Lecture 13: Optical properties of the parabola

11 March, 2019

Overview

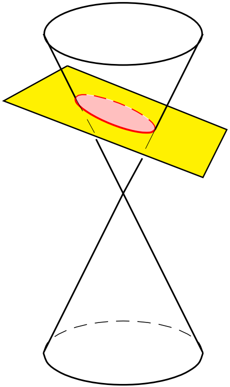
Last time.

- Introduction to conic sections
- Geometric properties of the ellipse and hyperbola.
- Optical properties of the ellipse and hyperbola.

Today.

- A geometric property of the parabola
- The optical property of the parabola
- Applications

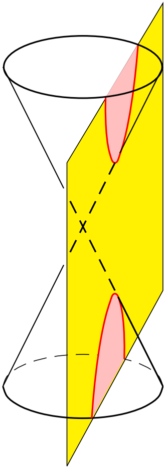
The ellipse as a cross-section of the cone



Last time we showed that an ellipse can be thought of as the intersection of a plane with a cone.

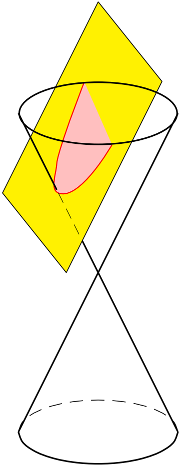
We then used this to prove the geometric and optical properties of the ellipse.

The hyperbola as a cross-section of the cone



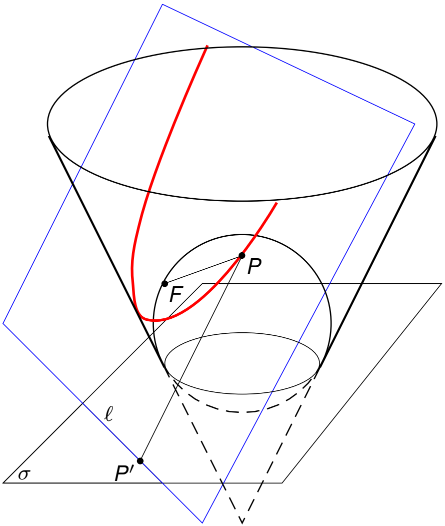
If the plane intersects both components of the cone then the intersection is a hyperbola.

The parabola as a cross-section of the cone



A plane parallel to a side of the double cone cuts the double cone in a parabola.

A geometric property of the parabola



Consider a sphere tangent to the cone which touches the plane of the parabola at one point F.

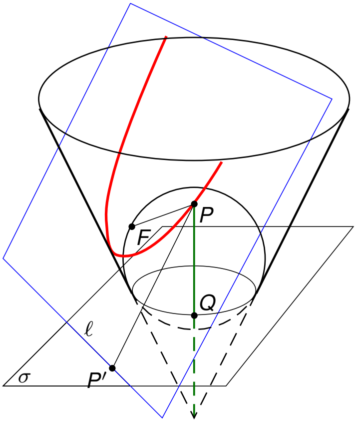
Let P be a point on the parabola.

Let ℓ be the intersection of the plane through the parabola with the plane through the circle where the sphere touches the cone.

Let P' be the orthogonal projection of P to the line ℓ.

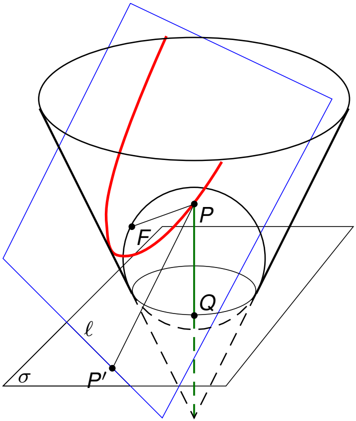
Exercise. Prove that $|FP| = |P'P|$.

A geometric property of the parabola



Solution.

A geometric property of the parabola



Solution. Let σ be the plane through the circle of intersection with the sphere.

Draw a line from *P* to the base of the cone. (Note that this line lies on the cone.) Let *Q* be the intersection of this line with the sphere.

Then $|FP| = |PQ|$ since both lines are tangent to the sphere.

The angle between PP' and the plane σ is equal to the angle between the two planes.

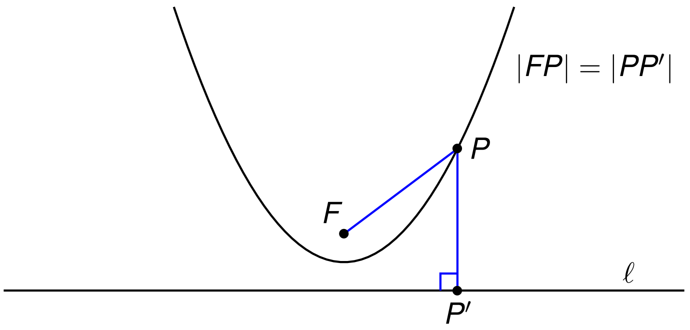
The angle between PQ and σ is equal to the angle between the sides of the cone and σ.

Since the plane through the parabola is parallel to a line on the cone through the base of the cone then PP' and PQ make the same angle with the plane σ. Therefore $|PP'| = |PQ| = |FP|$. ∎

Geometric definition of the parabola

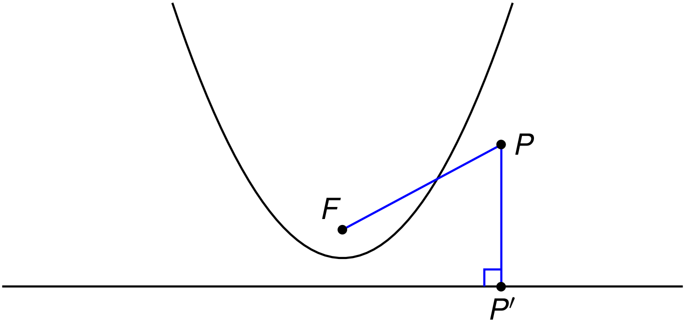
Therefore we can think of the parabola in the following way.

Let ℓ be a line in the plane and let F be a point not on ℓ. The parabola with focus F and directrix ℓ is the set of points P such that $|FP|$ is equal to the perpendicular distance from P to ℓ.



Exercise

Exercise. Consider a parabola with focus F and directrix ℓ.
Given a point P in the plane, let P' be the orthogonal projection of P onto the line ℓ.
Prove that if a point P lies outside the parabola (i.e. the line segment PF intersects the parabola) then $|PF| > |PP'|$.



Solution

Solution.

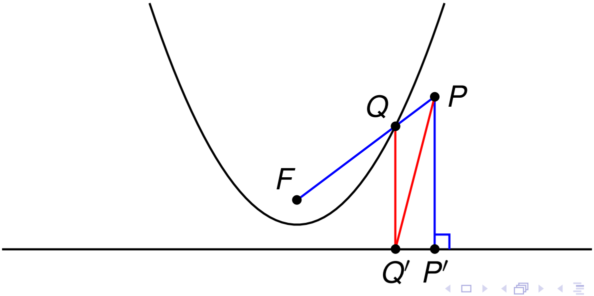
Solution

Solution. Let Q be the intersection of PF with the parabola. Then $|PF| = |FQ| + |QP|$.

Let Q' be the projection of Q onto the directrix. The geometric property of the parabola shows that $|FQ| = |QQ'|$.

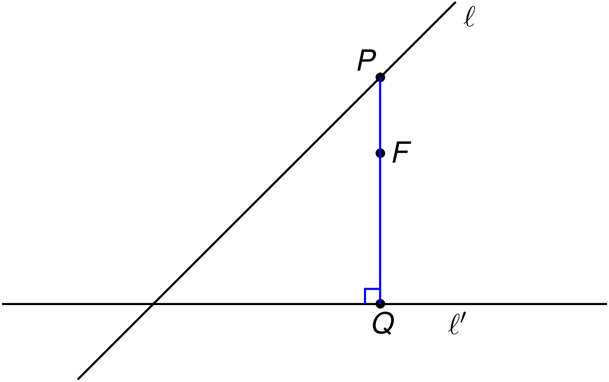
Then the triangle inequality applied to the triangle $\Delta QQ'P$ shows that $|FP| = |FQ| + |QP| = |QQ'| + |QP| \geq |PQ'|$.

We also know $|PQ'| > |PP'|$ since P' is the orthogonal projection of P onto ℓ. Therefore $|FP| > |PP'|$.



Optical property of the parabola (preliminary result)

Lemma 3. Let ℓ and ℓ' be two lines in the plane and let F be a point lying between the two lines. Given any point P on ℓ, let Q be the orthogonal projection to the line ℓ'. Then $|PQ| - |FP|$ is maximised if and only if F, P, Q are collinear.



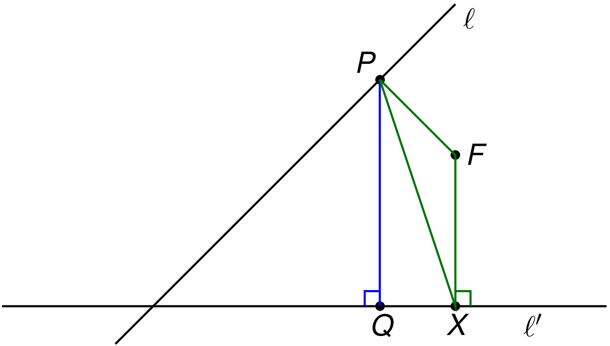
Exercise. Prove this!

Solution.

Optical property of the parabola (preliminary result)

Solution. Let X be the orthogonal projection of F to the line ℓ'. Note that $|FX|$ is constant (independent of P). Since Q is the projection of P to ℓ' then $|PQ| \leq |PX|$.

Note that $X = Q$ if and only if P, F, X are collinear.

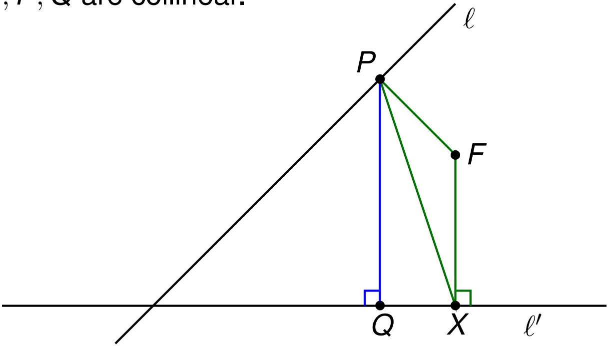


Optical property of the parabola (preliminary result)

Applying the triangle inequality to the triangle ΔFPX gives us $|PX| \leq |FP| + |FX|$ with equality if and only if P, F, X are collinear.

Therefore $|PQ| \leq |PX| \leq |FP| + |FX|$ with equality if and only if P, F, Q are collinear.

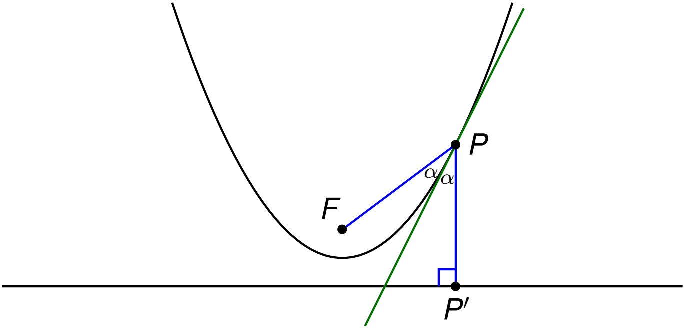
Since $|FX|$ is constant, then $|PQ| - |FP|$ is maximised if and only if P, F, Q are collinear. ∎



Optical property of the parabola

Now we are ready to prove the optical property of the parabola.

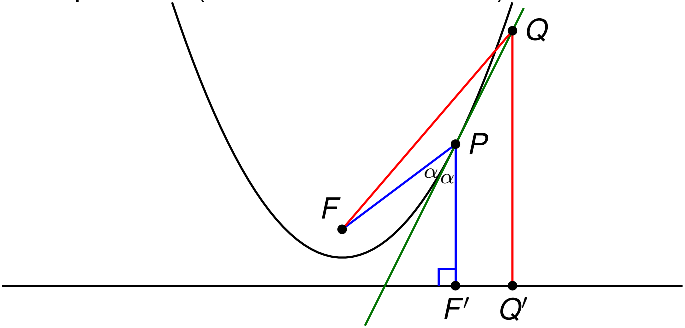
Theorem. (Optical property of the parabola) Let P be a point on a parabola with focus F and directrix ℓ. Let P' be the orthogonal projection of P to the directrix. Then the tangent to the parabola at P is the angle bisector of $\angle FPP'$.



Proof. Let F' be the reflection of F across the tangent line. We want to show that $F' = P'$.

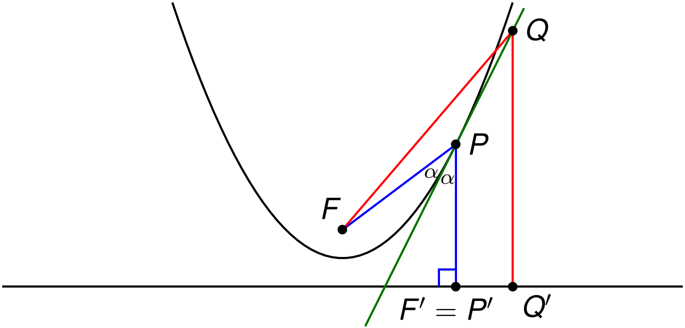
Let Q be any other point on the tangent line and let Q' be the projection to the directrix. Since F' is the reflection of F across a line containing Q then $|FQ| = |F'Q|$.

By the earlier exercise $|F'Q| = |FQ| \geq |QQ'|$ with equality iff Q lies on the parabola (which occurs iff $Q = P$).

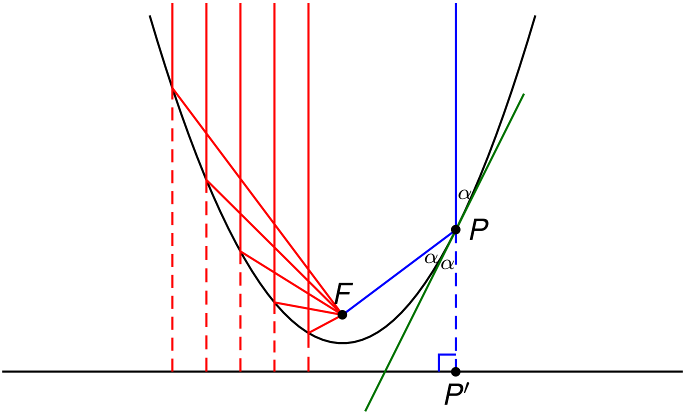


Therefore $|QQ'| - |F'Q| \leq 0$ with equality iff $Q = P$. Therefore $|QQ'| - |F'Q|$ is maximised iff $P = Q$.

Then we can use Lemma 3 to show that F', P, P' are collinear. Therefore, since $|F'P| = |FP| = |PP'|$ then $F' = P'$, which completes the proof. ∎



The optical property shows that if you shine a beam of light at the parabola in a direction perpendicular to the directrix then the light will be reflected through the focus.

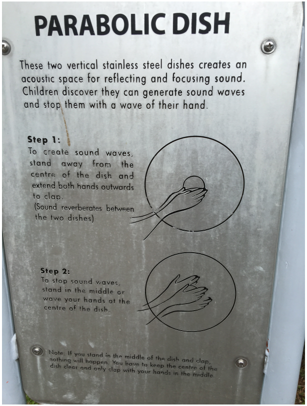


Applications

This property is used in the design of satellite dishes. If a signal is coming from outer space then we can assume that the rays from the source are almost parallel. Therefore a mirror with parabolic cross-sections will reflect the signal onto the focus.

Parabolic Dish at One North Park

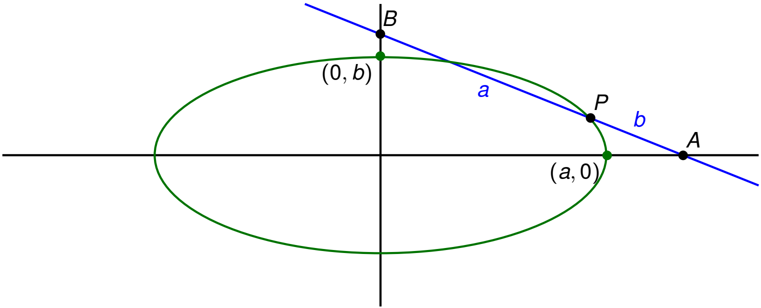
You can also see an example of a pair of parabolic reflectors at One North Park. The focal points of the two parabolic reflectors coincide. If you clap your hands at this focal point, the sound will reflect off one dish, onto the other dish, and then back to the focal point.



Constructing an ellipse: the trammel of Archimedes

Exercise. Consider the ellipse defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let ℓ be a line in $\mathbb{R}^2$ intersecting the x-axis at A, and intersecting the y-axis at B. Suppose that $|AB| = a + b$ and let $P = (x, y)$ be a point on ℓ such that $|AP| = b$, $|PB| = a$.

Prove that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and therefore P lies on the ellipse.



This construction is known as the Trammel of Archimedes.

Constructing an ellipse: the concentric circle method

Exercise. Consider the ellipse defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Draw two circles centred at the origin, one with radius a and the other with radius b.

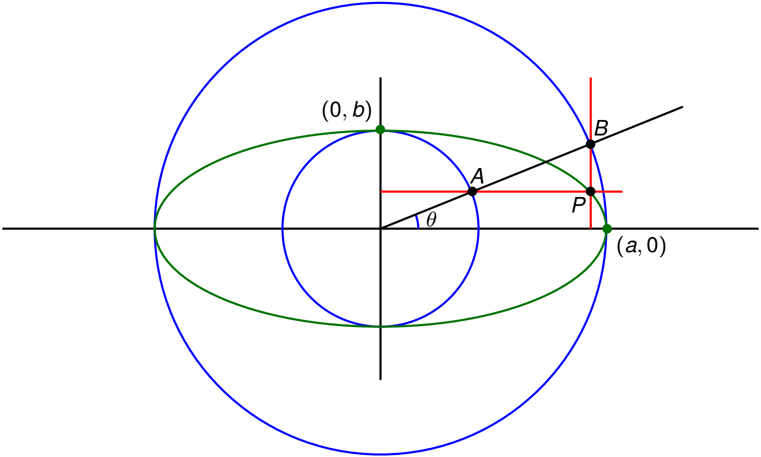
Now, given any ray from the origin, let A be the intersection of this ray with the inner circle and B the intersection with the outer circle.

Draw a line through A perpendicular to the minor axis, and a line through B perpendicular to the major axis.

Let $P = (x, y)$ be the intersection of these two lines.

Prove that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and therefore P lies on the ellipse.

Constructing an ellipse: the concentric circle method



Solution.

Solution. We can solve the problem by finding the coordinates of P using similar triangles (or any other method you prefer).

We know that $|AB| = a + b$, $|BP| = a$ and $|AP| = b$. Let $A = (x_0, 0)$ and $B = (0, y_0)$. Note that $x_0^2 + y_0^2 = (a + b)^2$ by Pythagoras' theorem.
Then $P = (x, y)$ satisfies

$$x = \frac{ax_0}{a + b}, \; y = \frac{by_0}{a + b}$$

and so

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2 x_0^2}{(a + b)^2 a^2} + \frac{b^2 y_0^2}{(a + b)^2 b^2} = \frac{x_0^2 + y_0^2}{(a + b)^2} = 1$$

Therefore $P = (x, y)$ satisfies the equation of the ellipse.

Solution.

Constructing an ellipse: the concentric circle method

Solution. Let θ be the angle between the ray AB and the x-axis.

In terms of θ, the point A has coordinates $(b\cos\theta, b\sin\theta)$ and the point B has coordinates $(a\cos\theta, a\sin\theta)$.

Therefore the point P has coordinates $(x, y) = (a\cos\theta, b\sin\theta)$ (the x coordinate from B and the y coordinate from A), and so

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta = 1$$

Therefore $P = (x, y)$ lies on the ellipse.