Lecture 2: Introduction to proofs in geometry

18 January, 2019

Overview

Last time.

- Applications of geometry
- What is geometry?
- Axioms of Euclidean geometry
- Some simple proofs in Euclidean geometry

Today.

- More examples of proofs in Euclidean geometry
- How to write proofs
- How to think geometrically
- Examples of geometric proofs vs other types of proofs

Some preliminaries

The process of proving a theorem falls naturally into two stages.

1. Developing the idea. The first stage is to develop the idea for the proof. This is where abstract concepts such as "thinking outside the box", "creativity" and "geometric intuition" become useful.

There is no formula or template for developing an idea, but there are things you can do to become better at this step.

Read lots of proofs. The more proofs you read, then the more new ideas you will see, which will help build your creativity and intuition.

When reading a proof, it is important to do more than just look at the proof. You will get the most out of reading proofs if you read "actively" by drawing a diagram and trying to follow the argument. You should also ask yourself "What is the key idea in the proof?" "Where does the idea come from?"

Some preliminaries (cont.)

Practice developing your own proofs. The more practice you do, the better you will get.

This is difficult at first, since you are still building your skills and knowledge in geometry. It may seem time-consuming, and it can be frustrating to be stuck on a problem, but the time you spend early in the semester will help make you better at proofs as the semester progresses.

Just as physical exercises make you stronger, mathematical exercises also make you better at mathematics, but you have to do them properly to get all the benefits.

Be prepared to make mistakes. If your original idea doesn't work, then you can try to modify it in some way to find the right idea. Often you have to repeat this process before you hit on the correct solution (and sometimes you have to start all over again from scratch).

A good description of this is available here.

Some preliminaries (cont.)

2. Writing proofs. The second stage of a proof is to write it up carefully. The key principles are

- Be rigorous. In geometry, this means checking that each step in the proof follows from a given axiom, a common notion or a previously proven proposition.
- Make it readable. English sentences are always preferable to a dense collection of symbols (but make sure that the sentence is still correct and unambiguous).
- Explain the intuition. In geometry, this usually means drawing a diagram to make the idea of the proof clear to the reader. If you can explain the ideas clearly then it will help people understand your proof.

We'll talk more below about good and bad ways to write proofs.

Some preliminaries (cont.)

In summary. Developing the idea for a proof is a *creative process*, while writing a proof and checking that a proof is correct is subject to rules and regulations. You can still be creative in writing your proof, but you must make sure that it is rigorous (see the first point on the previous slide).

To get better at developing your own ideas for proofs, read widely (and actively), get plenty of practice and be prepared to try new ideas, even if you're not sure that they will work.

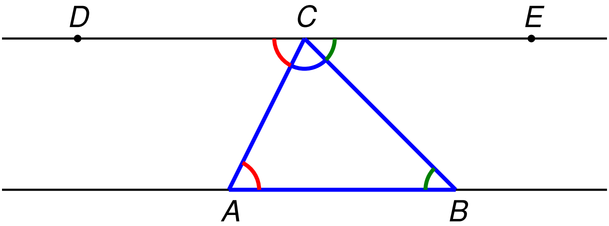
The best way to get started is to write something.

Sometimes you will only discover a mistake or gap in your proof when you are writing it up. Then you have to go back to the first stage and modify your original idea. This "back and forth" process can be frustrating, but it is something that everyone encounters when proving theorems. The more you work on this, the better you will get.

Theorem. In any triangle the sum of the angles is 180 degrees.

Remark. In calculus, it is more natural to use radians (360 degrees = 2π radians) to measure angles. In this course we will use degrees.

Proof. First, we will draw the picture of the proof.



Angle sum in a triangle (cont.)

The proof then uses the fact that for any pair of parallel lines, alternate angles are equal (Euclid Book I, Prop. 29; see also Lecture 1).

We want to prove that $\angle BAC + \angle ACB + \angle ABC = 180$.

Extend the line through AB indefinitely. (Axiom 1)

Through the point C, draw the unique line parallel to AB (Playfair's Axiom, which follows from Euclid's Axiom 5).

On this line, choose points D and E on either side of C.

Then $\angle BAC = \angle DCA$ and $\angle ABC = \angle ECB$, since they are pairs of alternate angles between parallel lines.

We also have $\angle DCA + \angle ACB + \angle BCE = \angle DCE = 180$.

Therefore $\angle BAC + \angle ACB + \angle ABC = 180$ also. ■

Remarks on the proof

Once you draw the picture, then the idea of the proof is clear.

By itself, the picture is not sufficient, since we still need to write down the steps in the proof and check that they are all correct.

The picture can help you write the proof.

Without the picture, we still have a proof, but the idea behind the proof is lost.

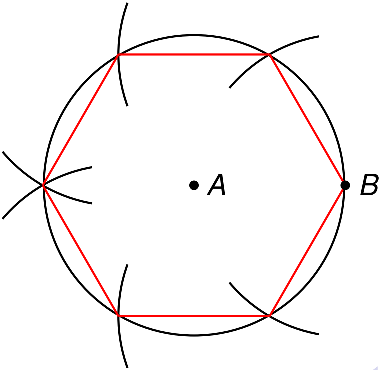
How did we know to use this idea to prove the theorem? There was no formula or list of steps that told us to do the proof this way. Now that you've seen this idea then you can add it to your list of techniques that you can use in future proofs.

An incomplete proof

It is possible to be fooled by a good picture, as we see in the following example.

Theorem. Given a circle, it is possible to construct a regular hexagon (all sides are equal and all internal angles are equal) inscribed in the circle.

Proof. The picture of the "proof" is as follows.



An incomplete proof (cont.)

To construct the hexagon, choose a point B on the circle. Draw a circle of radius $|AB|$ with centre at B.

It will intersect the original circle at two points. Draw another circle of radius $|AB|$ with center at each of these two points.

At the new points of intersection with the original circle, draw another circle of radius $|AB|$ and mark the point of intersection with the original circle.

We now have six points on the circle of equal distance. Therefore we have constructed a hexagon inscribed in the original circle.

Question. Can you find the gaps in the proof?

The construction of the hexagon is correct, but the proof is not complete.

Exercise

Exercise. Find the problems in the proof on the previous slide and correct them.

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Solution.

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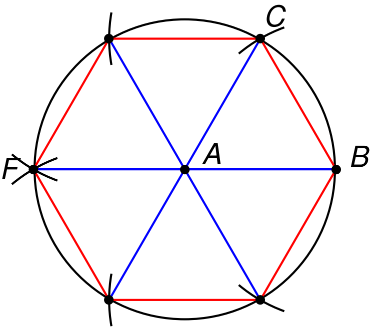
Solution. Two gaps in the proof are that (a) we have not proved that the hexagon is regular, and (b) we constructed the point F in two different ways, but we did not prove that they agree.



Exercise (cont.)

Proof. Continuing on from the end of the original proof, we notice that $|AB| = |AC|$ (since they are radii of the same circle) and $|AB| = |BC|$ (since we drew a circle of radius $|AB|$ centred at B).

Therefore the triangle ΔABC is an equilateral triangle, and so all of the angles are equal.

Since the angles add up to 180 degrees then each angle must be 60 degrees.

Continuing the process, the same argument shows that each triangle in the previous picture is an equilateral triangle, and so the angle subtended at the centre must be 60 degrees.

Therefore all of the internal angles add up to 360 degrees, and so the point F is uniquely defined.

We also see that the hexagon is regular since all the sides are equal to $|AB|$ and all the angles are 120 degrees (since each angle is formed from two angles of 60 degrees each). ∎

Side Angle Side (SAS) Congruence for triangles

Definition. Two triangles are congruent if and only if their corresponding angles and sidelengths are equal.

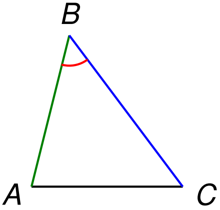
Proposition. (Euclid Book I, Prop. 4) If two triangles ΔABC and ΔDEF are such that

$$|AB| = |DE|, |BC| = |EF| \text{ and } \angle ABC = \angle DEF,$$

then

$$|AC| = |DF|, \angle BCA = \angle EFD \text{ and } \angle CAB = \angle FDE.$$

Therefore the two triangles are congruent.
(This is known as SAS congruence.)



Proof of SAS congruence

The idea behind the proof of SAS congruence is to superimpose the triangle ΔABC on the triangle ΔDEF and so that the vertices and sides are equal.

Proof. Move the triangle ΔABC so that the point A is placed on the point D and the line AB is placed on DE. Then the point B coincides with E since $|AB| = |DE|$ (by assumption).

If necessary, reflect the triangle ΔABC across the side AB so that the points C and F are both on the same side of the line $AB = DE$. Since $\angle ABC = \angle DEF$, then the line BC coincides with EF and so the point C coincides with F since $|BC| = |EF|$ (by assumption).

Since A coincides with D and C coincides with F then AC coincides with DF. Therefore the whole triangle ΔABC coincides with the triangle ΔDEF and so they are congruent.

Therefore, the remaining angles are also equal, i.e. $\angle BCA = \angle EFD$ and $\angle CAB = \angle FDE$. ∎

Some remarks about the proof

Again, when reading the proof it will help to draw a picture of each step and convince yourself that it makes sense.

Note that we needed to use some intuition here, since we are using the idea of superimposing one triangle over another. This relies on our intuition about what geometry in the plane should be, rather than deriving everything from a given list of axioms.

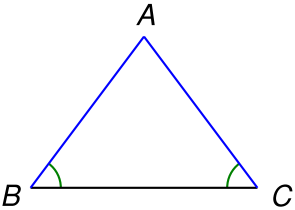
Euclid's idea of using translations, reflections and rotations to move a triangle around the plane was actually the beginning of a modern approach to geometry developed by Felix Klein, who described different kinds of geometries via their symmetries. We will talk more about this when we encounter different kinds of non-Euclidean geometries later in the semester.

Hilbert's axioms contain SAS congruence as an axiom (see Axiom C6 in the notes that I posted on IVLE), which makes everything rigorous by avoiding the need to prove this statement.

Isosceles triangles

Definition. A triangle ΔABC is isosceles if two of the sides have equal length.

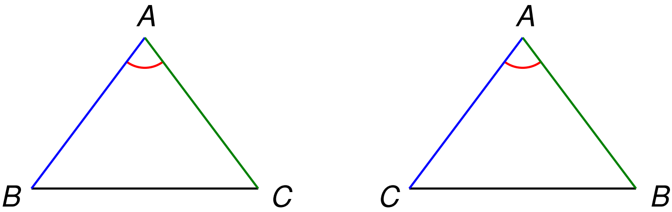
Proposition. (Euclid Book I, Prop. 5) Let ΔABC be an isosceles triangle, with $|AB| = |AC|$. Then $\angle ABC = \angle ACB$.



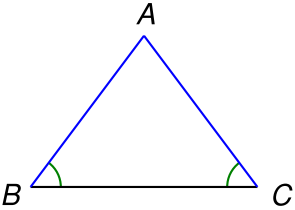
Note. Euclid's original proof follows only the axioms and the previously proven results (Propositions 1–4 from Book I). You can read about it by clicking on the link above. The proof on the next slide was originally given by Pappus, some years after Euclid's proof. It is much shorter, but requires some visualisation.

Isosceles triangles (cont.)

Proof. Consider the two triangles ΔABC and ΔACB below. Since $|AB| = |AC|$ then two pairs of sides are equal. Since $\angle BAC = \angle CAB$ then the angle in-between these sides is also equal. Therefore the two triangles are congruent by SAS, and so the corresponding angles are equal. Therefore $\angle ABC = \angle ACB$.



Proposition. (Euclid Book I, Prop. 6) Consider the triangle ΔABC, and suppose that $\angle ABC = \angle ACB$. Then $|AB| = |AC|$.

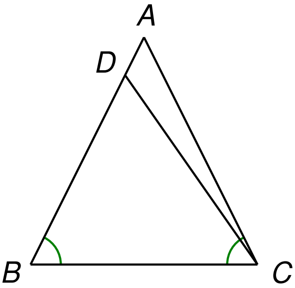


Proof. Suppose that the sides are not equal. Suppose without loss of generality that $|AB| > |AC|$. Let D be a point on AB such that $|DB| = |AC| < |AB|$. Note that ΔDBC is contained inside ΔACB.

Isosceles triangles (cont.)

Proof (cont.) In the two triangles ΔDBC and ΔACB, we have $|DB| = |AC|$, $|BC| = |CB|$ and $\angle DBC = \angle ABC = \angle ACB$ (note that this last equality is our original assumption that two angles are equal).

Therefore ΔDBC is congruent to ΔACB by SAS, which contradicts the fact that ΔDBC is contained inside ΔACB (the whole is greater than the part). Therefore we must have $|AB| = |AC|$.



What does "without loss of generality" mean?

Strictly, we should have considered both cases $|AB| > |AC|$ and $|AB| < |AC|$.

The proofs for both cases are essentially the same, only the labels on the points are different. We don't need to write out the same proof more than once, it is sufficient to choose one of the cases and say "without loss of generality".

Warning. You can only do this if the cases really are the same!!! You can't use "without loss of generality" as a shortcut to avoid a difficult part of the proof!!

Side Side Side (SSS) Congruence for triangles

Proposition. (Euclid Book I, Prop. 8) If two triangles ΔABC and ΔDEF have equal sides

$$|AB| = |DE|,\ |AC| = |DF| \text{ and } |BC| = |EF|$$

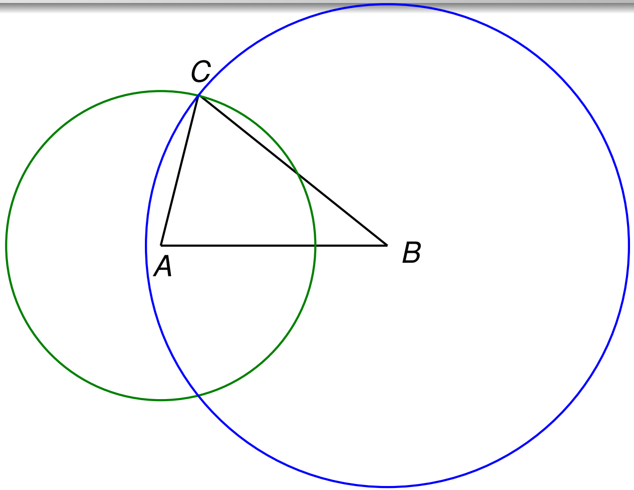
then they are congruent and so their corresponding angles are equal.

Proof. Move the triangle ΔABC so that the point A is placed on the point D and the line AB is placed on DE. Then the point B coincides with E since $|AB| = |DE|$.

Now draw two circles: one with centre $A = D$ and radius $|AC| = |DF|$, and the other with centre $B = E$ and radius $|BC| = |EF|$.

The circles intersect at two points (why?). These are the two possibilities for the points C and F. If they coincide then we are done. If not, then we can reverse the orientation of ΔDEF (reflect along the line DE) so that $F = C$. ∎

Side Side Side (SSS) Congruence for triangles



We already know that $A = D$ and $B = E$. We also know that the radii of the two circles are $|AC| = |DF|$ and $|BC| = |EF|$. Therefore the points C and F must be at one of the two intersection points.

What did we learn from reading these proofs?

- The picture helps to explain the idea of the proofs and the theorem. By itself, the picture is not enough to prove the theorem, but without it the idea of the proof is lost.
- It helps to add explanations to each statement. We could even add more explanation; for example we could say that we used Euclid's third axiom to draw the two circles in the previous proof.

Where did the idea for the proof come from?
The idea of superimposing one triangle onto another goes back to Euclid's original definition of congruence (he defined congruent triangles as those which are equal when one is placed over another).

More on writing proofs

How did we know to draw the two circles in the second proof? We wanted to see all the possibilities for the points C and F, and the only information we had was their distance from two other points A, B and D, E respectively. Any question like this can be solved by intersecting two circles.

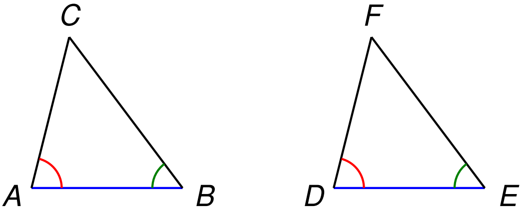
Now that you have seen this trick, you can use it again in future proofs.

ASA and AAS Congruence for triangles

Proposition. (Euclid Book I, Prop. 26) If two triangles ΔABC and ΔDEF have equal angles

$$\angle ABC = \angle DEF \text{ and } \angle CAB = \angle FDE$$

and one equal side, then they are congruent.



The picture is for the case where the equal side is between the two equal angles (ASA).

Proof. First we do the case where the equal side is between the two equal angles (ASA).

Suppose (for contradiction) that the two triangles are not congruent. Then one pair of sides is not equal. Without loss of generality assume that $|AC| > |DF|$.

Choose a point G on the line AC between A and C such that $|AG| = |DF|$.

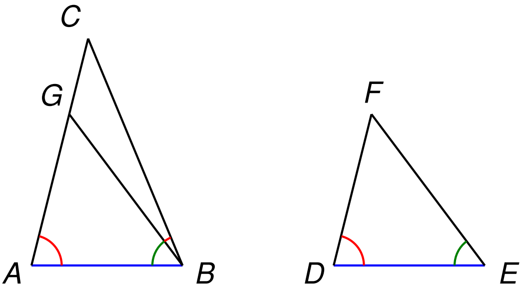
Since $|AG| < |AC|$ then $\angle ABG < \angle ABC$. (See Tutorial 1)

Then we have $|AB| = |DE|$, $|AG| = |DF|$ and $\angle GAB = \angle FDE$. Therefore the triangles ΔABG and ΔDEF are congruent (SAS).

Since the triangles are congruent, then $\angle ABG = \angle DEF = \angle ABC$, however we assumed that $\angle ABG < \angle ABC$.

Therefore our initial assumption that the triangles are not congruent leads to a contradiction, and therefore must be false. We conclude that the triangles are congruent. ■

Picture of the proof



The proof of the case where the equal sides are not between the two equal angles (AAS) is similar, but it requires an extra step using Euclid Book I, Prop. 16. You can read it here.

Next time

In the next class we will continue studying triangles and investigate

- Areas of parallelograms and triangles.
- Pythagoras' theorem
- Thales Theorem
- Similar triangles