Lecture 4: Similar triangles

25 January, 2019

Overview

Last time.

- Area in Geometry
- Areas of triangles and parallelograms
- Euclid's proof of Pythagoras' theorem
- Thales' theorem

Today.

- Similar triangles
- Another proof of Pythagoras' Theorem
- More applications of similar triangles
- Pappus' theorem and Desargues' theorem
- Constructions in Euclidean geometry

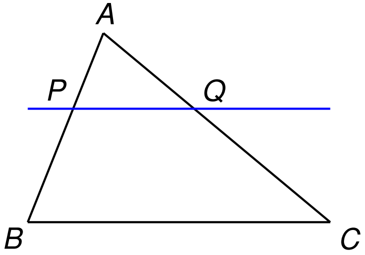
Thales Theorem

Last time we proved

Thales Theorem. (Euclid Book VI, Prop 2)

If a straight line is drawn parallel to one side of a triangle then it cuts the other two sides of the triangle proportionally.

Conversely, if a straight line cuts two sides of a triangle proportionally, then it is parallel to the other side.



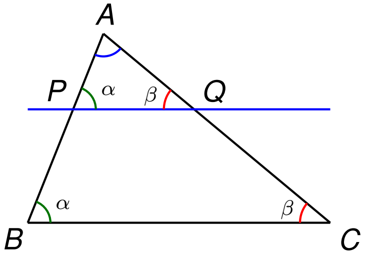
$$\frac{|AP|}{|PB|} = \frac{|AQ|}{|QC|} \text{ if and only if } PQ \text{ is parallel to } BC.$$

We know that two lines are parallel if and only if the corresponding angles are equal (Euclid Prop. I.27 and Euclid Prop. I.29).

Therefore, Thales' theorem can be rewritten as follows.

Theorem. If $\angle APQ = \angle ABC$ or $\angle AQP = \angle ACB$ then

$$\frac{|QC|}{|AQ|} = \frac{|PB|}{|AP|} \quad \Leftrightarrow \quad \frac{|AP|}{|AQ|} = \frac{|PB|}{|QC|}$$



Similar triangles

We can rewrite Thales' theorem in terms of the side lengths of the triangle ΔABC

$$\begin{aligned}
\frac{|QC|}{|AQ|} = \frac{|PB|}{|AP|} &\Leftrightarrow \frac{|QC|}{|AQ|} + 1 = \frac{|PB|}{|AP|} + 1 \\
&\Leftrightarrow \frac{|QC| + |AQ|}{|AQ|} = \frac{|PB| + |AP|}{|AP|} \\
&\Leftrightarrow \frac{|AC|}{|AQ|} = \frac{|AB|}{|AP|}
\end{aligned}$$

Therefore, Thales' theorem tells us that the triangles ΔAPQ and ΔABC have corresponding sides in the same ratio.

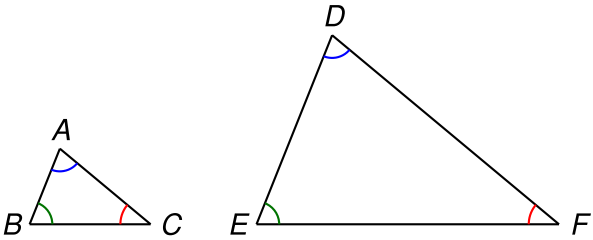
Similar triangles

Definition. Two triangles ΔABC and ΔDEF are similar if and only if

$$\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|CA|}{|FD|}$$

"Corresponding sides are in proportion"

Theorem. If $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$ and $\angle CAB = \angle FDE$ then the triangles ΔABC and ΔDEF are similar.



This theorem shows that we can use "corresponding angles are equal" as an equivalent definition of similar triangles.

Similar triangles

Proof. Move the triangle ΔABC onto ΔDEF so that the vertex A coincides with the vertex D and the line AB lies on DE. Since $\angle CAB = \angle FDE$ then (after possibly reflecting in the line AB) then the line AC lies on DF.

We also know that $\angle ABC = \angle DEF$. Therefore the lines BC and EF are parallel by Euclid Prop. I.27.

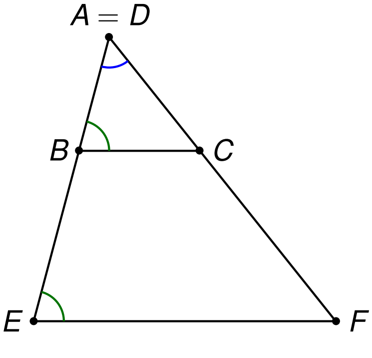
Thales' theorem then shows that $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|}$.

Repeating this process with the vertices B and E then shows that $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$ and so the triangles ΔABC and ΔDEF satisfy the definition of similar triangles. ∎

Remark. We only use the fact that $\angle ABC = \angle DEF$ and $\angle CAB = \angle FDE$. Of course, we know that the other angles are equal, since

$$\angle BCA = 180 - \angle ABC - \angle CAB = 180 - \angle DEF - \angle FDE = \angle EFD$$

Similar triangles



Exercise

Exercise. What happens if we only know that $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$? Can we conclude that $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|CA|}{|FD|}$ and hence the triangles are similar?
Or is there a counterexample?

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Solution.

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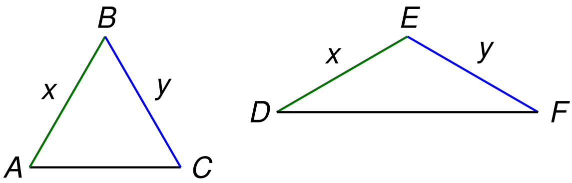
Or is there a counterexample?

Solution. We need more information than just $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$.
For example, we could have $|AB| = |DE|$ and $|BC| = |EF|$, but the triangles are not similar.



Side angle side similarity

Why can't we use Thales' theorem to solve the previous exercise?

Thales' theorem says that the triangles ΔAPQ and ΔABC are similar if and only if $\frac{|AB|}{|AP|} = \frac{|AC|}{|AQ|}$ (see slides 4 and 5 of today's lecture).

Thales' theorem also implicitly assumes that $\angle PAQ = \angle BAC$ (since the triangles ΔPAQ and ΔBAC coincide at the vertex A).

Therefore, we can add an extra assumption to the previous exercise and use Thales' theorem to prove the following theorem.

Theorem. (Euclid Book VI, Prop. 6) Triangles ΔABC and ΔDEF are similar if and only if they satisfy

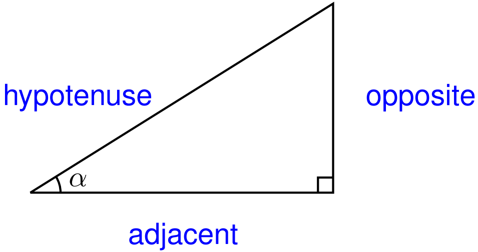
$$\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} \quad \text{and} \quad \angle ABC = \angle DEF$$

Proof. Use Thales' theorem **(Exercise)**.

A familiar result

Recall from trigonometry: for a right-angled triangle, the functions sin, cos and tan are defined by

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\alpha = \frac{\text{opposite}}{\text{adjacent}}$$

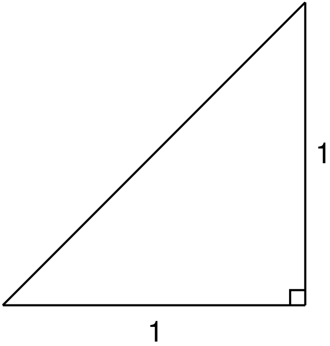


The third angle is $90 - \alpha$. Any other right-angled triangle with angle α has the same angles and is therefore similar.

Therefore sin, cos and tan are well-defined functions.

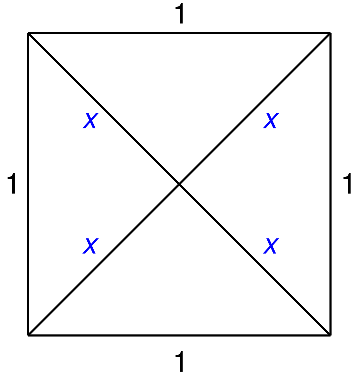
Constructing the square root of 2

Given a line segment of length 1, we can construct a right angled isosceles triangle using a ruler and compass (we will do this later).



What is the length of the hypotenuse? Pythagoras' theorem says that it is $\sqrt{2}$, however we can also see this using similar triangles.

Constructing the square root of 2



The triangles are all similar (**Exercise**) and therefore we have

$$\frac{2x}{1} = \frac{1}{x} \Leftrightarrow 2x^2 = 1 \Leftrightarrow x = \frac{\sqrt{2}}{2}$$

Irrationality of the square root of 2

The Greeks only believed in *rational numbers*, those numbers of the form $r = \frac{a}{b}$ for integers a and b.

It is believed that Hippasus was the first person to prove that the square root of 2 cannot be rational, and therefore it is possible to construct an irrational number using a ruler and compass.

You can read the story of Hippasus here.

Irrationality of the square root of 2

The idea of the proof is very simple (you may have seen it before).

Suppose (for contradiction) that $\sqrt{2} = \frac{a}{b}$ for some integers a and b. Assume also that a and b have no common factors (if they do then we can cancel them). Then

$$\sqrt{2}b = a \Rightarrow 2b^2 = a^2$$

and so a^2 is even, therefore a is even and so we can write $a = 2c$ for some integer c. Therefore

$$2b^2 = 4c^2 \Leftrightarrow b^2 = 2c^2$$

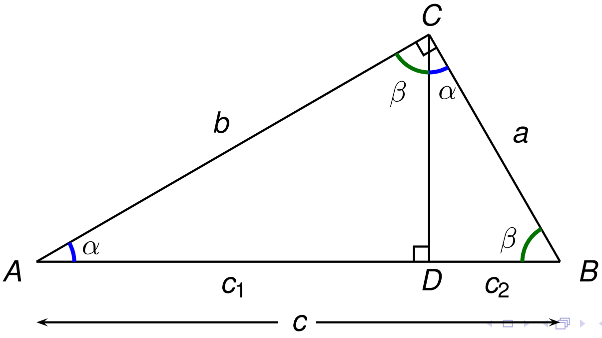
and so b^2 is even, therefore b is even. So a and b are both divisible by 2, contradicting our original assumption.

A new proof of Pythagoras' theorem

The idea behind this construction of $\sqrt{2}$ leads to a new proof of Pythagoras' theorem.

Theorem. (Pythagoras) In a right-angled triangle where c denotes the length of the hypotenuse and a, b denote the length of the legs, we have $a^2 + b^2 = c^2$.

Strategy of Proof. The idea is to divide up the triangle into two similar triangles and prove that $a^2 = c_2 c$ and $b^2 = c_1 c$.



A new proof of Pythagoras' theorem (cont.)

Proof. First note that each of the triangles ΔABC, ΔACD and ΔCBD have the angles α, β and 90 degrees. Therefore they are similar, and so their sides are in proportion.

This implies that

$$\frac{b}{c} = \frac{c_1}{b} \quad \text{and} \quad \frac{a}{c} = \frac{c_2}{a}$$

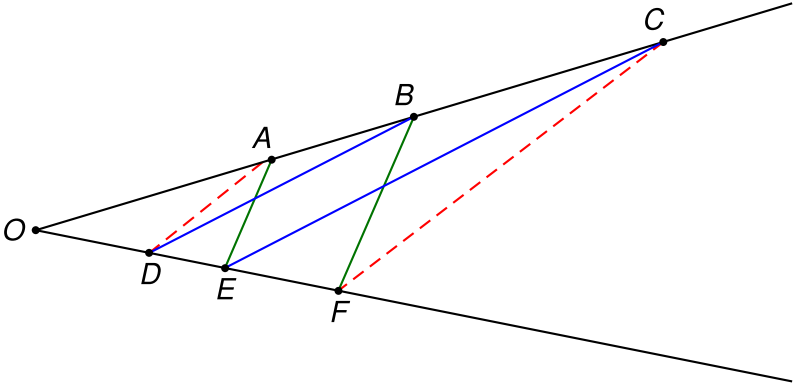
Therefore $b^2 = c_1 c$ and $a^2 = c_2 c$. Adding these together gives us

$$a^2 + b^2 = c_1 c + c_2 c = (c_1 + c_2)c = c^2$$

∎

A special case of Pappus' theorem

Theorem (Pappus). Suppose that A, B, C lie on the same line (collinear), and D, E, F lie on another line as in the diagram below. If AE is parallel to BF and DB is parallel to EC, then AD is parallel to CF.



Click here for an interactive picture.

Proof of a special case of Pappus' theorem

Proof. The goal of the proof is to use Thales' theorem.

We know that AE is parallel to BF, therefore (by Thales)

$$\frac{|OA|}{|OE|} = \frac{|OB|}{|OF|}$$

Similarly, we know that DB is parallel to EC and so (again by Thales)

$$\frac{|OD|}{|OB|} = \frac{|OE|}{|OC|}$$

Therefore

$$\frac{|OA|}{|OC|} = \frac{|OA|}{|OE|} \cdot \frac{|OE|}{|OC|} = \frac{|OB|}{|OF|} \cdot \frac{|OD|}{|OB|} = \frac{|OD|}{|OF|}$$

Thales' theorem (again) shows that AD is parallel to CF. ∎

Some comments on the proof

Note that we used Thales' theorem in two different ways.

- Statements about parallel lines lead to statements about ratios of lengths

$$AE \text{ parallel to } BF \Rightarrow \frac{|OA|}{|OE|} = \frac{|OB|}{|OF|}$$

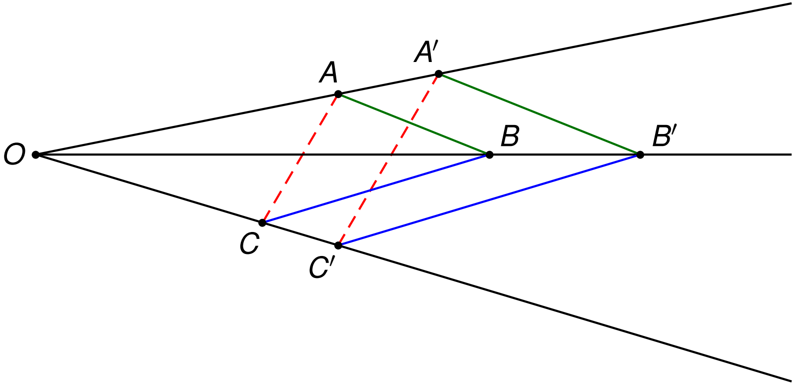
- Statements about ratios of lengths lead to statements about parallel lines

$$\frac{|OA|}{|OC|} = \frac{|OD|}{|OF|} \Rightarrow AD \text{ is parallel to } CF$$

In addition to knowing the statement of a theorem, it is also important to know the techniques for using the theorem.

A special case of Desargues' theorem

Theorem (Desargues). Suppose that A and A' are collinear, B and B' are collinear, and C and C' are collinear as in the diagram below. If AB is parallel to $A'B'$ and BC is parallel to $B'C'$ then AC is parallel to $A'C'$.



Click here for an interactive picture.

Proof of a special case of Desargues' theorem

Proof. We use a similar idea to the proof of Pappus' theorem (i.e. we use Thales' theorem and then solve an equation involving ratios of side lengths).

We know that AB is parallel to $A'B'$. Therefore (by Thales)

$$\frac{|OA|}{|OA'|} = \frac{|OB|}{|OB'|}$$

Similarly, we know that BC is parallel to $B'C'$. Therefore (Thales again)

$$\frac{|OB|}{|OB'|} = \frac{|OC|}{|OC'|}$$

Therefore

$$\frac{|OA|}{|OA'|} = \frac{|OB|}{|OB'|} = \frac{|OC|}{|OC'|}$$

and so Thales' theorem shows that AC is parallel to $A'C'$. ∎

Construction Exercises

Do the following exercises using either

- A ruler, compass and some scrap paper
- An iPad with the “Apollonius” or “Geogebra” app installed
- The construction problems page on the course website.

Bisecting a line. Given two points A and B and a line segment AB, construct a point C on AB such that $|AC| = |BC|$.

Bisecting an angle. Given points A, B, C and an angle $\alpha = \angle ABC$, construct a point D such that $\angle ABD = \angle CBD = \frac{1}{2}\alpha$.

Constructing a perpendicular through a given point I. Given a line AB and a point C on AB, construct a point D such that the line CD is perpendicular to AB.

Constructing a perpendicular through a given point II. Given a line AB and a point C not on AB, construct a point D on AB such that CD is perpendicular to AB.

Construction Exercises (cont.)

Dividing a line segment into equal lengths. Given a line segment AB and an integer n, construct points $A_1, \ldots, A_{n-1}$ on AB such that $|AA_1| = |A_1A_2| = \cdots |A_{n-1}B| = \frac{1}{n}|AB|$.

Multiplication in Euclidean Geometry. Given collinear points O, A, B and C such that $|OA| = 1$, $|OB| = x$ and $|OC| = y$, construct a point D collinear with O, A, B, C such that $|OD| = xy$.

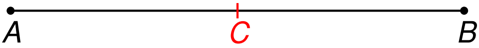
Commutativity and associativity of multiplication. Using the multiplication construction from the previous question, can you prove that $xy = yx$ (commutativity) and $(xy)z = x(yz)$ (associativity)?

In the following pictures, the question is asking you to construct the red points and lines. The blue points and lines are included as hints for intermediate steps in the construction.

You can also access the exercises on the website https://graemewilkin.github.io/Geometry/Constructions/

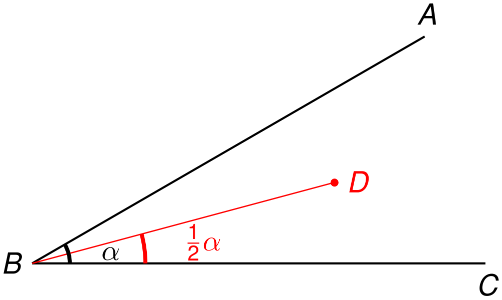
Bisecting a line segment

Given two points A and B and a line segment AB, construct a point C on AB such that $|AC| = |BC|$.



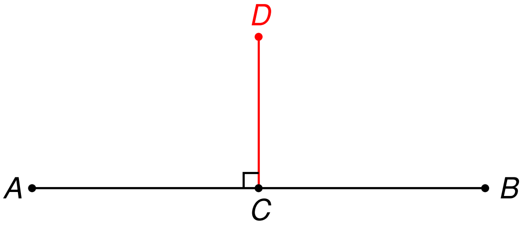
Bisecting an angle

Given points A, B, C and an angle $\alpha = \angle ABC$, construct a point D such that $\angle ABD = \angle CBD = \frac{1}{2}\alpha$.



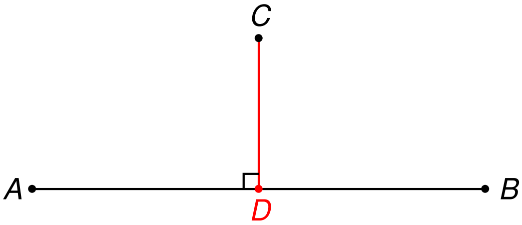
Constructing a perpendicular through a given point I

Given a line AB and a point C on AB, construct a point D such that the line CD is perpendicular to AB.



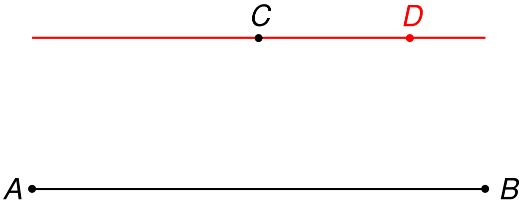
Constructing a perpendicular through a given point II

Given a line AB and a point C not on AB, construct a point D on AB such that CD is perpendicular to AB.



Constructing parallel lines

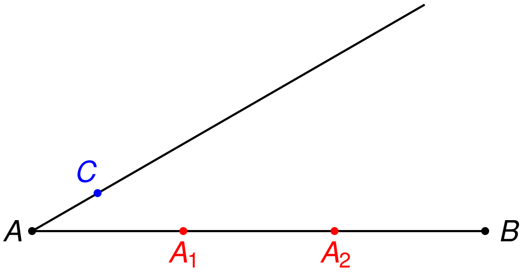
Given a line AB and a point C not on AB, construct a point D such that CD is parallel to AB.



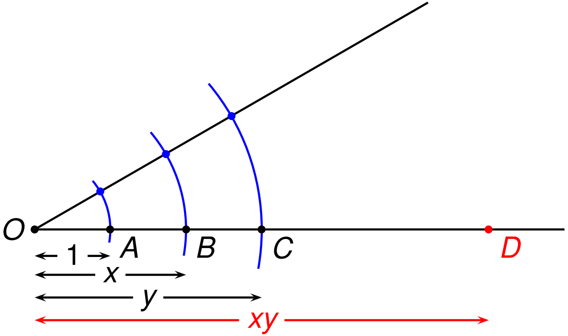
Dividing a line segment into equal lengths

Given a line segment AB and an integer n, construct points $A_1, \ldots, A_{n-1}$ on AB such that

$$|AA_1| = |A_1A_2| = \cdots |A_{n-1}B| = \frac{1}{n}|AB|$$



Given collinear points O, A, B and C such that $|OA| = 1$, $|OB| = x$ and $|OC| = y$, construct a point D collinear with O, A, B, C such that $|OD| = xy$.



Hint. Use Thales' theorem together with the configuration above.