Lecture 14: Introduction to drawing in perspective

15 March, 2019

Overview

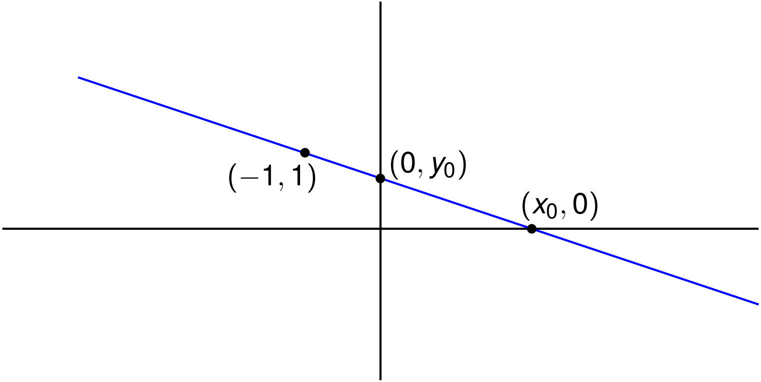
Last time.

- Introduction to conic sections
- Geometric properties of the ellipse, parabola and hyperbola.
- Optical properties of the ellipse, parabola and hyperbola.
- Applications

Today.

- Projection and its relation to drawing in perspective
- Introduction to the projective line and the projective plane
- Constructions related to drawing in perspective

Exercise. Consider a line through the point $(-1, 1)$ which intersects the x-axis at $(x_0, 0)$. Find the point $(0, y_0)$ where it intersects the y-axis.

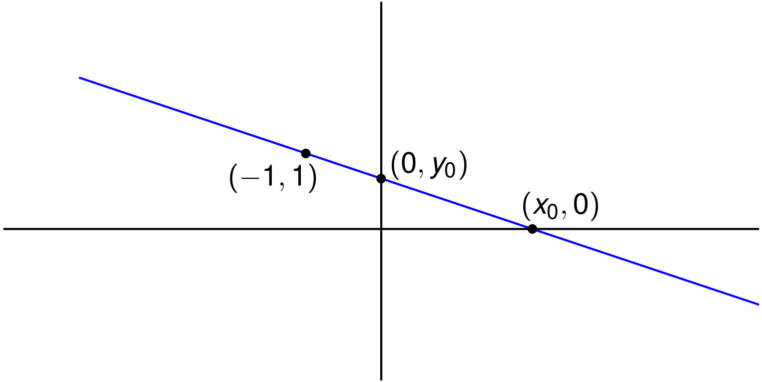


Solution

Solution.

Solution

Solution. The slope of the line is $-\frac{1}{x_0+1}$. Therefore it has equation $y - y_0 = \frac{x}{x_0+1}$. Substituting the point $(x, y) = (-1, 1)$ gives us $y_0 = 1 - \frac{1}{x_0+1} = \frac{x_0}{x_0+1}$.



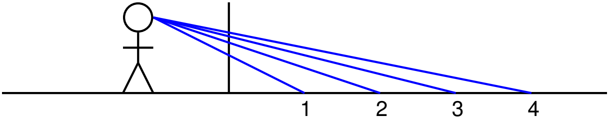
Projective transformations

We can think of the previous exercise as defining a transformation from the x-axis to the y-axis.

A person viewing the x-axis from the point $(-1, 1)$ will see it projected onto the y-axis by the function defined in the exercise.

Example. Suppose you want to draw a picture of the points $1, 2, 3, \ldots, n$ from the x-axis. Once you have finished, you would like to hang your drawing on a wall (the y-axis) and give the viewer (who is positioned at the point $(-1, 1)$) the illusion that they are seeing the x-axis.

Therefore, the points $1, 2, 3, \ldots, n$ (on the x-axis) should be positioned at the points shown on the y-axis.

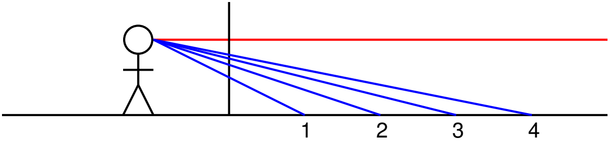


Any problem of this type can be thought of as a projection of one line onto another through a point of perspective.

Projective transformations

The function $y = \frac{x}{x+1}$ is an example of a projection of the x-axis onto the y-axis.

Question. What about $y = 1$? Which point on the x-axis projects to $y = 1$?



We can think of the point $y = 1$ as a point on the horizon. The red line above in the picture above does not intersect the x-axis. Instead we say that it corresponds to a point at infinity, or a point on the horizon.

The point at infinity on the x-axis projects to the point $y = 1$.

The projective plane

We are more used to seeing the horizon as a horizontal line in the two-dimensional plane.

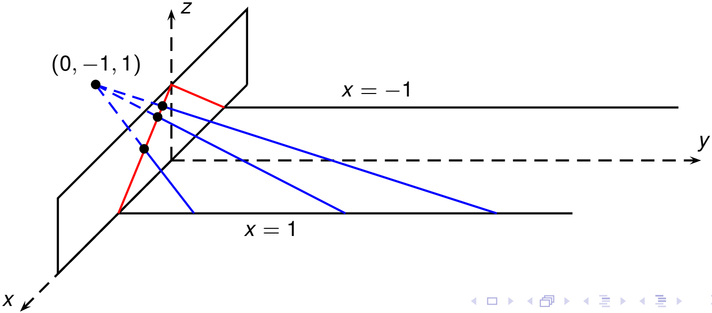
For example, imagine an airport runway as viewed from an airplane coming in to land.

The projective plane

The sides of the runway are supposed to be parallel, however we see them as two lines appearing to meet at the horizon.

Exercise. Imagine now that you are piloting a plane in $\mathbb{R}^3$. As you come in to land on the *xy* plane ($z = 0$), the runway consists of the lines $x = 1, z = 0$ and $x = -1, z = 0$.

From your position at $(0, -1, 1)$, you see the runway projected onto the *xz*-plane ($y = 0$). What are the equations describing the projection of the runway onto the *xz*-plane?



Solution.

Solution

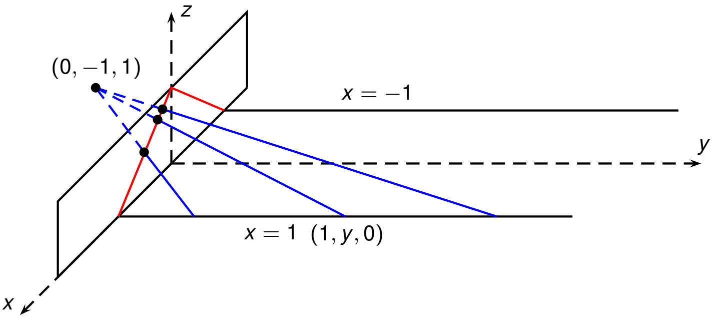
Solution. First consider the line $x = 1, z = 0$. A line from $(0, -1, 1)$ to a point $(1, y, 0)$ has parametric equations

$$(x(t), y(t), z(t)) = (t, t(1 + y) - 1, 1 - t)$$

Solution. First consider the line $x = 1, z = 0$. A line from $(0, -1, 1)$ to a point $(1, y, 0)$ has parametric equations

$$(x(t), y(t), z(t)) = (t, t(1+y) - 1, 1 - t)$$

This intersects the xz-plane $\{y = 0\}$ at $t = \frac{1}{1+y}$, which corresponds to the point $\left(\frac{1}{1+y}, 0, \frac{y}{1+y}\right)$ in the xz plane. Therefore the image is the line $y = 0, x + z = 1$.



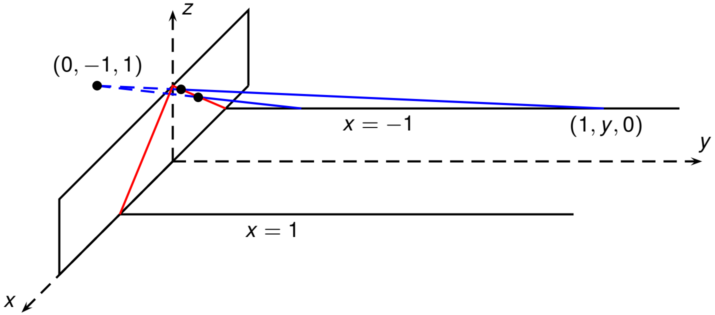
Solution (cont.)

Now consider the line $x = -1, z = 0$. A line from $(0, -1, 1)$ to a point $(-1, y, 0)$ has parametric equations

$$(x(t), y(t), z(t)) = (-t, t(1 + y) - 1, 1 - t)$$

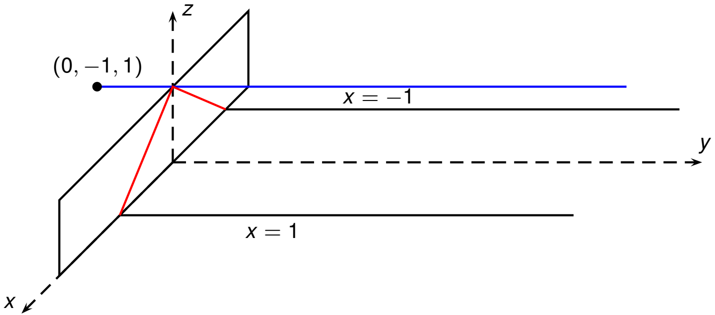
This intersects the plane $y = 0$ at $t = \frac{1}{1+y}$, which corresponds to the point $\left(-\frac{1}{1+y}, 0, \frac{y}{1+y}\right)$ in the xz plane.

Therefore the image is the line $y = 0, z - x = 1$.



The projective plane

We saw in the previous exercise that the two lines making up the sides of the airport runway project to two lines in the *xz* plane. These lines meet at the point $(0, 0, 1)$. The line through $(0, -1, 1)$ and $(0, 0, 1)$ is parallel to the *xy*-plane.



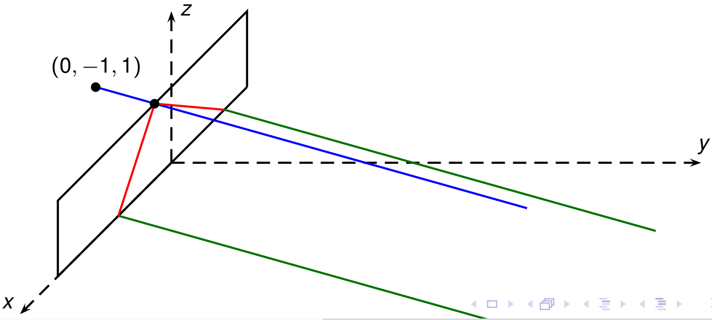
We can think of this line as meeting the *xy*-plane at infinity (which corresponds to the point on the horizon line where the two sides of the airport runway meet).

This illustrates an important principle in projective geometry: *parallel lines meet on the horizon at infinity*.

Question. What are the points on the horizon?

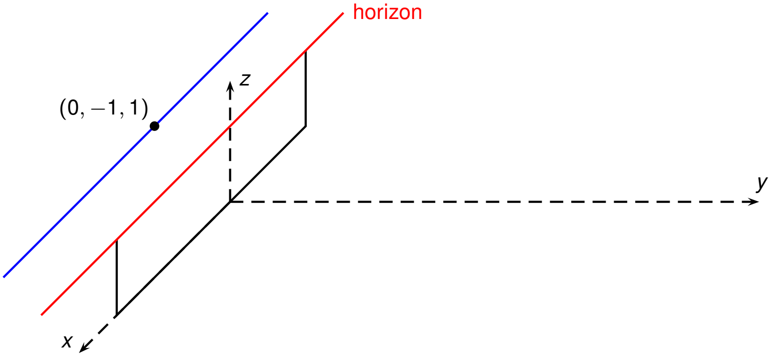
Unlike the example at the beginning of today's lecture (with only one point at infinity), on the projective plane there are many points on the horizon.

Parallel lines in a different direction will meet at a different point on the horizon.



The projective plane

In our picture, the horizon corresponds to the line $y = 0, z = 1$ (together with a point at infinity corresponding to a line through $(0, -1, 1)$ parallel to the x-axis).



The horizon corresponds to all of the lines through $(0, -1, 1)$ parallel to the xy-plane.

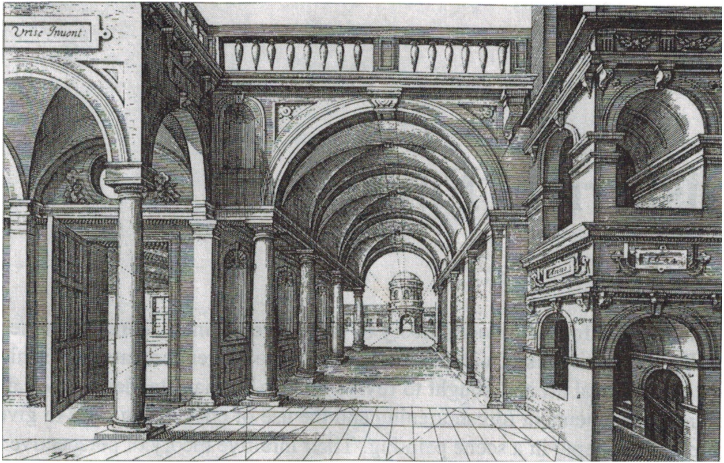
Origins of projective geometry in Renaissance art

This was a problem faced by the artists of the Renaissance. How to project an image onto a flat picture hanging on a wall?

The drawing on the left by Albrecht Durer is in perspective.

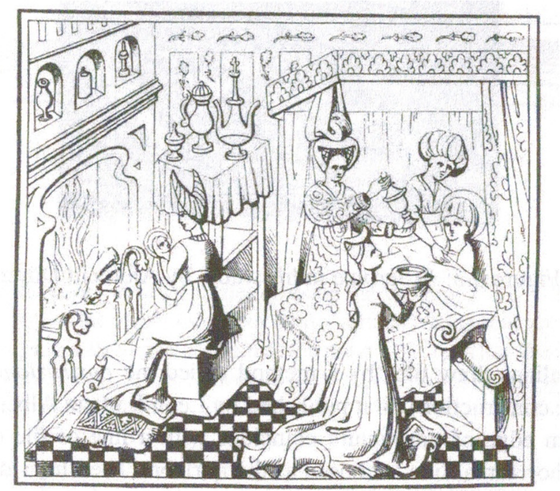
Note that the parallel lines on the window frame and the ceiling are drawn to meet at the horizon.

Origins of projective geometry in Renaissance art



Here is another picture drawn in perspective. Once again, you can see how the parallel lines meet on the horizon.

Origins of projective geometry in Renaissance art



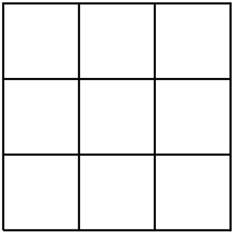
The above painting was drawn before people understood how to draw in perspective.

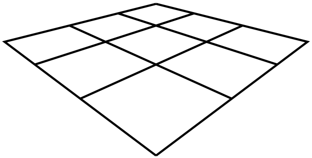
Note that the tiles on the floor are all the same size.

Drawing a tiled floor in perspective

A basic question in perspective drawing is to draw a floor tiled with rectangular tiles (such as the one on the previous slide) in perspective.

Original Picture | **Perspective drawing**





The sides of the tiles consist of two pairs of parallel lines. Each pair of parallel lines meets at a point on the horizon.

Click here for an interactive picture.

Drawing a tiled floor in perspective

First, consider some basic facts about perspective drawing based on our investigation so far.

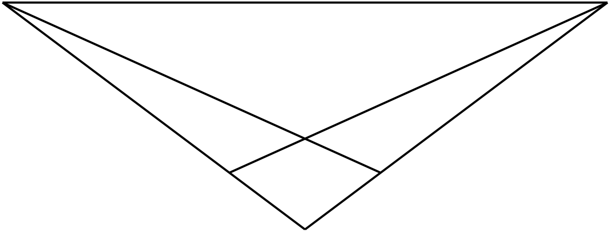
Basic rules of drawing in perspective.

- Straight lines in the original picture project to straight lines in the perspective drawing.
- Intersection points in the original picture project to intersection points in the perspective drawing.
- Parallel lines in the original picture project to lines which intersect on the horizon in the perspective drawing.

With just these rules, we have enough information to use a ruler to draw the tiled floor in perspective.

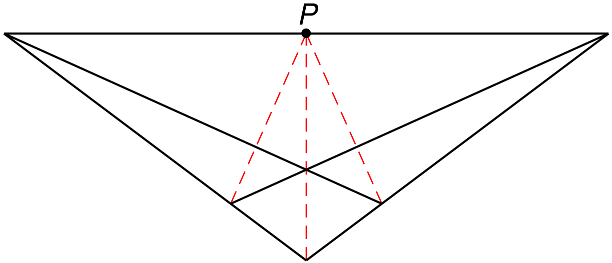
Drawing a tiled floor in perspective

Begin with a single square. Since the sides of the square are parallel, then they intersect on the horizon when we draw them in perspective.

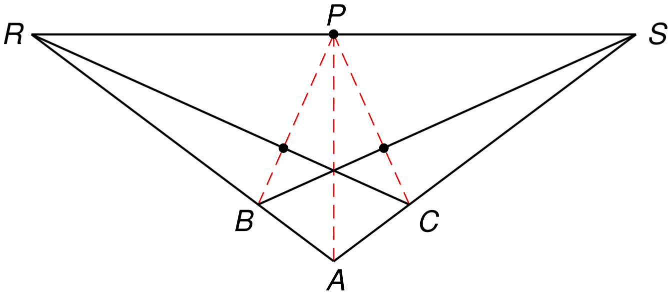


Drawing a tiled floor in perspective

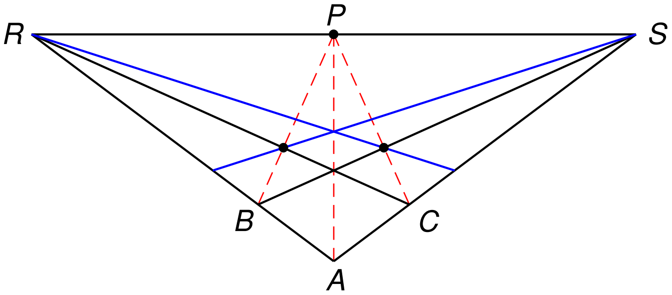
Since all of the diagonals in the original drawing are parallel then they must meet at a point P on the horizon when we draw them in perspective.



Therefore the top corner of the next tile in the picture is the intersection of the diagonal *BP* with the line *CR* in the picture below. The top corner of the other tile is the intersection of *BS* and *CP*.

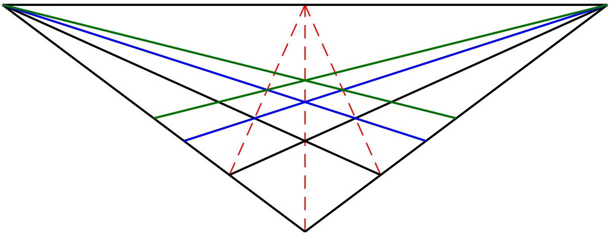


Now connect these points to R and S to draw the next tiles in the picture.



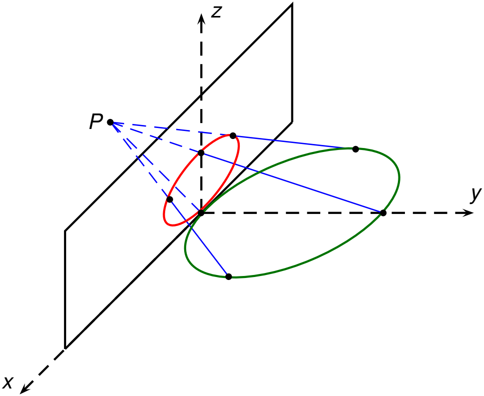
Drawing a tiled floor in perspective

Repeating the process again will gives us the next row of tiles in the picture.



Continuing in this way, we can draw as many tiles as we like.

Exercise. Consider the circle with equation $x^2 + (y-1)^2 = 1$ on the xy-plane. Find the equation describing the circle on the xz-plane using $P = (0, -1, 1)$ as a point of perspective. What happens when we use $Q = (0, 1, 1)$ as a point of perspective?



Solution.

Solution

Solution. First consider a line from $(0, -1, 1)$ to a point $(x_1, y_1, 0)$ in the xy-plane. This has parametric equation

$$(x(t), y(t), z(t)) = (x_1 t, (y_1 + 1)t - 1, (1 - t))$$

The intersection with the xz-plane occurs when $(y_1 + 1)t - 1 = 0$, i.e. $t = \frac{1}{y_1+1}$. Then the point of intersection (a, b, c) with the xz-plane is

$$(a, b, c) = \left(\frac{x_1}{y_1 + 1}, 0, \frac{y_1}{y_1 + 1}\right)$$

Now solve for (x_1, y_1) in terms of a and c and substitute into the equation for the circle.

Solution (cont.)

Solution. (cont.) Solving for (x_1, y_1) in terms of a and c gives us

$$c = \frac{y_1}{y_1 + 1} \Leftrightarrow y_1 = \frac{c}{1 - c} \Leftrightarrow y_1 + 1 = \frac{1}{1 - c}$$

and

$$a = \frac{x_1}{y_1 + 1} \Leftrightarrow x_1 = a(y_1 + 1) = \frac{a}{1 - c}$$

Therefore the equation $x_1^2 + (y_1 - 1)^2 = 1$ becomes

$$\frac{a^2}{(1 - c)^2} + \frac{(2c - 1)^2}{(1 - c)^2} = 1$$
$$a^2 + (2c - 1)^2 = (1 - c)^2$$

which simplifies to the equation of an ellipse.

(we get $a^2 + 3c^2 +$ lower-order terms $=$ const)

Solution. (cont.) Now suppose that we use $(0, 1, 1)$ as the point of perspective.

The parametric equations for the line from $(0, 1, 1)$ to $(x_1, y_1, 0)$ are

$$(x(t), y(t), z(t)) = (x_1 t, (y_1 - 1)t + 1, 1 - t)$$

and so the intersection with the xz-plane is at $t = -\frac{1}{y_1 - 1}$

$$(a, b, c) = \left(-\frac{x_1}{y_1 - 1}, 0, \frac{y_1}{y_1 - 1}\right)$$

Solving for x_1, y_1 in terms of a, c gives us

$$x_1 = \frac{a}{1 - c}, \quad y_1 = \frac{c}{c - 1}$$

and substituting these into the equation for the circle gives us

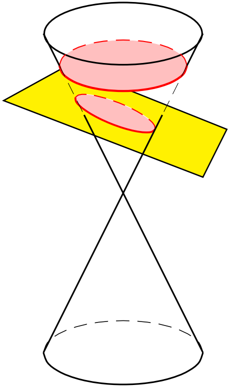
$$a^2 + 1 = (1 - c)^2 \Leftrightarrow a^2 - (c - 1)^2 = -1$$

which is the equation of a hyperbola.

Projecting a circle onto another plane

We have seen (by explicit computation) in this example that **the projection of the circle onto another plane is an ellipse or a hyperbola**.

This fits in with our earlier definition of the ellipse and hyperbola as conic sections.



Using the vertex of the cone as a point of perspective, we see that the projection of the circle onto the yellow plane is an ellipse.

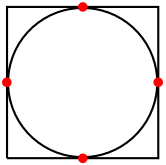
Click here for an interactive picture of a general projection of a circle onto another plane.

Construction exercises

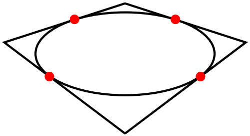
Consider a circle inscribed in a square, as in the diagram below. In the next few exercises we will see how to draw this in perspective.

First, we need to construct the points where the circle touches the square. In the picture below, we see the square drawn in perspective. In the first exercise you should construct the points where the ellipse touches the sides of the quadrilateral. You can do this using only a ruler.

Original Picture



Perspective drawing



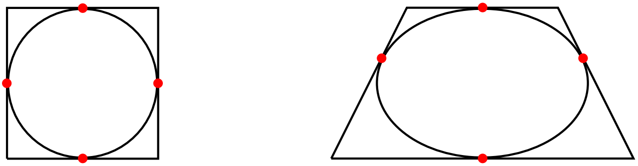
Hint. Remember the exercise from Lecture 10.

Construction exercises

Now consider a special case, where the square projects to a symmetric trapezoid, as in the diagram below.

Again, construct the points where the circle (which projects to an ellipse) touches the sides of the trapezoid. Since one pair of sides is parallel then you need to use a compass to draw parallel lines to do this construction.

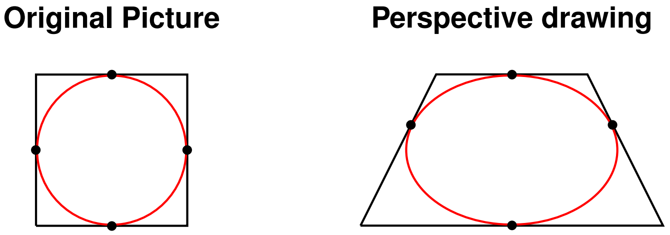
Original Picture **Perspective drawing**



Construction exercises

Now show how to construct the ellipse that is tangent to the symmetric trapezoid at the four points that you constructed in the previous exercise.

Hint. Use one of the constructions from the previous lecture.



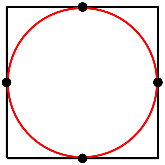
Construction exercises

Now consider the case where the quadrilateral has an axis of symmetry along one diagonal (such a figure is called a kite).

Can you construct the ellipse which is tangent to the given points?

Hint. Try to construct the focal points by using the optical property of the ellipse.

Original Picture



Perspective drawing