Lecture 15: Introduction to inversive geometry

18 March, 2019

Overview

Last time.

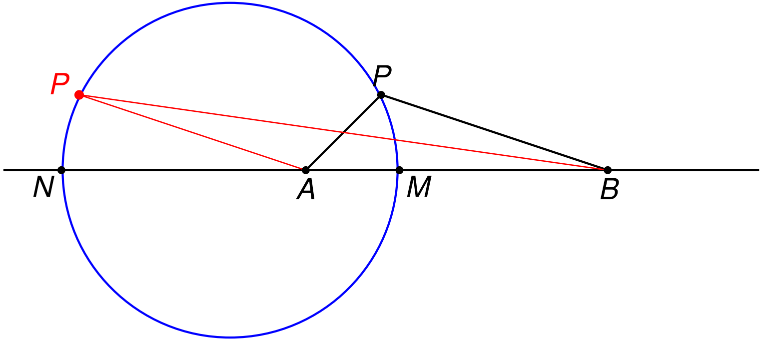
- Drawing in perspective

Today.

- The inverse of a point through a circle
- Inverting lines and circles through a fixed circle
- Properties of inversion
- Inversion preserves angles
- What happens to the centre of a circle under inversion?
- Circles fixed under inversion

Harmonic division

Recall. (Tutorial 6)
Let A and B be two distinct points in the plane. For any $\lambda > 0$, the locus of points P such that $|PA| = \lambda|PB|$ is a circle.

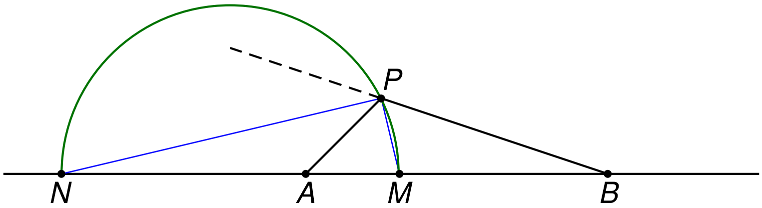


The points M and N satisfy $\frac{AM}{MB} = \lambda$ and $\frac{AN}{NB} = -\lambda$. We say that M and N are harmonic conjugates with respect to A and B.

Construction of the harmonic conjugate

The points M and N in the figure are called harmonic conjugates with respect to A and B.

M and N are said to harmonically divide the interval AB.



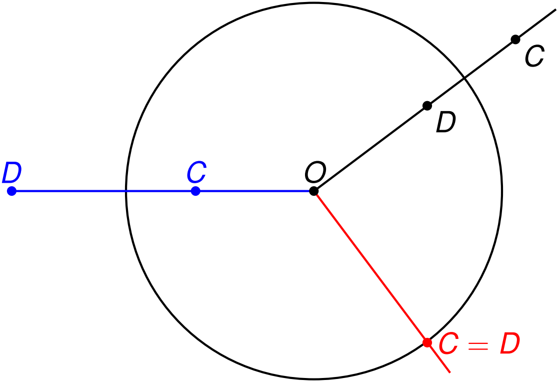
We can construct the harmonic conjugates by drawing a circle of radius μ centred at B and a circle of radius $\lambda\mu$ centred at A. As long as μ is not too small and not too large then the circles will intersect at a point P satisfying $|PA| = \lambda|PB|$.

The angle bisectors give us the harmonic conjugates M, N.

The locus of all such points P is a circle with diameter MN.

Definition of the inverse

Definition. Given a circle with centre O and radius r, let C be a point not equal to O. The inverse of C with respect to the circle is the point D on the ray $\overrightarrow{OC}$ such that $|OD| \cdot |OC| = r^2$.



Intuition behind inversion

Preliminary Observations. First note that if C is inside the circle (i.e. $|OC| < r$) then $|OD| = \frac{r^2}{|OC|} > r$, so D is outside the circle. Conversely, if $|OC| > r$ then $|OD| < r$.

Therefore, the process of inversion "turns the circle inside out" by mapping points on the inside to points on the outside and vice versa.

If C is on the circle then $|OC| = r$ and so the inverse D satisfies $|OD| = r$. Therefore the inverse of C is the same point C, and so inversion fixes all the points on the circle.

As the point C gets closer to the center O of the circle then the point D satisfies $|OD| = \frac{r^2}{|OC|}$ which becomes very large. Therefore points close to the centre map to points "far away" from the centre.

The same argument shows that points close to the edge of the circle maps to points close to the edge of the circle.

Experiment for yourself on the inversive geometry webpage.

Inversion is an involution

Proposition. If D is the inverse of C, then C is the inverse of D.

Proof. Since D is the inverse of C then $|OC| \cdot |OD| = r^2$. To find the inverse of D, we look for the unique point E on the ray $\overrightarrow{OD}$ such that $|OE| \cdot |OD| = r^2$. Since this point is unique then we must have $E = C$.

Question. What happens to the centre of the circle?

We define the point at infinity to be the inverse of the centre of the circle. The inverse of the point at infinity is defined to be the centre of the circle.

There is only one point at infinity in inversive geometry.

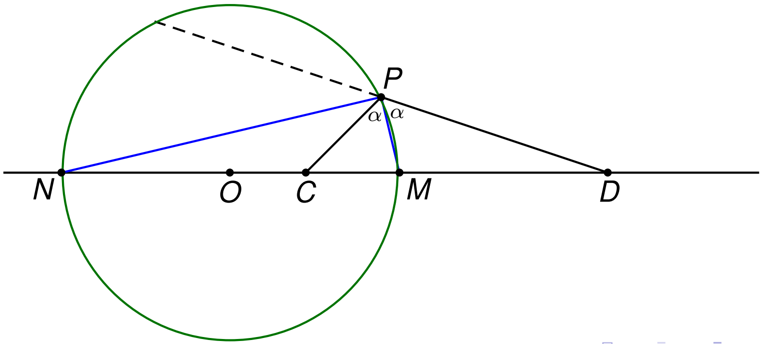
The extended plane $\mathbb{R}^2 \cup \{\infty\}$ is the Euclidean plane together with the point at infinity.

Therefore inversion is an *involution* of the extended plane (i.e. if we apply inversion twice then we get the same point back).

Relationship to the harmonic conjugate construction

Exercise. Consider a circle with centre O and let NM be a diameter of the circle. Let P be any point on the circle and let C be a point on the line between O and M. Construct D such that $\angle CPD$ is bisected by PM.

Prove that D is the inverse of C with respect to the circle.



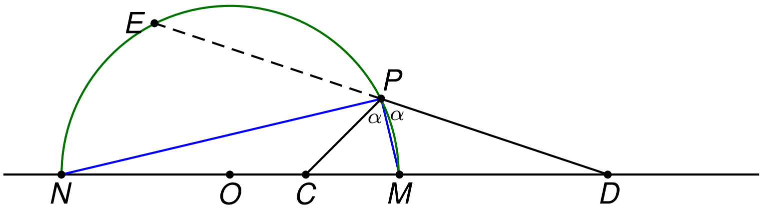
Solution.

Solution. Let E be a point on the extension of the line through P, as in the diagram below. Then PN is the angle bisector of $\angle EPC$, since $\angle NPC = 90 - \alpha$ and
$\angle EPN = 180 - (2\alpha + 90 - \alpha) = 90 - \alpha$.

Therefore we can use the angle bisector theorem to show that

$$\frac{|CM|}{|MD|} = \frac{|PC|}{|PD|} = \frac{|CN|}{|ND|}$$

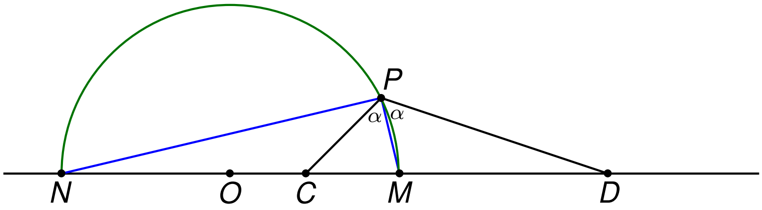
and so $|CM| \cdot |ND| = |CN| \cdot |MD|$.



Solution. (cont.) We also have $|CM| = |OM| - |OC|$, $|ND| = |NO| + |OD|$, $|CN| = |CO| + |ON|$ and $|MD| = |OD| - |OM|$. Let $r = |OM| = |ON|$. Then

$$\begin{aligned}|CM| \cdot |ND| &= (r - |OC|)(r + |OD|) \\ &= r^2 - |OC| \cdot |OD| - r(|OC| - |OD|)\end{aligned}$$

and similarly, $|CN| \cdot |MD| = |CO| \cdot |OD| - r^2 - r(|CO| - |OD|)$. Then the equation $|CM| \cdot |ND| = |CN| \cdot |MD|$ simplifies to $r^2 = |OC| \cdot |OD|$.



Which geometric objects are preserved by inversion?

In Euclidean geometry we studied lines and circles (ruler and compass geometry). We were allowed to translate and rotate these lines and circles (for example, we did this in the proof of SAS congruence of triangles).

In inversive geometry we use inversions instead of translations/rotations.

We would like to find out which geometric structures are preserved by inversion.

Then inversive geometry will involve studying these geometric structures.

First question. What happens to a line after we apply an inversion?

First, we study the simple case of a line through the centre of the circle. By definition, the inverse of any point on this line remains on the line. Therefore this line is preserved by inversion.

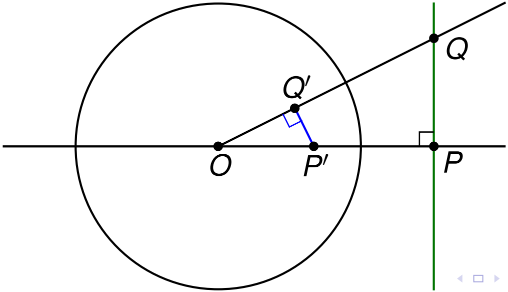
The inverse of a line

What if the line does not pass through the centre?

Consider the green line in the picture below. Draw a perpendicular from the centre of the circle to a point P on the line and let Q be any other point on the line. The inverse points P' and Q' satisfy

$$|OP| \cdot |OP'| = r^2 = |OQ| \cdot |OQ'| \quad \Leftrightarrow \quad \frac{|OP|}{|OQ|} = \frac{|OQ'|}{|OP'|}$$

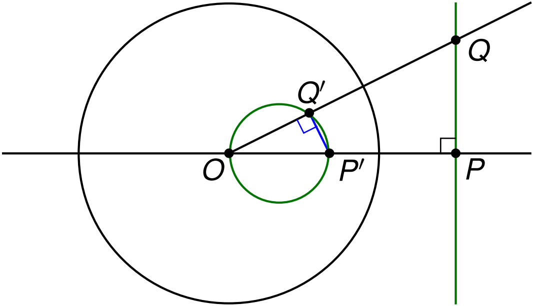
and so the triangles ΔOPQ and $\Delta OQ'P'$ are similar.



The inverse of a line

Since ΔOPQ and $\Delta OQ'P'$ are similar then we see that $\angle OQ'P' = \angle OPQ = 90^\circ$.

This is true for any point Q on the line and so Q' lies on a circle with diameter OP'.



Proposition. The inverse of a line PQ not passing through the centre O of the circle is a circle which passes through O. The centre of this circle lies on the line through O perpendicular to the original line PQ.

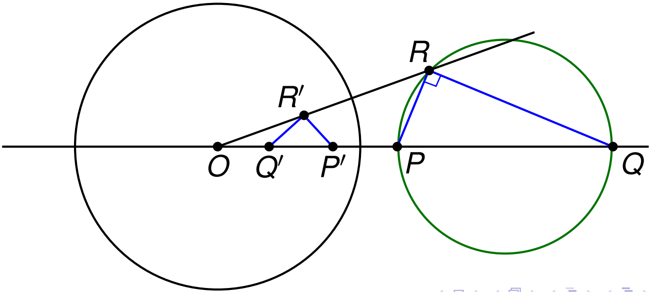
The inverse of a circle

Since inversion is an involution, we can reverse the previous proposition to show that the inverse of a circle passing through the centre O is a line (the line does not pass through O).

Second question. What is the inverse of a circle that does not pass through the centre of the original circle?

Let P, Q be a diameter such that O, P, Q are collinear, and let R be any other point on the circle.

Let P', Q', R' be the inverses of P, Q, R respectively.



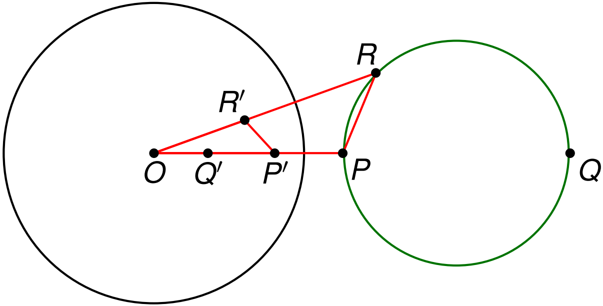
The inverse of a circle

We are going to use the same idea as before to show that $\angle P'R'Q' = 90^\circ$.

Since P', R' are the inverses of P, R respectively then

$$|OP| \cdot |OP'| = r^2 = |OR| \cdot |OR'| \quad \Leftrightarrow \quad \frac{|OP|}{|OR|} = \frac{|OR'|}{|OP'|}$$

Since $\angle POR = \angle R'OP'$ then we have $\Delta POR \sim \Delta R'OP'$ and therefore $\angle OR'P' = \angle OPR$ and $\angle OP'R' = \angle ORP$.

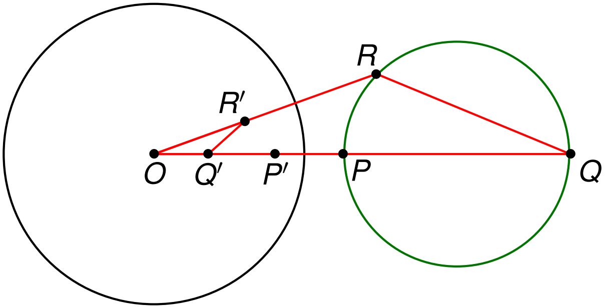


The inverse of a circle

Since Q', R' are the respective inverses of Q, R then the same argument shows that

$$|OQ| \cdot |OQ'| = r^2 = |OR| \cdot |OR'| \quad \Leftrightarrow \quad \frac{|OQ|}{|OR|} = \frac{|OR'|}{|OQ'|}$$

Since $\angle QOR = \angle R'OQ'$ then we have $\Delta QOR \sim \Delta R'OQ'$ and therefore $\angle OR'Q' = \angle OQR$ and $\angle OQ'R' = \angle ORQ$.



The inverse of a circle

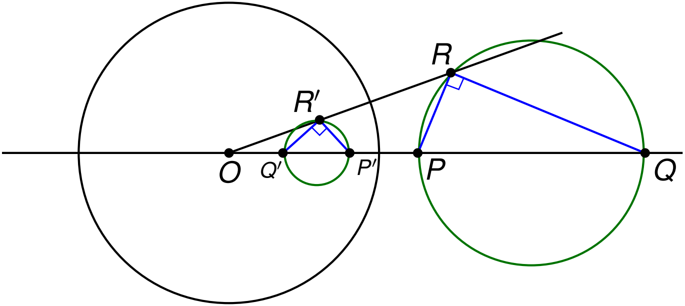
Therefore we can conclude that

$$\angle Q'R'P' = \angle OR'P' - \angle OR'Q' = \angle OPR - \angle OQR$$

Since the angle sum of a triangle is 180° then we also have

$$\angle OPR - \angle OQR = 180 - \angle QPR - \angle PQR = \angle PRQ = 90^\circ$$

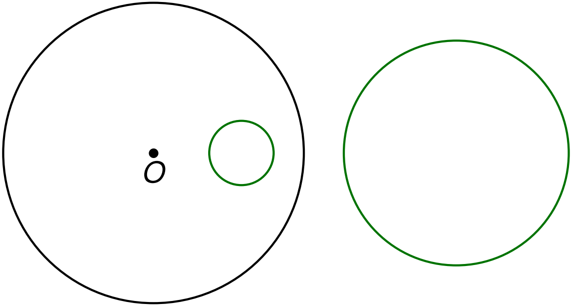
Therefore $\angle Q'R'P' = 90^\circ$ and so the point R' lies on a circle with diameter $Q'P'$.



The inverse of a circle

Together with our previous result on the inverse of a line, this tells us how lines and circles behave under inversion.

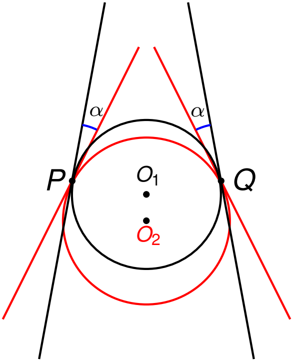
Proposition. Consider a circle in the plane with centre O. With respect to this circle, the inverse of another circle which passes through O is a line not passing through O.
The inverse of a circle which does not pass through O is another circle not passing through O.



Click here to investigate this on the Geometry Website.

Exercise

Exercise. Suppose that two circles intersect at points P and Q as in the diagram below. Show that the angle between the tangent lines is the same at both P and Q.



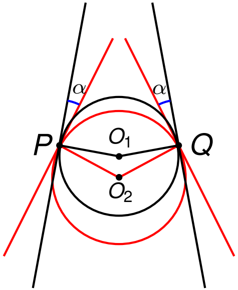
Solution

Solution.

Solution

Solution. Recall that the orthogonal to the tangent passes through the centre of the circle. It is sufficient to show that the angle between the orthogonals is the same.

Let O_1 and O_2 be the centres of the circles. Since the triangles ΔPO_1Q and ΔPO_2Q are isosceles, then $\angle QPO_1 = \angle PQO_1$ and $\angle QPO_2 = \angle PQO_2$. Therefore $\angle O_1PO_2 = \angle O_1QO_2$ and so the angle between the orthogonals is the same. Therefore the angle between the tangents is the same also.

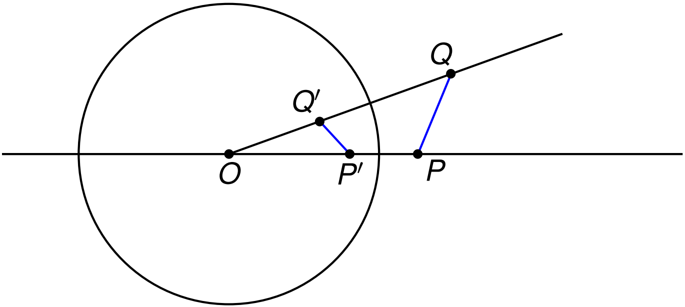


What happens to distance under inversion?

Intuitively, we can see that the distance between two points P and Q will change when we apply an inversion to get two new points P' and Q' respectively.

Question. Can we write down a formula for $|P'Q'|$ in terms of $|PQ|$ and the radius of the circle?

We can answer this by using similar triangles as before.



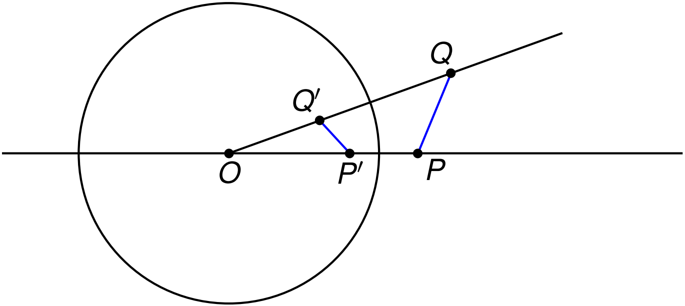
What happens to distance under inversion?

Since $|OP| \cdot |OP'| = r^2 = |OQ| \cdot |OQ'|$ then $\frac{|OP|}{|OQ|} = \frac{|OQ'|}{|OP'|}$.

Since $\angle POQ = \angle Q'OP'$ then $\Delta POQ \sim \Delta Q'OP'$.

Therefore

$$\frac{|Q'P'|}{|PQ|} = \frac{|OQ'|}{|OP|} = \frac{|OQ| \cdot |OQ'|}{|OQ| \cdot |OP|} = \frac{r^2}{|OQ| \cdot |OP|}$$

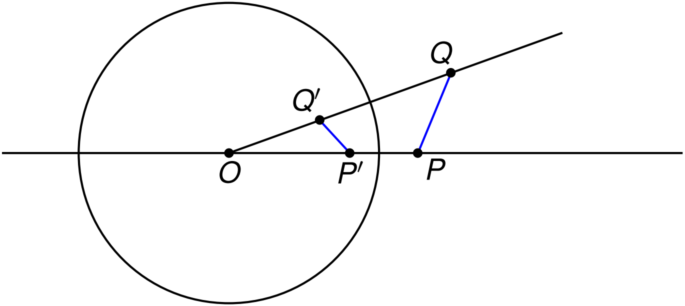


What happens to distance under inversion?

We have proven the following result.

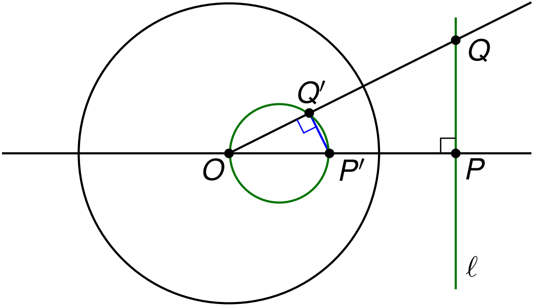
Proposition. Consider a circle with centre O and radius r. Let P and Q be two points distinct from O. Then the distance between the inverse points P' and Q' is

$$|Q'P'| = \frac{r^2|PQ|}{|OQ| \cdot |OP|}$$



Recall from earlier. Consider a circle with centre O and radius r. Let ℓ be any line in the plane. Then either

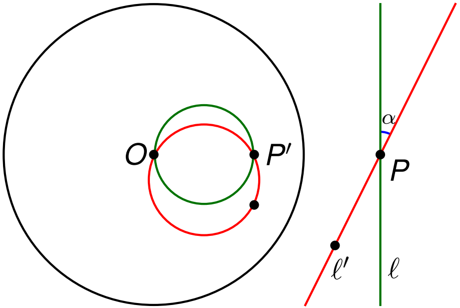
- ℓ passes through O, in which case the inverse of ℓ is the same line ℓ, or
- ℓ does not pass through O, in which case the inverse of ℓ is a circle passing through O.



From our construction, we see that the tangent to the image circle at O is parallel to the original line.

Angles are preserved by inversion

Question. Suppose two lines ℓ and ℓ' intersect at a point P. What happens to the angle α at P under inversion?

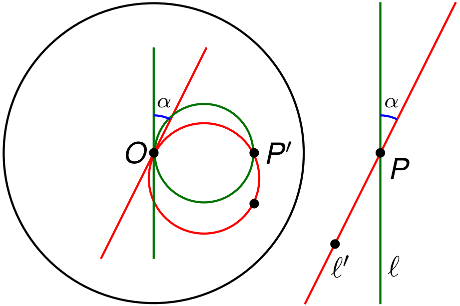


We know that the inverse of ℓ' is a circle passing through O. It also passes through P' (since ℓ' passes through P) and the tangent at O is parallel to ℓ'.

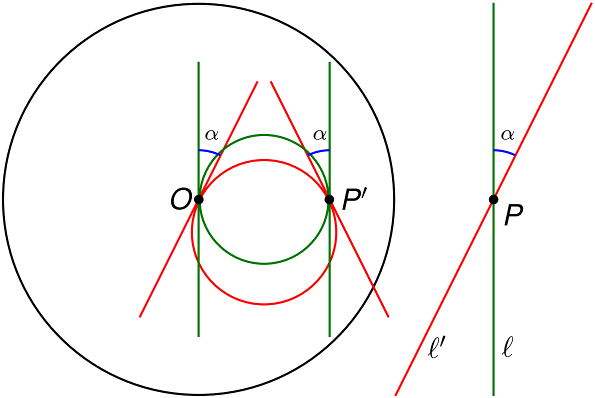
Angles are preserved by inversion

Since the tangents to the circles at O are parallel to the original lines, then the angle between the tangents must be equal to α.

Now the exercise from earlier today shows that the angle between the tangents at the point P' is also α.



Angles are preserved by inversion (picture)



Proposition. Let ℓ and ℓ' be two lines intersecting at P. The inverses are circles ω and ω' intersecting at P'. Then the angle between the lines at P is equal to the angle between the tangents to the circles at P'.

Click here to investigate this on the Geometry Website.

Next time

We will continue to study inversive geometry

- Circles fixed by inversion
- Applications of inversive geometry: The problem of Apollonius, Steiner's porism
- Constructing inversions using a ruler and compass.

Some problems (e.g. the problem of Apollonius) are difficult if we consider them in general position. One of the main applications of inversive geometry is to simplify such problems so that they are easy to solve.