Moment map flows and the Hecke correspondence for quivers

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Two ways to think of the Grassmannian

- $\mathrm{Gr}(k, n)$ denotes the Grassmannian of k-dimensional planes in $\mathbb{C}^n$ modulo equivalence.
- $\mathrm{GL}(k, \mathbb{C})$ and $U(k)$ act on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by $g \cdot A = Ag^{-1}$.

$$\begin{aligned}\mathrm{Gr}(k, n) &= \left\{ A \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \ : \ A \text{ is injective} \right\} / \, \mathrm{GL}(k, \mathbb{C}) \\ &= \left\{ A \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \ : \ A^* A = \mathrm{id}_{\mathbb{C}^k} \right\} / \, U(k).\end{aligned}$$

- The first is a *Geometric Invariant Theory (GIT) quotient* of $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by $\mathrm{GL}(k, \mathbb{C})$ (A injective $\Leftrightarrow$ A stable).
- The second is a *symplectic quotient* of $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by $U(k)$ ($\sqrt{-1} A^* A$ is the moment map for the $U(k)$ action).

Irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$

- Recall that $\mathfrak{sl}(2,\mathbb{C})$ is generated by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

  Relations $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$.
- An irreducible rep. $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(\mathbb{C}^{n+1})$ defines a decomposition into weight spaces for H

$$\mathbb{C}^{n+1} = \bigoplus_{k=0}^{n} \mathbb{C}_{(2k-n)}, \text{where } \rho(H)v = \lambda v \text{ for } v \in \mathbb{C}_{(\lambda)}.$$

- $E : \mathbb{C}_{(\lambda)} \to \mathbb{C}_{(\lambda+2)}$ ("raising operator")
  $F : \mathbb{C}_{(\lambda)} \to \mathbb{C}_{(\lambda-2)}$ ("lowering operator")

Representations of $\mathfrak{sl}(2, \mathbb{C})$ on the cohomology of the Grassmannian

- Now consider $\mathbb{C}^{n+1} \cong \bigoplus_{k=0}^{n} H_{top}(\mathrm{Gr}(k, n))$.
- $H_{top}(\mathrm{Gr}(k, n))$ will be the weight space for the action of H with weight $2k - n$.
- Construct the operators E and F using the flag variety

$$\mathcal{F}(k, k+1, n) = \left\{ \mathbb{C}^k \subset \mathbb{C}^{k+1} \subseteq \mathbb{C}^n \right\} / \sim .$$

- The flag variety is a *correspondence variety* for the two Grassmannians.

$$\begin{tikzcd} & \mathcal{F}(k, k+1, n) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Gr}(k, n) & & \mathrm{Gr}(k+1, n) \end{tikzcd}$$

- The raising and lowering operators are constructed via pullback and pushforward (in Borel-Moore homology) in the diagram below.

$$\begin{tikzcd}
 & T^* \mathrm{Gr}(k,n) \times T^* \mathrm{Gr}(k+1,n) \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"] & \\
T^* \mathrm{Gr}(k,n) & \mathcal{F}(k,k+1,n) \arrow[u, hook] \arrow[dl, "p_1"] \arrow[dr, "p_2"] & T^* \mathrm{Gr}(k+1,n) \\
\mathrm{Gr}(k,n) \arrow[u, hook] & & \mathrm{Gr}(k+1,n) \arrow[u, hook]
\end{tikzcd}$$

$$E(\omega) = (\pi_2)_* \left(\pi_1^*(\omega) \cap [\mathcal{F}(k,k+1,n)] \right)$$
$$F(\omega) = (\pi_1)_* \left(\pi_2^*(\omega) \cap [\mathcal{F}(k,k+1,n)] \right)$$

(ω is the fundamental class of the Grassmannian)

Hecke correspondence for bundles

- Let Σ be a compact Riemann surface. Given a holomorphic bundle $E \to \Sigma$, choose $p \in \Sigma$ and $v_p \in E_p^*$.
- Evaluation $\mathcal{E}(U) \ni s \mapsto v_p(s(p)) \in \mathbb{C}$ gives an exact sequence of sheaves

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathbb{C}_p \to 0$$

- The pair $(\mathcal{E}, \mathcal{E}')$ gives the data of a quasi-parabolic bundle. $\deg E' = d - 1$. Choose weights to get a parabolic structure.

$$\begin{tikzcd} & \mathcal{M}_{par}(r,d) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{M}(r, d-1) & & \mathcal{M}(r,d) \end{tikzcd}$$

Hecke correspondence for quivers

- Motivation: Hecke modifications of bundles on hyperkähler ALE (Asymptotically Locally Euclidean) 4-manifolds
- Γ a finite subgroup of $SU(2)$.
- Construct minimal resolution $p : \widetilde{\mathbb{C}^2/\Gamma} \to \mathbb{C}^2/\Gamma$.
- Exceptional divisor: $p^{-1}(0) = \cup \mathbb{C}P^1$.
- Given an instanton $E \to \widetilde{\mathbb{C}^2/\Gamma}$, construct Hecke modifications over the exceptional divisor.

$$\begin{tikzcd} & \mathcal{B}_k(Q, \mathbf{v}, \mathbf{w}) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{M}^{HK}(Q, \mathbf{v} - \mathbf{e_k}, \mathbf{w}) & & \mathcal{M}^{HK}(Q, \mathbf{v}, \mathbf{w}) \end{tikzcd}$$

Flow line correspondence

- Let M be a manifold and $f : M \to \mathbb{R}$ a smooth function (plus extra conditions, e.g. M compact, f Morse-Bott).
- Label the critical sets $\{C_j\}_{j=1}^n$. Define the gradient flow

$$\gamma : M \times \mathbb{R} \to M$$

$$\gamma(x, 0) = x, \quad \left. \frac{\partial}{\partial t} \right|_{t=0} \gamma(x, t) = \operatorname{grad} f(x)$$

- Upwards limit: $\gamma(x, \infty) = \lim_{t \to \infty} \gamma(x, t)$
- Downwards limit: $\gamma(x, -\infty) = \lim_{t \to -\infty} \gamma(x, t)$
- Given two critical sets C_j, C_k, define the *space of flow lines*

$$\mathcal{F}(C_j, C_k) := \{x \in M \mid \gamma(x, \infty) \in C_j, \gamma(x, -\infty) \in C_k\}$$

$$\tilde{\mathcal{F}}(C_j, C_k) := \mathcal{F}(C_j, C_k)/\mathbb{R} \quad (\mathbb{R}\text{-action defined by flow})$$

Flow line correspondence (cont.)

The flow defines projection maps

$$\begin{tikzcd} & \tilde{\mathcal{F}}(C_j, C_k) \arrow[dl, "u"'] \arrow[dr, "\ell"] & \\ C_j & & C_k \end{tikzcd}$$

- When M is compact and f is Morse-Bott-Smale, the projection maps give $\tilde{\mathcal{F}}$ the structure of a sphere bundle.
- Construct a complex $MB^\bullet$ from $H^*(C_j)$ for each crit. set C_j.
- The differentials are $u_* \circ \ell^* : H^*(C_k) \to H^{*-\lambda_{jk}+1}(C_j)$.
- **Theorem.** (Austin-Braam) $H^*(MB^\bullet) \cong H^*(M)$.

(Cup product has a similar push-pull construction)

Morse correspondence

Given two critical sets C_j, C_k, define the *Morse correspondence*

$$\mathcal{M}(C_j, C_k) := \{(x_1, x_2) \in C_j \times C_k \mid \exists x \in M \, s.t. \, \gamma(x, \infty) = x_1, \gamma(x, -\infty) = x_2\}$$

$$\begin{tikzcd} & \mathcal{M}(C_j, C_k) \arrow[dl, "u"'] \arrow[dr, "\ell"] & \\ C_j & & C_k \end{tikzcd}$$

Goal of the talk. In the setting of Nakajima quiver varieties, the Morse correspondence is homeomorphic to the Hecke correspondence.

Quiver varieties (definition)

Let Q be a directed graph ("quiver" = "bunch of arrows").
Edges E, vertices I, head/tail maps $h, t : E \to I$.

A *framed complex representation* of a quiver consists of

- Complex vector spaces V_k and W_k at each vertex $k \in I$, and
- $\mathbb{C}$-linear homomorphisms $A_a : V_{t(a)} \to V_{h(a)}$ for each edge $a \in E$ and $i_k : V_k \to W_k$ for each $k \in I$.

Quiver varieties definition (cont.)

- Define $\mathbf{v} = (\dim_{\mathbb{C}} V_k)_{k \in I}$ and $\mathbf{w} = (\dim_{\mathbb{C}} W_k)_{k \in I}$ to be the *dimension vectors* of the representation.
- Let $\operatorname{Rep}(Q, \mathbf{v}, \mathbf{w})$ be the space of all representations with fixed dimension vectors $\mathbf{v}$ and $\mathbf{w}$.
- The groups $G_{\mathbf{v}} = \prod_{k \in I} GL(V_k, \mathbb{C})$ and $K_{\mathbf{v}} = \prod_{k \in I} U(V_k)$ act on $\operatorname{Rep}(Q, \mathbf{v}, \mathbf{w})$.

Examples:

$$\begin{tikzcd} \bullet V \arrow[d, "i"] \\ \bullet W \end{tikzcd} \qquad \begin{tikzcd} \bullet V \arrow[loop above, "A"] \arrow[d, "i"] \\ \bullet W \end{tikzcd} \qquad \begin{tikzcd} \bullet V_1 \arrow[r, bend left, "A"] \arrow[d, "i_1"] & \bullet V_2 \arrow[d, "i_2"] \\ \bullet W_1 & \bullet W_2 \end{tikzcd}$$

Hyperkähler quotients (Nakajima quiver varieties)

- "Double" the edges of the quiver. $\mathrm{Rep}(Q, \mathbf{v}, \mathbf{w})$ becomes $T^* \mathrm{Rep}(Q, \mathbf{v}, \mathbf{w})$. Induced action of $G_{\mathbf{v}}$ and $K_{\mathbf{v}}$.
- Quaternionic structure on $T^* \mathrm{Rep}(Q, \mathbf{v}, \mathbf{w})$.
  Complex structures I, J, K.
  There are three moment maps μ_I, μ_J, μ_K for action of $K_{\mathbf{v}}$.
  Stability parameter $\alpha_{\mathbf{v}} \in Z(\mathfrak{k}_{\mathbf{v}}^*)$.
- The *Nakajima quiver variety* is

$$\mathcal{M}^{HK}(Q, \mathbf{v}, \mathbf{w}) := \mu_I^{-1}(\alpha_{\mathbf{v}}) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0) / K_{\mathbf{v}}$$
$$= \left(\mu_J^{-1}(0) \cap \mu_K^{-1}(0) \right)^{\alpha-stable} / G_{\mathbf{v}}.$$

Energy minimising representations

$$\mathcal{M}^{HK}(Q, \mathbf{v}, \mathbf{w}) := \mu_I^{-1}(\alpha_{\mathbf{v}}) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)/K_{\mathbf{v}}.$$

Consider the function $f : \mu_J^{-1}(0) \cap \mu_K^{-1}(0) \to \mathbb{R}$ given by

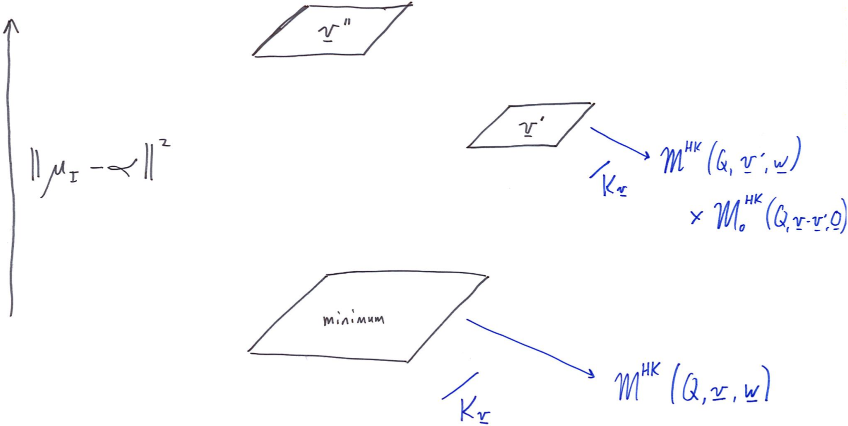
$$f(x) = \|\mu_I(x) - \alpha_{\mathbf{v}}\|^2$$

Then $\mathcal{M}^{HK}(Q, \mathbf{v}, \mathbf{w}) = f^{-1}(0)/K_{\mathbf{v}}$.

Question. What are the critical sets of f?
At a critical point, the representation splits into two subrepresentations $x = x' + x''$.

- $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$.
- x' is stable for $\alpha_{\mathbf{v}'}$ and $G_{\mathbf{v}''} \cdot x''$ is closed.
  Both x' and x'' are energy minimising.
- Modulo $K_{\mathbf{v}}$ we get $\mathcal{M}^{HK}(Q, \mathbf{v}', \mathbf{w}) \times \mathcal{M}_0^{HK}(Q, \mathbf{v}'', \mathbf{0})$.

Critical sets of $f = \|\mu_I - \alpha\|^2 : \mu_{\mathbb{C}}^{-1}(0) \to \mathbb{R}$



Construction of hyperkähler ALE spaces (Kronheimer)

Consider the case where Q has the structure of an affine Dynkin diagram of ADE type and $\mathbf{w} = 0$.

- $\mathbf{v}$ is determined by the kernel of the Cartan matrix C.
- $\Gamma \subset \mathrm{SU}(2)$ is determined by the McKay correspondence.

Example.

$$\begin{tikzcd} & \bullet_{\mathbb{C}} \arrow[dl, bend right] \arrow[dr] & \\ \bullet_{\mathbb{C}} \arrow[ur] \arrow[rr, bend right] & & \bullet_{\mathbb{C}} \arrow[ul, bend right] \arrow[ll, bend right] \end{tikzcd}$$

Type $\hat{A}_3$,

$\Gamma = \mathbb{Z}/3\mathbb{Z}$

$$C = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Theorem (Kronheimer, 1989)

$\mathcal{M}^{HK}(Q, \mathbf{v}, \mathbf{0})$ *is a hyperkähler asymptotically locally Euclidean 4-manifold. Moreover,* $\mathcal{M}^{HK}(Q, \mathbf{v}, \mathbf{0}) \cong \widetilde{\mathbb{C}^2/\Gamma}$.

Instantons on ALE spaces (Kronheimer & Nakajima)

Kronheimer showed that all ALE hyperkähler four manifolds can be constructed in this way.

Given a bundle E over an ALE space $\mathcal{M}^{HK}(Q, \mathbf{v}, 0) \cong \widetilde{\mathbb{C}^2/\Gamma}$ with a fixed framing at infinity, can construct new dimension vectors $\mathbf{v}'$ and $\mathbf{w}'$ determined by the Chern classes of E and the framing at infinity.

Theorem (Kronheimer & Nakjima, 1990)

$\mathcal{M}^{HK}(Q, \mathbf{v}', \mathbf{w}')$ *is the moduli space of framed instantons on* E*.*

(Generalisation of the ADHM construction of instantons on S^4.)

Nakajima's constructions

Theorem (Nakajima)

Let $\hat{\mathfrak{g}}$ be an affine Kac-Moody algebra, and let Q be the quiver associated to the Dynkin diagram of $\hat{\mathfrak{g}}$. Then there is a representation of $U(\hat{g})$ on $\bigoplus_v H_{top}(\mathcal{M}^{HK}(Q, \mathbf{v}, \mathbf{w})_x)$, and this is the irreducible representation with highest weight $\mathbf{w}$.

The representation is constructed via the *Hecke correspondence* $\mathcal{B}_k(Q, \mathbf{v}, \mathbf{w})$, consisting of pairs

$$(x_1, x_2) \in \mathcal{M}^{HK}(Q, \mathbf{v} - \mathbf{e_k}, \mathbf{w}) \times \mathcal{M}^{HK}(Q, \mathbf{v}, \mathbf{w})$$

related by an injective "intertwining" homomorphism.

Later constructions of Nakajima use similar ideas for other algebras (quantum affine algebras, etc.)

Hecke correspondence example

- Let Q be the quiver with one vertex and no edges. For $k = 1, 2$, choose framed representations of Q, consisting of vector spaces V_k and W_k with homomorphisms $i_k : V_k \to W_k$ and $j_k : W_k \to V_k$.
- A homomorphism $A : V_1 \to V_2$ *intertwines* the representations if the following diagram commutes.

$$\begin{tikzcd} \bullet V_1 \arrow[r, bend left, "A"] \arrow[d, bend right, "i_1"'] & \bullet V_2 \arrow[d, bend right, "i_2"'] \\ \bullet W \arrow[u, bend right, "j_1"'] \arrow[r, bend right, "\mathrm{id}"] & \bullet W \arrow[u, bend right, "j_2"'] \end{tikzcd}$$

Choose $\dim V_1 = \dim V_2 - 1$. The Hecke correspondence consists of all pairs $((i_1, j_1), (i_2, j_2))$ for which there exists such an intertwining homomorphism.

Critical points for the square of the moment map

- The critical sets are classified by dimension vectors $0 \leq \mathbf{v}' \leq \mathbf{v}$.
- Modulo $K_{\mathbf{v}}$ we have

$$C_{\mathbf{v}'} := \mathcal{M}^{HK}(Q, \mathbf{v}', \mathbf{w}) \times \mathcal{M}_0^{HK}(Q, \mathbf{v} - \mathbf{v}', \mathbf{0}).$$

- $\mathcal{M}_0^{HK}(Q, \mathbf{v} - \mathbf{v}', \mathbf{0})$ is contractible.
- Each critical set deformation retracts onto the subset $C_{\mathbf{v}'}^0 := \mathcal{M}^{HK}(Q, \mathbf{v}', \mathbf{w}) \times \{0\}$.

Questions. What do the spaces of flow lines look like? What about the Morse correspondence?
Want to answer these questions for the subset $C_{\mathbf{v}'}^0 \subset C_{\mathbf{v}'}$.

Algebraic description of gradient flow convergence

Theorem (Harada, W., 2011)

- *The downwards gradient flow of $\|\mu - \alpha\|^2$ converges to a critical point.*
- *The limit of the flow is isomorphic to the graded object of the Harder-Narasimhan-Jordan-Hölder filtration of the initial condition.*

In other words, the limit of the flow is determined (up to isomorphism) by the algebraic geometry of the initial condition.

This is an analog of theorems for the Yang-Mills functional (Daskalopoulos '92, Daskalopoulos-Wentworth '04, Sibley '13) and Yang-Mills-Higgs functional (W. '08, Li-Zhang '11).

Flow lines for quivers

Given a critical point x, let S_x^- denote the exponential image of the negative eigenspace of the Hessian.

(S_x^- is the tangent space to the unstable manifold)

Definition. Two critical points x_1 and x_2 are *connected by an approximate flow line* if and only if there exists $\delta x \in S_{x_1}^-$ such that $x_1 + \delta x$ flows down to x_2.

$\mathcal{M}(Q, \mathbf{v_1}, \mathbf{v_2}, \mathbf{v}, \mathbf{w})$ denotes the space of pairs $x_1 \in C_{\mathbf{v_1}}^0$ and $x_2 \in C_{\mathbf{v_2}}^0$ s.t. x_1 and x_2 are connected by an approximate flow line.

($\mathcal{M}$ is the Morse correspondence for this example)

Critical points connected by a flow line

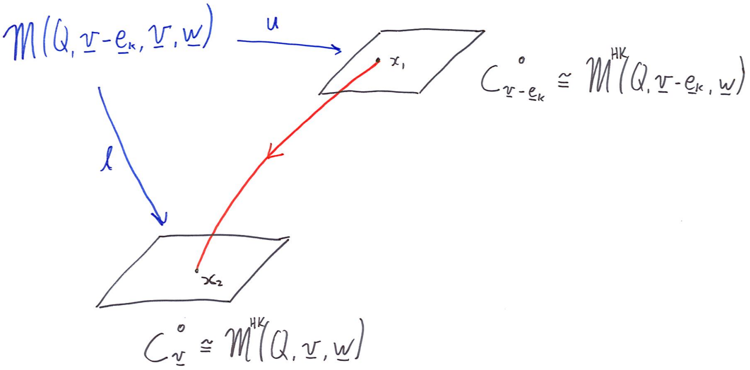
Combining this with the earlier theorem (algebraic description of the limit of the downwards gradient flow), we now have a relationship between Nakajima's constructions and the gradient flow picture.

Theorem (W.)

A pair $[x_1, x_2]$ *is in the Hecke correspondence* $\mathcal{B}_k(Q, \mathbf{v}, \mathbf{w})$ *iff* $[x_1, x_2] \in \mathcal{M}(Q, \mathbf{v} - \mathbf{e_k}, \mathbf{v}, \mathbf{w})$.

(Also true for handsaw quiver varieties)

Hecke Correspondence = Morse Correspondence



Lagrangian subvariety

Let $0 \leq \mathbf{v_1}, \mathbf{v_2} \leq \mathbf{v}$. Nakajima defines a Lagrangian subvariety

$$\mathcal{L}(Q, \mathbf{v_1}, \mathbf{v_2}, \mathbf{w}) \subset \mathcal{M}^{HK}(Q, \mathbf{v_1}, \mathbf{w}) \times \mathcal{M}^{HK}(Q, \mathbf{v_2}, \mathbf{w})$$

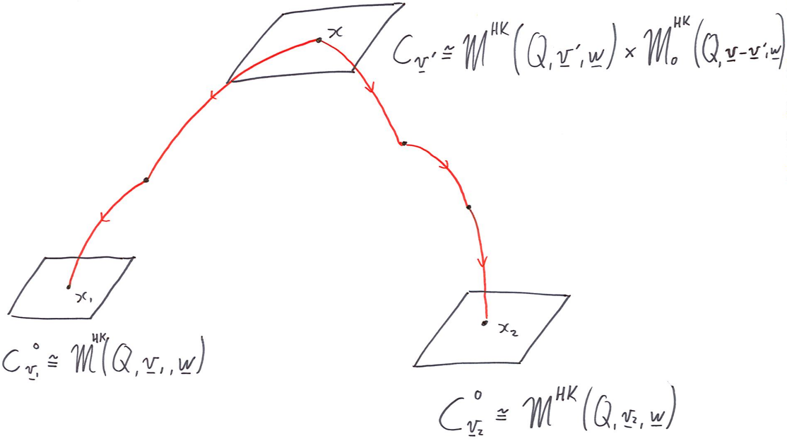
consisting of pairs whose orbit closures intersect in $\mu_{\mathbb{C}}^{-1}(0) \subset T^* \operatorname{Rep}(Q, \mathbf{v}, \mathbf{w})$.

Theorem (W.)

A pair $[x_1, x_2]$ is in the Lagrangian subvariety iff there exists a critical point x such that x_1 and x_2 are connected by (possibly broken) flow lines to x.

(x is isomorphic to the graded object of the Jordan-Hölder filtration of x_1 and x_2.)

Lagrangian Subvariety



Kashiwara's operators

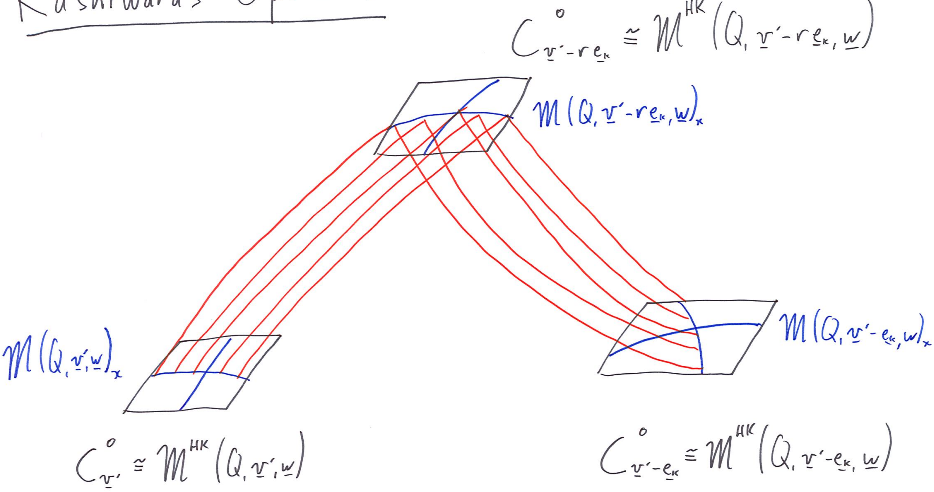
Given $x \in \mathcal{M}_0^{HK}(Q, \mathbf{v}, \mathbf{w})$, Nakajima also studies the subvariety $\mathcal{M}^{HK}(Q, \mathbf{v}', \mathbf{w})_x \subset \mathcal{M}^{HK}(Q, \mathbf{v}', \mathbf{w})$ consisting of points whose orbit closures contain x.

He defines an operator $\tilde{E}_k$ mapping irreducible components of $\mathcal{M}^{HK}(Q, \mathbf{v}', \mathbf{w})_x$ to irreducible components of $\mathcal{M}^{HK}(Q, \mathbf{v}' - \mathbf{e_k}, \mathbf{w})_x$. Another operator $\tilde{F}_k$ maps in the opposite direction.

Kashiwara and Saito prove that the set of all irreducible components with these operators is isomorphic to the crystal of the highest weight module of the modified universal enveloping algebra of $\hat{\mathfrak{g}}$.

The methods of the previous two theorems give a "flow-up, flow-down" construction of Kashiwara's operators. (See picture)

Kashiwara's Operators



Questions

- Can we see the Hecke correspondence for holomorphic bundles or Higgs bundles via the Yang-Mills flow?
- Can we prove that approximate flow lines are the same as flow lines?
- Nakajima constructs representations of affine Kac-Moody algebras, etc. using the Hecke correspondence and a push-pull operation. Is there a Morse-theoretic interpretation of this?