Lecture 6: More special circles associated to triangles

1 February, 2019

Overview

Last time.

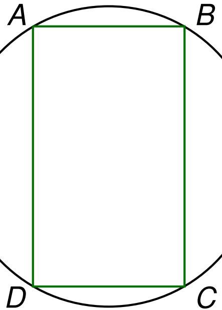
- Ceva's Theorem
- The centroid of a triangle
- The circumcentre of a triangle
- The incentre of a triangle
- The orthocentre of a triangle

Today.

- The Euler line
- Cyclic quadrilaterals
- The nine-point circle
- The orthic triangle
- More constructions in Euclidean geometry

Exercise

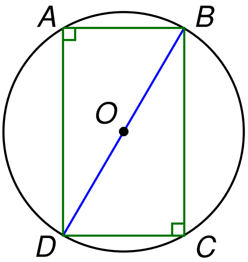
Exercise. Let $ABCD$ be a rectangle. Show that it is *cyclic*, i.e. that there is a circle which passes through all four points A, B, C, D.



Solution 1.

Exercise Solution

Solution 1. Since $\angle BCD = 90^\circ$, then C lies on the circle with diameter BD. The same idea shows that A also lies on the circle with diameter BD. Therefore $ABCD$ is cyclic.



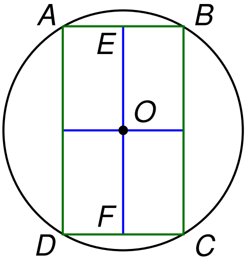
Exercise Solution

Solution 2.

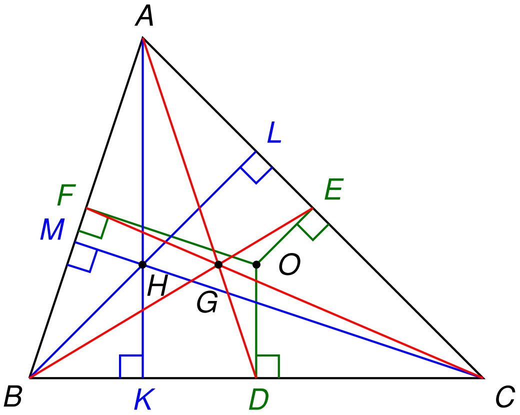
Exercise Solution

Solution 2. Construct the perpendicular bisector EF of AB (see the diagram below). Since EF is parallel to AD, then $AEFD$ is a rectangle, so $|AE| = |DF|$. Similarly, $|EB| = |FC|$. Therefore EF is also the perpendicular bisector of CD.

Using the same idea, we can show that the perpendicular bisector of AD is also the perpendicular bisector of BC. Therefore their intersection point (labelled O in the diagram below) is equidistant from A, B, C, D (use isosceles triangles to prove this as in the last lecture). Therefore $ABCD$ is cyclic.



Recall from last lecture the construction of the circumcentre O, the centroid G and the orthocentre H.



The Euler line

The diagram on the previous slide suggests that H, G and O are *collinear* (lie on the same line).

Is this true in general?

Theorem. (Euler) The orthocentre H, centroid G and circumcentre O are collinear. Moreover, $|GH| = 2|OG|$.

The line through H, G and O is called the **Euler line.**

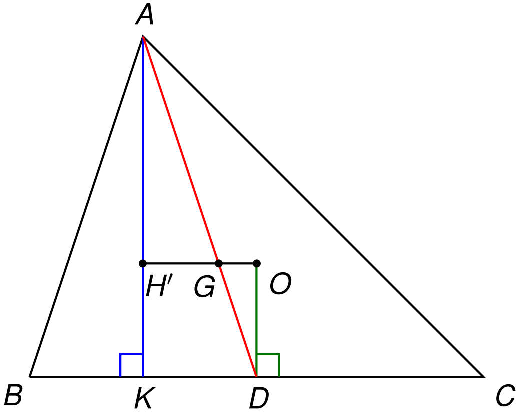
You can visit the Geometry webpage to see an interactive picture of the Euler Line.

Strategy of Proof. The idea of the proof is to draw a line through the circumcentre O and centroid G. Let H' be the point where this line intersects one of the altitudes.

A priori the construction of H' depends on the choice of altitude.

We aim to show that this point H' is independent of the choice of altitude, and therefore lies on the intersection of all of the altitudes, which is the orthocentre.

The Euler line (diagram)



Proof. The lines *OD* and *AK* are both perpendicular to *BC*. Therefore *OD* is parallel to *AK*.

Therefore (see the exercise at the beginning of Lecture 5) the triangles $\Delta AH'G$ and ΔDOG are similar.

Proof. (cont.)

Now, it is a property of the centroid G that $|AG| = 2|DG|$.
(We will prove this in Tutorial 3.)

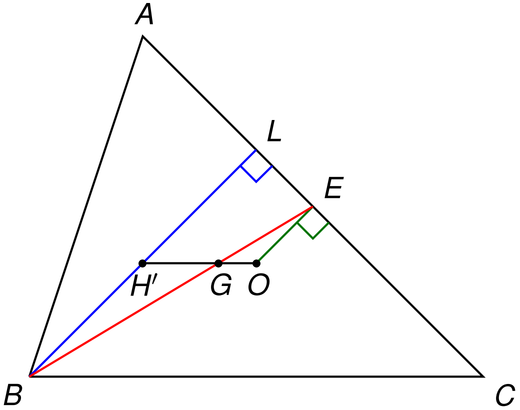
Therefore $|H'G| = 2|OG|$ (since $\Delta AH'G \sim \Delta DOG$).

We have shown that for any choice of altitude, you can construct H' by extending OG in a line and defining $|H'G| = 2|OG|$.

Therefore H' is the same point for any choice of altitude, and so it must lie on the intersection of all of the altitudes, which is the orthocentre.

Therefore the orthocentre is collinear with the circumcenter O and centroid G. ∎

The Euler line (picture of proof using a different altitude)



Again, BH' is parallel to OE, and so $\Delta BH'G \sim \Delta EOG$. By the property of the centroid (see Tutorial 3), $|BG| = 2|EG|$.

Therefore, once again we have $|H'G| = 2|OG|$.

Cyclic quadrilaterals

Definition. A cyclic quadrilateral is a set of four points A, B, C, D (in that order) lying on a circle together with the lines AB, BC, CD and DA joining them.
The *diagonals* of the quadrilateral are AC and BD.

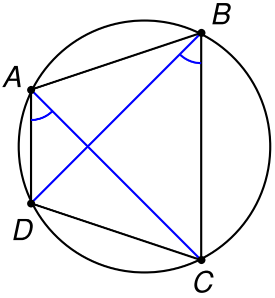
Question. How do we know if a quadrilateral is cyclic? We proved that every triangle is cyclic (and has a unique circumcircle), but this is not true for quadrilaterals in general.
(Easy) Exercise. Construct a quadrilateral which is not cyclic.

Proposition. Let A, B, C and D be four points in the plane, with A and B both on the same side of the line CD. Then the quadrilateral $ABCD$ is cyclic if and only if $\angle DAC = \angle DBC$.

Proof. $\Rightarrow$ Since $ABCD$ is cyclic then Euclid III.21 (which we proved last lecture) shows that $\angle DAC = \angle DBC$.

Therefore it only remains to show that $\angle DAC = \angle DBC$ implies that $ABCD$ is cyclic.

Cyclic quadrilaterals (cont.)



$\angle DAC = \angle DBC$

Proof. $\Leftarrow$ Now suppose that $\angle DAC = \angle DBC$. Consider the circumcircle of the triangle ΔADC. We aim to show by contradiction that B lies on this circle.

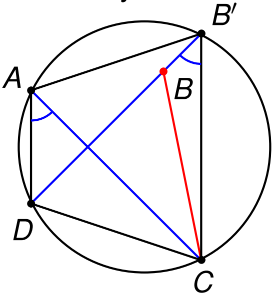
Suppose that B does not lie on the circle, and let B' be the intersection of DB with the circle.

Cyclic quadrilaterals (cont.)

Therefore $\angle DB'C = \angle DAC = \angle DBC$, since the angle subtended by the sector DC at A is the same as that at B' by Euclid III.21 again.

Since we assume that B does not lie on the circle, then either it lies inside the circle (and so $\angle DBC > \angle DB'C$) or it lies outside the circle (and so $\angle DBC < \angle DB'C$).

Therefore we have a contradiction, and so $\angle DAC = \angle DBC$ implies that $ABCD$ is cyclic. ∎



$\angle DBC > \angle DB'C$

The nine-point circle

Theorem. For any triangle ΔABC, let D, E, F be the midpoints of the three sides, K, L, M the feet of the three altitudes and P, Q, R the midpoints of the line segments joining the three vertices to the orthocentre. Then $D, E, F, K, L, M, P, Q, R$ all lie on the same circle, called the nine-point circle.

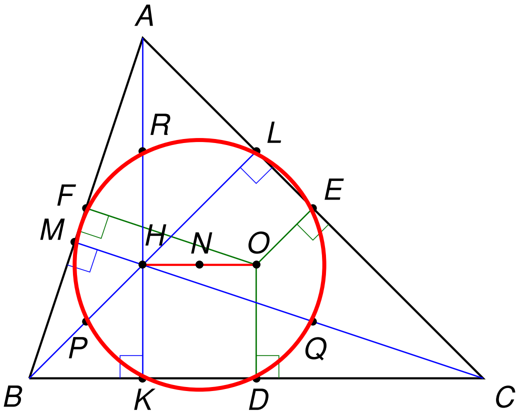
Moreover, the centre N of the nine-point circle is the midpoint of the line segment OH joining the circumcentre to the orthocentre, and the radius of the nine-point circle is half of the radius of the circumcircle.

You can visit the Geometry webpage to see an interactive picture of the Nine Point Circle.

We will use cyclic quadrilaterals to prove this theorem.

The nine-point circle (diagram)

The circumcentre is O, the orthocentre is H and the centre N of the nine-point circle is the midpoint of OH.

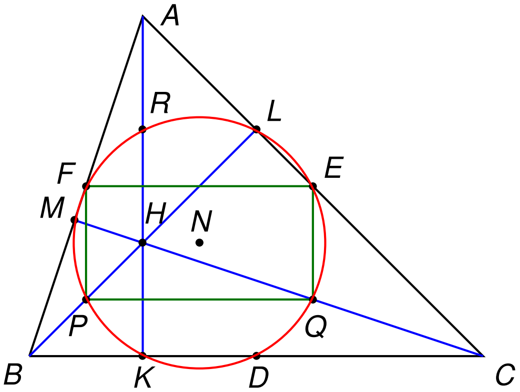


D, E, F are the midpoints of the sides and K, L, M are the feet of the altitudes.

The nine-point circle (proof)

Proof. Consider the quadrilateral *EFPQ*. We want to show that this is a rectangle.

By Thales' Theorem (Euclid Prop. VI.2), since E is the midpoint of AC and F is the midpoint of AB, then EF is parallel to BC and $2|EF| = |BC|$.

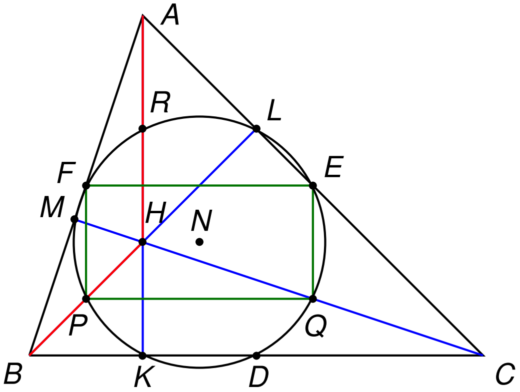


The nine-point circle (cont.)

Similarly, P is the midpoint of BH and Q is the midpoint of CH. Therefore, Thales' theorem applied to the triangle ΔBCH shows that PQ is parallel to BC and $2|PQ| = |BC| = 2|EF|$.

Therefore $EFPQ$ is a parallelogram.

Now consider the triangle ΔABH.



Since F is the midpoint of AB and P is the midpoint of BH then Thales' theorem again shows that PF is parallel to AH and $2|PF| = |AH|$.

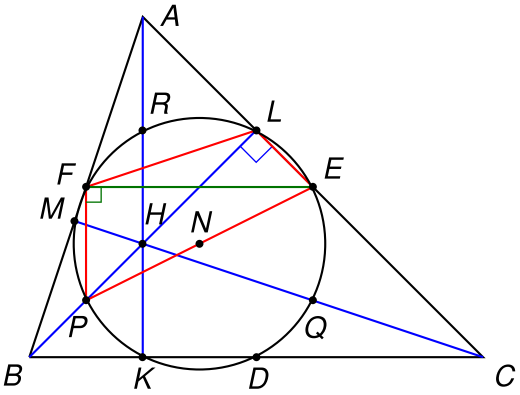
The same argument applied to ΔCAH shows that EQ is parallel to AH and $2|EQ| = |AH|$.

Since AH is perpendicular to BC, then FP and EQ are also perpendicular to BC.

Therefore $EQPF$ is a rectangle, which (by the exercise at the beginning of class today) is a cyclic quadrilateral. The centre of the circle through E, Q, P, F is the midpoint of the diagonal EP.

Next we show that $FLEP$ is also a cyclic quadrilateral. Since it has three points in common with $EQPF$ then the circle must be the same for both quadrilaterals (Euclid Prop. III.10).

The angle subtended by *EP* at *F* is a right angle. Similarly, the angle subtended by *EP* at *L* is a right angle. Therefore *FLEP* is cyclic.



The nine-point circle (cont.)

The same idea shows that the quadrilateral $FMEQ$ is cyclic, and therefore we have shown that E, F, L, M, P, Q all lie on the same circle.

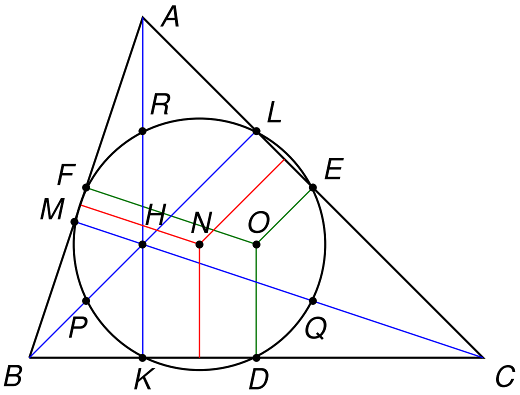
To show that the remaining points D, K, R also lie on the same circle, we switch our point of view to one of the other sides AB or AC and repeat the argument. **(Exercise)**

Finally, we want to show that the centre N is the midpoint of OH.

Since D and K lie on the nine-point circle, then the centre N must lie on their perpendicular bisector. This perpendicular bisector must pass through the midpoint of OH.

Repeating the argument for the pairs of points E, L and F, M, we see that N must lie on the intersections of all these perpendicular bisectors, which is the midpoint of OH. ■

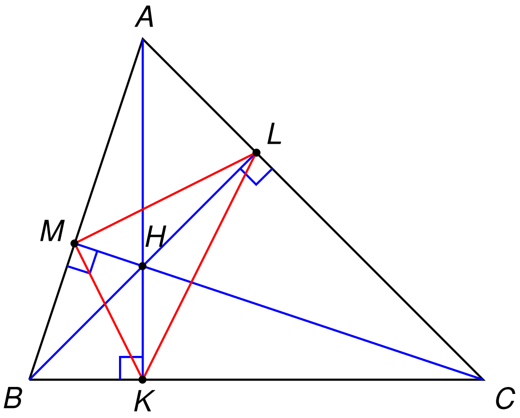
The nine-point circle (cont.)



The centre N lies on the intersection of the perpendicular bisectors of DK, EL and FM.

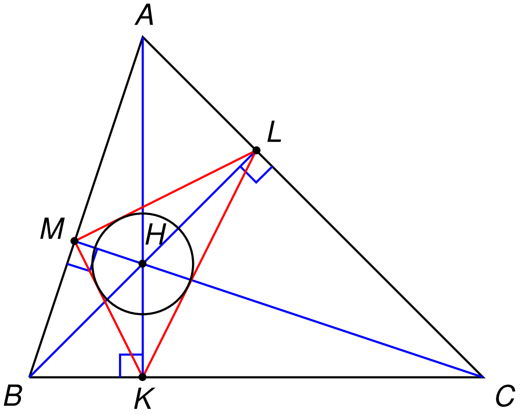
The orthic triangle

Definition. Given a triangle ABC the orthic triangle is the triangle with vertices at the feet of the altitudes of ABC.



The orthic triangle (cont.)

Theorem. Let ΔABC be an acute triangle. The incentre of the orthic triangle KLM is the orthocentre of ΔABC.



The proof uses cyclic quadrilaterals.

The orthic triangle (cont.)

Proof. First note that $LKBA$ is cyclic, since AB subtends a right-angle at both L and K. Therefore $\angle AKL = \angle ABL$.

Similarly, $MLCB$ is cyclic, since BC subtends a right-angle at both L and M. Therefore $\angle ABL = \angle MBL = \angle MCL = \angle MCA$.

Finally, $KMAC$ is cyclic, since AC subtends a right-angle at both K and M. Therefore $\angle MCA = \angle MKA$.

Combining all of these results gives us $\angle AKL = \angle MKA$, and so the altitude AK is the angle bisector of $\angle MKL$.

Repeating the same argument for the vertices shows that the other two altitudes BL and MC are angle bisectors for the orthic triangle.

Therefore, the incentre of the orthic triangle ΔKLM is the intersection point of all the altitudes of ΔABC, which is the orthocentre of ΔABC. ∎

Construction exercises

Construct the orthocentre, centroid, circumcentre and nine-point circle of a triangle with an obtuse angle.

Construct the orthic triangle of a triangle with an obtuse angle. How can we modify the previous theorem to work in this situation?

Given a line segment AB**, construct a square with side** AB**.**

Construct the square root of a number. Given a line segment AC and a point B such that $x = |AB|$ and $|BC| = 1$, construct a point D such that $|AD| = \sqrt{x}$.
Hint. Construct a right-angled triangle and use similar triangles.

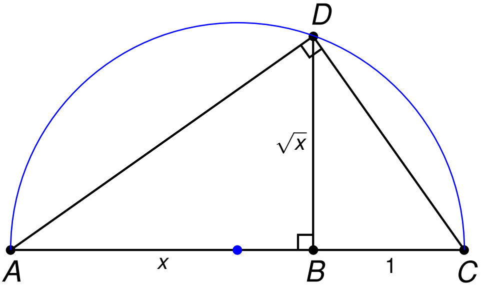
Construct the *golden ratio*. Given a line segment AB, construct a rectangle $ABCD$ such that $\frac{|AB|}{|AC|} = \frac{|AC|}{|AB|-|AC|}$.

Hint. Try to solve algebraically for $\frac{|AB|}{|AC|}$ and then construct a right-angled triangle with one side equal to this length.

Constructing the square root of a number

If we construct a right-angled triangle ΔACD as in the diagram below, with $|AB| = x$ and $|BC| = 1$, then the line segment $|BD|$ will have length $\sqrt{x}$. Prove this using similar triangles.

You can construct this triangle by bisecting AC and then drawing the circle with diameter AC. Now draw the perpendicular to AC through B and define D to be the point where the circle intersects this perpendicular.

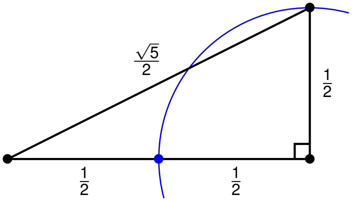


Constructing the golden ratio

Let x be the golden ratio. Then x satisfies the equation $\frac{x}{1} = \frac{1}{x-1}$. This is equivalent to the quadratic equation $x^2 - x - 1 = 0$, which has solutions $x = \frac{1}{2}(1 \pm \sqrt{5})$.

Therefore we can construct the golden ratio if we can construct $\frac{1}{2}(1 + \sqrt{5})$. Begin with a line segment of length 1. Bisect to get a line segment of length $\frac{1}{2}$.

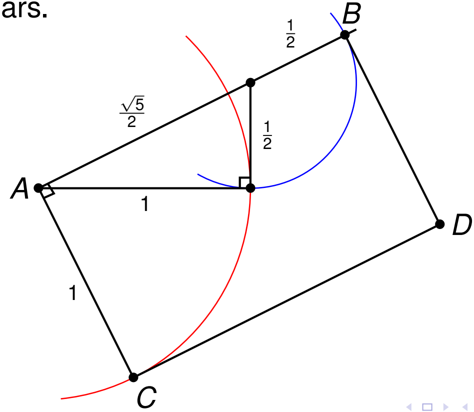
Now draw a right-angled triangle with legs of length 1 and $\frac{1}{2}$. The hypotenuse will then have length $\frac{\sqrt{5}}{2}$.



Constructing the golden ratio (cont.)

Now extend the line and use your compass to mark off a segment of length $\frac{1}{2}$, as in the diagram below. The segment AB will have length $\frac{1}{2}(1+\sqrt{5})$.

Use your compass to draw a perpendicular to AC of length 1. Then draw the rest of the rectangle by constructing perpendiculars.



Next time

We will move on to study quadrilaterals and their properties.

- A formula for the circumradius
- More cyclic quadrilaterals
- The Simson line
- Ptolemy's Theorem
- Heron's formula for the area