Lecture 7: Properties of cyclic quadrilaterals

4 February, 2019

Overview

Last time.

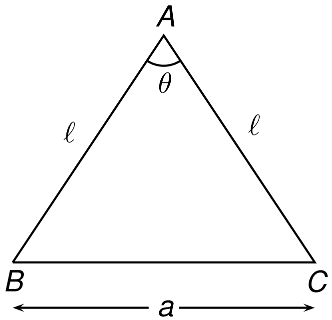
- The Euler line
- Cyclic quadrilaterals
- The nine-point circle
- The orthic triangle

Today.

- Formulae for the inradius and circumradius of a triangle
- Heron's formula for the area of a triangle
- Recall basic properties of cyclic quadrilaterals
- The Simson line of a triangle
- Ptolemy's theorem

Exercise. Let ΔABC be an isosceles triangle with sidelengths $|AB| = \ell$, $|AC| = \ell$, $|BC| = a$ and angle $\angle BAC = \theta$.

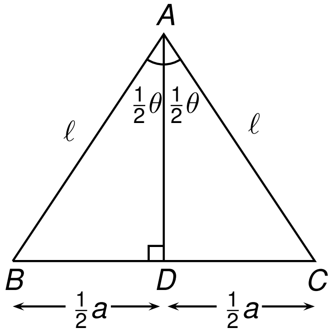
Prove that $a = 2\ell \sin\left(\frac{\theta}{2}\right)$.



Solution.

Exercise Solution

Solution. Draw a perpendicular from A to BC. We have shown earlier that this bisects BC (use RASS).

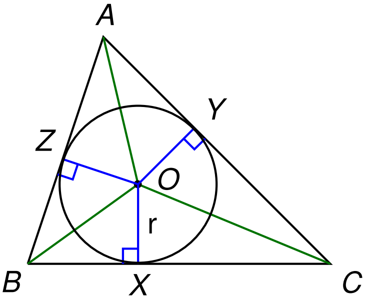


Since $\Delta DBA \cong \Delta DCA$ then $\angle DAB = \angle DAC$. Therefore

$$\frac{1}{2}a = \ell \sin\left(\frac{\theta}{2}\right).$$

What is the circumradius of a triangle?

In Lecture 5 we computed the inradius of a triangle in terms of the area and the sidelengths.



Let $a = |BC|$, $b = |AC|$ and $c = |AB|$. Then

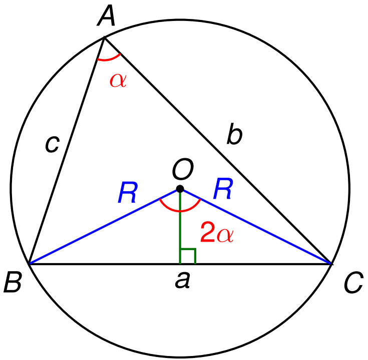
$$\text{Area}(\Delta ABC) = \frac{1}{2} r(a + b + c)$$

Question. Is there a nice formula for the circumradius?

What is the circumradius of a triangle?

Let R be the circumradius. In the diagram below, we use Euclid Prop. III.20 to show that $\angle BOC = 2\angle BAC$. Then

$$\text{Area}(\Delta ABC) = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}bc\sin\alpha$$



The exercise from the beginning of today's class shows that

$$a = 2R\sin\alpha$$

Eliminating $\sin\alpha$ then shows that

$$\text{Area}(\Delta ABC) = \frac{abc}{4R}$$

Summary

The inradius r and circumradius R are related to the Area A and sidelengths a, b, c by the equations

$$r = \frac{2A}{a+b+c}, \quad R = \frac{abc}{4A}$$

Remark. The proof of the formula for the inradius is purely geometric (it involves adding up areas of triangles). The intuition is clear from the proof, and we do not need to use trigonometry.

The formula for the circumradius does not involve trigonometric functions (sin, cos, etc.), however we used them in the proof.

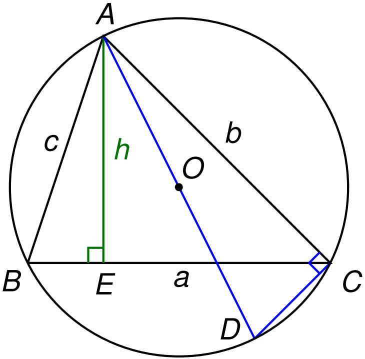
It is natural to ask: Is there a purely geometric proof of this formula that does not use trigonometry?

A geometric proof of the circumradius formula

First note that Area(ΔABC) $= \frac{1}{2}ah$.

Draw the line AD through the centre of the circle. Since AD is a diameter then $\angle ACD = 90^\circ$ (Euclid Prop. 3.20).

Euclid Prop. 3.21 says that the angles subtended by AC are equal. Therefore $\angle ABC = \angle ADC$.



A geometric proof of the circumradius formula (cont.)

Therefore, by AAA, the triangles ΔAEB and ΔACD are similar and so

$$\frac{c}{h} = \frac{|AB|}{|AE|} = \frac{|AD|}{|AC|} = \frac{2R}{b}$$

Rearranging this formula gives us

$$R = \frac{bc}{2h} = \frac{a}{4\,\mathrm{Area}(\Delta ABC)} \cdot bc = \frac{abc}{4\,\mathrm{Area}(\Delta ABC)}$$

which matches the earlier formula that we derived using trigonometry.

Heron's formula for the area of a triangle

We have seen how to relate the inradius r and circumradius R to the area of a triangle.

Is there a formula which is stated purely in terms of the side lengths of the triangle?

Theorem. (Heron's formula) Let ΔABC be a triangle with side lengths a, b, c. Let $s = \frac{1}{2}(a+b+c)$ be the *semiperimeter*. Then

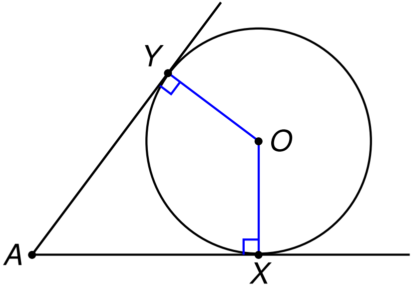
$$\text{Area}(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$$

There are a number of proofs of this formula using trigonometry.

We would like to give a purely geometric proof, since this will give us some insight into why the semiperimeter appears in the formula.

Tangents from the same point have equal length

Recall. Given a circle and a point A outside the circle, there are two tangents AX and AY as in the diagram below. We proved in Lecture 5 that $|AX| = |AY|$.



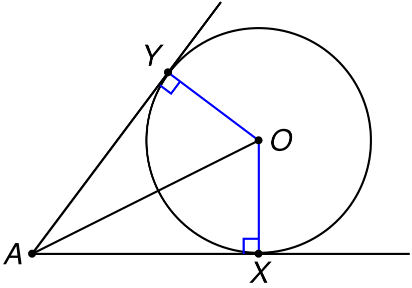
This uses Euclid Prop. III.18 which says that $\angle AXO = 90^\circ = \angle AYO$.

Tangents from the same point have equal length

We want to show that $\Delta AOX \cong \Delta AOY$.

To see this, note that $|OY| = |OX|$ (since they are both radii of the same circle), OA is common to both triangles, and $\angle AXO = 90 = \angle AYO$.

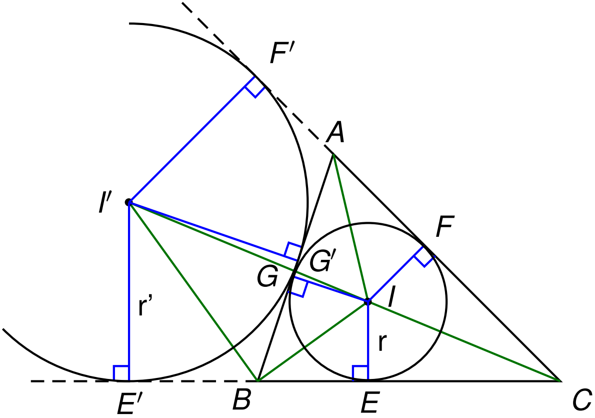
Therefore RASS shows that $\Delta AOX \cong \Delta AOY$ and therefore $|AX| = |AY|$.



Note that we also proved that $\angle YAO = \angle XAO$.

Heron's formula for the area of a triangle

We have already investigated the *incircle* of a triangle. We can also define three *excircles*, which are tangent to the three sides, but which lie on the outside of one side.



Heron's formula for the area of a triangle (cont.)

The previous exercise gives us lots of relations among the side lengths. On the smaller circle we have

$$|CE| = |CF|, \; |AF| = |AG|, \; |BE| = |BG|$$

and the sidelengths of the triangle are

$$a = |BE| + |EC|, \quad b = |CF| + |FA|, \quad c = |AG| + |GB|.$$

Adding the first three relations shows that

$$\begin{aligned} 2s = a + b + c = |BE| + |EC| + |CF| + |FA| + |AG| + |GB| \\ \Rightarrow \; s = |BE| + |EC| + |AG| = a + |AG|. \end{aligned}$$

Therefore $|AG| = s - a = |AF|$ and similarly we can prove that $|BE| = s - b = |BG|$, $|CE| = s - c = |CF|$.

Heron's formula for the area of a triangle (cont.)

Applying the result of the exercise to the larger circle shows that

$$|CE'| = |CF'|, \ |AF'| = |AG'|, \ |BE'| = |BG'|$$

Adding these relations gives us

$$\begin{aligned} |CE'| + |CF'| &= |CE| + |EB| + |BE'| + |CF| + |FA| + |AF'| \\ &= |CE| + |EB| + |BG'| + |CF| + |FA| + |AG'| = 2s \end{aligned}$$

Therefore $|CE'| = s = |CF'|$, so $|E'B| = |E'C| - |BC| = s - a$.

Finally, we also know that $\angle CBI = \angle ABI$ and $\angle ABI' = \angle E'BI'$ (angle bisectors). Moreover,

$$\angle E'BI' + \angle ABI' + \angle ABI + \angle CBI = 180^\circ$$

and therefore $\angle ABI = 90^\circ - \angle E'BI'$.

We also know that $\angle BEI = 90^\circ = \angle BE'I'$, therefore $\Delta EBI \sim \Delta E'I'B$ and so $\frac{r'}{s-a} = \frac{s-b}{r}$, which gives us the formula

$$r'r = (s-a)(s-b)$$

Heron's formula for the area of a triangle (cont.)

Using the result of the exercise again, we can show that $\angle ICE = \frac{1}{2}\angle ACB = \angle I'CE'$, and so $\Delta CIE \sim \Delta CI'E'$. Therefore

$$\frac{r}{r'} = \frac{|CE|}{|CE'|} = \frac{s-c}{s} \quad \text{and} \quad r'r = (s-a)(s-b)$$

Multiplying the two formulae, we can solve for r

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}$$

We also have the inradius formula $r^2 = \frac{\text{Area}(\Delta ABC)^2}{s^2}$. Therefore

$$\text{Area}(\Delta ABC)^2 = s(s-a)(s-b)(s-c)$$

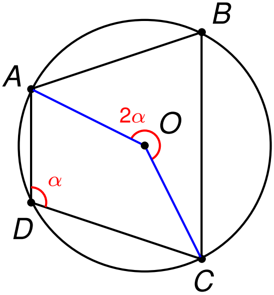
We have proved the following theorem.

Theorem. (Heron's formula) Let ΔABC be any triangle. Then

$$\text{Area}(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}.$$

Exercise

Exercise. Prove the case of Euclid Prop. III.20 where the angle is obtuse.



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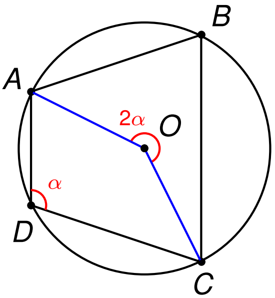

Solution.

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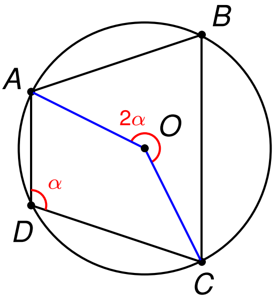
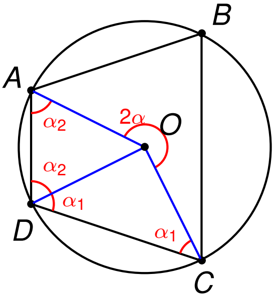


Solution. The idea is very similar to the case where the angle α is acute. The line segments $|OA|, |OC|, |OD|$ are all radii of the same circle and are therefore equal. Therefore ΔODC and ΔOAD are both isosceles.

Solution. (cont.)
Let $\alpha_1 = \angle CDO$ and $\alpha_2 = \angle ADO$. Then $\angle OCD = \alpha_1$ and so $\angle DOC = 180 - 2\alpha_1$. Similarly, $\angle DOA = 180 - 2\alpha_2$.

Adding these together gives us
$\angle COA = 360 - 2\alpha_1 - 2\alpha_2 = 360 - 2\alpha$ and so $\angle AOC = 2\alpha$.



Cyclic quadrilaterals

Last time we used cyclic quadrilaterals to give a proof of the existence of the nine-point circle and the properties of the Euler line.

We also learned an important technique: we can use properties of cyclic quadrilaterals to show that a given point lies on a circle.

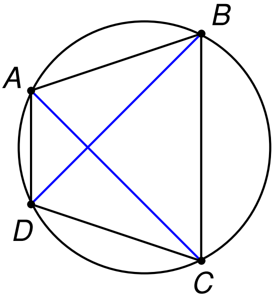
Today we will show that a quadrilateral has certain properties if and only if it is cyclic. We then have a number of different ways to prove that a quadrilateral is cyclic.

We will then use cyclic quadrilaterals to prove the existence of the *Simson line* of a triangle and then prove Ptolemy's theorem which gives a relation on the side lengths of cyclic quadrilaterals.

Cyclic quadrilaterals (definition)

Definition. A cyclic quadrilateral is a set of four points A, B, C, D (in that order) lying on a circle together with the lines AB, BC, CD and DA joining them. In this case, the points A, B, C, D are called *concyclic*.

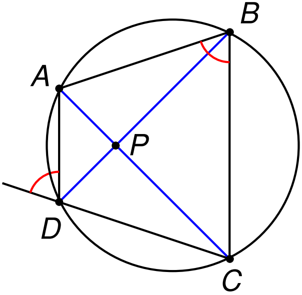
Remark. Any collection of three distinct points (non-collinear) lies on a unique circle, called the circumcircle. It is very unlikely that a fourth point will also lie on the same circle. Therefore the fact that a quadrilateral is cyclic is a very special property.



Cyclic quadrilaterals (defining properties)

Theorem. Let $ABCD$ be a convex quadrilateral. Then the following statements are equivalent (TFAE)

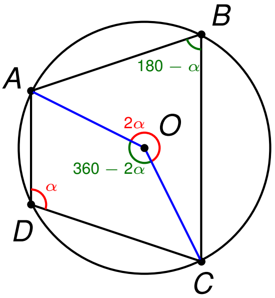
(a) $ABCD$ is a cyclic quadrilateral.
(b) $\angle DBC = \angle DAC$ (and the same for the angles subtended by the other line segments AB, BC and DA).
(c) The opposite angles add up to 180°.
(d) Given any vertex, the external angle is equal to the angle opposite the vertex (see the diagram).



Cyclic quadrilaterals (defining properties)

Proof. Last time we proved that (*a*) $\Leftrightarrow$ (*b*). To finish the proof, we will show that (*a*) $\Leftrightarrow$ (*c*) $\Leftrightarrow$ (*d*).

First we prove (*c*) $\Leftrightarrow$ (*a*). Let *O* be the centre of the circle.



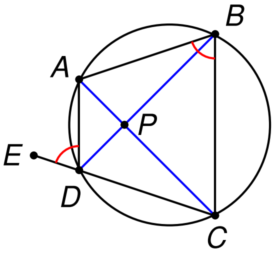
Suppose that *ABCD* is cyclic. Then $\angle AOC = 2\alpha$ by Euclid Prop. III.20, therefore $\angle COA = 360 - 2\alpha$, and so another application of Euclid Prop. III.20 shows that $\angle ABC = 180 - \alpha$.

Cyclic quadrilaterals (defining properties)

To prove the converse, note that if B lies inside the circle then $\angle ABC > \frac{1}{2}\angle COA = 180 - \angle ADC$ by the converse to Euclid Prop. III.20. Similarly, if B lies outside the circle then $\angle ABC < \frac{1}{2}\angle COA = 180 - \angle ADC$. Therefore $\angle ABC = 180 - \angle ADC$ implies that $ABCD$ is cyclic.

(See also Lecture 6 for a similar argument)

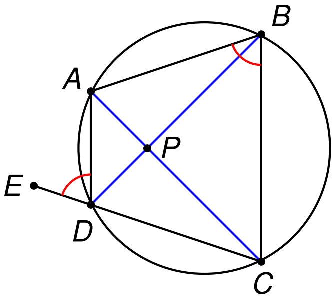
Now we prove $(c) \Rightarrow (d)$. In the diagram below, note that $\angle EDA = 180 - \angle ADC$. We also know that $\angle ADC = 180 - \angle ABC$ (since we're assuming that (c) is true) and so $\angle EDA = \angle ABC$.



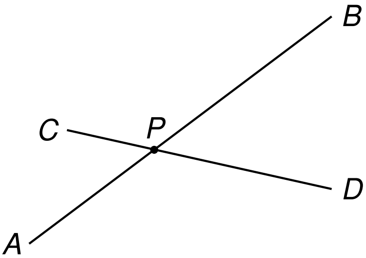
Next we show that $(d) \Rightarrow (c)$. Since we're assuming (d), then $\angle ABC = \angle EDA$. Therefore

$$\angle ABC = \angle EDA = 180 - \angle ADC$$

and so the opposite angles add up to 180°. ■



Exercise. Let AB be a straight line, let P be a point on AB and let C and D be points on opposite sides of AB. Show that C, P, D are collinear if and only if $\angle CPA = \angle DPB$ if and only if $\angle CPB = \angle DPA$.



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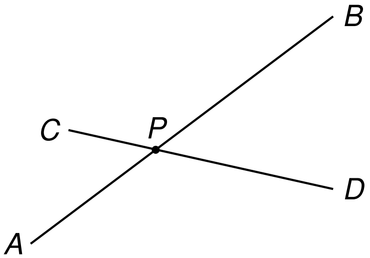
Solution.



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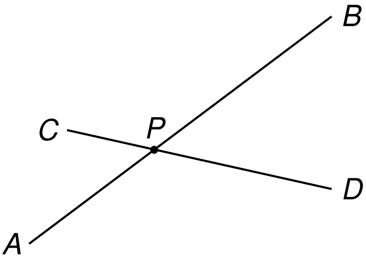


Solution. If the points C, P, D are collinear then

$$\angle CPA = 180 - \angle BPC = \angle DPB$$

For the same reason C, P, D collinear implies that

$$\angle CPB = \angle DPA$$

If C, P, D are not collinear then we will have an inequality in the above statements.

The Simson line of a triangle

Now consider a triangle ΔABC and let P be any point in the plane. We can drop altitudes from P to the three sides of the triangle as in the diagram below. (Note that we may have to extend some of the sides of the triangle in order to do this.)

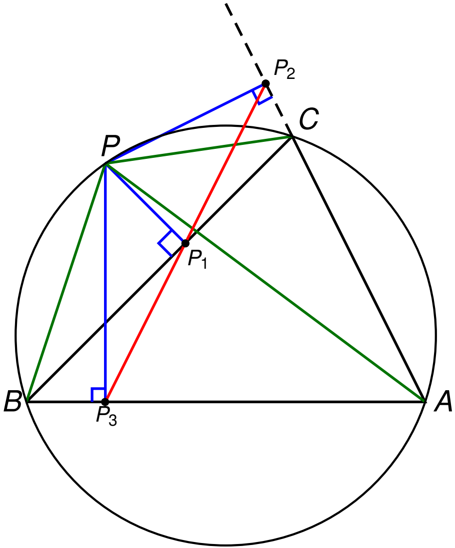
Let P_1, P_2, P_3 be the three intersections of the altitudes with the sides of the triangle.

Theorem. (Simson) P_1, P_2 and P_3 are collinear if and only if P lies on the circumcircle of ΔABC.

Strategy of Proof. The proof makes use of cyclic quadrilaterals. We will use the fact that if the opposite angles of a quadrilateral are both 90° then the quadrilateral must be cyclic (condition (c) of the previous theorem).

We aim to show that P_1, P_2 and P_3 are collinear by using the previous exercise. We want to show that $\angle BP_1P_3 = \angle CP_1P_2$.

The Simson line of a triangle (picture)



You can go to the Geometry Website for an interactive picture.

The Simson line of a triangle (proof)

Proof. Let $\alpha = \angle BP_1P_3$, $\beta = \angle CP_1P_2$ and $\gamma = \angle P_2PB$.

We want to show that $\alpha = \beta$, and therefore the result of the previous exercise will show that P_1, P_2 and P_3 are collinear.

Since the line BP subtends angles of 90° at P_3 and P_1 then BP_3P_1P is cyclic. Therefore $\angle BPP_3 = \angle BP_1P_3 = \alpha$.

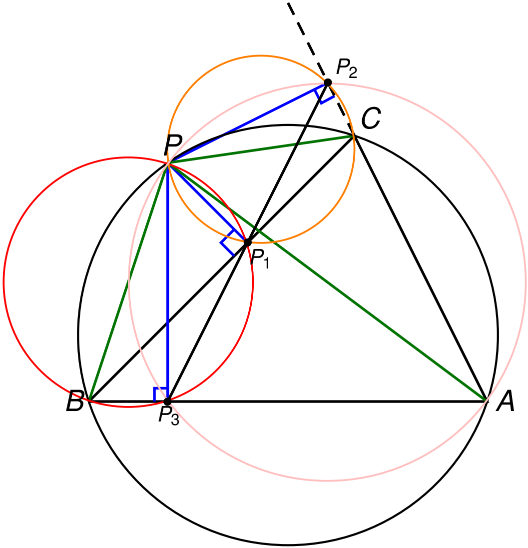
CP_2PP_1 is cyclic (opposite angles are 90°) and so $\angle CPP_2 = \angle CP_1P_2 = \beta$.

The quadrilateral AP_2PP_3 is also cyclic (again, a pair of opposite angles is 90°), and so $\angle BAC = 180 - \angle P_3PP_2 = 180 - (\gamma - \alpha)$.

Finally, we originally assumed that $ACPB$ is cyclic and so $\angle BAC = 180 - \angle BPC = 180 - (\gamma - \beta)$.

Therefore $180 - \gamma + \beta = 180 - \gamma + \alpha$ and so $\alpha = \beta$. Therefore P_1, P_2 and P_3 are collinear. ∎

The cyclic quadrilaterals in the Simson line proof

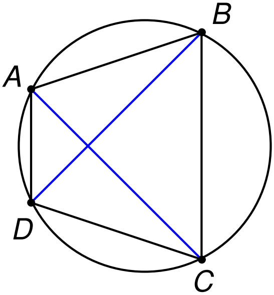


Ptolemy's Theorem

Theorem. Let *ABCD* be a cyclic quadrilateral. Then

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |AD|$$

The product of the diagonals is the sum of the products of the opposite sides

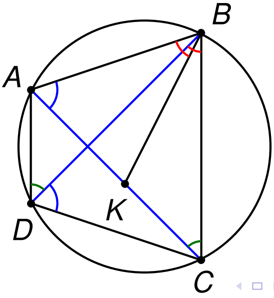


Ptolemy's Theorem

Proof. Choose K on AC such that $\angle ABK = \angle CBD$. Then (since $ABCD$ is cyclic) we have $\angle CAB = \angle CDB$, so $\Delta ABK \sim \Delta CBD$ (by AAA). Therefore $\frac{|AK|}{|AB|} = \frac{|CD|}{|BD|}$.
Since $ABCD$ is cyclic then $\angle ADB = \angle ACB$. We also have

$$\angle CBD + \angle CBK = \angle ABK + \angle CBK = \angle ABC = \angle CBD + \angle ABD$$

and so $\angle CBK = \angle ABD$.
Therefore $\Delta CBK \sim \Delta ABD$ and so $\frac{|CK|}{|BC|} = \frac{|AD|}{|BD|}$.



Ptolemy's Theorem

The equations $\frac{|AK|}{|AB|} = \frac{|CD|}{|BD|}$ and $\frac{|CK|}{|BC|} = \frac{|AD|}{|BD|}$ are equivalent to

$$|BD| \cdot |AK| = |AB| \cdot |CD| \quad \text{and} \quad |BD| \cdot |CK| = |BC| \cdot |AD|$$

and adding these equations gives us

$$|BD| \cdot |AC| = |BD| \left(|AK| + |CK|\right) = |AB| \cdot |CD| + |BC| \cdot |AD|$$

which is the required formula. ∎

Next time

We will prove more properties of cyclic quadrilaterals.

- Brahmagupta's formula for the area of a cyclic quadrilateral.
- Properties of non-cyclic quadrilaterals

We will also do more construction problems in Euclidean geometry.