Lecture 8: More quadrilaterals

8 February, 2019

Overview

Last time.

- Formulae for the inradius and circumradius of a triangle
- Heron's formula for the area of a triangle
- Recall basic properties of cyclic quadrilaterals
- The Simson line of a triangle
- Ptolemy's theorem

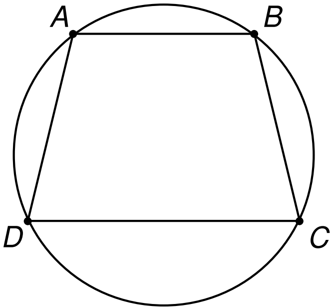
Today.

- Brahmagupta's formula for the area of a cyclic quadrilateral.
- Properties of non-cyclic quadrilaterals
- More constructions in Euclidean geometry

Exercise

Let $ABCD$ be a cyclic quadrilateral and suppose that AB is parallel to CD. Prove that $\angle ABC = \angle DAB$ and $\angle CDA = \angle BCD$.

Then prove that $|AD| = |BC|$.



Solution

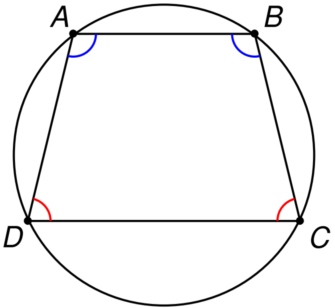
Solution.

Solution. Since $ABCD$ is cyclic then $\angle ABC = 180 - \angle CDA$.

Moreover, since AB is parallel to CD then $\angle CDA = 180 - \angle DAB$.

Combining these two results gives us $\angle ABC = \angle DAB$.

Similarly, $\angle DCB = 180 - \angle ABC$ (since AB is parallel to CD) and so $\angle DCB = \angle CDA$.

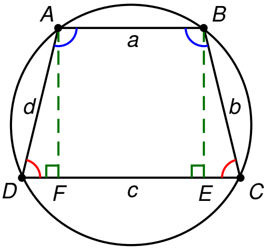


Solution. (cont.)
Now we want to show that $|AD| = |BC|$. Suppose without loss of generality that $|AB| < |CD|$ (so that $\angle CDA = \angle DCB$ are acute and $\angle DAB = \angle CBA$ are obtuse).

Drop altitudes from A and B to points E and F on CD.

Since $ABEF$ is a rectangle, then $|AF| = |BE|$. Since $\angle ECB = \angle FDA$, $\angle CEB = 90^\circ = \angle DFA$ and $|AF| = |BE|$ then $\Delta ADF \cong \Delta BCE$ by ASA.

Therefore we can conclude that $|AD| = |BC|$.



Brahmagupta's formula

Recall Heron's formula from the previous lecture.

Theorem. (Heron) Let ΔABC be a triangle with sidelengths a, b, c and let $s = \frac{1}{2}(a + b + c)$ be the *semiperimeter*. Then

$$\text{Area}(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$$

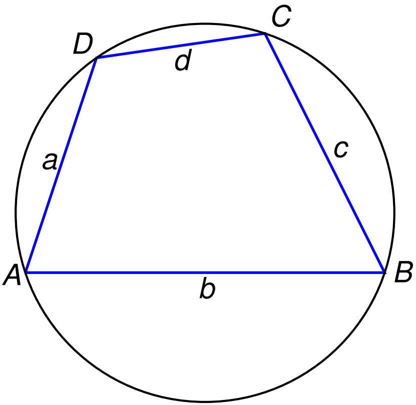
Question. Does a similar formula exist for quadrilaterals?

Theorem. (Brahmagupta) Let $ABCD$ be a cyclic quadrilateral with sidelengths a, b, c, d and let $s = \frac{1}{2}(a + b + c + d)$ be the semiperimeter. Then

$$\text{Area}(ABCD) = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Remark. Heron's formula is the special case of a cyclic quadrilateral with one sidelength d equal to zero (i.e. two of the vertices coincide).

Brahmagupta's formula (picture)



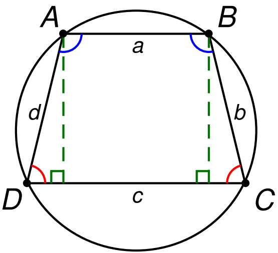
Brahmagupta's formula (picture)

A full proof can be found in the lecture notes on IVLE taken from a previous version of the course (this proof uses trigonometry).

I will sketch the idea of a geometric proof of Brahmagupta's formula and I will post the details on IVLE.

First consider the case where $ABCD$ is a trapezoid (AB is parallel to CD).

We proved in the exercise at the beginning of class that $\angle DAB = \angle ABC$ (and hence $\angle BCD = \angle CDA$ also).



Brahmagupta's formula (trapezoid case)

Therefore, using the notation in the picture, we have $b = d$ and so $s = b + \frac{1}{2}(a + c)$. The height of the trapezoid is

$$h = \sqrt{b^2 - \frac{1}{4}(a - c)^2} \quad \text{(by Pythagoras)}$$

and so the area is

$$\mathrm{Area}(ABCD) = \frac{1}{2}(a + c)h = \frac{1}{2}(a + c)\sqrt{b^2 - \frac{1}{4}(a - c)^2}.$$

From the equation $s = b + \frac{1}{2}(a + c)$, we have

$$s - a = b - \frac{1}{2}(a - c), \; s - b = \frac{1}{2}(a + c) = s - d,$$
$$s - c = b + \frac{1}{2}(a - c)$$

Brahmagupta's formula (trapezoid case)

Therefore, Brahmagupta's formula becomes

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$
$$= \frac{1}{2}(a+c) \cdot \sqrt{\left(b - \frac{1}{2}(a-c)\right)\left(b + \frac{1}{2}(a-c)\right)}$$

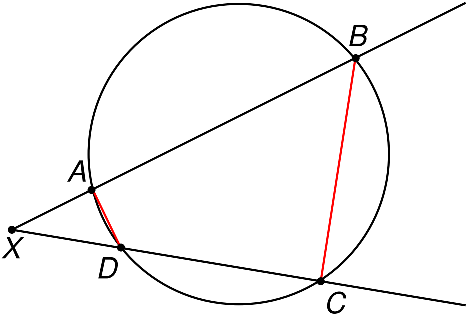
Using the formula $(x-y)(x+y) = x^2 - y^2$ we obtain

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$
$$= \frac{1}{2}(a+c) \cdot \sqrt{b^2 - \frac{1}{4}(a-c)^2} = \text{Area}(ABCD)$$

which is exactly the formula for the area which was derived earlier.

Brahmagupta's formula (non-trapezoid case)

Now suppose that at least one pair of sides is not parallel. If we extend this pair of opposite sides then they will meet at a point, call it X, as in the diagram below



The idea of the proof is to show that $\Delta XDA \sim \Delta XBC$ (this uses the fact that the quadrilateral is cyclic). **See Lecture 9**

Brahmagupta's formula (non-trapezoid case)

We can then compute the area of the triangles ΔXAD and ΔXCD from Heron's formula.

The fact that they are similar means that

$$\text{Area}(\Delta XAD) = \frac{|XA|^2}{|XC|^2} \cdot \text{Area}(\Delta XCB)$$

and so

$$\begin{aligned}\text{Area}(ABCD) &= \text{Area}(\Delta XCB) - \text{Area}(\Delta XAD)\\ &= \left(1 - \frac{|XA|^2}{|XC|^2}\right)\text{Area}(\Delta XCB)\end{aligned}$$

After a lot of algebra (see the handout on IVLE), the expression for Area(ΔXCB) using Heron's formula reduces to Brahmagupta's formula.

General quadrilaterals

What happens if the quadrilateral is non-cyclic? In general, it is hard to find nice geometric results that are valid for all quadrilaterals.

The following two theorems show that we can find some structure within general quadrilaterals.

Theorem. (Varignon) Let W, X, Y, Z be the midpoints of the respective sides AB, BC, CD, DA of a quadrilateral $ABCD$. Then the quadrilateral $WXYZ$ is a parallelogram.

Theorem. The angle bisectors of a quadrilateral form a cyclic quadrilateral (see the picture).

We will prove these theorems after the next exercise.

Exercise

Exercise. Let AB, CD and EF be three distinct lines and suppose that AB is parallel to CD and that CD is parallel to EF. Prove that AB is parallel to EF.

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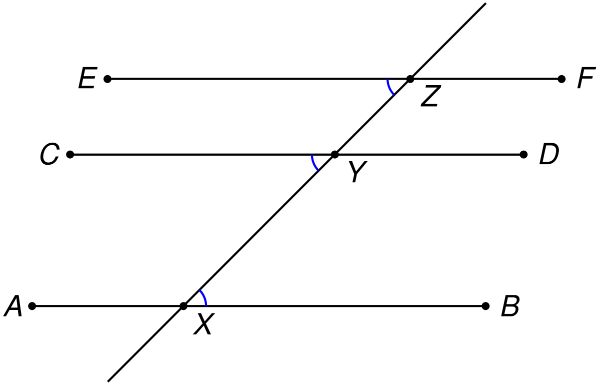
Solution.



Solution

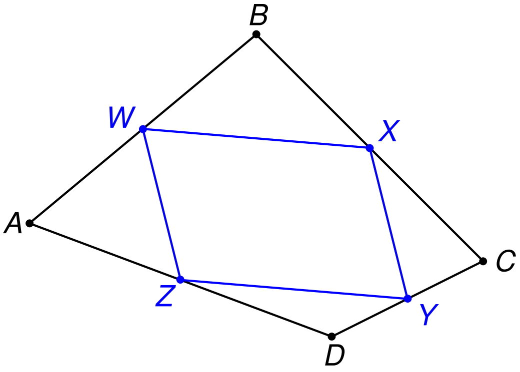
Solution. Let XY be a line crossing the parallel lines AB and CD. Since AB is parallel to CD then the alternate angles are equal, therefore $\angle BXY = \angle CYX$ by Euclid Prop. I.29.

Since EF is parallel to CD and XY crosses CD then XY also crosses EF (for example, this follows from Playfair's Axiom, which is equivalent to Euclid's Parallel Axiom). Let Z be the intersection point. Then $\angle EZY = \angle CYX$ by Euclid Prop. I.29.

Therefore $\angle BXY = \angle EZY$ and so the alternate angles are equal for the lines AB and EF. Therefore they are parallel by Euclid Prop. I.27.

Varignon's theorem

Theorem. (Varignon) Let W, X, Y, Z be the midpoints of the respective sides AB, BC, CD, DA of a quadrilateral $ABCD$. Then the quadrilateral $WXYZ$ is a parallelogram.



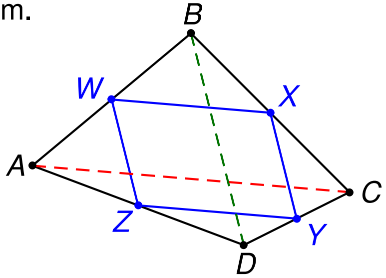
Proof of Varignon's theorem

Proof. We have

$$1 = \frac{|AW|}{|BW|} = \frac{|CX|}{|BX|}$$

Therefore Thales' theorem says that WX is parallel to AC. Similarly, $1 = \frac{|AZ|}{|ZD|} = \frac{|CY|}{|YD|}$, and so ZY is also parallel to AC. The result of the exercise then says that WX is parallel to ZY.

The same idea shows that XY is parallel to WZ. Therefore $WXYZ$ is a parallelogram. ∎

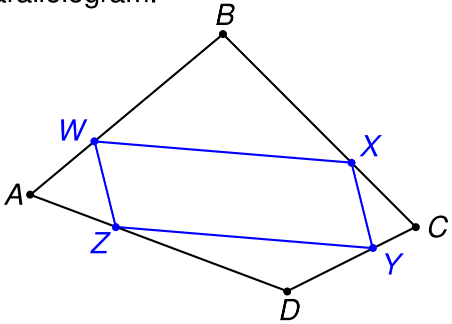


The same idea leads to a more general version of Varignon's theorem.

Theorem. Let $ABCD$ be a quadrilateral, and let W, X, Y, Z be points on the respective sides AB, BC, CD, DA such that

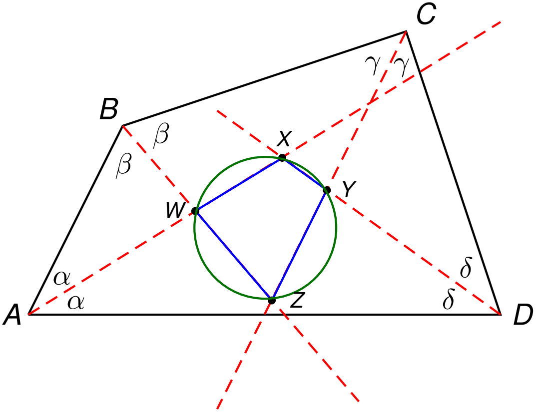
$$\lambda = \frac{|AW|}{|WB|} = \frac{|CX|}{|XB|} = \frac{|CY|}{|YD|} = \frac{|AZ|}{|ZD|}$$

Then $WXYZ$ is a parallelogram.



The quadrilateral formed by the angle bisectors

Theorem. The angle bisectors of a quadrilateral form a cyclic quadrilateral.



You can go to the Geometry Website for an interactive picture.

Proof of theorem

Proof. The idea of the proof is to write the angles in the quadrilateral *WXYZ* in terms of $\alpha, \beta, \gamma, \delta$. To do this, we use the fact that the angle sum in a triangle is 180°.

First, note that since the internal angles of a quadrilateral add up to 360° then

$$2\alpha + 2\beta + 2\gamma + 2\delta = 360 \Rightarrow \alpha + \beta + \gamma + \delta = 180$$

From the diagram, we have

$$\angle ZWX = \angle AWB = 180 - \alpha - \beta, \quad \angle ZYX = \angle DYC = 180 - \gamma - \delta$$
$$\angle WZY = 180 - \beta - \gamma, \quad \angle WXY = 180 - \alpha - \delta$$

Therefore, when we add the opposite pairs of angles we obtain

$$\angle ZWX + \angle ZYX = 360 - \alpha - \beta - \gamma - \delta = 180$$
$$\angle WZY + \angle WXY = 360 - \beta - \gamma - \alpha - \delta = 180$$

and so the quadrilateral *WXYZ* is cyclic, since the pairs of opposite angles add up to 180°. ∎

What happens to Brahmagupta's formula in general?

Let 2θ be the sum of a pair of opposite angles in a quadrilateral *ABCD*.

Then *ABCD* is cyclic $\Leftrightarrow\ 2\theta = 180\ \Leftrightarrow\ \cos\theta = 0$.

Moreover, in general (even if *ABCD* is non-cyclic) notice that $\cos 2\theta$ (and hence $\cos^2\theta$) is independent of the choice of the pair of opposite angles (since all the angles add up to 360°).

Theorem. (Bretschneider's formula) The area of a quadrilateral *ABCD* with sidelengths a, b, c, d and sum of opposite angles equal to 2θ is

$$\text{Area}(ABCD) = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\theta}$$

Corollary. Given sidelengths a, b, c, d (satisfying a generalisation of the triangle inequality so that we can form a quadrilateral), the quadrilateral with the largest area formed from these sidelengths must be cyclic.

Construction exercises

You can find the Week 4 problems on the Geometry Website.

1. Let A, B, C be non-collinear points in the plane, and let DE be a given line segment. Construct F so that $\angle DEF = \angle ABC$.

2. Let AB be a given line segment and C a given point such that A, B, C are non-collinear. Construct D so that $|CD| = |AB|$.

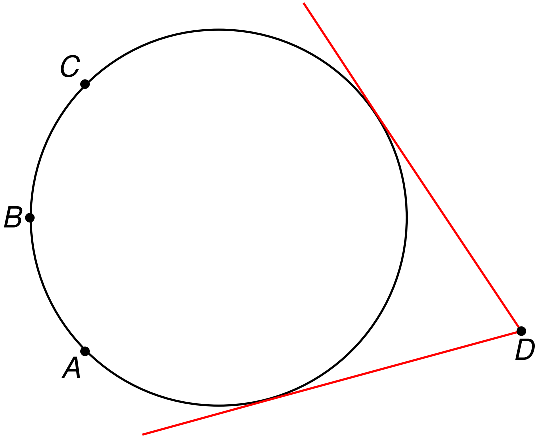
3. Repeat the previous exercise for the case where A, B, C are collinear.

4. Let ABC be a given circle and D a point outside the circle.
   - (a) Construct the centre of the circle.
   - (b) Draw a line from D which touches the circle at one point.

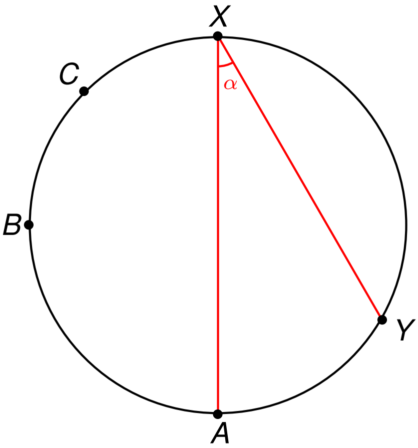
5. Now let ABC be a given circle, and D, E, F non-collinear points outside the circle. Construct an arc on the circle which subtends an angle equal to $\angle DEF$. Can you construct an arc with one endpoint at A?

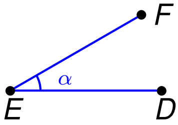
6. Given a circle ABC and a triangle ΔDEF, construct points G, H on the circle such that $\Delta DEF \sim \Delta AGH$.

4 (b) The circle *ABC* and a point *D* outside the circle are given. Construct a line through *D* which is tangent to the circle.



5. The circle *ABC* is given. Given the angle $\alpha = \angle DEF$, construct points *X* and *Y* on the circle such that $\angle AXY = \alpha$.





Next time

Next time we will begin the topic of collinearity and study

- Menelaus' theorem (which gives a criterion for points to be collinear)
- Desargues' theorem
- Pappus' theorem
- Pascal's hexagon theorem