Lecture 1: Introduction to geometry

14 January, 2019

What is geometry? Why study geometry?

Geometry is the study of shapes, sizes, patterns and positions. The most basic objects in our 3-dimensional world are points, lines, planes and solid shapes.

In ancient times geometry was used in the study of

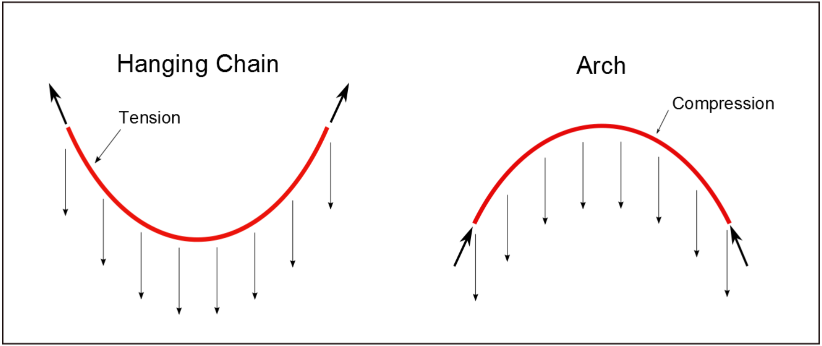
- buildings and architecture
- engineering (building machines, especially for battle)
- mapping and navigation
- art and drawing

Modern applications of geometry include computer graphics, the design of satellite dishes, GPS navigation, etc.

To build an arch using a material like brick or wood, architects use the fact that these materials are strong under compression. Unfortunately they are weaker under tension and shear forces. Therefore, the architect wants the forces to go along the length of the archway. What is the ideal shape?

Buildings and architecture (cont.)

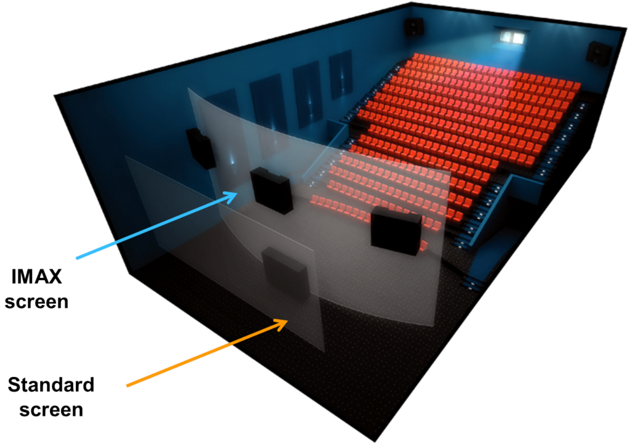
Consider an upside-down version of the problem. A rope hanging between two points has the shape of a catenary. All of the force passes along the length of the rope (no shear forces).



The ideal shape for an archway is therefore an upside-down catenary. (You may have encountered these shapes in your calculus classes.)

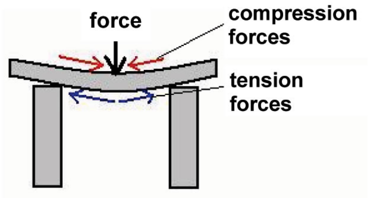
Buildings and architecture (cont.)

IMAX theatres use geometry to optimize the viewing experience.

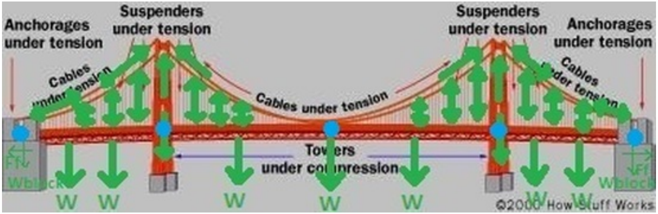


You can watch a video explaining this here.

Engineers who design bridges also use geometry to make sure that the bridge keeps its shape under load.



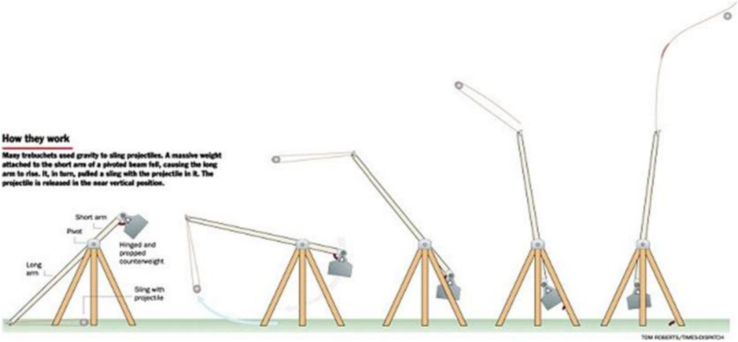
The downwards force of gravity causes the bridge to sag. The engineer needs to build some extra structure into the bridge to prevent this.



You can read more about it here.

Engineering and designing machines

In ancient Greece, many geometers were involved in building machines of war.
One example of a machine built by geometers in ancient Greece is the *trebuchet*.



Geometers not only designed the machine, but answered questions like “What is the optimal length of the sling?” and ”What is the optimal weight for the counterweight?”

Engineering and designing machines (cont.)

A story from ancient Greece involves Archimedes using a parabolic mirror or an array of mirrors to focus the sun's rays on an approaching Roman fleet, thus burning the ships and saving the city.

There is some doubt over the truth of this story, but there is no doubt about the geometric principles involved.

Today these ideas are used in the design of satellite dishes (we will study this later in the semester) and proposals for solar power generation. You can read more about Archimedes here and more about efficient energy generation here.

Engineering and designing machines (cont.)

Another example of a machine using levers to save effort is the *shaduf*. This was used in ancient Egypt and in China to lift water from a river or lake. The counterweight balances the weight of the water and makes moving the water much easier.

A modern example of a machine using counterweights is the elevator.

Mapping and navigation

The Greeks used geometry to navigate across the Mediterranean Sea.

They used trigonometry in the plane to navigate (a flat plane is a reasonable approximation on the Mediterranean, which is small relative to the size of the earth).

Later, once people started travelling longer distances, they used spherical trigonometry to navigate.

The basic questions are: If a boat travels a given distance on a given heading, what is the position of the boat? What heading should the boat take to reach its destination?

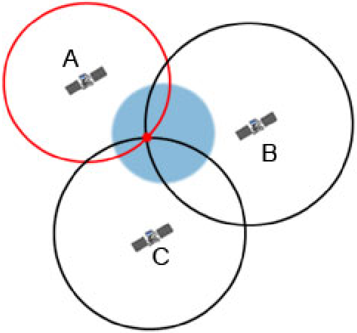
You can read more about applications of trigonometry to navigation here.

Explorers in the Middle Ages used geometry to navigate using the sun, moon and stars. Geometers designed navigational instruments for sailors to use on their voyages.

Mapping and navigation (cont.)

Today's GPS system uses geometry to determine your position on the ground based on satellite information.

A GPS receiver (for example, your phone) can measure the distance to any GPS satellites in its range. Based on information about the position of the satellites, the receiver then then determine your position on the ground.



This process is called *trilateration*. The receiver needs to connect to at least three satellites for this to work.

Geometrically, this is the problem of determining the point of intersection of three spheres.

Modern applications (computer graphics)

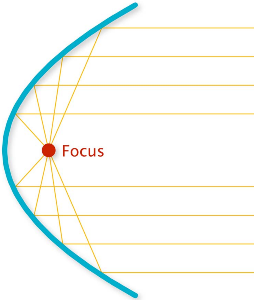
Geometry is used in computer graphics, for example

- Using Bezier curves to draw smooth curves based on a small number of given control points.
- Using projective geometry to understand how to draw in perspective.

Fractals are used to draw lifelike images using the self-similarity properties of objects in nature.

Modern applications (satellite dishes)

Satellite dishes take signals from space (these are far away so that we can assume the rays are parallel lines) and focus them to a single point. What is the correct shape?



We will see later that a parabola is the correct shape to focus the signal. We can also reverse the process and take a light source at the focal point. The light will then leave the parabola in parallel rays.

What if we want the light to focus at another point? (This is used to design spotlights) The correct shape is now an ellipse.

We will study all of these properties in the chapter on conic sections.

What is Euclidean geometry?

Euclidean geometry is the study of points, lines, circles and related shapes in the two-dimensional plane.

These must satisfy Euclid's axioms.

The Greeks had an intuitive understanding of geometry for hundreds of years before Euclid's time, but it was Euclid who introduced a systematic approach to geometry using axioms and rigorous proofs.

In the first half of this course we will learn how to use Euclid's axioms to prove theorems in Euclidean geometry, how to construct shapes in Euclidean geometry and we will also see some applications of Euclidean geometry.

In the second half of the course we will study non-Euclidean geometries.

Euclid's axioms

Axiom 1. One can draw a straight line segment between any two points.

Axiom 2. One can extend a straight line segment indefinitely.

Axiom 3. One can draw a circle with a given centre and radius.

Axiom 4. All right angles are equal to each other.

Axiom 5. If a straight line crossing two straight lines makes the interior angles on one side less than two right angles, then the two straight lines, if produced indefinitely, will meet on that side on which the angles are less than the two right angles.

Euclid's axioms (cont.)

The first three axioms are construction axioms. In other words, given a collection of points, lines and circles in the plane, the construction axioms allow us to construct new points, lines and circles.

The first two axioms describe the possible lines we can draw using a ruler. The third operation describes the circles we can draw using a compass.

As a consequence, constructions in Euclidean geometry are also called ruler and compass constructions.

Construction problems in Euclidean geometry

An important problem in Euclidean geometry is to determine whether one can construct a desired shape. For example

- Can you construct an equilateral triangle with a given sidelength?
- Given a line segment, can you construct a new line segment with length $\sqrt{2}$ times the original segment?
- Given a circle, can you construct a new square with area equal to that of the original circle? (Squaring the circle)
- Given an angle of x degrees, can you construct a new angle of $\frac{x}{3}$ degrees? (Trisecting the angle)
- Given a cube, can you construct a new cube with twice the volume? (Doubling the cube)

It turns out that the first two constructions are quite easy (we will encounter them soon), however the last three constructions are impossible using a ruler and compass. A rigorous proof of their impossibility wasn't found until the 1800s (you may learn more about this in the Galois theory class).

Euclid's axioms (cont.)

We will come back to these construction problems later in the course. Something interesting (and enjoyable!) that you can try now are the following websites containing a number of construction puzzles and games that you can solve using a ruler and compass.

- Euclid the Game.
  (http://www.euclidthegame.com/)
- Ancient Greek Geometry.
  (http://www.sciencevsmagic.net/geo/)

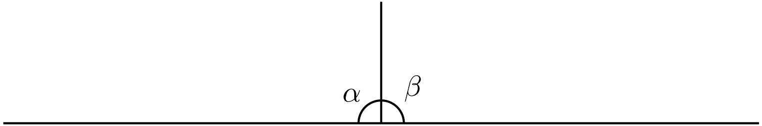
More about Axiom 5

Axiom 5. If a straight line crossing two straight lines makes the interior angles on one side less than two right angles, then the two straight lines, if produced indefinitely, will meet on that side on which the angles are less than the two right angles.

First, we need to define what we mean by a *right angle*.

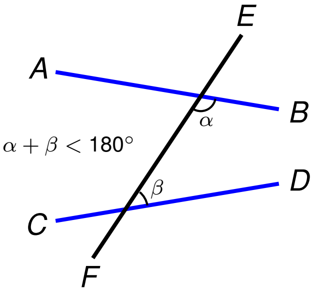
Definition. A right angle is equal to half of the angle on a straight line.

Consider the diagram below. If the two angles α and β are equal, then they are each equal to a right angle.



We define 90° to be the numerical quantity associated to a right angle. (We can use this as a definition of degree.)

More about Axiom 5



In the diagram on the left, since $\alpha + \beta < 180°$ (two right angles), then Axiom 5 says that the lines *AB* and *CD* will eventually intersect on the side of the angles α and β.

Axiom 5 is also known as the Parallel Axiom. Before discussing this, we need the definition of a parallel line.

Definition. Parallel lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

A consequence of the parallel axiom is that if $\alpha + \beta = 180°$ then the lines are parallel in the diagram above.

The parallel axiom

There are also a number of equivalent statements to Axiom 5 that are easier to use. For example, there is

Playfair's Axiom If P is a point not on a line ℓ then there is exactly one line through P that is parallel to ℓ.

We'll talk more about Playfair's Axiom later in the course.

Ever since the days of the ancient Greeks, many people have tried to prove Axiom 5 using only the other axioms. It turns out that one cannot do this and it has been proved that Axiom 5 is independent of the other axioms.

We will see later in the course that changing the statement of Axiom 5 will change the underlying geometry to a non-Euclidean geometry.

For example, Axiom 5 is not true in spherical geometry, since great circles on the sphere always intersect (and so there are no parallel lines). Again, we'll talk more about this in the second half of the course.

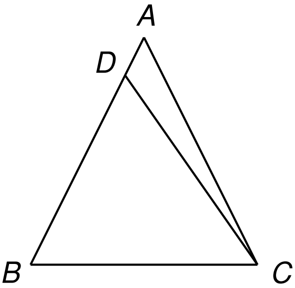
Common notions in Euclidean geometry

Before we try to prove theorems, we need the following Common Notions. We have already been using these in previous mathematics courses, and now we also use them for geometric quantities such as angle, length and area.

- Things which are equal to the same thing are also equal to one another.
  If $a = b$ and $b = c$ then $a = c$.
- If equals are added to equals then the wholes are equal.
  If $a = b$ and $x = y$ then $a + x = b + y$.
- If equals are subtracted from equals then the remainders are equal.
  If $a = b$ and $x = y$ then $a - x = b - y$.
- Things which coincide with one another are equal to one another.
  For example, two line segments are equal if they coincide.
- The whole is greater than the part.
  If $a > 0$ then $x + a > x$ for any $x \in \mathbb{R}$.

Common notions in Euclidean geometry (cont.)

A geometric example of the last common notion “the whole is greater than the part” is given in the following picture.



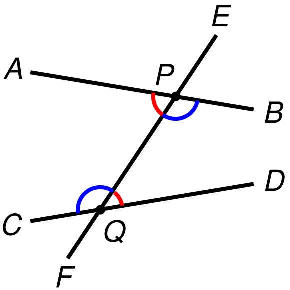
We see that the triangle ΔDBC (the “part”) is strictly contained inside the triangle ΔABC (the “whole”). Therefore ΔABC is “greater” than ΔDBC.

A useful consequence of this is that the two triangles cannot be congruent.

An example of a proof in Euclidean geometry

The axioms form the basis of our proofs. The common notions are the logical steps that allow us to progress from one step to the next.

Here are two examples of proofs. In the next lecture we will talk more about proofs and how to write them.

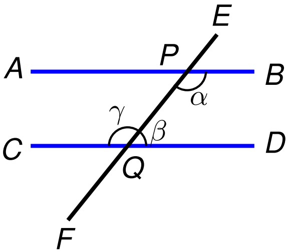


Definition. Let AB and CD be two lines, and let EF be a line intersecting them at distinct points P and Q. The alternate angles in the diagram are the two pairs of angles $(\angle BPF, \angle CQE)$ and $(\angle APF, \angle DQE)$.

Theorem. If *AB* and *CD* are parallel, then any pair of alternate angles is equal.

Proof. Since *AB* and *CD* are parallel, then the angles α and β in the diagram below must satisfy $\alpha + \beta = 180$ (if not, then the lines will intersect by Axiom 5, and therefore they will not be parallel).

Since the sum of the angles β and γ is the angle $\angle CQD = 180$ (since *CD* is a straight line) then we have $\beta + \gamma = 180$.



Therefore $\alpha + \beta = 180 = \gamma + \beta$ and so $\alpha = \gamma$ (here we have used the first and third Common Notion). ∎

Some comments about Euclid's axioms

Unfortunately, Euclid's axioms are not enough to rigorously prove all of the theorems that we need. For example, the proof of the theorem on the next slide is not complete (can you see the gap in the proof?)

There is a complete set of axioms, known as Hilbert's axioms. Rather than spending weeks working through all of these axioms, instead we will rely on some geometric intuition as well as the axioms.

As we will see in the next lecture, when developing a proof, having good geometric intuition is just as important as being able to use the axioms and definitions.

You can read more about Hilbert's axioms in Chapter 2 of Hartshorne's book "Geometry: Euclid and beyond" (available through the library website). I have also posted a copy of this chapter on IVLE.

Some basic results in Euclidean geometry (cont.)

We use $|AB|$ to denote the length of the line AB.

The next theorem is an example of a *construction theorem* (it says that we can construct a particular shape). In the proof we will use the construction axioms.

Theorem. Given a line segment AB we can construct an equilateral triangle with one side equal to AB.

Proof. First, draw a circle centred at A and with radius AB.

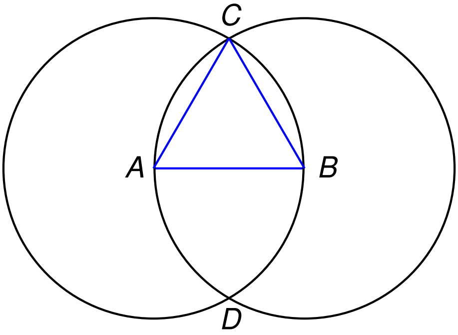
Next, draw a circle centred at B and with radius $|AB|$.

The two circles will intersect (why?) at two points C and D. Draw the line segments $|AC|$ and $|BC|$.

Note that $|AC| = |AB|$ (since they are both radii of the circle centred at A) and $|BC| = |AB|$ (since they are both radii of the circle centred at B).

Therefore the three sides of the triangle are all equal $|AC| = |AB| = |BC|$, and so the triangle is an equilateral triangle. ■

Constructing an equilateral triangle (Picture)



You can try doing this yourself, either with a ruler and compass, by visiting the Constructions Webpage or by going to the Ancient Greek Geometry website.

Next time

The topic of the next class will be “Introduction to proofs in geometry”. We will cover

- Some examples of proofs
- How to write proofs (good and bad examples of proofs)
- Proofs of some of Euclid’s basic propositions.

Extra references

- The internet version of Euclid’s “The Elements” by David Joyce is available at http://aleph0.clarku.edu/ djoyce/java/elements/toc.html.
- The page for Book I contains links to Euclid’s Definitions, Postulates (or Axioms) and Common Notions.
- The theorem on slide 22 is Proposition 29 in Book I of the elements. The converse is Proposition 27.