Lecture 19: Geometry on the Poincare disk

8 April, 2019

Overview

Last time. (Spherical Geometry)

- Introduction to spherical geometry
- Defining a geometry. Rigid motions on the sphere.
  - What happens when we change the parallel axiom?
  - How do Euclid's propositions change in spherical geometry?
- The angle sum of a triangle in spherical geometry
- Spherical trigonometry

Today. (Introduction to hyperbolic geometry)

- The Poincaré disk model for hyperbolic geometry
- How do Euclid's propositions change in hyperbolic geometry?
- Arclength, lines and circles on the Poincaré disk

Recall that to define a geometry, we need to define the objects of the geometry.

Euclidean Geometry.

For example, in Euclidean geometry the objects of the geometry were lines and circles. Rigid motions of the Euclidean plane (translations, rotations, reflections and glide reflections) map lines to lines and circles to circles.

These rigid motions also preserve angles, lengths and area, so we can talk about measuring these quantities in Euclidean geometry.

Objects and rigid motions of geometries studied so far (cont.)

Inversive Geometry.

The objects of inversive geometry are circles.

The rigid motions of inversive geometry are inversions. These preserve circles, but they do not preserve the radius of the circle, nor do they preserve the centre of the circle (Tutorial 9).

An inversion may map a straight line (a circle of infinite radius) to a circle of finite radius. Therefore straight lines are not preserved by inversions.

Inversions preserve angles, and so we can talk about measuring angles in inversive geometry.

We cannot talk about length as a meaningful quantity since this is not preserved by inversions.

Instead we can measure the *cross-ratio* of four points (see Lecture 16). This is invariant under inversions and so the cross ratio is a meaningful quantity in inversive geometry.

Objects and rigid motions of geometries studied so far (cont.)

Spherical Geometry.

The lines of spherical geometry are great circles on the sphere.

The rigid motions are rotations of the sphere and reflections through any great circle on the sphere.

The distance is the arclength of the shortest path along a great circle connecting two points on the sphere. A circle is defined as the set of all points a constant distance from a given point (the centre of the circle).

The angle between two lines is the angle between the tangents to the great circles defining the two lines.

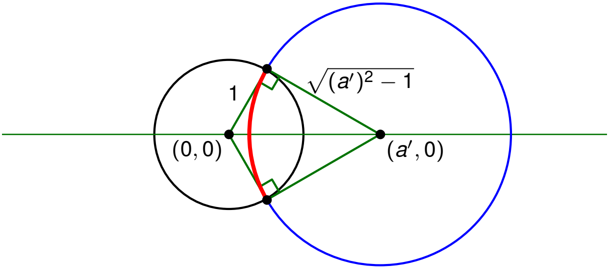
Rigid motions preserve lines and circles (the objects of the geometry) and angle, distance and area (the measurable quantities).

The Poincaré disk model

Now we define the Poincaré disk model of hyperbolic geometry by defining the objects and rigid motions of the geometry.

Definition. The *Poincaré disk* is a disk in the plane. For convenience (if we need to use coordinates) we normally use the unit disk centred at the origin.

The *lines* in the Poincaré disk are given by intersections of the disk with circles orthogonal to the boundary of the disk.



The red arc in the picture above is an example of a line in the Poincaré disk.

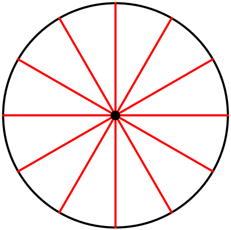
Lines through the origin in the Poincaré disk

What about lines through the origin? Any circle which intersects the boundary of the Poincaré disk orthogonally will not pass through the origin.

Instead, we define a line through the origin to be a diameter of the circle.

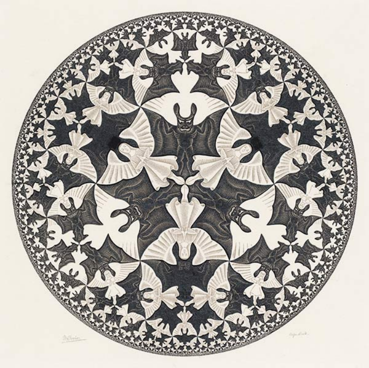
Note that a diameter always intersects a circle orthogonally. Moreover, from the exercise in Tutorial 9, we can think of a line as a circle of infinite radius with centre at the point at infinity in the extended plane.

Examples of lines through the origin in the Poincaré disk.



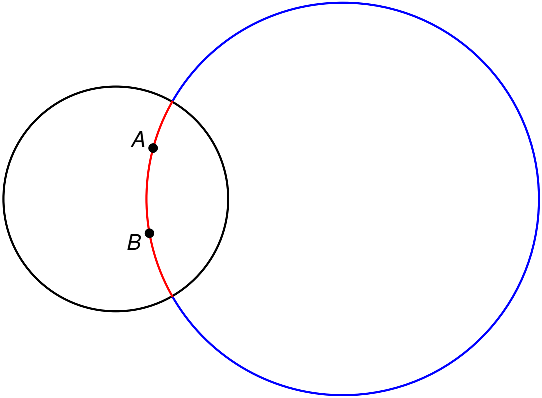
Pictures of lines in the Poincaré disk

The Poincaré disk shows up in the art of M.C. Escher. Here are two pictures where you can see the straight lines (also called "geodesics").



You can read more about it here.

Exercise. Show that any two points A and B in the Poincaré disk are connected by a unique hyperbolic line. Equivalently, show that any two points on the disk are on a unique circle orthogonal to the boundary of the disk.



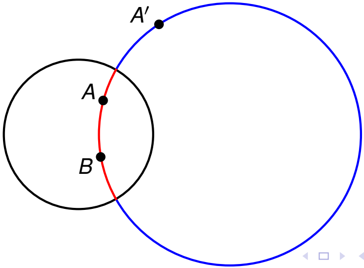
Solution.

Solution

Solution. Let A' be the inverse of A with respect to the boundary circle of the Poincaré disk. Then the three points A, A' and B lie on a unique circle. Moreover, this circle must be orthogonal to the boundary of the Poincaré disk, since it contains both A and the inverse of A.

Conversely, if another circle orthogonal to the boundary passes through A and B, then it also contains the inverse A' and so it must be the circle constructed above.

Therefore there is a unique line in the Poincaré disk through the two points A and B.



Which of Euclid's axioms are valid in hyperbolic geometry?

Axiom 1. One can draw a straight line segment between any two points.
This is valid in hyperbolic geometry.

Axiom 2. One can extend a straight line segment indefinitely.
This is valid in hyperbolic geometry

Axiom 3. One can draw a circle with a given centre and radius.
This is true in hyperbolic geometry.
We will see some examples of circles later today

Axiom 4. All right angles are equal to each other.
This is true in hyperbolic geometry

Which of Euclid's axioms are valid in hyperbolic geometry?

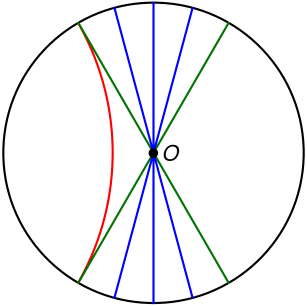
Axiom 5. If a straight line crossing two straight lines makes the interior angles on one side less than two right angles, then the two straight lines, if produced indefinitely, will meet on that side on which the angles are less than the two right angles.
This is not true in hyperbolic geometry.

One way to see that this is true is to recall that Axiom 5 implies that a line through a point parallel to a given line must be unique.

In hyperbolic geometry there are infinitely many parallel lines through a given point.

Examples of parallel lines

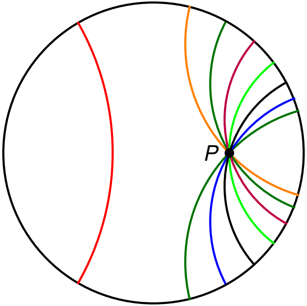
Infinitely many lines through the origin O are parallel to the red line in the figure below.



The green lines do not intersect the red line on the interior of the Poincaré disk, and so they are also considered parallel.

Examples of parallel lines

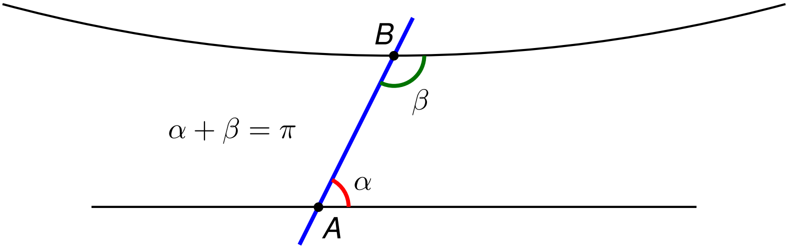
Here is an example where none of the lines pass through the origin. Again, there are infinitely many lines through P which are parallel to the red line.



Consequences of the failure of the parallel axiom

All of Euclid's propositions from I.1-I.28 are still valid, since they only depend on Axioms 1-4.

In particular (Euclid Prop I.28), if a line falling across two straight lines makes the sum of the interior angles equal to π radians then the two lines are parallel.

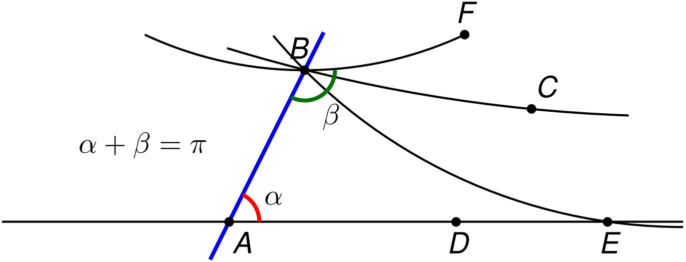


From I.29 onwards Euclid uses the parallel axiom. In particular, this is used in I.32 (the angle sum of a triangle is equal to π radians).

Consequences of the failure of the parallel axiom

If we have more than one line through B which is parallel to AD then the angles of a triangle may no longer sum to π radians. Draw another line BC which is parallel to AD. Using a similar idea to the solution to Q3 from Tutorial 1, we can show that there exists E on AD such that $\angle AEB < \angle FBC$. Then the sum of the angles in triangle ΔAEB is

$$\angle BAE + \angle EBA + \angle AEB = \alpha + \angle EBA + \angle AEB < \alpha + \beta = \pi$$



Distance on the Poincaré disk

Question. How do we measure the distance between two points?

Since the lines in the Poincaré disk are different to Euclidean lines, then you might expect the distance between two points to be different to the Euclidean distance.

This is true. As you approach the boundary of the Poincaré disk, the distance becomes more and more distorted.

We can describe this precisely using the formula below, and we can also see this intuitively by looking again at the pictures of M.C. Escher.

Definition. Let $P = (a, 0)$ be a point in the Poincaré disk. The distance from P to the origin O is

$$d_{hyp.}(O, P) = \int_0^a \frac{2dr}{1 - r^2}$$

Distance on the Poincaré disk (cont.)

Definition. Let $P = (a, 0)$ be a point in the Poincaré disk. The distance from P to the origin O is

$$d_{hyp.}(O, P) = \int_0^a \frac{2dx}{1 - x^2}$$

We can solve this integral to obtain

$$d = \int_0^a \frac{2dx}{1 - x^2} = \int_0^a \left(\frac{1}{1 + x} + \frac{1}{1 - x} \right) dr$$
$$= \log(1 + a) - \log(1 - a) = \log \left(\frac{1 + a}{1 - a} \right)$$

If a is very close to zero (i.e. P is close to the origin), then this function is well approximated by the linear function $2a$ (you can make this rigorous by doing a Taylor expansion).

If a is very close to 1 (i.e. P is close to the boundary of the Poincaré disk), then the distance approaches infinity. Therefore the boundary is infinitely far from the origin.

Distance on the Poincaré disk (cont.)

Equivalently, we can solve this for a in terms of the distance d

$$a = \frac{e^d - 1}{e^d + 1} = \frac{\frac{1}{2}(e^{\frac{d}{2}} - e^{-\frac{d}{2}})}{\frac{1}{2}(e^{\frac{d}{2}} + e^{-\frac{d}{2}})}$$

We can rewrite this formula in terms of the *hyperbolic* functions.

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}), \quad \cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

and $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$. Therefore

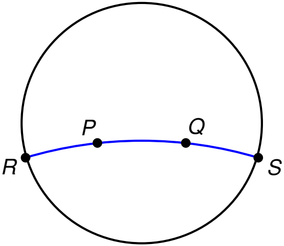
$$a = \frac{\frac{1}{2}(e^{\frac{d}{2}} - e^{-\frac{d}{2}})}{\frac{1}{2}(e^{\frac{d}{2}} + e^{-\frac{d}{2}})} = \frac{\sinh\frac{d}{2}}{\cosh\frac{d}{2}} = \tanh\frac{d}{2}.$$

The hyperbolic functions will appear again in Lecture 20 when we describe the geometry of the hyperboloid.

Distance between two general points on the Poincaré disk

Question. What is the distance between two general points P and Q on the Poincaré disk?

Given two points P, Q in the Poincaré disk, let R and S be the ends of the hyperbolic line PQ (i.e. the intersection of this line with the unit disk) as in the diagram below.



A formula for the hyperbolic distance from P to Q is given by

$$d_{hyp.}(P,Q) = \left|\log\left(\frac{|PS|\cdot|QR|}{|PR|\cdot|QS|}\right)\right|$$

Distance between two general points on the Poincaré disk

General Formula. (Not tested)

One way to prove this formula is to use a path integral along the hyperbolic line from P to Q. Instead of using Euclidean arclength

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

in hyperbolic geometry we use

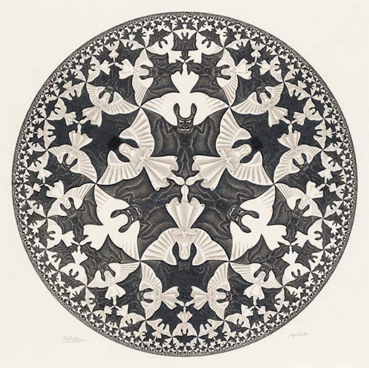
$$ds_{hyp.} = \frac{2}{1 - (x^2 + y^2)} \sqrt{dx^2 + dy^2} = \frac{2\,ds}{1 - r^2}.$$

The factor $\frac{2}{1-r^2}$ measures the "distortion" in the arclength caused by the different geometry in the Poincaré disk.

You may learn more about this in further courses on differential geometry.

Hyperbolic distance and M.C. Escher

Returning to Escher's pictures, the objects are all the same size, but they appear to get smaller as they approach the boundary of the disk, since the lengths are distorted by the function $t = \log\left(\frac{1+r}{1-r}\right)$.



You can read more about it here.

Rigid motions on the Poincaré disk

It is possible to prove this distance formula by integrating the distance measure on the Poincaré disk, but it is easier to understand it in terms of symmetries of the disk.

Geometers use symmetries to simplify a problem.

Question. What are the symmetries of the Poincaré disk?

It is easy to see that our definition of distance is invariant under rotation about the centre of the disk.

Therefore rotation is a *symmetry* of the Poincaré disk.

We can then see that rotation is a *rigid motion* (it preserves the hyperbolic distance between two points).

In a similar way, we see that reflection across a line through the origin is also a rigid motion.

Rigid motions on the Poincaré disk (cont.)

Are there any other rigid motions?

We will prove in **Tutorial 11** that inversion through a circle orthogonal to the boundary of the Poincaré disk is also a rigid motion (this is analogous to reflecting across a line in Euclidean geometry).

We will then use these inversions to prove the formula for hyperbolic distance.
(Note the appearance of the cross product in the formula for $d_{hyp.}(P, Q)$ given previously.)

Recall from Lecture 15 that inversions preserve angles. Therefore the rigid motions that we have defined also preserve the angle between two hyperbolic lines.

You can experiment with rigid motions on the Geometry Website page rigid motions on the Poincaré disk.

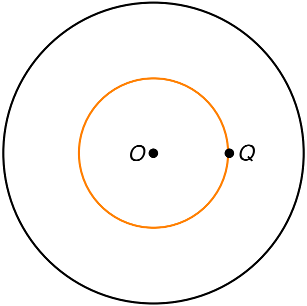
Again, if we look at Escher's picture below, we can see that each fish is preserved after reflection across each hyperbolic line. These reflections are examples of rigid motions.

Circles on the Poincaré disk

We can define a circle in an analogous way to Euclidean geometry.

Definition. A *circle* with centre P and radius r is the set of all points Q such that $d_{hyp.}(P, Q) = r$.

It is clear that a circle centred at the origin with radius r will be a regular Euclidean circle containing the point $Q = \left(\tanh \frac{r}{2}, 0\right)$.



Circles on the Poincaré disk (cont.)

How do we describe the other circles on the Poincaré disk?

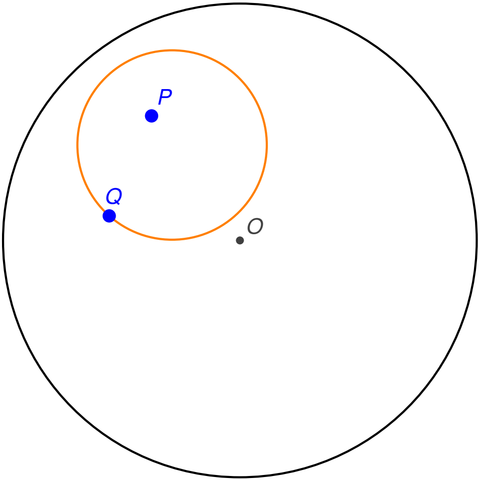
Consider a circle with centre P and radius r (note that r is the *hyperbolic distance*). Using a rigid motion, we can map the centre P to the origin (we will see how to do this in Tutorial 11).

Since rigid motions preserve *hyperbolic* lengths, then all of the points on the circle (which are a hyperbolic distance of r from the centre) map to a circle of hyperbolic radius r (Euclidean radius $\tanh \frac{r}{2}$) centred at the origin.

Therefore, our original hyperbolic circle is the inverse image of a circle centred at the origin by a rigid motion. Since our rigid motions are either rotations (which map circles to circles) or inversions (which also map circles to circles) then our original hyperbolic circle is also a circle.

Circles on the Poincaré disk (cont.)

Here is an example of a hyperbolic circle with centre P and a point Q on the circle, so the hyperbolic radius is $d_{hyp.}(P, Q)$.



Next time

In Friday's lecture we will see how the geometry of the Poincaré disk arises from the geometry of the *pseudosphere*. We will also learn about hyperbolic trigonometry and the angle sum of a hyperbolic triangle.

I will also give some information about the final exam on Friday.