$ and $x_\ell = L_1^\ell \oplus L_2^\ell$ be critical points and suppose that $\deg L_1^u > \deg L_1^\ell > \deg L_2^\ell > \deg L_2^u$. (Therefore $\mathrm{YM}(x_u) > \mathrm{YM}(x_\ell)$.)

Theorem 1. If the upwards flow with initial condition $\bar{\partial}_A$ on E converges to x_u then E admits an extension

$$0 \to L_2^u \to E \to L_1^u \to 0$$

Conversely, if E admits such an extension then it is isomorphic to an initial condition which flows up to x_u.

Theorem 2. x_u and x_ℓ are connected by a flow line iff there exists $\bar{\partial}_A$ on E admitting extensions as in the diagram below.

$$\begin{tikzcd}
0 \arrow[r] & L_2^u \arrow[r] & E \arrow[r] \arrow[d, leftrightarrow, "\cong"] & L_1^u \arrow[r] & 0 \\
0 & L_2^\ell \arrow[l] & E \arrow[l] & L_1^\ell \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Example: Rank 2 Yang-Mills flow (cont.)

How to determine whether an extension $0 \to L_2^u \to E \to L_1^u \to 0$ flows down to a given critical point?

The embedding $X \hookrightarrow \mathbb{P}H^1(L_1^* L_2)$ determines the limit of the downwards flow.

Suppose $\deg L_1^u - \deg L_2^u > 2n$. Let

$$\mathrm{Sec}^n(X) := \bigcup_{p_1, \ldots, p_n \in X} \mathrm{span}\{p_1, \ldots, p_n\} \subset \mathbb{P}H^1((L_1^u)^* L_2^u)$$

be the n^{th} *secant variety* of X in $\mathbb{P}H^1((L_1^u)^* L_2^u)$.

Theorem 3. If $e \in \mathrm{span}\{p_1, \ldots, p_n\} \subset \mathrm{Sec}^n(X) \setminus \mathrm{Sec}^{n-1}(X)$, then $0 \to L_2^u \to E \to L_1^u \to 0$ with extension class e flows down to

$$L_1[-p_1 - \cdots - p_n] \oplus L_2[p_1 + \cdots + p_n]$$

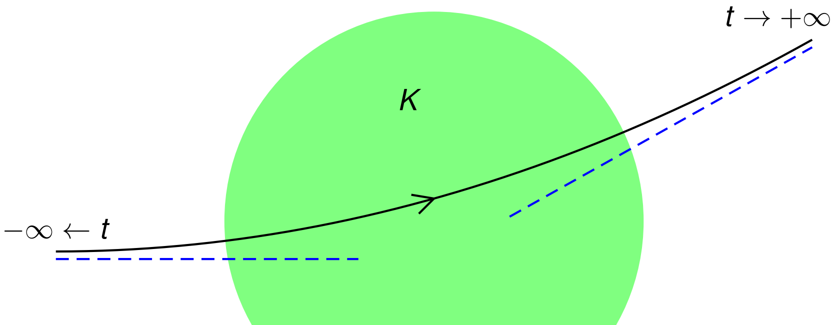
This describes *unbroken flow lines* between critical sets.

The scattering construction

An important problem in scattering theory is to study an interacting system on a scale which is large compared to the scale of the interaction itself.

For example, in the system below a particle passes through a force field which is very weak outside some compact set K.

We would like to relate the asymptotic behaviour of the particle as $t \to -\infty$ to the asymptotic behaviour as $t \to +\infty$.



Structure of critical points

The methods used to solve the scattering problem can also be applied to the problem of constructing flow lines for YMH.

At a critical point $x = (\bar{\partial}_A, \phi)$, the Higgs bundle splits as a direct sum of YMH minima $(E_1, \phi_1) \oplus \cdots \oplus (E_\ell, \phi_\ell)$ ordered so that $\text{slope}(E_j) < \text{slope}(E_k)$ if $j < k$.

The moment map $\beta = *(F_A + [\phi, \phi^*])$ is then determined by the slope of the subbundles E_j.

$$i\beta = \text{diag}(\lambda_1, \ldots, \lambda_n), \quad \lambda_j = \frac{2\pi \cdot \text{slope}(E_j)}{\text{vol}(X)}.$$

The *slice* S_x at a critical point x consists of all nearby Higgs pairs y such that $y - x$ is orthogonal to the $\mathcal{G}^{\mathbb{C}}$-orbit through x.

If $x = (\bar{\partial}_A, \phi)$ and $y = x + (a, \varphi) \in S_x$, then

$$\bar{\partial}_A^* a - *[\phi, *\varphi] = 0, \quad \bar{\partial}_A \varphi + [a, \phi] + [a, \varphi] = 0 \quad \text{(slice equations)}$$

Structure of critical points (cont.)

Given x critical, all the points in the orbit $\mathcal{G} \cdot x$ are also critical.

The unstable set of x (denoted W_x^-) consists of all points y such that the reverse flow exists and converges to x in the C^∞ topology.

For $y_0 \in S_x$, define the linearised flow by $y_t = e^{-i\beta t} \cdot y_0$.

The unstable set of the linearised flow is a subset of the slice called the negative slice and denoted S_x^-.

$$S_x^- := \{y_0 \in S_x \, : \, \lim_{t \to -\infty} e^{-i\beta t} \cdot y_0 = x\}$$

The analytic part of the theorem is that S_x^- and W_x^- are related by the action of $\mathcal{G}^{\mathbb{C}}$.

The algebraic part of the theorem involves analysing the Harder-Narasimhan types of isomorphism classes in S_x^-.

The action of the linearised flow

Example. Suppose the critical point x corresponds to a direct sum $(E_1, \phi_1) \oplus (E_2, \phi_2)$.

$$i\beta = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_1 < \lambda_2$$

If $y_0 \in S_x$ has the form

$$y_0 - x = (a, \varphi) = \left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix} \right) \in \Omega^{0,1}(\mathrm{End}(E)) \oplus \Omega^{1,0}(\mathrm{End}(E))$$

then the linearised flow acts by $y_t - x = e^{(\lambda_1 - \lambda_2)t}(y_0 - x)$.

In this case the point $y_0 \in S_x$ corresponds to a Higgs extension

$$0 \to (E_1, \phi_1) \to (E, \phi) \to (E_2, \phi_2) \to 0$$

The slice equations reduce to the condition that (a, φ) is harmonic.

The linearized flow acts by scalar multiplication on the extension class.

The action of the linearised flow (cont.)

More generally, if a critical point x has the form

$$(E_1, \phi_1) \oplus \cdots \oplus (E_\ell, \phi_\ell)$$

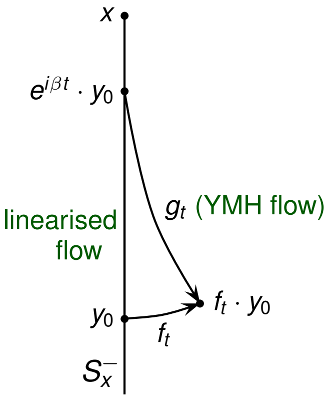
with $\text{slope}(E_j) < \text{slope}(E_k)$ iff $j < k$, and a Higgs pair $y \in S_x$ corresponds to a filtration of Higgs bundles

$$(F_1, \phi_1) \subset \cdots \subset (F_\ell, \phi_\ell)$$

such that $(F_j, \phi_j)/(F_{j-1}, \phi_{j-1}) \cong (E_j, \phi_j)$, then the linearised flow "collapses" the filtration and converges to x as $t \to \infty$.

The scattering method is to flow upwards for time t using the linearised flow (action of $e^{i\beta t}$), and then downwards for time t using the YMH heat flow (action of $g_t \in \mathcal{G}^{\mathbb{C}}$).

The goal is then to show that this process converges as $t \to \infty$.



Theorem. (W.)

- $h_t = f_t^* f_t$ converges to a smooth metric h_∞.
- There exists a smooth gauge transformation f_∞ such that $f_\infty^* f_\infty = h_\infty$ and $f_\infty \cdot y_0$ is in the unstable set of x.

Idea of proof.

0. Given a self-adjoint endomorphism h of E, define

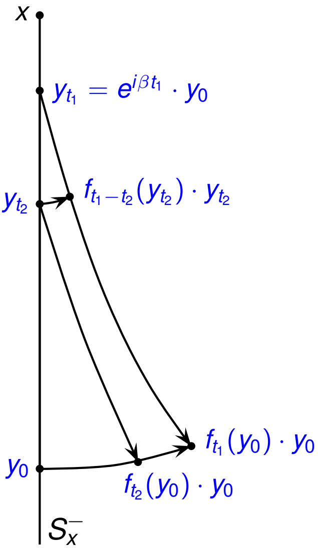
$$\sigma(h) = \mathrm{tr}(h) + \mathrm{tr}(h^{-1}) - 2\,\mathrm{rank}(E)$$

1. Prove the uniform bound

$$\sigma(f_t(y)^* f_t(y)) \leq C\|y - x\|^2_{C^0} \quad \forall y \in S_x^-$$

2. Use Donaldson's distance decreasing formula for the metric flow to show that

$$\begin{aligned}\sigma((f_{t_2}(y_0)^{-1} f_{t_1}(y_0))^* f_{t_2}(y_0)^{-1} f_{t_1}(y_0)) \\ \leq \sigma(f_{t_1 - t_2}(y_{t_2})^* f_{t_1 - t_2}(y_{t_2})) \\ \leq C\|y_{t_2} - x\|^2 \leq Ce^{-Kt_2}\end{aligned}$$



Convergence of the scattering method (cont.)

Therefore the sequence of metrics $h_t = f_t(y_0)^* f_t(y_0)$ has the Cauchy property with respect to σ

$$\sigma(h_{t_1} h_{t_2}^{-1}) \leq Ce^{-KT} \quad \text{for all } t_1, t_2 \geq T$$

Can then show that it converges in the C^0 norm.

The smoothness of the limit metric h_∞ then follows from the smoothing properties of the heat equation (derivatives of the curvature are bounded).

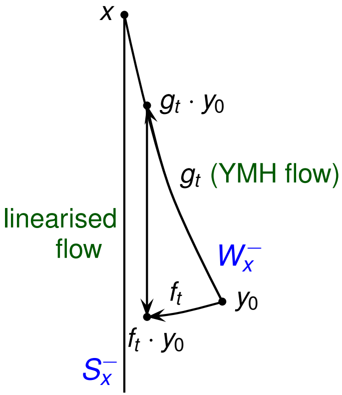
Given any smooth gauge transformation $f \in \mathcal{G}^{\mathbb{C}}$ such that $f^* f = h_\infty$, can construct a solution to the reverse YMH heat flow from $f \cdot y_0$ which converges in the C^∞ topology to a critical point $g \cdot x \in \mathcal{G} \cdot x$.

The $\mathcal{G}$-equivariance of the flow then implies that there is a flow line from $g^{-1} \cdot f \cdot y_0$ to x.

Every point in the negative slice S_x^- is complex gauge equivalent to a point in the unstable set W_x^-.

Convergence of the scattering method (cont.)

Can then reverse the process to show that every point y_0 in the unstable set W_x^- is complex gauge equivalent to a point in the negative slice S_x^-.



Again, we can show that

- $h_t = f_t^* f_t$ converges to a smooth metric h_∞.
- There exists a smooth gauge transformation f_∞ such that $f_\infty^* f_\infty = h_\infty$ and $f_\infty \cdot y_0$ is in the negative slice S_x^-.

Every point in W_x^- is complex gauge equivalent to a point in S_x^-.

Classification of points in the unstable set

Therefore, the isomorphism classes of points in the unstable set W_x^- are completely classified by the isomorphism classes of points in the negative slice S_x^-.

The isomorphism classes in the negative slice can be classified in terms of filtrations.

Definition. Let $x \in \mathcal{B}$ be a critical point of YMH corresponding to a direct sum decomposition $(E_1, \phi_1) \oplus \cdots \oplus (E_\ell, \phi_\ell)$ ordered so that slope$(E_j) <$ slope(E_k) for all $j < k$. A Higgs pair $y \in \mathcal{B}$ admits a filtration upwards compatible with x iff y admits a filtration $(F_1, \varphi_1) \subset \cdots \subset (F_\ell, \varphi_\ell)$ such that $(F_j, \varphi_j)/(F_{j-1}, \varphi_{j-1}) \cong (E_j, \phi_j)$.

Theorem. (W.) Let x be a critical point of YMH. Then each point $y \in W_x^-$ is complex gauge equivalent to a Higgs bundle admitting a filtration upwards compatible with x.

Conversely, if $y \in \mathcal{B}$ admits a filtration upwards compatible with x then it is complex gauge equivalent to a point in W_x^-.

An algebraic criterion for critical points to be connected by flow lines

Corollary. (W.)
Two critical points $x_u, x_\ell \in \mathcal{B}$ with $\mathrm{YMH}(x_u) > \mathrm{YMH}(x_\ell)$ are connected by a YMH flow line if and only if there exists $y \in \mathcal{B}$ admitting a filtration upwards compatible with x_u, and such that the graded object of the Harder-Narasimhan-Seshadri double filtration of y is isomorphic to the lower critical point x_ℓ.

Simple case. Consider a critical point x corresponding to a direct sum of stable Higgs bundles $(E_1, \phi_1) \oplus (E_2, \phi_2)$.

Then points in the negative slice correspond to Higgs extensions with harmonic extension class

$$0 \to (E_1, \phi_1) \to (E, \phi) \to (E_2, \phi_2) \to 0$$

The previous theorem says that a Higgs bundle is isomorphic to a point in W_x^- if and only if it is isomorphic to such an extension.

A geometric criterion for convergence

Question. What are the possible Harder-Narasimhan types of such extensions? Can we describe these using the geometry of extension classes?

First consider the simple case of the rank 2 Yang-Mills flow (E_1, E_2 are line bundles, slope$(E_1) <$ slope(E_2) and $\phi = 0$).

A point in the negative slice corresponds to an extension

$$0 \to E_1 \to E \to E_2 \to 0$$

If E' is the maximal semistable subbundle of E then slope$(E') >$ slope(E_1), so $\mathrm{Hom}(E', E_1) = 0$, therefore E' is a subsheaf of E_2.

$$\begin{tikzcd} & & & E' \arrow[dl] \arrow[d, dashed] & \\ 0 \arrow[r] & E_1 \arrow[r] & E \arrow[r] & E_2 \arrow[r] & 0 \end{tikzcd}$$

Conversely, if E' is a locally free subsheaf of E_2 then a result of Narasimhan and Ramanan shows that E' lifts to a subsheaf of E iff the extension class $e \in H^1(E_2^* E_1)$ pulls back to $0 \in H^1((E')^* E_1)$.

$$
\begin{tikzcd}
 & & & E' \arrow[dl, dashed] \arrow[d] & \\
0 \arrow[r] & E_1 \arrow[r] & E \arrow[r] & E_2 \arrow[r] & 0
\end{tikzcd}
$$

Therefore, we can ask the following questions.

Question 1. Given a subsheaf $E' \subset E_2$, which extension classes $e \in H^1(E_2^* E_1)$ are in the kernel of the pullback map $H^1(E_2^* E_1) \to H^1((E')^* E_1)$?

Question 2. Given an extension class $e \in H^1(E_2^* E_1)$, for which subsheaves $E' \subset E_2$ is $e \in \ker(H^1(E_2^* E_1) \to H^1((E')^* E_1))$? Which of these subsheaves lifts to the maximal semistable subbundle of E?

When is E' the maximal semistable subbundle of E?

Suppose E' is destabilising ($\deg(E') > \frac{1}{2}(\deg(E_1) + \deg(E_2))$). E' is a locally free subsheaf of E_2, therefore $E' = E_2[-D]$ for some effective divisor $D = \sum n_i p_i$ on X.

Then (following Lange-Narasimhan) e is in the linear span of $D \subset X$ via the embedding $X \hookrightarrow \mathbb{P}H^1(E_2^* E_1)$.
(Recall $\deg E_2^* E_1 < \deg(E')^* E_1 < -\frac{1}{2}(\deg(E') - \deg E_1) < 0$.)

Take minimal subset D_{min} of D such that e is in linear span of D_{min}. Then max. semistable subbundle of E is $E_2[-D_{min}]$.
Inequality on $\deg(E')$ $\Rightarrow$ nondegeneracy of $2|D|^{th}$ secant variety

Therefore

- Extension classes for (possibly broken) Yang-Mills flow lines from $E_1 \oplus E_2$ to $E_1' \oplus E_2[-D]$ are determined by a subspace of the $|D|^{th}$ secant variety of X in $\mathbb{P}H^1(E_2^* E_1)$.
- The unbroken flow lines correspond to an open subset of this subspace.

Higher rank case

Now consider an extension E of a line bundle by a rank n semistable bundle E_2. After tensoring the whole setup by a line bundle, we can reduce to the case of an extension of $\mathcal{O}$ by E_2.

The same argument as before shows that the maximal semistable subbundle of E is a subsheaf of E_2.

$$
\begin{tikzcd}
 & & & E' \arrow[dl, dashed] \arrow[d] & \\
0 \arrow[r] & \mathcal{O} \arrow[r] & E \arrow[r] & E_2 \arrow[r] & 0
\end{tikzcd}
$$

Conversely, a locally free subsheaf $E' \subset E_2$ lifts to a subsheaf of E iff the extension class is in the kernel of $H^1(E_2^*) \to H^1((E')^*)$.

If $\operatorname{rank}(E') = \operatorname{rank}(E_2)$ then E' is a Hecke modification of E_2.

Therefore we can construct Hecke modifications using Yang-Mills flow lines.

Constructing Hecke modifications

More precisely, using the same idea as before (and results of Choe-Hitching on secant varieties in $\mathbb{P}H^1(E_2^*)$), we can prove the following result.

Theorem. (W.)
Let $\mathcal{O} \oplus E_2$ be a critical point of the Yang-Mills functional. Then a semistable bundle E' is a Hecke modification of E_2 if and only if there is a (possibly broken) flow line from $\mathcal{O} \oplus E_2$ to a critical point isomorphic to $L \oplus E'$ for some line bundle L.

The unbroken flow lines are determined by secant varieties of $\mathbb{P}E_2^*$ in $\mathbb{P}H^1(X, E_2^*) \cong \mathbb{P}H^0(\mathbb{P}E_2^*, \pi^* K_X \otimes \mathcal{O}_{\mathbb{P}E_2^*}(1))^*$.

Describing the unbroken flow lines explicitly as before requires a bound on the Segre invariant of E_2 so that the secant varieties are nondegenerate.

YMH flow lines with nonzero Higgs field

Now we would like to carry out an analogous process when the Higgs field is non-zero.

One complication is that a Hecke modification of the holomorphic bundle may introduce poles into the Higgs field.

To preserve the holomorphicity of the Higgs field ϕ, we need the Hecke modification to be compatible with ϕ.

More precisely, given $v \in E_p^*$ and $\mu \in \mathbb{C}$, want to impose the condition that $v(\phi(s)) = \mu v(s)$. Equivalently, $\phi(s) - \mu s \in \ker v$ and so ϕ preserves the locally free subsheaf $E' = \ker v$.

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r] \arrow[d, "\phi'"] & E \arrow[r, "v"] \arrow[d, "\phi"] & \mathbb{C}_p \arrow[r] \arrow[d, "\mu"] & 0 \\ 0 \arrow[r] & E' \otimes K \arrow[r] & E \otimes K \arrow[r, "v"] & \mathbb{C}_p \arrow[r] & 0 \end{tikzcd}$$

When the discriminant of ϕ has simple zeros then Witten shows that the space of Hecke modifications is the spectral curve.

Which Hecke modifications correspond to extensions?

Now we can introduce the Higgs field into the previous argument for the Yang-Mills flow.

Suppose first that $\deg E' = \deg E_2 - 1$.

Let (E', ϕ_2') be a Hecke modification of (E_2, ϕ_2) over $p \in X$.

As before, if the Higgs extension class pulls back to zero then (E', ϕ_2') is a subsheaf of (E, ϕ).

$$\begin{tikzcd} & & & (E', \phi_2') \arrow[dl, dashed] \arrow[d] & \\ 0 \arrow[r] & (\mathcal{O}, \phi_1) \arrow[r] & (E, \phi) \arrow[r] & (E_2, \phi_2) \arrow[r] & 0 \end{tikzcd}$$

Conversely, we would like to construct a Higgs extension class which pulls back to zero. Can do this if $\phi_1(p)$ matches the eigenvalue of the Hecke modification.

Can find such ϕ_1 since the canonical system is basepoint free.

Therefore we see that the spectral curve determines the flow lines between generic points of adjacent critical sets.

Theorem. (W.)
Let (E_2, ϕ_2) be a stable Higgs bundle with deg $E_2 > 2$.
Then (E', ϕ_2') with deg $E' =$ deg $E_2 - 1$ is a Hecke modification of (E_2, ϕ_2) if and only if there exists $\phi_1 \in H^0(K)$ and a YMH flow line between $(E_2, \phi_2) \oplus (\mathcal{O}, \phi_1)$ and $(E', \phi_2') \oplus (L, \phi_L)$ for some line bundle L.

$$\begin{tikzcd} & & & (E', \phi_2') \arrow[dl, dashed] \arrow[d] & \\ 0 \arrow[r] & (\mathcal{O}, \phi_1) \arrow[r] & (E, \phi) \arrow[r] & (E_2, \phi_2) \arrow[r] & 0 \end{tikzcd}$$

If the discriminant of ϕ_2 has simple zeros then the flow lines are determined by the spectral curve inside the projectivisation of the space of Higgs extensions, where ϕ_1 is allowed to vary.

In the case $\deg E' = \deg E - 1$, let C_ℓ be the lower critical set and C_u be the upper critical set. Suppose $\deg E$ is coprime to rank E.

There are canonical projections $C_\ell \to \mathcal{M}_{Higgs}^{ss}(E')$ and $C_u \to \mathcal{M}_{Higgs}^{st}(E)$.

Let $\mathcal{F}_{\ell,u}$ denote the space of flow lines connecting C_ℓ and C_u.

Let $\mathcal{P}_{\ell,u} \subset C_\ell \times C_u$ denote the space of pairs of critical points connected by a flow line.

$$\begin{tikzcd} & \mathcal{F}_{\ell,u} \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & \mathcal{P}_{\ell,u} \arrow[dl] \arrow[dr] & \\ C_\ell & & C_u \end{tikzcd}$$

Corollary. (W.) The induced correspondence $\mathcal{H}_{\ell,u}$ is the Hecke correspondence.

$$
\begin{tikzcd}
 & \mathcal{F}_{\ell,u} \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \mathcal{P}_{\ell,u} \arrow[dl] \arrow[d] \arrow[dr] & \\
C_\ell \arrow[d] & \mathcal{H}_{\ell,u} \arrow[dl] \arrow[dr] & C_u \arrow[d] \\
\mathcal{M}^{ss}_{Higgs}(E') & & \mathcal{M}^{st}_{Higgs}(E)
\end{tikzcd}
$$