Lecture 3: Area in Euclidean Geometry

21 January, 2019

Overview

Last time.

- Proofs in Geometry
- Congruence of triangles (SSS, SAS, ASA, AAS)

Today.

- Area in Euclidean Geometry
- Pythagoras' Theorem
- Thales' Theorem
- Similar Triangles

Exercise (Why is there no SSA theorem?)

Exercise. Give an example of two triangles ΔABC and ΔDEF, which are not congruent, but which have $|AB| = |DE|$, $|BC| = |EF|$ and $\angle BCA = \angle EFD$.

This an example of a question where you are asked to show that something exists. There are a number of ways to approach it, for example

- you could explicitly construct the two triangles (writing everything out in coordinates), or
- you could describe how to construct two such triangles.

Hint. Fix the side BC and the angle $\angle BCA$ and consider the different possible ways to draw AB to form a triangle.

Solution

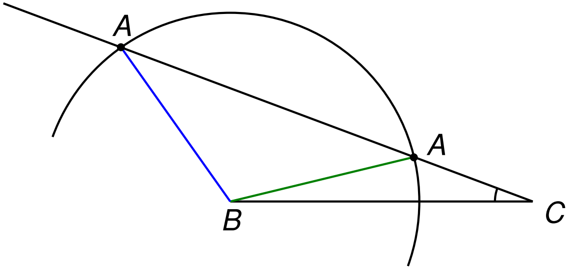
Solution.

Solution. An example is given in the diagram below.

The side BC and the angle $\angle BCA$ are fixed.

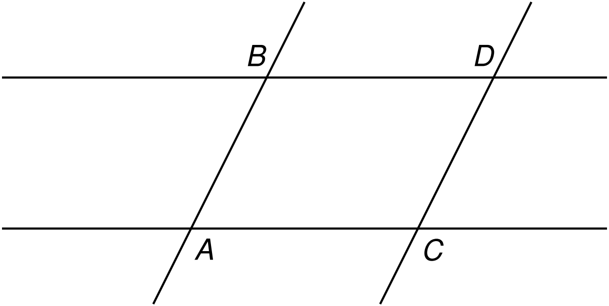
The line segment AB is free to move, with the restriction that the point B is fixed.

The point A moves on a circle with centre B and radius $|AB|$.



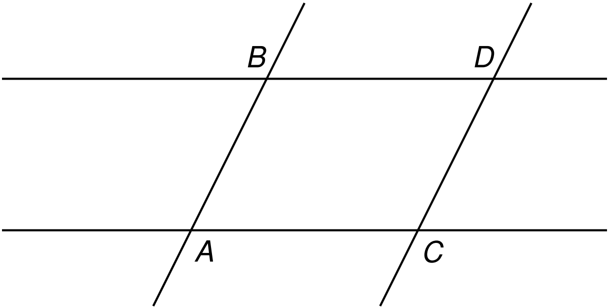
Depending on the angle $\angle BCA$, the line AC will intersect the circle at two points, one point or zero points.

Exercise. Consider two pairs of parallel lines, intersecting in the points A, B, C and D as in the diagram below.



Prove that $|AB| = |CD|$ and $|AC| = |BD|$.

Exercise. Consider two pairs of parallel lines, intersecting in the points A, B, C and D as in the diagram below.

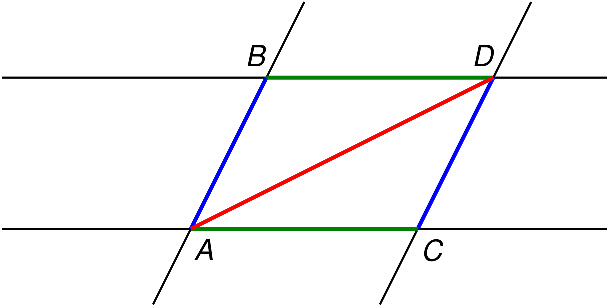


Prove that $|AB| = |CD|$ and $|AC| = |BD|$.
Hint. Consider the diagonal AD and prove that the triangle ΔACD is congruent to ΔDBA.

Exercise Solution

Solution.

Solution.



Proof. (see also Euclid Book I, Prop. 34)

The angles $\angle CAD$ and $\angle BDA$ are alternate angles and therefore equal.

The angles $\angle DAB$ and $\angle ADC$ are alternate angles and therefore equal.

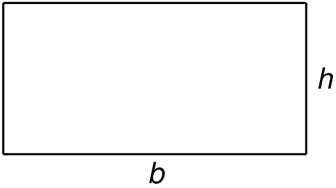
The line segment AD is common to both triangles. Therefore, by ASA, the two triangles are congruent and hence the corresponding sides are equal.

Therefore $|AC| = |BD|$ and $|AB| = |CD|$. ∎

The notion of area in Euclidean geometry

What is the area of a figure in the plane? We already have an intuitive notion of area. We would like to make this precise.

In calculus, we begin with the statement that the area of a rectangle is Area $= bh$.



and then move on to areas of general objects by filling them up with "small" rectangles and then taking the limit as the size of the rectangles goes to zero.

In Euclidean geometry there is no notion of limit. How do we define area?

The notion of area in Euclidean geometry (cont.)

Axiomatic Definition. Area is a function associating a positive real number to each figure in the plane.

It should satisfy the following properties (compare these with the "common notions" from Lecture 1).

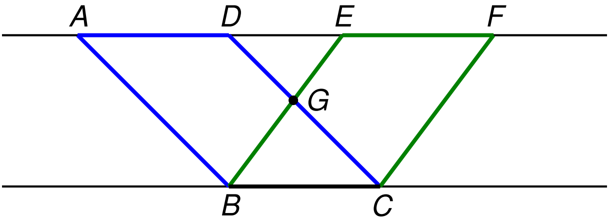
- Congruent figures have equal area.
- If two figures have the same area as a third, then they all have equal area.
- If pairs of figures with equal area are added together ("added" means "joined without overlap") to make new figures, then these new figures have equal area.
- If pairs of figures with equal area are subtracted ("subtracted" means "removed from the interior") then the new figures have equal area.
- Halves of figures with equal area have equal area.
- If one figure is contained in another figure then the larger figure has greater area ("the whole is greater than the part").

What is the meaning of the axioms for area?

Each axiom matches our intuition (so they are reasonable assumptions to make).

Let's try to use the axioms to prove a theorem about the areas of triangles and parallelograms.

Proposition. (Euclid Book I, Prop. 35) Any two parallelograms with the same base and which lie between the same parallels have the same area.



Proof. The idea of the proof is to find congruent triangles in the picture. Then we use the axioms for area to show that the parallelograms have equal area.

First we claim that the triangles ΔABE and ΔDCF are congruent (and therefore have the same area). First note that $|AD| = |BC|$ and $|EF| = |BC|$ (by our exercise at the beginning of today's lecture). Therefore $|AE| = |AD| + |DE| = |DE| + |EF| = |DF|$.

The angles $\angle EAB$ and $\angle FDC$ are corresponding angles (the line AF cuts the parallel lines AB and DC), and therefore $\angle EAB = \angle FDC$.

Similarly, the angles $\angle AEB$ and $\angle DFC$ are corresponding angles, and therefore $\angle AEB = \angle DFC$.

By ASA, we can then conclude that the triangles ΔABE and ΔDCF are congruent.

Proof. (cont.) Since the triangles ΔABE and ΔDCF are congruent, then the sum of the triangles ΔBGC and ΔABE has the same area as the sum of the triangles ΔBGC and ΔDCF.

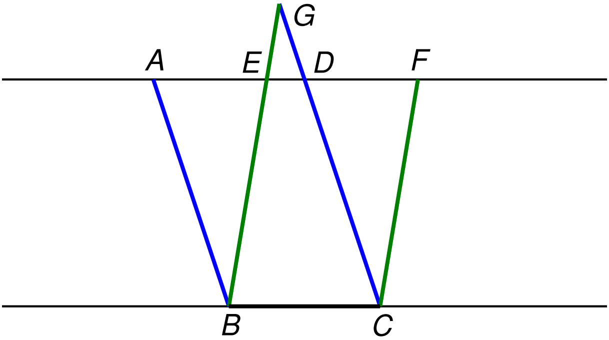
Both of these figures contain the triangle ΔDEG. If we subtract ΔDEG from the sum of ΔBGC and ΔABE then we are left with the parallelogram $ABCD$.

If we subtract ΔDEG from the sum of ΔBGC and ΔDCF then we are left with the parallelogram $BCEF$.

Therefore the parallelograms $ABCD$ and $BCEF$ have the same area. ∎

The area of a parallelogram (cont.)

The case where the lines BE and CD intersect above AF.



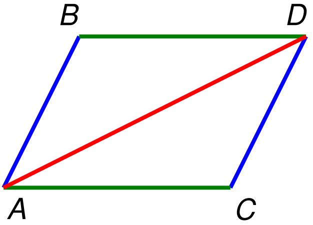
- the triangles ΔABE and ΔDCF are still congruent,
- the parallelogram $ABCD$ is the sum of the triangles ΔABE and ΔBCG minus the triangle ΔDEG, and
- the parallelogram $BCEF$ is the sum of the triangles ΔDCF and ΔBCG minus the triangle ΔDEG.

The area of a triangle

Before computing the area of a triangle, let's return to the axiom "Halves of figures with equal area have equal area".

What is "half" of a figure? We say that a figure is divided into halves if it is divided into two congruent pieces. Each piece is half of the original figure.

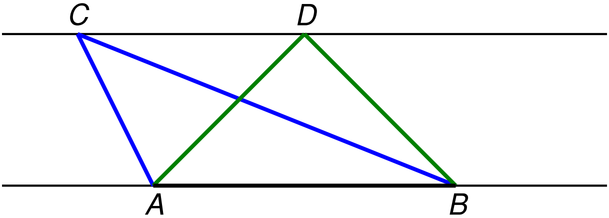
Example. As part of the first exercise today, we proved that the diagonal of a parallelogram divides it into halves.



The triangles ΔACD and ΔDBA are congruent, and therefore they are halves of the parallelogram $ABCD$.

The area of a triangle (cont.)

Proposition. (Euclid Book I, Prop. 37) Triangles which have the same base and are between the same parallels have equal area.



Proof. We proved earlier that each triangle is half of a parallelogram which lies between the parallels in the diagram.

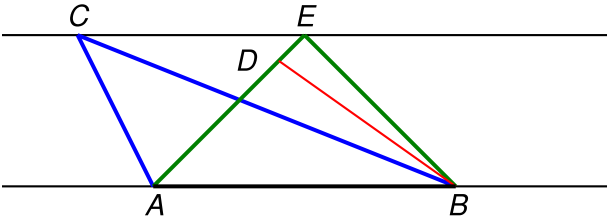
Since the bases of these parallelograms are equal and they are between the same parallels then they have equal areas.

Therefore the triangles have equal area, since they are halves of figures with equal area. ∎

The area of a triangle (cont.)

What if triangles have the same base and equal area? Do they lie between the same parallels?

Proposition. (Euclid Book I, Prop. 39) Triangles of equal area which are on the same base and on the same side also lie between the same parallels.



Suppose (for contradiction) that ΔABC and ΔABD are triangles of equal area with the same base (and on the same side of AB).

The area of a triangle (cont.)

Draw a line through C parallel to AB, and let E be the intersection of this line with AD.

Then $\text{Area}(\Delta ABE) = \text{Area}(\Delta ABC)$ since they have the same base and lie between the same parallels, and (by assumption) we also have $\text{Area}(\Delta ABC) = \text{Area}(\Delta ABD)$.

Therefore $\text{Area}(\Delta ABE) = \text{Area}(\Delta ABD)$.

We claim that the point E must coincide with D. Either ΔABD is contained in ΔABE (this is the situation in the picture on the last slide) or ΔABE is contained in ΔABD.

If $\text{Area}(\Delta ABD) \leq \text{Area}(\Delta ABE)$ then $\Delta ABD \subset \Delta ABE$. If $\text{Area}(\Delta ABE) \leq \text{Area}(\Delta ABD)$ then $\Delta ABE \subset \Delta ABD$.

Since $\text{Area}(\Delta ABE) = \text{Area}(\Delta ABD)$ then both $\Delta ABD \subset \Delta ABE$ and $\Delta ABE \subset \Delta ABD$ simultaneously. Therefore $\Delta ABD = \Delta ABE$ and so $D = E$.

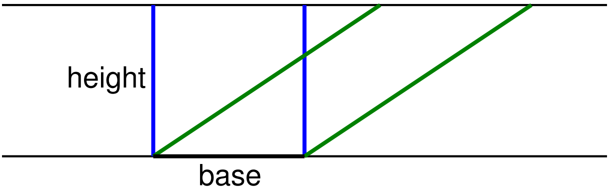
Therefore the triangles lie between the same parallels. ∎

What is the area?

The axioms do not uniquely define the area function. To do this, we need to fix the area of some standard figure.

It is natural to fix the area of a rectangle as
Area $=$ base $\times$ height.

Therefore, by Prop I.35, any paralleogram has area equal to the base times the height.

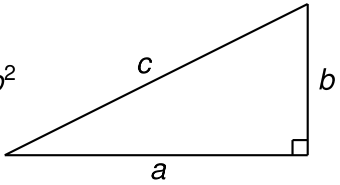


The "height" of the parallelogram is the shortest distance between the parallels.

Now we can prove one of the most famous results in geometry.

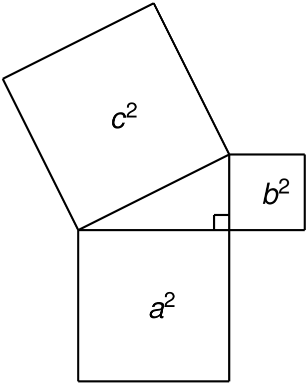
Pythagoras' Theorem. (Euclid Book I, Prop. 47) In right-angled triangles, the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

$$c^2 = a^2 + b^2$$



How can we prove this? Proofs existed before the time of Euclid. These were based on the theory of similar figures (we'll talk about this soon). Euclid's proof involves translating the formula into a statement about the areas of squares.

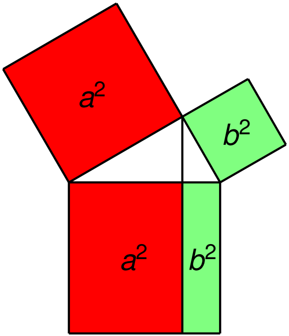
Picture of Euclid's proof



The idea of the proof is to show that the sum of the areas of the two smaller squares is equal to the area of the larger square.

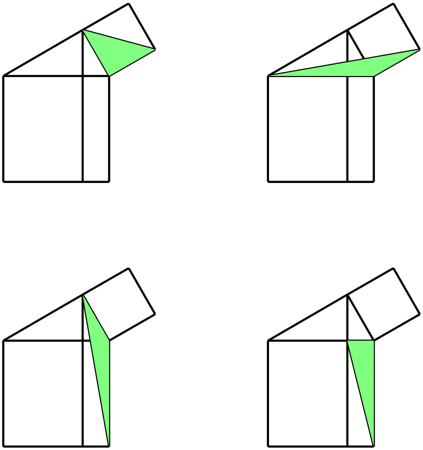
Picture of Euclid's proof

We can accomplish this by showing that the area of the red rectangle is equal to the area of the red square and that the area of the green rectangle is equal to the area of the green square.



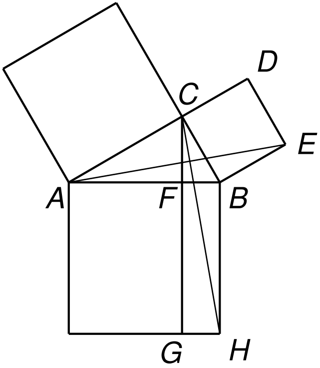
Picture of Euclid's proof (cont.)

Use congruent triangles and Euclid Prop. I.37.



We want to show that each green triangle has the same area.

Euclid's proof of Pythagoras' theorem



Proof. First note that the triangle ΔABE has the same base as the square $BCDE$, and they both lie between the same parallels. Therefore the area of ΔABE is half the area of $BCDE$.

Proof. (cont.) Next we claim that the triangles ΔABE and ΔHBC are congruent. To see this, first note that $|AB| = |HB|$ and $|BE| = |BC|$ (since each pair consists of two sides of the same square).

Next, note that $\angle ABE = 90 + \angle ABC$ and that $\angle HBC = 90 + \angle ABC$. Therefore $\angle ABE = \angle HBC$.

Therefore the triangles ΔABE and ΔHBC are congruent by *SAS*, and so they have the same area.

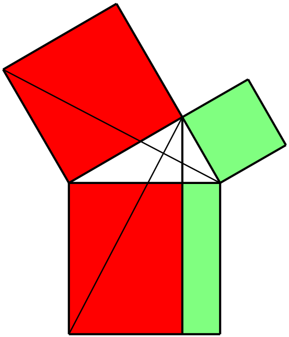
Finally, note that the triangle ΔHBC has the same base as the rectange *HBFG*, and they both lie between the same parallels. Therefore the area of ΔHBC is half of the area of *HBFG*.

In summary, we have shown that

$$\text{Area}(BCDE) = 2\,\text{Area}(\Delta ABE) = 2\,\text{Area}(\Delta HBC) = \text{Area}(HBFG)$$

Euclid's proof of Pythagoras' theorem

To complete the proof, use the same idea to show that the area of the red rectangle is equal to the area of the red square. **(Exercise)**



Thales' Theorem

Recall that we proved that the area of a triangle is $\frac{1}{2}bh$.

Another way to interpret this is as follows.

Proposition. If two triangles have the same height then their areas are in proportion to their bases.

Remark. We can actually prove this without defining the area by a specific formula. This is done in Euclid Book VI, Prop 1.

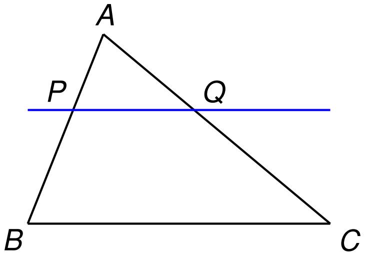
Exercise. Euclid's proof implicitly assumes that the ratio of the bases is rational. Can you see where this is used?

We'll talk more about this later in the semester.

This theorem depends on the results of Book V (which don't depend on previous books) and previously proven results in Book I together with some exercises based on Book I.
(We will do these exercises in Tutorial 2)

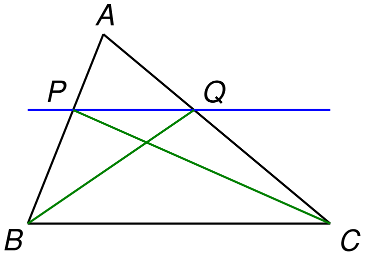
Thales Theorem. (Euclid Book VI, Prop 2)
If a straight line is drawn parallel to one side of a triangle then it cuts the other two sides of the triangle proportionally.

Conversely, if a straight line cuts two sides of a triangle proportionally, then it is parallel to the other side.



$$\frac{|AP|}{|PB|} = \frac{|AQ|}{|QC|}$$ if and only if PQ is parallel to BC.

Proof of Thales' Theorem



Proof. ($\Leftarrow$). Let PQ be parallel to BC. The triangles ΔPQB and ΔPQC have the same base PQ and lie between the same parallels, so they have the same area by Euclid Prop. I.37.

Therefore the triangle ΔAQB (which is the union of the triangles ΔAPQ and ΔPQB) has the same area as the triangle ΔAPC (which is union of the triangles ΔAPQ and ΔPQC).

Proof. (cont.)
The triangles ΔAPQ and ΔPQB have base on the same line AB, and they have the same height (since the third point Q is common to both triangles). Therefore

$$\frac{\text{Area}(\Delta APQ)}{\text{Area}(\Delta PQB)} = \frac{|AP|}{|PB|}$$

by Euclid Prop VI.1. The same idea applied to the triangles ΔAPQ and ΔPQC shows that

$$\frac{\text{Area}(\Delta APQ)}{\text{Area}(\Delta PQC)} = \frac{|AQ|}{|QC|}$$

We showed that ΔPQB and ΔPQC have the same area. Therefore

$$\frac{|AP|}{|PB|} = \frac{\text{Area}(\Delta APQ)}{\text{Area}(\Delta PQB)} = \frac{\text{Area}(\Delta APQ)}{\text{Area}(\Delta PQC)} = \frac{|AQ|}{|QC|}$$

Proof of Thales' theorem (cont.)

Now let's proof the converse direction $\Rightarrow$.

Suppose that $\frac{|AP|}{|PB|} = \frac{|AQ|}{|QC|}$. The same argument as before shows that

$$\frac{\text{Area}(\Delta APQ)}{\text{Area}(\Delta PQB)} = \frac{|AP|}{|PB|} = \frac{|AQ|}{|QC|} = \frac{\text{Area}(\Delta APQ)}{\text{Area}(\Delta PQC)}$$

Therefore $\text{Area}(\Delta PQB) = \text{Area}(\Delta PQC)$.

Now the triangles ΔPQB and ΔPQC have the same base. Since their areas are equal, then they must lie between the same parallels by (Euclid Prop. I.39).

Therefore PQ is parallel to BC. ∎

Remark. This is one method you can use to prove that two lines are parallel.

Next time

In the next class we will continue by using Thales' theorem to prove results about similar triangles, and then look at some applications of similar triangles (including a new proof of Pythagoras' theorem).

We will also begin doing some ruler and compass constructions. You should bring either:

- a ruler, a compass and some scrap paper, or
- an iPad with the "Apollonius" app installed, or
- a computer with Geogebra, or
- any device able to access the constructions on the webpage https://graemewilkin.github.io/Geometry/Constructions/