Lecture 11: Pencils of circles and Poncelet's theorem

18 February, 2019

Overview

Last time.

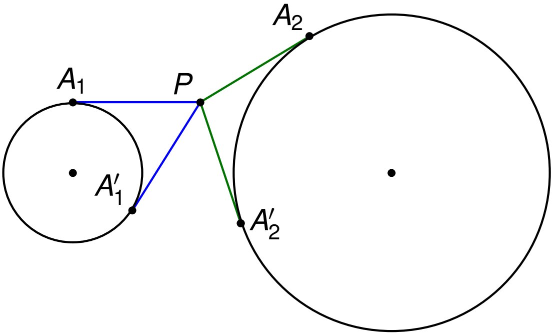
- Power of a point with respect to a circle
- Radical axis of a pair of circles
- Radical centre of three circles
- Pencils of circles

Today.

- Degenerate points of a hyperbolic pencil
- Orthogonal pencils of circles
- Poncelet's theorem
- The problem of Apollonius

Exercise. Let C_1 and C_2 be two circles in the plane. For any point P outside both circles, choose A_1 on C_1 such that A_1P is tangent to C_1, and choose A_2 on C_2 such that A_2P is tangent to C_2. (There are two choices for each circle)

Describe the locus of points P such that $|A_1P| = |A_2P|$.



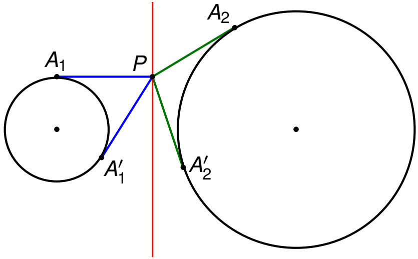
Solution

Solution

For $i = 1, 2$, the power of P with respect to C_i is $\mathcal{P}_C(P) = |A_i P|^2$ (if P is outside C_i) or $\mathcal{P}_C(P) = -|A_i P|^2$ (if P is inside C_i).

Then $|A_1 P| = |A_2 P|$ if and only if P has equal power with respect to both circles.

Therefore the locus of points such that $|A_1 P| = |A_2 P|$ is the radical axis of the two circles, which is a line.

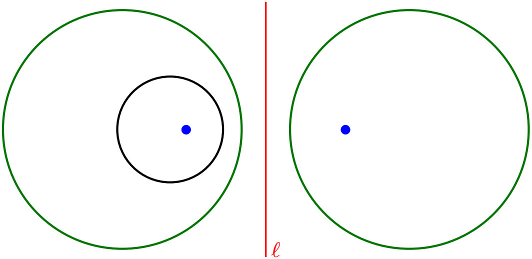


In particular, we have shown that if P is on the radical axis of a pencil of circles, then the quantity $|AP|$ is invariant (it doesn't depend on the choice of circle in the pencil).

More about hyperbolic pencils

Suppose that a pair of circles is non-intersecting. We saw last time that the associated pencil of circles is hyperbolic (i.e. none of the circles in the pencil intersect each other).

Given a circle C_1, a radical axis ℓ (not intersecting the circle) and a radius $r \geq 0$, we can construct two circles C_2 and C_2' of radius r (one for each side of the radical axis) such that ℓ is the radical axis of C_1 and C_2. Moreover, these are the only circles in the pencil with that particular radius.

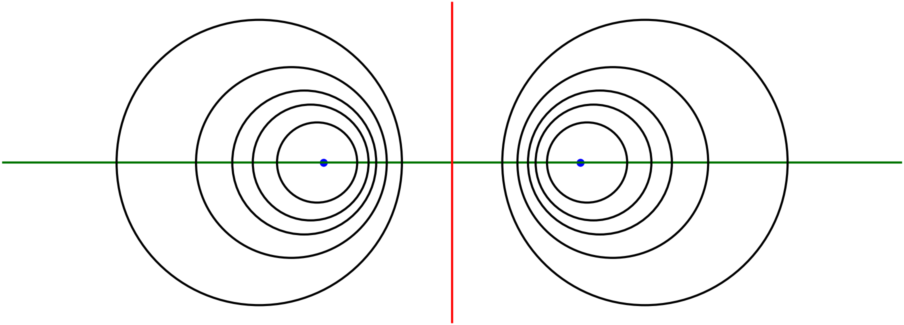


Constructing pencils from two given points

In particular, we can construct a circle of radius zero.

Conversely, given two distinct points, we can think of them as two circles of radius zero (degenerate circles) and use them to define a hyperbolic pencil.

The radical axis is then the perpendicular bisector of these two points and the line through these two points is the line through the centres of the circles in the pencil.



Constructing pencils from two given points

Instead of thinking of the two points as degenerate circles, we can think of them as intersection points of an elliptic pencil.

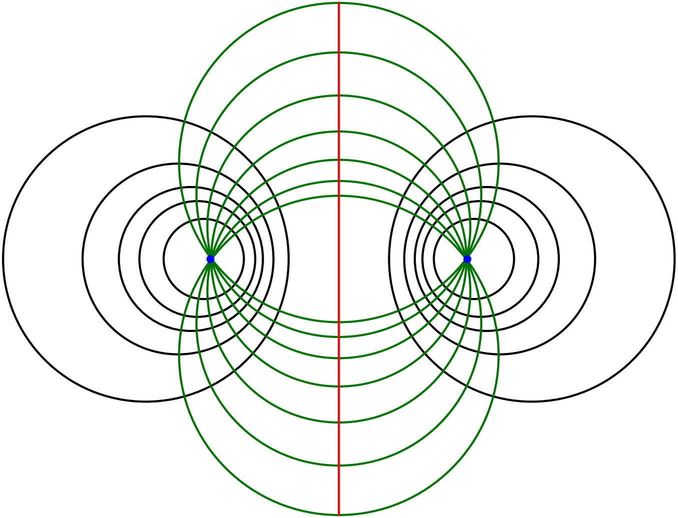
Then the radical axis is the line through the two points and the perpendicular bisector is the line through the centres of all the circles in this elliptic pencil.

Therefore, given two distinct points in the plane, we have two ways to construct pencils. One way gives a hyperbolic pencil and the other gives an elliptic pencil.

We say that the pencils are the “orthogonal duals” of each other. (The reason for the word “orthogonal” will be made apparent a few slides from now)

Exercise. What happens if the two points coincide? Can we construct a pencil? Is it unique?

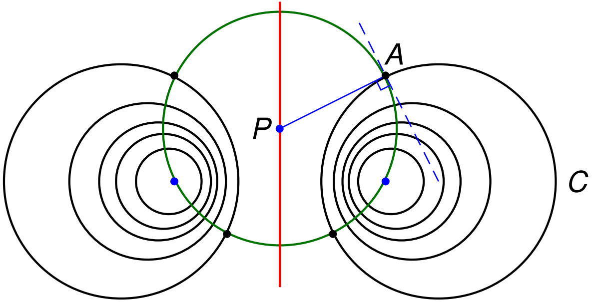
Orthogonal pencils of circles



Orthogonal pencils of circles

Given any point P and a line AP tangent to a circle C, the circle of radius $|AP|$ centred at P is orthogonal to the line AP and hence also orthogonal to the circle C.

Therefore, if P lies on the radical axis of a pencil of circles, then if we choose any circle and a tangent line AP, the circle of radius $|AP|$ centred at P is orthogonal to every circle in the pencil, since P has the same power $\mathcal{P}_C(P) = |AP|^2$ with respect to every circle in the pencil.



Orthogonal pencils of circles

Each point P on the radical axis defines such a circle orthogonal to all the circles in the original pencil.

Therefore we have constructed a one-parameter family of circles, each one centred on the radical axis of the original pencil.

Each circle in this family is orthogonal to every circle in the original pencil.

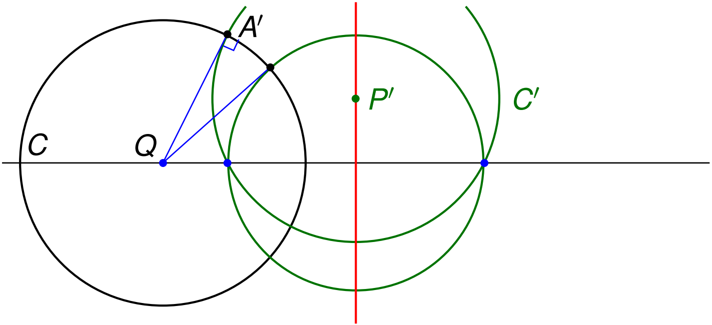
Question. Is this new family of circles a pencil?

To answer the question, we show that this orthogonal family of circles is coaxial with radical axis equal to the line through the centres of the circles of the original pencil.

Orthogonal pencils of circles

Let Q be a point on the line through the centres of the circles of the original pencil and let C' be a circle in the orthogonal family of circles centred at P'.

To compute the power of Q with respect to C', we draw a tangent from Q to A' on C.Then A' must lie on the circle centred at Q in the original pencil.



Orthogonal pencils of circles

Therefore $\mathcal{P}_{C'}(Q) = |QA'|^2$ is independent of the circle C' (since $|QA'|$ is the radius of the circle C). This is true for all circles in the orthogonal family, and so this forms a pencil with radical axis the line through the centres of the circles in the original pencil.

The new pencil is called the orthogonal pencil to the original pencil of circles.

If the original pencil was hyperbolic, then the orthogonal pencil is elliptic. The intersection points of the new pencil are exactly the degenerate circles of the original pencil.

If the original pencil was elliptic, then the orthogonal pencil is hyperbolic. The degenerate circles are exactly the intersection points of the original pencil.

If the original pencil was parabolic, then the orthogonal pencil is parabolic with the same tangency point as the original pencil.

Orthogonal pencils of circles

Proposition. Given a pencil of circles, the orthogonal pencil of the orthogonal pencil (the “double orthogonal”) is the original pencil.

Proof. The orthogonal pencil consists of all circles that are orthogonal to every circle in the original pencil.

Therefore, all of the circles in the original pencil are orthogonal to all the circles in the orthogonal pencil.

Therefore the original pencil is the orthogonal pencil of the orthogonal pencil. ∎

Remark. This proof seems tautological, but we do have to make sure that the definitions match up. What makes the proof work is that an orthogonal line to an orthogonal line through a given point is the original line.

The locus where the ratio of powers is constant

Given two circles C_1 and C_2, we showed that the locus of points P such that $\mathcal{P}_{C_1}(P) = \mathcal{P}_{C_2}(P)$ is a line.

What if we require instead that $\mathcal{P}_{C_1}(P) = \lambda \mathcal{P}_{C_2}(P)$ for some constant λ?

Case 1. Clearly if $\lambda = 0$ then the locus is the circle C_1. We can also think of the circle C_2 as the locus where $\lambda = \infty$ (or $\frac{1}{\lambda} = 0$). (The constant λ should really take values in a projective line $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$. We'll talk more about this when we study projective geometry)

Case 2. If $\lambda > 0$ then $\mathcal{P}_{C_1}(X)$ and $\mathcal{P}_{C_2}(X)$ have the same sign, and so the locus consists of points which are either inside both circles (both $\mathcal{P}_{C_1}(X)$ and $\mathcal{P}_{C_2}(X)$ are negative), or outside both circles (both $\mathcal{P}_{C_1}(X)$ and $\mathcal{P}_{C_2}(X)$ are positive).

Case 3. If $\lambda < 0$ then $\mathcal{P}_{C_1}(X)$ and $\mathcal{P}_{C_2}(X)$ have different signs, and so the locus consists of points which are inside one circle and outside another circle.

The locus where the ratio of powers is constant

Another special case. If the two circles intersect at a point X, then $\mathcal{P}_{C_1}(X) = 0 = \lambda \mathcal{P}_{C_2}(X)$. Therefore X is in this locus.

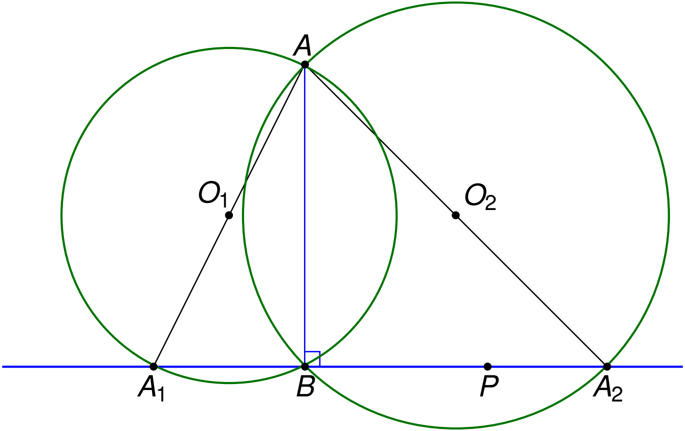
Proposition. Let $\lambda \neq 0$ be a constant and C_1 and C_2 be two given circles. Then the locus of points P such that $\mathcal{P}_{C_1}(P) = \lambda \mathcal{P}_{C_2}(P)$ is a circle in the pencil defined by C_1 and C_2.

Remark. The radical axis is the case where $\lambda = 1$. We can think of the radical axis as a circle of infinite radius which is in the pencil. The right way to think of this is as a circle in the projective plane $\mathbb{R}P^2$ (more about this later in the semester).

We will prove the elliptic case (C_1 and C_2 intersect) using a geometric method. The hyperbolic case can be done algebraically once we derive some more formulas for the power of a point with respect to a circle. (**See Tutorial 6**)

Let A, B be the points of intersection of the two circles. Then the perpendicular to AB through B intersects C_1 at the point A_1 and C_2 at the point A_2.

The locus where the ratio of powers is constant



On the line A_1A_2, choose the point P such that $\frac{PA_1}{PA_2} = \lambda$, and therefore $\mathcal{P}_{C_1}(P) = PB \cdot PA_1 = \lambda PB \cdot PA_2 = \lambda \mathcal{P}_{C_2}(P)$.

Then P lies on the required locus. We then want to show that the locus is a circle with diameter AP.

The locus where the ratio of powers is constant

The goal of the rest of the proof is to show that any other point X such that $\mathcal{P}_{C_1}(X) = \lambda\mathcal{P}_{C_2}(X)$ must lie on the circle with diameter AP.

Now let X be a point such that $\mathcal{P}_{C_1}(X) = \lambda\mathcal{P}_{C_2}(X)$. Draw the line AX and let X_1 be the intersection of this line with C_1 and X_2 the intersection with C_2.

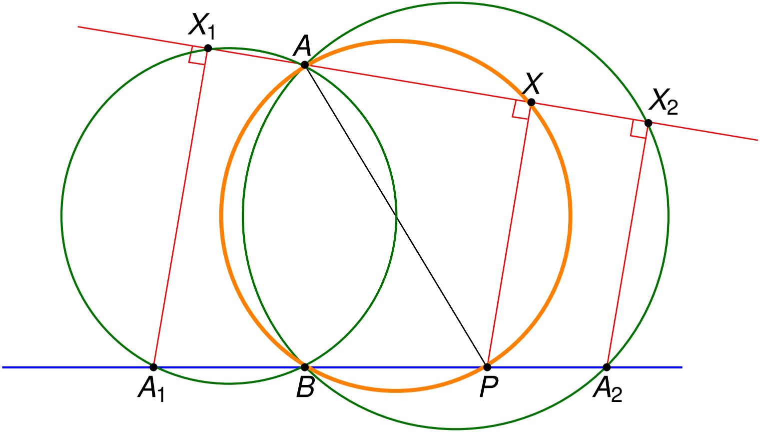
Since AA_1 is a diameter of the circle C_1, then we have $\angle AX_1A_1 = 90^\circ$, and the same argument shows that $\angle AX_2A_2 = 90^\circ$.

Then $XX_1 \cdot XA = \mathcal{P}_{C_1}(X) = \lambda\mathcal{P}_{C_2}(X) = \lambda\, XX_2 \cdot XA$.

Therefore $\frac{XX_1}{XX_2} = \lambda = \frac{PA_1}{PA_2}$ and so Thales' theorem shows that PX is parallel to A_2X_2 and A_1X_1.

Therefore PX makes the same angle with X_1X_2 as the lines A_2X_2 and A_1X_1.

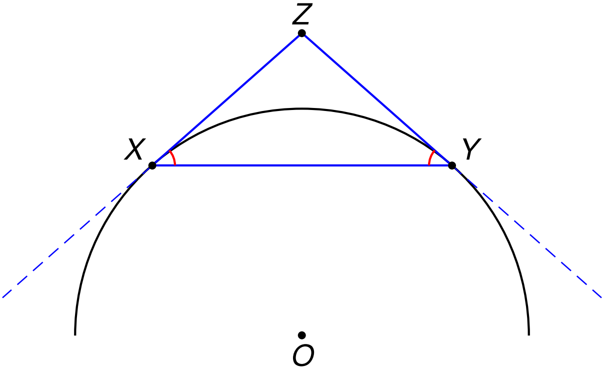
The locus where the ratio of powers is constant



Therefore $\angle AXP = 90^\circ$. Therefore X lies on the circle with diameter AP.

Since $\angle ABP = 90^\circ$ then B also lies on this circle and so the circle must be in the elliptic pencil defined by C_1 and C_2. ∎

Exercise. Let ΔXYZ be an isosceles triangle with $\angle ZXY = \angle ZYX$. Show that there exists a circle which is simultaneously tangent to XZ at X and tangent to YZ at Y.

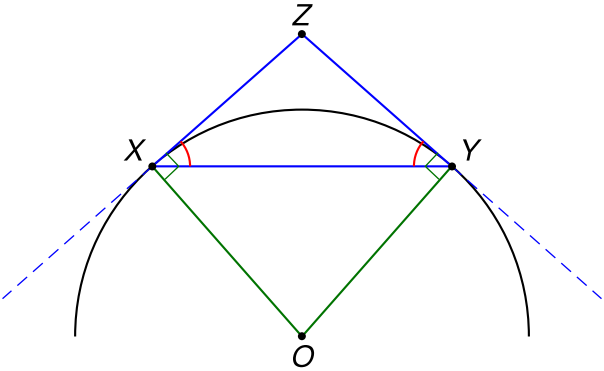


Solution

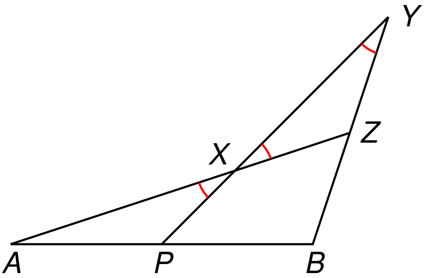
Solution.

Solution. Draw a perpendicular to XZ at X and to YZ at Y. Then these intersect at a point O.

We also know that $\angle OYX = 90 - \angle ZYX = 90 - \angle ZXY = \angle OXY$. Therefore ΔOXY is isosceles and the circle of radius $|OX| = |OY|$ is tangent to both XZ and YZ.



Exercise. Consider the diagram below with $\angle APX > 90^\circ$. Suppose also that $\angle ZXY = \angle ZYX$. Show that $\frac{|AX|}{|AP|} = \frac{|BY|}{|BP|}$.



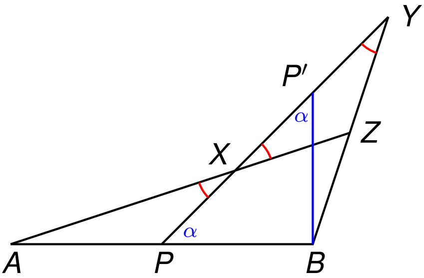
Solution

Solution.

Solution

Solution. Construct a point P' on PY such that $|BP| = |BP'|$. Let $\alpha = \angle BPY$. Then $\Delta BPP'$ is isosceles, so $\angle BP'X = \alpha$, and so $\angle APX = 180 - \alpha = \angle YP'B$. Moreover, $\angle AXP = \angle ZXY = \angle ZYX$. Therefore the triangles ΔAPX and $\Delta BP'Y$ are similar, and so

$$\frac{|AX|}{|AP|} = \frac{|BY|}{|BP'|} = \frac{|BY|}{|BP|}$$



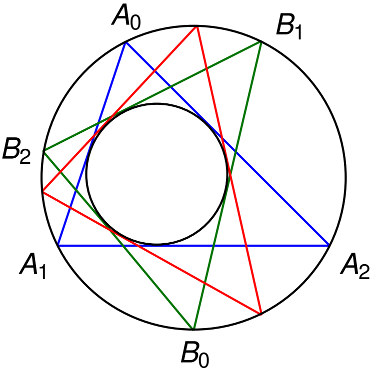
Theorem. (Poncelet) Let C_0 and C_1 be two non-intersecting circles with C_1 inside C_0. Given A_0 on C_0, construct a line from A_0 tangent to C_1 (going counterclockwise around the circle to make the line unique). Let A_1 be the intersection of this line with C_0.

Repeating the process, construct $A_2, \ldots, A_n, \ldots$.

Now repeat the same construction with another initial point B_0 on C_0.

Then $A_n = A_0$ if and only if $B_n = B_0$.

If the polygon closes up for one initial point A_0 then it closes up for every initial point.



Click here for an interactive picture of Poncelet's theorem.

Poncelet's theorem

Remark. A theorem of this type is called a *porism*. It says that either the points join up for *every* initial point on the circle, or that the points *never* join up.

Proof. Begin by choosing B_0 between A_0 and A_1 (if we prove the theorem for all such B_0 then we can continue the process to prove it for all points on the circle).

Let X be the point of intersection of the tangent A_0A_1 with the circle C_1, and let Y be the point of intersection of the tangent B_0B_1 with the circle C_1. Let Z be the intersection of A_0A_1 and B_0B_1.

Then the triangle ΔXYZ is isosceles, and so $\angle XYZ = \angle YXZ$.

Extend the line XY and let P be the intersection with A_0B_0, and let Q be the intersection with A_1B_1.

Moreover, $\angle B_1A_1A_0 = \angle B_1B_0A_0$ since they are subtended by the same arc B_1A_0.

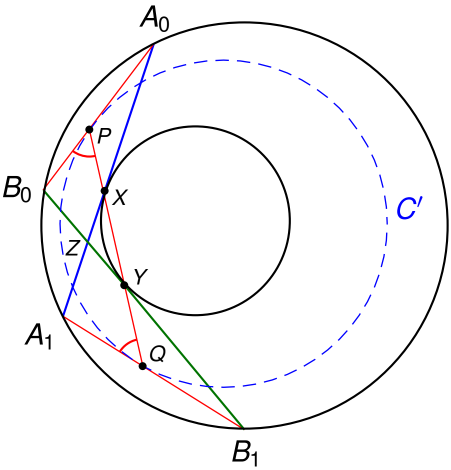
Therefore $\Delta QA_1X \sim \Delta PB_0Y$.

Poncelet's theorem

$\angle XYZ = \angle YXZ$

$\angle B_1 A_1 A_0 = \angle B_1 B_0 A_0$

$\Delta QA_1 X \sim \Delta PB_0 Y$



Since the triangles are similar, then $\angle XQA_1 = \angle XPB_0$ and so (by the result of the first exercise) there is a circle passing through P and Q which is tangent to $A_1 B_1$ and $A_0 B_0$.

Poncelet's theorem

We want to show that this circle C' is in the pencil defined by C_0 and C_1.

To see this, note that we can use similar triangles to show

$$\frac{\mathcal{P}_{C_1}(B_1)}{\mathcal{P}_{C'}(B_1)} = \frac{|B_1Y|^2}{|B_1Q|^2} = \frac{|A_0X|^2}{|A_0P|^2} = \frac{\mathcal{P}_{C_1}(A_0)}{\mathcal{P}_{C'}(A_0)}$$

and

$$\frac{\mathcal{P}_{C_1}(B_0)}{\mathcal{P}_{C'}(B_0)} = \frac{|B_0Y|^2}{|B_0P|^2} = \frac{|A_1X|^2}{|A_1Q|^2} = \frac{\mathcal{P}_{C_1}(A_1)}{\mathcal{P}_{C'}(A_1)}$$

The result of the second exercise shows that

$$\frac{\mathcal{P}_{C_1}(B_0)}{\mathcal{P}_{C'}(B_0)} = \frac{|B_0Y|^2}{|B_0P|^2} = \frac{|A_0X|^2}{|A_0P|^2} = \frac{\mathcal{P}_{C_1}(A_0)}{\mathcal{P}_{C'}(A_0)}$$

and so we have four points A_0, A_1, B_0, B_1 on the circle C_0 such that the ratios of the powers are constant. Therefore the same is true for every point on the circle C_0 and this circle is in the pencil defined by C_1 and C'.

Poncelet's theorem

Conversely, since any pair of circles in a pencil defines all of the circles in the rest of the pencil, then this shows that C' is in the pencil defined by C_0 and C_1.

If we carry out the same process for the lines A_1A_2 and B_1B_2 then we want to show that A_2B_2 is tangent to the same circle as A_1B_1.

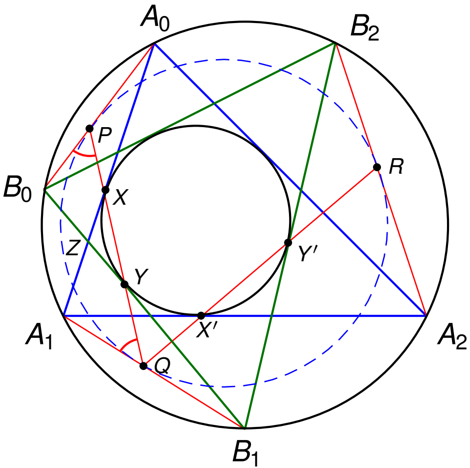
Let X' be the intersection of A_1A_2 with the inner circle C_1 and let Y' be the intersection of B_1B_2 with the inner circle C_1. Let Q' be the intersection of $X'Y'$ with A_1B_1 and let R be the intersection of $X'Y'$ with A_2B_2.

Applying the second exercise twice (**see Tutorial 6**) shows that $\frac{|A_1Q|}{|B_1Q|} = \frac{|A_1Q'|}{|B_1Q'|}$ and so $Q = Q'$.

Poncelet's theorem

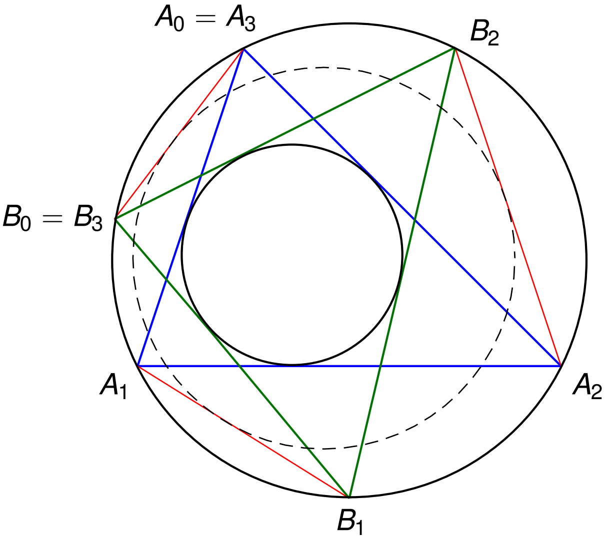
Therefore, if we repeat the process from before we see that A_2B_2 is tangent to the *same* circle C' at the point R.

We have shown that *given A_i we can construct B_i by taking the (anticlockwise) tangent line to the circle C'*.



We also know that $A_n = A_0$, and there is only one anticlockwise line from A_0 tangent to the circle C'. Therefore $A_0B_0 = A_nB_n$, which shows that $B_0 = B_n$. ∎

Poncelet's theorem



Apollonius' problem

We will finish with a problem to think about over the break.

Problem. Suppose that you hear a bolt of lightning strike somewhere in Singapore and you note down the precise time t_1 that you hear the thunderclap.

Since the speed of sound is approx. 330m/s, then determining the exact time T of the lightning strike is the same as determining the distance to the location of the strike.

Suppose also that two friends (in different locations) do the same thing and measure times t_2 and t_3.

How can you determine the location and time of the lightning strike?

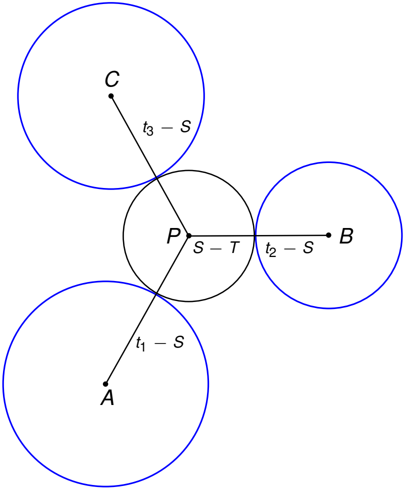
This type of problem is useful in navigation, when you want to determine your location from the given data of the difference in distances from your location to three given points. You can read more about hyperbolic navigation by clicking the link.

Apollonius' problem

Let A, B, C denote the location of the three observers and let P denote the location of the lightning strike.

If we guess the time of the lightning strike as S, then by considering circles of radius $t_1 - S$, $t_2 - S$ and $t_3 - S$ around each observer, you can show that the point P must be at the centre of a circle of radius $S - T$ which is tangent to all three circles.

The problem of Apollonius is to determine P and T from this data. Is there a unique solution?



Next time

This type of problem is useful in navigation, where a ship on the ocean knows the *difference in distances* to three different points. This was used in navigation systems before GPS, such as the Omega navigation system.

We will discuss it again after the break when we learn about conic sections.

Next time

On Friday the lecture will be devoted to the construction problems throughout the first half of the course. You can ask questions about any of the problems (either how to do the construction, or how to prove that the construction works).

On the Tuesday after the break we will begin the study of conic sections. This also serves as an introduction to projective geometry, after which we will study other non-Euclidean geometries such as inversive geometry, spherical geometry and hyperbolic geometry.

- Definition of conic sections
- Description of conics sections as cross-sections of a cone
- Geometric properties
- Applications of conics

Parabolic Dish

If you have time over the break, then you can check out the "parabolic dish" at the playground in One-North Park. We will talk more about parabolic reflectors after the break.