Lecture 20 - Hyperbolic geometry on the pseudosphere

12 April, 2019

Overview

Last time.

- Introduction to geometry on the Poincaré disk
- Rigid motions on the Poincaré disk
- What happens when we change the parallel axiom?

Today.

- Defining hyperbolic geometry via the hyperboloid
- Different models for hyperbolic geometry
- Hyperbolic trigonometry
- The angle sum of a triangle in hyperbolic geometry
- Curvature and the angle defect of a triangle

Information about the final

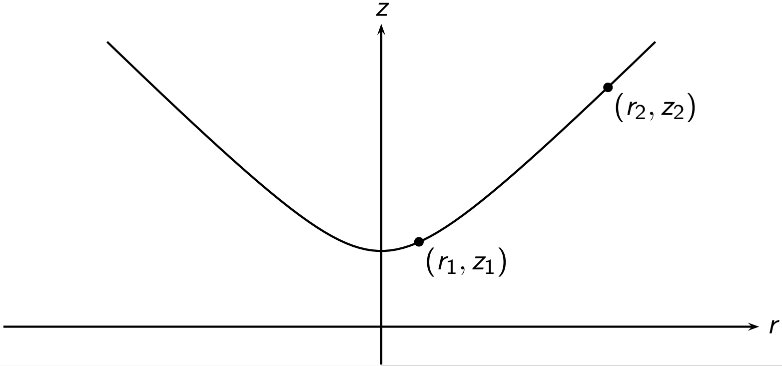
- The final exam is on Wednesday 8 May.
- Approx. two-thirds of the questions will be based on the second half of the course (conics, projective geometry, inversive geometry, spherical and hyperbolic geometry) and approx. one-third will come from the first half of the course (Euclidean geometry).
- There will be 6 questions.
- The questions will be of varying difficulty.
- I will post final exams from the past three years next Thursday (18 April) and I will post solutions one week before the exam (1 May).

Exercise

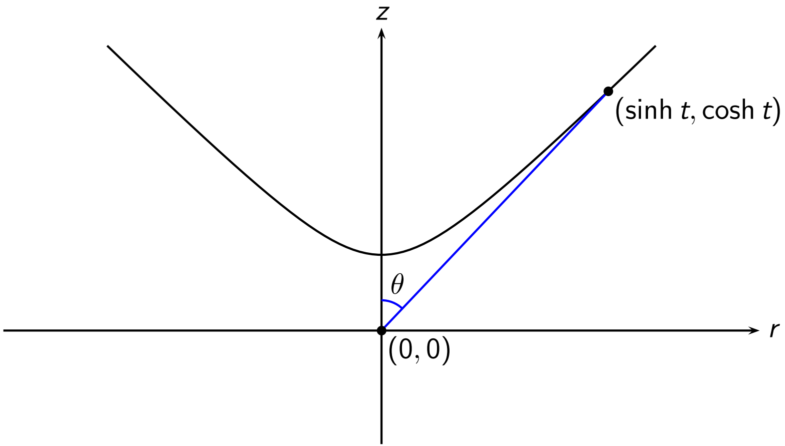
Exercise 1. Compute the arclength of the hyperbola $z^2 = r^2 + 1$ between the points (r_1, z_1) and (r_2, z_2) using the formula

$$d((r_1, z_1), (r_2, z_2)) = \int_{r_1}^{r_2} \sqrt{1 - \left(\frac{dz}{dr}\right)^2}\, dr \quad \text{(hyperbolic distance)}$$

Hint. Parametrise the curve using $r = \sinh t, z = \cosh t$ and use the formula $\cosh^2 t - \sinh^2 t = 1$.



Exercise 2. Use your previous answer to express the hyperbolic arclength from $(0,1)$ to $(\sinh t, \cosh t)$ in terms of the angle θ in the diagram below.



Solution

Solution 1. Substituting $r = \sinh t$, $z = \cosh t$ gives us

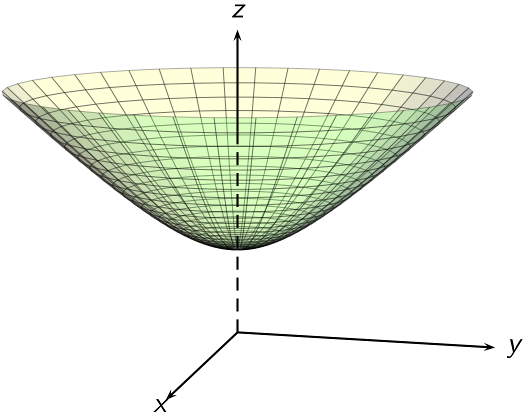
$$\begin{aligned} d((r_1, z_1), (r_2, z_2)) &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dr}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\cosh^2 t - \sinh^2 t}\, dt \\ &= \int_{t_1}^{t_2} 1\, dt = (t_2 - t_1) \\ &= \sinh^{-1} r_2 - \sinh^{-1} r_1 \end{aligned}$$

Solution 2. When $t_1 = 0$ and $t_2 = t$ then the hyperbolic distance is given by t. Therefore

$$\tan\theta = \frac{\sinh t}{\cosh t} = \tanh t$$

So $\theta = \arctan(\tanh t)$.

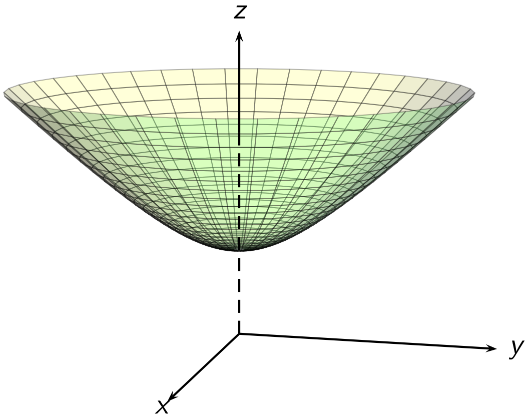
A model for hyperbolic geometry



Consider the surface $z = \sqrt{x^2 + y^2 + 1}$ (one sheet of a hyperboloid). This has a parametrisation given by

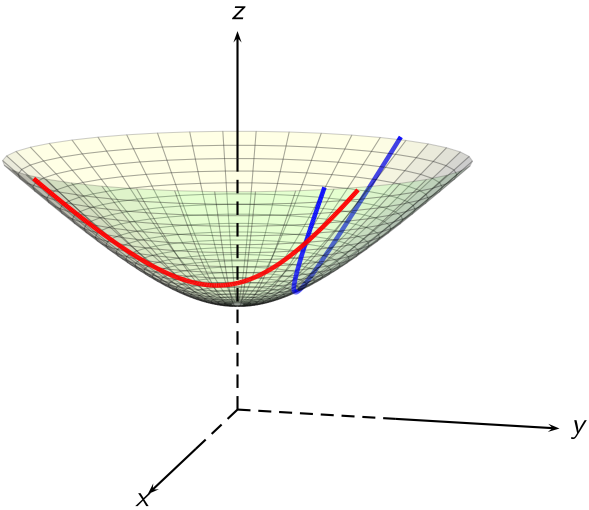
$$(\theta, t) \mapsto (\cos\theta \sinh t, \sin\theta \sinh t, \cosh t), \quad \theta \in [0, 2\pi),\ t \in [0, \infty)$$

A model for hyperbolic geometry



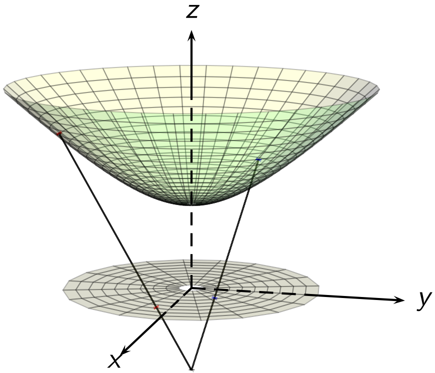
The *points* of hyperbolic space are the points on the surface of the hyperboloid. The *lines* of hyperbolic space are the intersections of this surface with a plane through the origin.
(In spherical geometry we did the same thing, but using the sphere instead of the hyperboloid.)

Picture of lines in hyperbolic space



It is often easier to visualise the lines by projecting them onto a disk.

The Poincaré disk model is obtained by projecting the lines from the hyperboloid to the xy-plane using the point $(0, 0, -1)$ as a point of perspective.



Lines in the Poincaré disk model

Question. What happens to the hyperbolic lines after we project to the Poincaré disk?

First consider a general plane through the origin $ax + by + cz = 0$. If $c = 0$ then this plane is perpendicular to the xy-plane and therefore projects to a straight line in the xy-plane, which passes through the origin.

If $c \neq 0$ then we can multiply the equation of the plane by $\frac{1}{c}$ to get $z = a_0 x + b_0 y$ where $a_0 = -\frac{a}{c}$ and $b_0 = -\frac{b}{c}$. Then rotate clockwise by an angle θ so that $\tan\theta = \frac{b_0}{a_0}$ to get new coordinates

$$(x_1, y_1) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$$

The equation of the plane then becomes

$$z = x_1(a_0\cos\theta + b_0\sin\theta) = a' x_1$$

Lines in the Poincaré disk model

Substituting $z = x_1(a_0 \cos\theta + b_0 \sin\theta) = a' x_1$ into the equation $z^2 = (x_1)^2 + (y_1)^2 + 1$ gives us an equation satisfied by the (x_1, y_1)-coordinates of a point on the intersection of the plane and the hyperbola

$$0 = (1 - (a')^2)(x_1)^2 + (y_1)^2 + 1$$

The projection onto the xy-plane is

$$(u, v) = \left(\frac{x_1}{z+1}, \frac{y_1}{z+1}\right) = \left(\frac{x_1}{a' x_1 + 1}, \frac{y_1}{a' x_1 + 1}\right)$$

Solving for x_1 in terms of u gives us $x_1 = \frac{u}{1 - a' u}$, which we can then solve to get $\frac{1}{a' x_1 + 1} = 1 - a' u$.

Lines in the Poincaré disk model

The equation of the projection of the line onto the xy-plane then becomes

$$\begin{aligned} 0 &= \frac{(1-(a')^2)(x_1)^2}{(1+a'x_1)^2} + \frac{(y_1)^2}{(1+a'x_1)^2} + \frac{1}{(1+a'x_1)^2} \\ &= (1-(a')^2)u^2 + v^2 + (1-a'u)^2 \\ &= u^2 + v^2 + 1 - 2a'u \end{aligned}$$

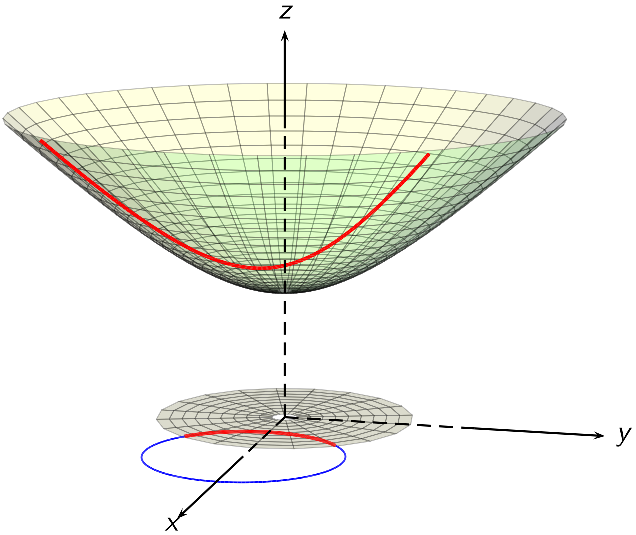
We can then complete the square to obtain

$$(u-a')^2 + v^2 = (a')^2 - 1$$

which is the equation of a circle of radius $\sqrt{(a')^2-1}$ which is centred at $(a', 0)$.

We can then rotate anticlockwise by θ to get the projection of the original hyperbolic line. Since circles are preserved by rotation then this projection is a circle of radius $\sqrt{(a')^2-1}$ centred at $(a'\cos\theta, a'\sin\theta)$.

The image of a hyperbolic line in the Poincaré disk.

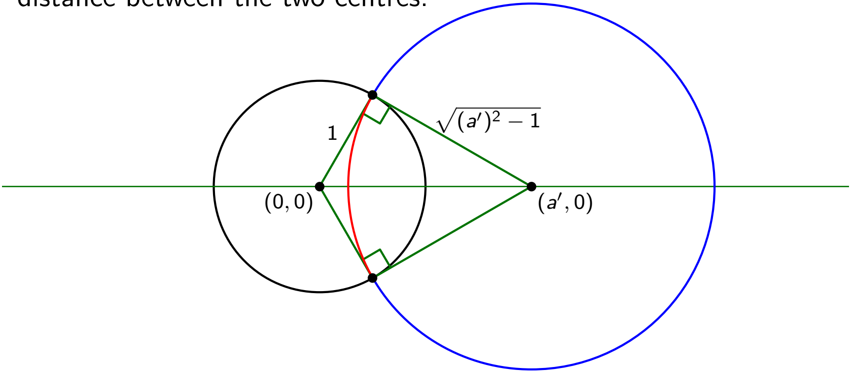


The Poincaré disk model

Recall that the equation of the circle is

$$(u - a')^2 + v^2 = (a')^2 - 1$$

which has centre $(a', 0)$ and radius $\sqrt{(a')^2 - 1}$. Therefore it is orthogonal to the circle with centre $(0, 0)$ and radius 1, since the sum of the squares of the radii is equal to the square of the distance between the two centres.



The Poincaré disk model

Therefore we have seen that hyperbolic lines (on the hyperbola) correspond to circles orthogonal to the unit circle (on the Poincaré disk).

A limiting case occurs when the plane in $\mathbb{R}^3$ is parallel to the z-axis. This corresponds to a hyperbolic line passing through the base of the hyperboloid, which projects to a line passing through the origin.

By thinking of lines through the origin as the limiting case of a circle as the radius becomes infinitely large (see Tutorial 9) we can think of all lines as circles orthogonal to the original circle.

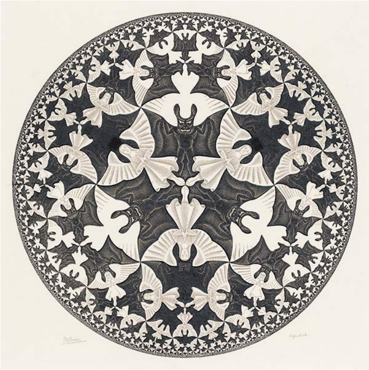
Definition. A *line* in the Poincaré disk is the intersection of the unit disk with a circle orthogonal to the boundary of the unit disk.

It is much easier to analyse lines on the Poincaré disk.

Definition. The *angle* between two lines in the Poincaré disk is the Euclidean angle between the Euclidean tangent lines.

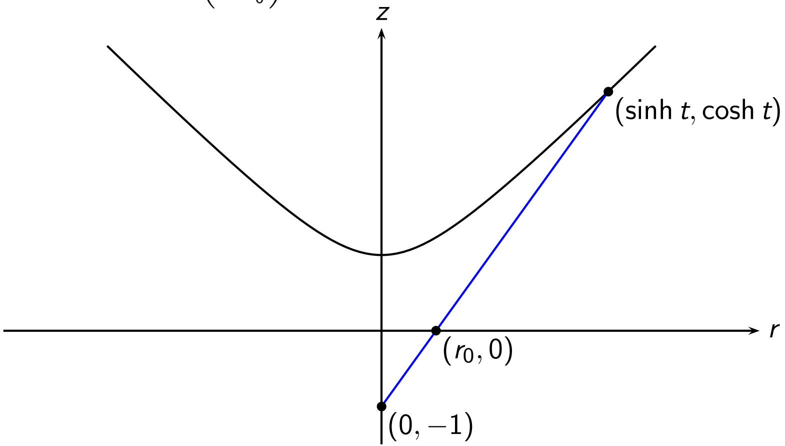
Pictures of lines in the Poincaré disk

Once again, recall the following pictures of M.C. Escher. In the first picture you can clearly see the lines in the Poincaré disk, and in the second picture you can trace the lines through the head and feet of the angels and devils.



You can read more about it here.

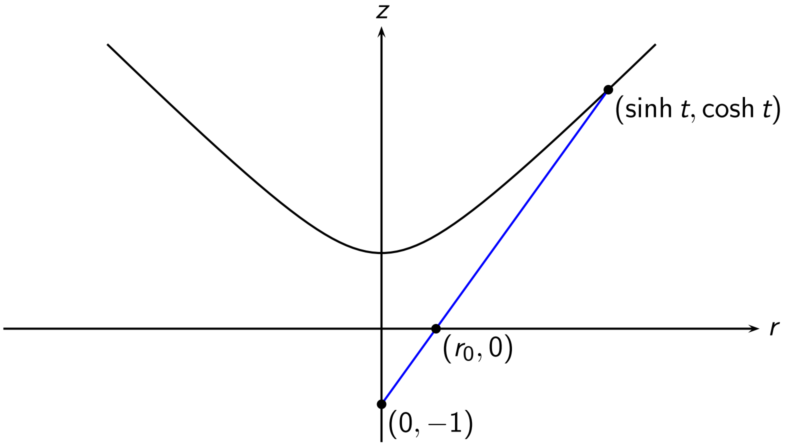
Exercise. Let $(\sinh t, \cosh t)$ be a point on the hyperbola $z^2 = r^2 + 1$. Find the coordinates $(r_0, 0)$ of the projection and prove that $t = \log\left(\frac{1+r_0}{1-r_0}\right)$.



This is the hyperbolic distance from 0 to r_0 in the Poincaré disk.

Solution

Solution. Using similar triangles, we see that $\frac{r_0}{1} = \frac{\sinh t}{\cosh t + 1}$.



We can then substitute $\sinh t = \frac{1}{2}(e^t - e^{-t})$ and $\cosh t = \frac{1}{2}(e^t + e^{-t})$ and solve the resulting equation for t.

Solution. (cont.) Therefore we have

$$\frac{1+r_0}{1-r_0} = \frac{1+\frac{\sinh t}{\cosh t+1}}{1-\frac{\sinh t}{\cosh t+1}} = \frac{\cosh t + 1 + \sinh t}{\cosh t + 1 - \sinh t}$$

Substituting $\sinh t = \frac{1}{2}(e^t - e^{-t})$ and $\cosh t = \frac{1}{2}(e^t + e^{-t})$ gives us

$$\cosh t + \sinh t = e^t, \quad \cosh t - \sinh t = e^{-t}$$

and so

$$\frac{1+r_0}{1-r_0} = \frac{1+e^t}{1+e^{-t}} = \frac{e^{t/2}}{e^{-t/2}} \cdot \frac{e^{-t/2}+e^{t/2}}{e^{t/2}+e^{-t/2}} = e^t$$

Therefore the hyperbolic distance from $r = 0$ to a point of radius $r = r_0$ in the Poincaré disk model is

$$t = \log\left(\frac{1+r_0}{1-r_0}\right)$$

Hyperbolic distance (preliminary observations)

We just showed that the hyperbolic distance (denoted $d_{hyp.}$) from $(0,0,1)$ to a point $(\sinh t, 0, \cosh t)$ is given by t.

Equivalently

$$\cosh d_{hyp.}((0,0,1),(\sinh t, 0, \cosh t)) = \cosh t$$

In the previous exercise we showed that (on the Poincaré disk) the hyperbolic distance from $(0,0)$ to $(r,0)$ is also $t = \log\left(\frac{1+r}{1-r}\right)$.

By rotating the coordinate system (a rigid motion), we see that the hyperbolic distance from $(0,0)$ to $(r\cos\theta, r\sin\theta)$ is also $t = \log\left(\frac{1+r}{1-r}\right)$.

The same idea (rotating the coordinate system) shows that (on the hyperbola) the hyperbolic distance from $(0,0,1)$ to $(\sinh t\cos\theta, \sinh t\sin\theta, \cosh t)$ is given by t.

Therefore we can compute the distance from the basepoint $(0,0,1)$ of the hyperbola to any point on the hyperbola

Hyperbolic distance (preliminary observations)

Theorem. In general, the hyperbolic distance $d_{hyp.}(P, Q)$ between two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ on the hyperboloid is

$$\cosh d_{hyp.}(P, Q) = z_1 z_2 - x_1 x_2 - y_1 y_2$$

We won't prove the theorem, but you can quote the result

Remark. In spherical geometry we have

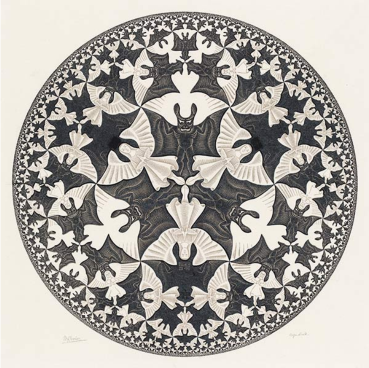
$$\cos d_{sph.}(P, Q) = z_1 z_2 + x_1 x_2 + y_1 y_2$$

This is just the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$ for the dot product of two vectors $\mathbf{v}$ and $\mathbf{w}$.

We can think of $z_1 z_2 - x_1 x_2 - y_1 y_2$ as the *hyperbolic dot product* of two vectors (x_1, y_1, z_1) and (x_2, y_2, z_2) on the hyperbola.

Hyperbolic distance and M.C. Escher

Returning to Escher's pictures, the objects are all the same size, but they appear to get smaller as they approach the boundary of the disk, since the lengths are distorted by the function $t = \log\left(\frac{1+r}{1-r}\right)$.



You can read more about it here.

Hyperbolic trigonometry

Definition. A hyperbolic triangle ΔPQR consists of three vertices P, Q, R and three hyperbolic lines PQ, QR and PR joining the vertices.

The hyperbolic angle at Q between the two lines PQ and QR is the angle between the two planes through the origin defining PQ and QR respectively (use the hyperbolic dot product).

Equivalently, on the Poincaré disk, the hyperbolic angle between two lines is the angle between the tangents to these lines.

Theorem. (Hyperbolic cosine formula) Let ΔPQR be a hyperbolic triangle and let $a = d_{hyp.}(Q, R)$, $b = d_{hyp.}(P, Q)$ and $c = d_{hyp.}(P, R)$. Let $\alpha = \angle RPQ$, $\beta = \angle QRP$ and $\gamma = \angle PQR$. Then

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha$$

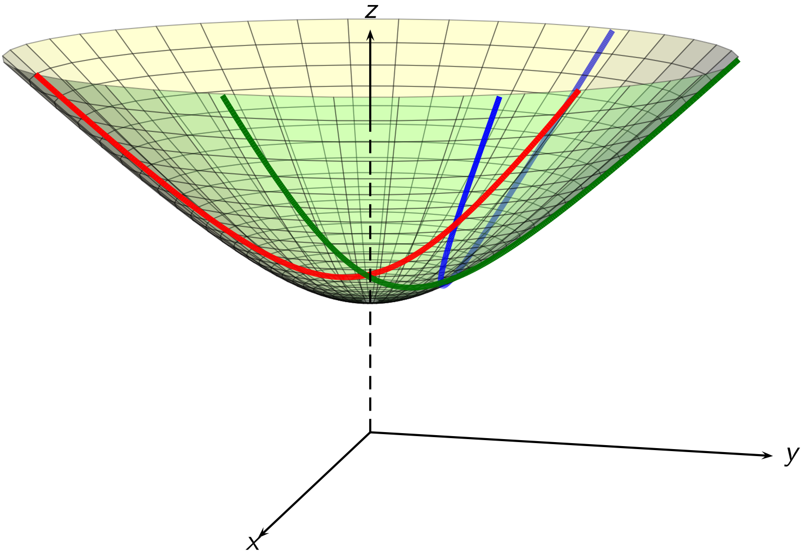
Hyperbolic trigonometry

Remark. Compare the hyperbolic cosine formula with the spherical cosine formula

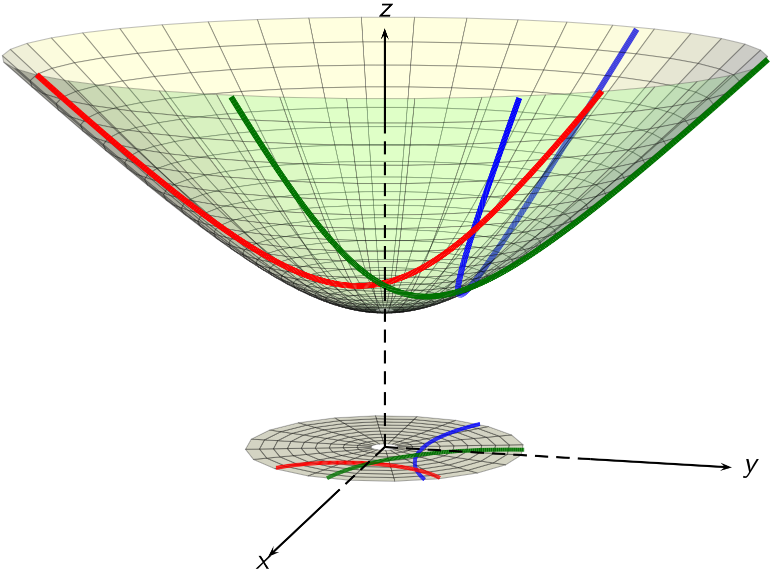
$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha \quad (\text{hyperbolic})$$
$$\cos a = \cos b \cos c + \sin b \sin c \cos\alpha \quad (\text{spherical})$$

We see that in hyperbolic geometry we use the hyperbolic functions cosh and sinh instead of the trigonometric functions cos and sin to determine distance.

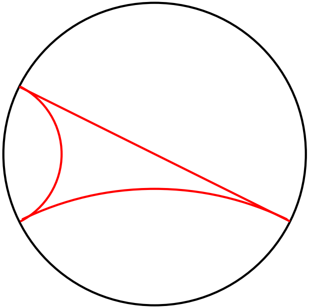
Picture of a hyperbolic triangle on the hyperboloid



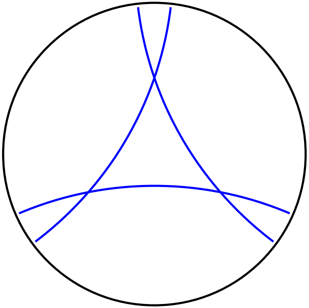
The image of a hyperbolic triangle in the Poincaré disk.



More examples of triangles on the Poincaré disk

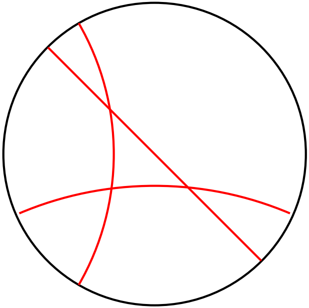


An ideal triangle
(All vertices are on the boundary)

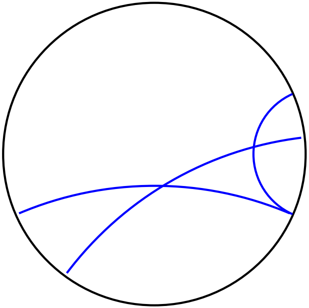


An equilateral triangle

More examples of triangles on the Poincaré disk



A triangle with one side passing through the origin.



A triangle with one ideal vertex. Note that this triangle does not contain the origin.

Proof of hyperbolic cosine formula

Proof. After applying a rigid motion (which preserves angles and lengths), we can assume that the (x, y, z) coordinates of the points on the hyperboloid are

$$\begin{aligned} P &= (0, 0, 1) \\ Q &= (\sinh b, 0, \cosh b) \\ R &= (\sinh c \cos \alpha, \sinh c \sin \alpha, \cosh c) \end{aligned}$$

Recall that the hyperbolic distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\cosh d_{hyp.}(Q, R) = z_1 z_2 - x_1 x_2 - y_1 y_2.$$

Therefore

$$\cosh a = \cosh d_{hyp.}(Q, R) = \cosh b \cosh c - \sinh b \sinh c \cos \alpha$$

Remark. Most of the work in proving this formula is done in proving the distance formula. Once we have this formula then the cosine rule follows easily.

The angle sum of a triangle in hyperbolic geometry

Theorem. Let ΔPQR be a hyperbolic triangle with angles $\alpha = \angle RPQ$, $\beta = \angle PQR$ and $\gamma = \angle QRP$. Then

$$\alpha + \beta + \gamma = \pi - \text{Area}(\Delta PQR)$$

Remark.

- To prove this formula properly requires a formula for the area, which requires doing a double integral over the hyperbola. Alternatively, one could derive a formula for the area in terms of a double integral on the Poincaré disk.
  Unfortunately we don't have time to prove this in detail.
- Since the area is positive, then we immediately see that the angle sum is always less than π.
  Conversely, since the angle sum is always positive then the area is always less than π.
- We can also take one or more the vertices to be at infinity on the hyperbola (equivalently, they are on the boundary of the Poincaré disk). The angle at these vertices is zero.

Consequences of the angle sum formula

We have seen the angle sum formula in three different geometries

$$\alpha + \beta + \gamma = \pi + \text{Area}(\Delta PQR) \quad \text{(Spherical Geometry)}$$
$$\alpha + \beta + \gamma = \pi \quad \text{(Euclidean Geometry)}$$
$$\alpha + \beta + \gamma = \pi - \text{Area}(\Delta PQR) \quad \text{(Hyperbolic Geometry)}$$

Therefore triangles in Spherical Geometry are "fatter" than triangles in Euclidean geometry, and triangles in Hyperbolic Geometry are "thinner" than triangles in Euclidean geometry.

In differential geometry we express this by saying that the curvature of spherical geometry is positive (angle excess) and the curvature of hyperbolic geometry is negative (angle defect).

We saw in Lecture 17 that this has consequences for mapping projections (there is no nice way to map the spherical Earth onto a Euclidean plane).

Consequences of the angle sum formula (for interest)

Another consequence that is easy to see explicitly is the impossibility of deforming a flat piece of paper (Euclidean $\Rightarrow$ zero angle defect) into a saddle shape (which has negative curvature $\Rightarrow$ angle defect).

You can see this in the pictures of the slices of pizza on the next two slides. Even though the pizza is very weak structurally, if you hold it the right way it will support its own weight.

Consequences of the angle sum formula (for interest)

If you hold the pizza by the crust then the end of the pizza will sag, making it difficult to eat.

Consequences of the angle sum formula (for interest)

If you squeeze the middle of the crust in the right way then the pizza will support its own weight. For the end to sag, the pizza would have to deform into a saddle shape and so the area of triangles on the surface will have to change (which can't happen without tearing the pizza).

If you are interested in reading more then you can click here or here.

Good luck!

No lecture next week.

Consultations also by appointment (just drop me an e-mail).

Watch out for past exams on IVLE next Thursday 18 April. I will post solutions on Wednesday 1 May.

Exam. Wednesday 8 May