Lecture 17: Introduction to spherical geometry

29 March, 2019

Overview

Last time. (Inversive Geometry)

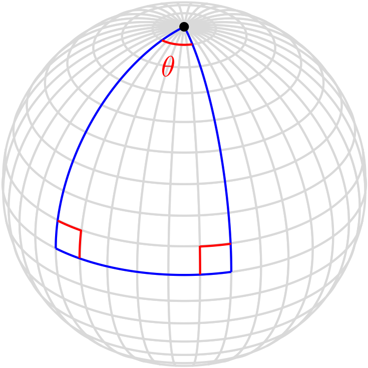
- Inversion preserves the cross-ratio
- What happens to the centre of a circle under inversion?
- The problem of Apollonius
- Steiner's porism
- Peaucellier's linkage

Today. (Spherical Geometry)

- A model for spherical geometry
- Euclid Book I revisited.
    - What happens when we change the parallel axiom?
- The angle sum of a triangle in spherical geometry
- Spherical trigonometry

Exercise

Exercise. Consider a sphere of radius r with a spherical triangle with two right-angles (one side is a segment of the equator and the other two sides join this segment to the north pole). Let θ be the angle (in radians) at the north pole. Compute the area of the triangle.



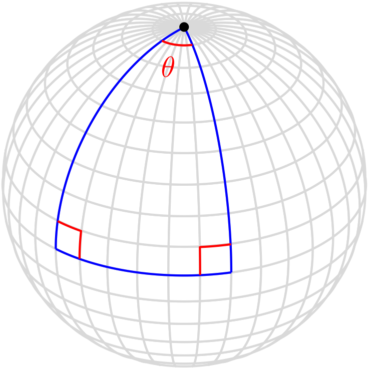
This week we will use radians for angles.

Solution.

Solution

Solution. The area of the northern hemisphere is $2\pi r^2$. The triangle makes up a proportion $\frac{\theta}{2\pi}$ of the northern hemisphere. Therefore the area is

$$\frac{\theta}{2\pi} \times 2\pi r^2 = \theta r^2$$



Motivation for spherical geometry

One example application of spherical geometry is for navigation on the surface of the earth.

- The Mediterranean sea and the Aegean sea near Greece are small enough that we can think of them as flat. Therefore Euclidean geometry is a good approximation to solve the Greeks' questions about navigation.
- Later, when people began sailing further around the world, the curvature of the earth became a factor in navigation. Therefore they needed a geometry that takes into account the fact that the earth is spherical.
- The curvature of the Earth also becomes a factor when trying to draw an accurate map of the world. You can read more about map projections and the difficulties in creating accurate map projections by clicking the links. Another website is www.thetruesize.com which will give you an idea of how the standard map projection distorts area around the north and south pole.

Defining a geometry

To define a geometry, we first need to define the objects of the geometry.

Examples. In Euclidean geometry we had points, lines and circles (“ruler and compass geometry”). From these we could derive the notion of angle.

In perspective drawing we had lines and points (“ruler geometry”), but no circles, distances or angles.

In inversive geometry we had points, circles (we think of a line as a circle of infinite radius - see Question 1 on Tutorial 9) and angles, but no distances.

We can also think of a geometry in terms of the transformations preserving the geometric objects. For example, projections preserve the objects of perspective drawing and inversions preserve the objects of inversive geometry.

This idea is due to Felix Klein (we will talk more about this later).

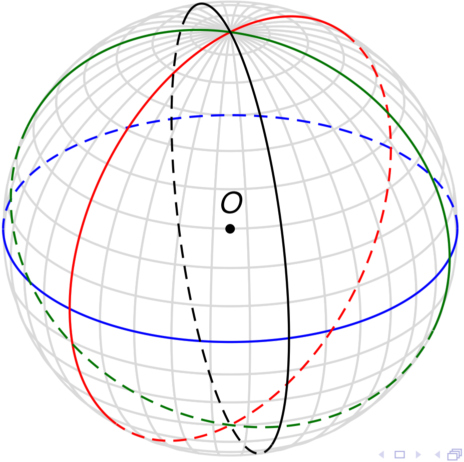
Definition.

- The *points* of spherical geometry are the points on the unit sphere in $\mathbb{R}^3$.
- The *lines* of spherical geometry are the "great circles" of the unit sphere, corresponding to the intersection of the sphere with a plane through the origin in $\mathbb{R}^3$.
- The *circles* of spherical geometry correspond to the intersection of the sphere with a plane (a line is a special case of a circle). The centre of the circle is one of the intersection points of the sphere with the normal to the plane.
- An *angle* between two lines at their intersection P is the angle between the two planes corresponding to those lines. Equivalently, it is the angle between the tangents to the great circles at P.

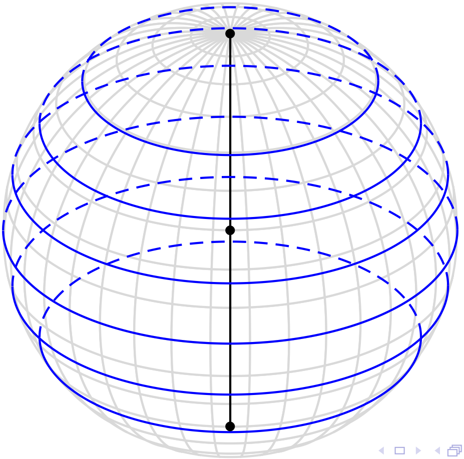
  For simplicity, in the rest of the lecture we will assume that angles are less than π radians.

Lines in spherical geometry

Here are some examples of lines (great circles) on the sphere. These examples correspond to the familiar examples of the equator and the lines of constant longitude on a map of the earth.

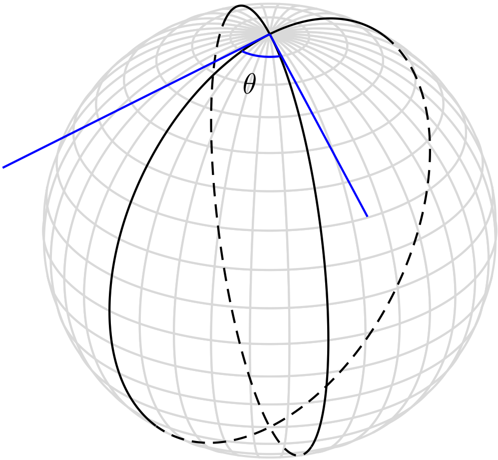


Here are some examples of circles on the sphere. These examples correspond to the familiar examples of the lines of constant latitude on a map of the earth. The centre is either the north or south pole.



Angles in spherical geometry

Here is an example of the angle between two lines on the sphere. The angle between the curves on the surface of the sphere is defined to be the angle between the tangent lines.



Which of Euclid's axioms are valid in spherical geometry?

Axiom 1. One can draw a straight line segment between any two points.
This is true, but the line segment is not unique

Axiom 2. One can extend a straight line segment indefinitely.
If we extend a line segment AB it will eventually pass through the points A and B again.

Axiom 3. One can draw a circle with a given centre and radius.
This is true if the radius is small enough (less than π).
If the radius is larger than π then we still get a circle, but this is the same as one of the circles with radius less than π.

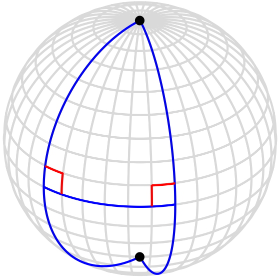
Axiom 4. All right angles are equal to each other.
This is true in spherical geometry

Which of Euclid's axioms are valid in spherical geometry?

Axiom 5. If a straight line crossing two straight lines makes the interior angles on one side less than two right angles, then the two straight lines, if produced indefinitely, will meet on that side on which the angles are less than the two right angles.

This is not true in spherical geometry. This axiom implies that if a line crosses two other lines and makes two right angles, then the other lines do not intersect.

In spherical geometry, every pair of lines intersects in two points.



Which of Euclid's results are valid in spherical geometry? (Congruent triangles)

Recall our basic results about congruent triangles in Euclidean geometry.

Proposition. (Euclid Prop. I.4 (SAS congruence))
If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they are congruent.

Proposition. (Euclid Prop. I.8 (SSS congruence))
If two triangles have two sides equal to two sides respectively, and also have the base equal to the base, then they are congruent.

Our proof in Euclidean geometry used the idea of superimposing one triangle over another.

Which of Euclid's results are valid in spherical geometry? (Congruent triangles)

Proof of (SAS congruence).

Let ΔABC and ΔDEF be two triangles such that $|AB| = |DE|$, $|AC| = |DF|$ and $\angle BAC = \angle EDF$.
If ΔABC is superimposed on ΔDEF and if the point A is placed on the point D and the straight line AB on DE then the point B coincides with E, since $|AB| = |DE|$.
Then (after reflecting if necessary) the straight line AC also coincides with DF because $\angle BAC = \angle EDF$. Therefore the point C coincides with the point F because $|AC| = |DF|$.
Therefore BC coincides with EF and so $|BC| = |EF|$. Similarly, $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$. ∎

Question. If we move a triangle around in the plane, is it still the same triangle?
We say that it is congruent to the original triangle as long as we preserve the sidelengths and angles

Rigid motions

Definition. (Euclidean rigid motion)
A rigid motion of the Euclidean plane is a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|AB| = |f(A)f(B)|$ for every pair of points A, B in the plane.

Rigid motions preserve distance and angle

Examples.

- Translation: $f(x, y) = (x, y) + (a, b)$ for some fixed vector $(a, b) \in \mathbb{R}^2$.
- Rotation: $f(x, y) = (\cos\theta x + \sin\theta y, \cos\theta y - \sin\theta x)$ for some angle θ.
- Reflection: $f(x, y) = (x, -y)$ (reflection across x-axis)
- Glide reflection: $f(x, y) = (x + a, -y)$ (reflection across x-axis followed by a translation in the direction of the x-axis)

You can construct a general reflection by composing a reflection across the x-axis with translations and rotations.

Rigid motions (cont.)

So the Euclidean rigid motions are those which preserve the objects of Euclidean geometry (lines, circles, distances, angles).

Euclid's idea of "superimposing one triangle over another" in the proof of (SAS congruence) involves applying rigid motions to ΔABC until A coincides with the point D (you can do this with translations) and then AB coincides with the line DE (you can do this with rotations and reflections).

Question. What are the rigid motions of spherical geometry? Once we define a rigid motion, can we use them to prove SAS congruence in spherical geometry?

If we look at the proof of SAS congruence again, then we see that **we need the rigid motions to preserve length and angle**.

Rigid motions of the sphere

Definition. The distance $|AB|$ between any two points A, B on the sphere is the distance along the shortest great circle arc between A and B.

Even though the great circle arc of shortest length may not be unique (e.g. A=North Pole and B=South Pole) the distance is always unique.

Definition. (Spherical rigid motion)
A rigid motion of the sphere S is a map $f : S \to S$ such that $|AB| = |f(A)f(B)|$.

Theorem. The rigid motions of the sphere are all compositions of rotations of the sphere and reflections through a great circle (equivalently, reflections in $\mathbb{R}^3$ across a plane through the origin).

Theorem. Rigid motions of the sphere map great circles to great circles, map circles to circles and preserve angles.

Rigid motions of the sphere (Exercise)

Exercise. Show how to construct the rigid motion $f(x, y, z) = (-x, -y, -z)$ as a composition of reflections.

This particular rigid motion is known as the antipodal map. It takes each point to its opposite point on the sphere.

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Solution.

Rigid motions of the sphere (Exercise)

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This particular rigid motion is known as the antipodal map. It takes each point to its opposite point on the sphere.

Solution. First reflect across the *xy* plane.
This maps $(x, y, z) \mapsto (x, y, -z)$.

Next reflect across the *xz*-plane.
This maps $(x, y, -z) \mapsto (x, -y, -z)$.

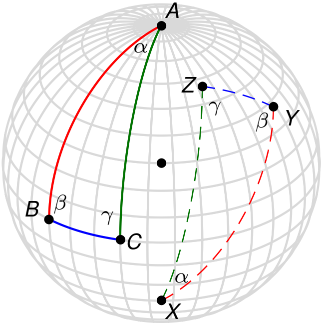
Finally, reflect across the *yz*-plane.
This maps $(x, -y, -z) \mapsto (-x, -y, -z)$.

By composing these three reflections, we get a map $(x, y, z) \mapsto (-x, -y, -z)$.

Therefore we can now complete the proof of SAS congruence and SSS congruence in spherical geometry by applying a rigid motion to ΔABC so that the image of A coincides with D and the line AB coincides with DE. After reflecting if necessary, the line AC then coincides with DF.

Example. Let ΔABC be a triangle on the sphere and let X, Y, Z be the respective images of the points A, B, C under the antipodal maps defined earlier.

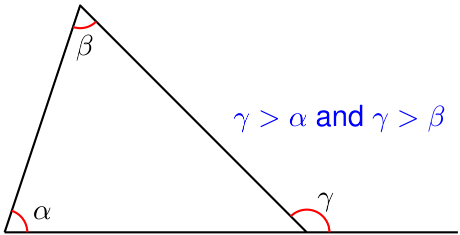
Then $\Delta ABC \cong \Delta XYZ$.



Which of Euclid's results are valid in spherical geometry? (External angles)

Recall Euclid Prop. I.16.

Proposition. In any triangle, the exterior angle is greater than either of the other two interior angles.



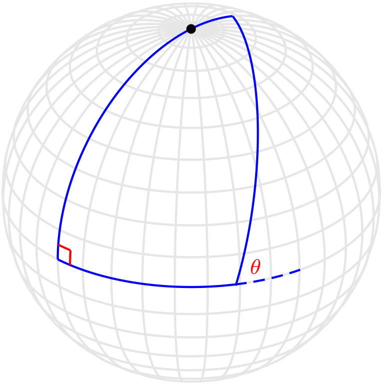
Exercise. Give an example in spherical geometry where this proposition fails.

Solution

Solution.

Solution. In our example from the exercise at the beginning of the lecture, the external angle of the triangle was equal to one of the interior angles.

Below is an example where the external angle is smaller than one of the other interior angles.



$\theta < 90°$

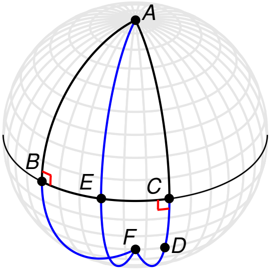
What goes wrong with the proof?

Here is the translation of the proof into spherical geometry for our earlier example where the external angle equals one of the internal angles.

Extend AC to a point D. Bisect BC at E. Extend AE to make a straight line to F such that $|EF| = |AE|$. Draw FC.

By SAS congruence, since $|BE| = |EC|$, $|AE| = |EF|$ and $\angle BEA = \angle CEF$ then $\Delta BAE \cong \Delta CFE$. Therefore $\angle ABE = \angle FCE$.

But the angle $\angle ECD$ is greater than the angle $\angle ECF$, therefore $\angle BCD > \angle ABE$.

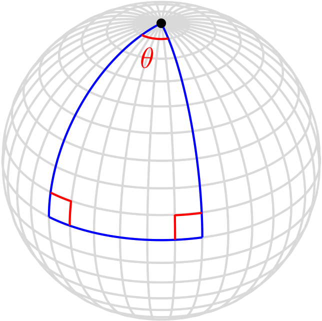


In spherical geometry the lines AB and AC intersect at another point beyond the line BC. Therefore it is possible that we may have $\angle ECD \leq \angle ECF$, and so the above statement in red is not always true.

Which of Euclid's results are valid in spherical geometry?

Therefore Euclid Prop. I.17 fails also (the sum of any two angles of a triangle is less than two right angles).

We have seen that there exist triangles whose angles add up to more than 180° (or π radians). The first exercise today gives an example of such a triangle.



By changing the angle in the triangle from the first exercise today, we see that we can make the angle sum of the triangle equal to any number from π to 3π.

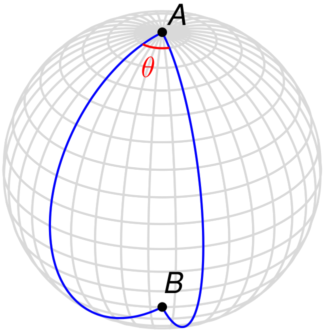
Therefore the angle sum of a triangle cannot be a constant.

Question. Can we derive any useful formula for the angle sum of a triangle?

Area of a lune

Definition. Let A be a point on the sphere and let B be the antipodal point of A (the image of A under the antipodal map). A lune is the region enclosed by two lines from A to B.

The angle of the lune is the angle between the two lines at A (which is equal to the angle at B).



Using our earlier exercise, we see that the area of a lune of angle θ is

$$\frac{\theta}{2\pi} \cdot 4\pi r^2 = 2\theta r^2$$

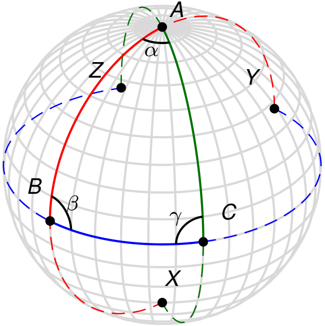
Angle excess of a spherical triangle

Theorem. (Angle excess of a spherical triangle)
Let ΔABC be a triangle on the sphere of radius r with angles α, β, γ at the vertices A, B, C respectively. Then

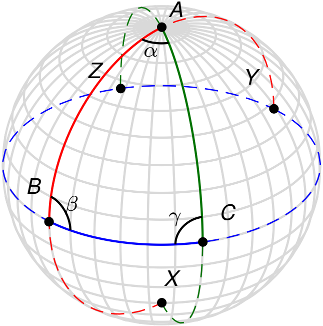
$$(\alpha + \beta + \gamma - \pi) r^2 = \text{Area}(\Delta ABC)$$

Proof. Let X, Y, Z be the antipodal points of A, B, C respectively.

The triangle is defined by three great circles. Two of the great circles pass through A and therefore define a lune of angle α. Similarly, there is a lune of angle β with one vertex at B and a lune of angle γ with one vertex at C.



Proof. (cont.) Each lune is the union of two triangles. We can now write the area of the three lunes in terms of the areas of these triangles.

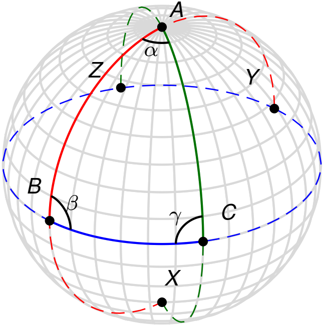


$$\text{Area}(BCYAB) = 2\beta r^2 =$$
$$= \text{Area}(\Delta ABC) + \text{Area}(\Delta CYA)$$
$$\text{Area}(CAZBC) = 2\gamma r^2 =$$
$$= \text{Area}(\Delta ABC) + \text{Area}(\Delta AZB)$$
$$\text{Area}(ABXCA) = 2\alpha r^2 =$$
$$= \text{Area}(\Delta ABC) + \text{Area}(\Delta BXC)$$

Therefore

$$2(\alpha + \beta + \gamma)r^2$$
$$= 3\,\text{Area}(\Delta ABC) + \text{Area}(\Delta CYA) + \text{Area}(\Delta AZB) + \text{Area}(\Delta BXC)$$

Proof. (cont.) We also know that $\Delta YAZ \cong \Delta BXC$ since they are antipodal triangles (the antipodal map is a rigid motion).
Finally, the hemisphere bounded by the great circle $BCYZB$ which contains the triangle ΔABC is the union of the triangles ΔABC, ΔACY, $\Delta AYZ \cong \Delta XBC$ and ΔAZB.



Therefore, the sum of the areas of the three lunes is

$$2(\alpha+\beta+\gamma)r^2$$
$$= 3\,\text{Area}(\Delta ABC) + \text{Area}(\Delta CYA) + \text{Area}(\Delta AZB) + \text{Area}(\Delta BXC)$$
$$= 3\,\text{Area}(\Delta ABC) + \text{Area}(\Delta CYA) + \text{Area}(\Delta AZB) + \text{Area}(\Delta YAZ)$$
$$= 2\,\text{Area}(\Delta ABC) + 2\pi r^2$$

and so $\text{Area}(\Delta ABC) = (\alpha+\beta+\gamma-\pi)r^2$.

What are the implications of the angle excess?

Suppose you want take a flat sheet of paper and you want to use it to cover a spherical object. Then any straight line on the flat paper will become a straight line on the sphere.

We just saw that triangles on the sphere have different angle sums to triangles on a flat sheet of paper. The difference is measured by the area of the triangle.

Therefore, to cover the sphere with a flat sheet of paper, we have to allow the triangles to expand in area (i.e. you will have to tear some holes in the paper).

You can see this when looking at a flat map of the earth. To map the (spherical) earth on a flat sheet of paper we must introduce distortions to account for the fact that the angle sum of a triangle on the sphere is different to the angle sum of a triangle on a flat sheet of paper.

This is due to the **curvature of the sphere** (more next week).

You can read more about map projections here, here and here.

Next time

Next Monday 1 April we will talk about trigonometry on the sphere and how to use this to solve problems in spherical navigation.

- The sine and cosine rule on the sphere
- Solving triangles on the sphere
- Applications to navigation.

The class on Friday 5 April will be devoted to construction problems related to spherical geometry (as well as any questions that you may have about the construction problems from the rest of the course).