Lecture 12: Introduction to conic sections

4 March, 2018

Last time.

- Pencils of circles
- Orthogonal pencils of circles
- Poncelet's theorem
- The problem of Apollonius

Today.

- Introduction to conic sections
- Definition of the ellipse, parabola and hyperbola in terms of second order polynomials
- Geometric properties of the ellipse and hyperbola
- Optical properties of the ellipse and hyperbola
- Application to the problem of Apollonius

Curves of degree 2

The topic of today's lecture is conic sections. Conic sections admit many beautiful geometric structures and we will see a sample of these today.

First we will describe the theory of curves defined by polynomials of order 2.
(Some of this may be familiar from calculus)

Consider the solutions to the equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2b_1x + 2b_2y + c = 0$$

We can remove the term $a_{12}xy$ by rotating the coordinate system, i.e. changing variables

$$(x, y) \mapsto (x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta)$$

(this is a clockwise rotation by angle θ).

Curves of degree 2

This gives us a new equation

$$Ax^2 + 2Bx + Cy^2 + 2Dy + E = 0$$

If $A \neq 0, C \neq 0$ then we can complete the square to obtain

$$A(x + B/A)^2 + C(y + D/C)^2 + E - B^2/A - D^2/C = 0$$

and then translating, i.e. changing variables $(x, y) \mapsto (x - B/A, y - D/C)$ gives us

$$Ax^2 + Cy^2 = F$$

If $C = 0$ then instead of completing the square on y we can directly obtain

$$Ax^2 + 2Dy = F$$

and similarly if $A = 0$ then $2Bx + Cy^2 = F$.

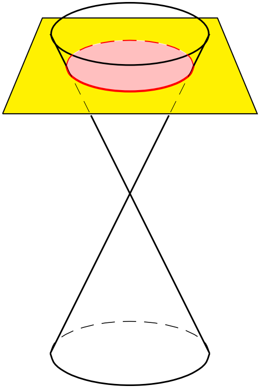
Curves of degree 2

We can now classify all of the different solution curves (see the next few slides for pictures)

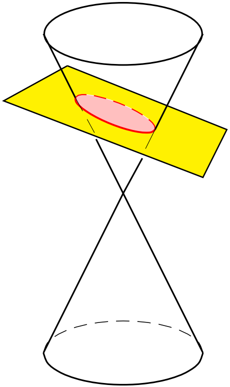
- If $AC > 0$ and F has the same sign as A and C then we obtain an ellipse.
  (The circle is the special case where $A = C$)
- If $AC < 0$ and $F \neq 0$ then we obtain a hyperbola.
- If $AC < 0$ and $F = 0$ then we obtain two intersecting lines.
- If one of A or C is zero then we obtain a parabola.
- If $A = C = 0$ then we obtain a straight line.

These curves are also called conic sections since we can construct them as intersections of a plane with the double cone in $\mathbb{R}^3$ given by $ax^2 + by^2 = cz^2$.

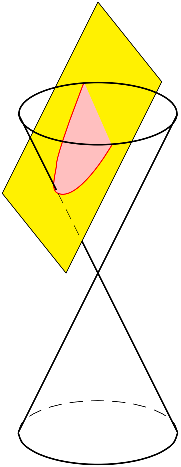
We could do this algebraically, but in keeping with the spirit of this course we will give a geometric proof of this later in today's lecture.



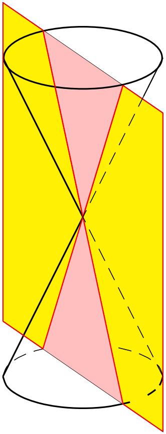
A plane parallel to a cross-section of the double cone cuts the double cone in a circle.



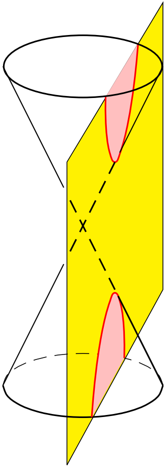
A plane cuts the double cone in an ellipse.



A plane parallel to a side of the double cone cuts the double cone in a parabola.

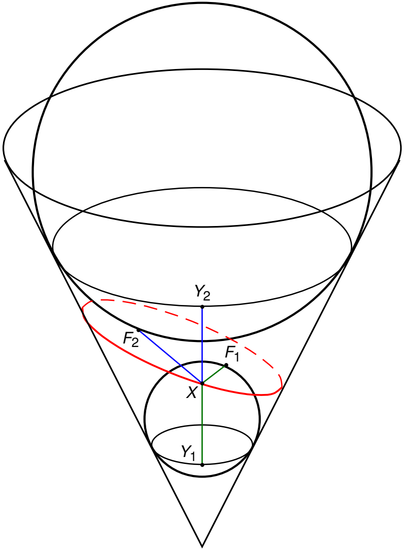


A plane through the vertex of the double cone cuts the double cone in a pair of straight lines.



A plane cuts the double cone in a hyperbola.

A geometric property of the ellipse



Consider the intersection of the cone with a plane (in the picture this is the red ellipse). Let S_1 and S_2 be two spheres resting inside the cone so that they just touch the plane at two points F_1 and F_2.

Choose any point X on the ellipse. Draw straight lines along the surface of the cone from X to intersect the spheres S_1 and S_2 at points Y_1 and Y_2.

Then XY_1 and XF_1 are both tangent to the sphere S_1, so $|XY_1| = |XF_1|$.

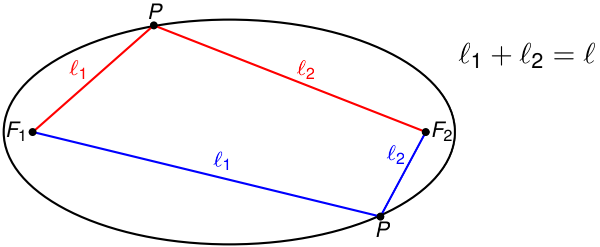
Also XY_2 and XF_2 are both tangent to the sphere S_2, so $|XY_2| = |XF_2|$.

$$|XF_1| + |XF_2| = |XY_1| + |XY_2| = |Y_1 Y_2| = \text{constant}$$

A geometric property of the ellipse

Therefore we have proved a geometric property of the ellipse, and our proof was based purely on the description as a conic section (i.e. we didn't need to use coordinates).

Let F_1 and F_2 be two points in the plane, and let $\ell > |F_1 F_2|$. The ellipse with foci F_1 and F_2 and length ℓ is the set of points P such that $|PF_1| + |PF_2| = \ell$.



This proof using the cone and the two spheres was originally due to Dandelin.

Click the following link for an interactive picture.

Exercise

Exercise. Prove that a point P lies inside the ellipse if and only if $|F_1P| + |F_2P| < \ell$ and that P lies outside the ellipse if and only if $|F_1P| + |F_2P| > \ell$.

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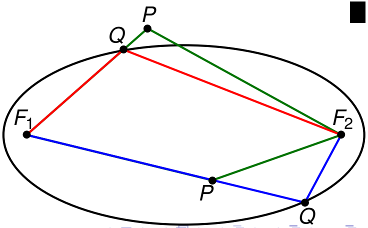
Solution.

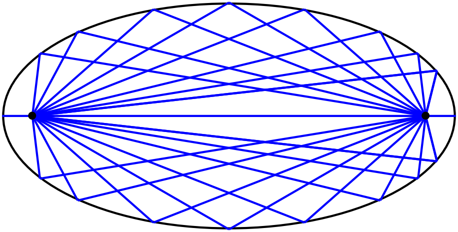
Exercise

Exercise. Prove that a point P lies inside the ellipse if and only if $|F_1P| + |F_2P| < \ell$ and that P lies outside the ellipse if and only if $|F_1P| + |F_2P| > \ell$.

Solution. Let Q be a point on the ellipse lying on the line F_1P. Since P is on the line F_1Q, then P lies inside the ellipse if and only if P lies inside the triangle ΔF_1QF_2. The triangle inequality shows that this is true if and only if
$|F_1P| + |F_2P| < |F_1Q| + |F_2Q| = \ell$
(see Q5, Tutorial 2 for a similar problem).
A similar proof shows that P lies outside the ellipse if and only if
$|F_1P| + |F_2P| > |F_1Q| + |F_2Q| = \ell$. ∎



Construction. Imagine a board with two nails hammered in at F_1 and F_2. Attach a piece of string of length ℓ to the nails. If you pull the string tight as in the diagram below, then $|PF_1| + |PF_2| = \ell$. Therefore the string traces out an ellipse.



Next week we will see how to construct points on the ellipse using a ruler and compass.

Formula for the ellipse in Cartesian coordinates

Proposition. Consider an ellipse with foci at $(\pm a, 0)$ and length $\ell > 2a$. Then the equation in Cartesian coordinates is

$$\frac{x^2}{\ell^2} + \frac{y^2}{\ell^2 - 4a^2} = \frac{1}{4}$$

The idea of the proof is to consider a point $P = (x, y)$ on the ellipse. Then

$$|(x, y) - (-a, 0)| = \sqrt{(x + a)^2 + y^2}$$
$$|(x, y) - (a, 0)| = \sqrt{(x - a)^2 + y^2}$$

Our equation is then

$$\sqrt{(x + a)^2 + y^2} + \sqrt{(x - a)^2 + y^2} = \ell$$

and the goal is to rearrange this into the form given above. (**Exercise.** See Tutorial 7)

A geometric property of the hyperbola

A similar idea to the construction given earlier for the ellipse shows that if a plane intersects the cone in a hyperbola, then there are two focal points F_1 and F_2 such that every point P on the hyperbola satisfies $|PF_1| - |PF_2| = \pm\ell$ for some fixed number $\ell < |F_1 F_2|$.

The two focal points are the points of tangency of the plane with a sphere resting inside the cone.

Again, we can use this property to write the equation of the hyperbola with foci at $(\pm a, 0)$ and length $\ell < 2a$ in Cartesian coordinates

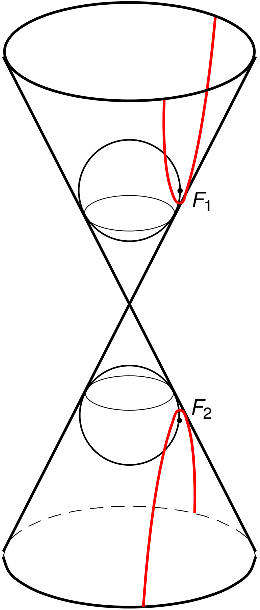
$$\frac{x^2}{\ell^2} + \frac{y^2}{\ell^2 - 4a^2} = \frac{1}{4}$$

(**Exercise.** See Tutorial 7)

Remark. Note that $\ell^2 - 4a^2$ is now negative.

The advantage of this formulation is that we can immediately see the length ℓ and the focal points $(\pm a, 0)$ from the equation.

A geometric property of the hyperbola



Consider two spheres resting inside the cone so that they are tangent to the plane passing through the hyperbola.

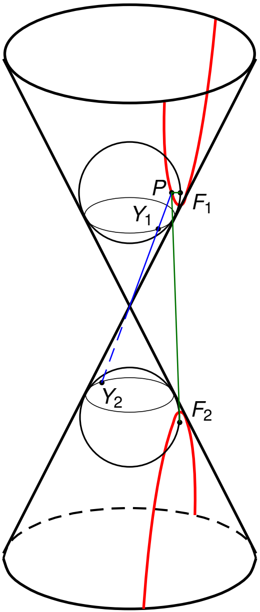
Let F_1 and F_2 be the points of intersection.

Exercise. Prove that every point P on the hyperbola satisfies

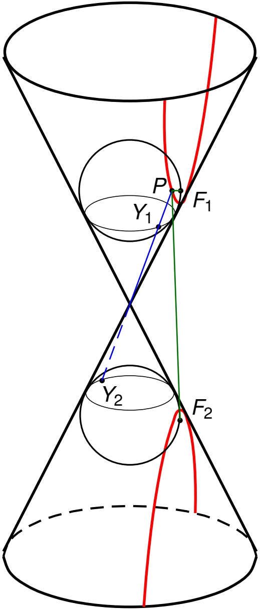
$$|PF_1| - |PF_2| = \pm\ell$$

for some constant ℓ.

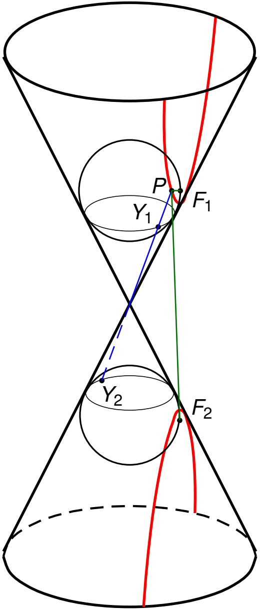
Exercise Solution



Exercise Solution



Exercise Solution.



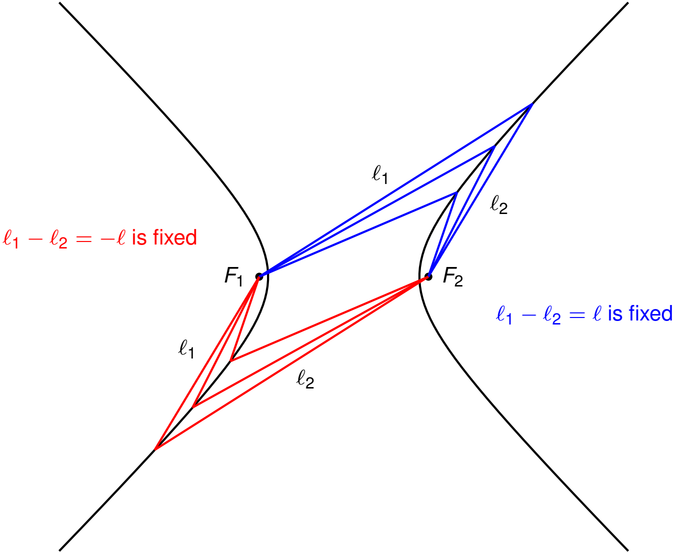
Exercise Solution. Let P be a point on the hyperbola. Draw a line from P through the vertex of the cone and let Y_1 and Y_2 be the intersection of this line with the two spheres.

Then $|PY_1| = |PF_1|$ and $|PY_2| = |PF_2|$. Moreover, $|Y_1Y_2|$ is fixed and (since P is outside the segment Y_1Y_2) then $|PY_2| - |PY_1| = |Y_1Y_2|$ is constant.

Therefore $|PF_2| - |PF_1|$ is constant as well. ∎

Click the link for an interactive picture.

A geometric property of the hyperbola (cont.)



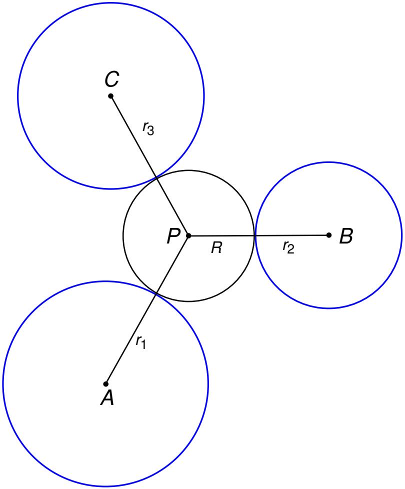
Application to the problem of Apollonius

Let A, B, C denote the centre of three circles of radius r_1, r_2 and r_3 respectively. How to find the centre P and radius R of a circle tangent to all three circles?

We know that $|PA| - |PB| = r_1 - r_2$. Therefore P lies on the hyperbola with focal points A and B and length $r_1 - r_2$.

We also have $|PB| - |PC| = r_2 - r_3$. Therefore P lies on the hyperbola with focal points B and C and length $r_2 - r_3$.

We can then find the point P as the intersection of these two hyperbolas. The radius is then $R = |PA| - r_1 = |PB| - r_2 = |PC| - r_3$.



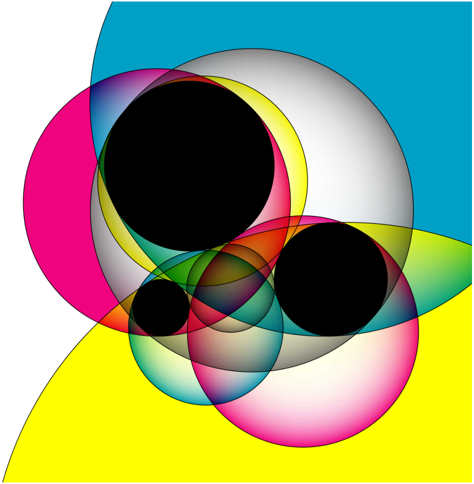
Is the solution unique?

We can instead ask for a point P such that

$$|PA| - |PB| = r_1 + r_2$$

This corresponds to the case where the second circle is inside the circle with centre P.

If we consider all of the possible combinations of inside and outside then (in general) we get eight circles.



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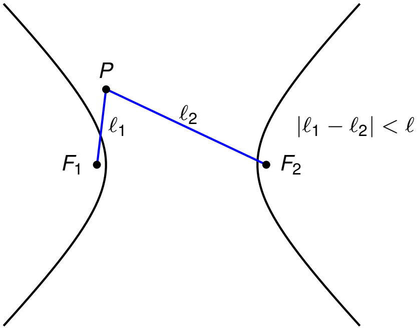
A geometric property of the hyperbola

We will use the following result later when we study the optical properties of the hyperbola.

Lemma. Consider a hyperbola as in the diagram below with focal points F_1, F_2 and length ℓ. Then a point P lies between the two components of the hyperbola if and only if the absolute value of $|F_1P| - |F_2P|$ is less than ℓ.



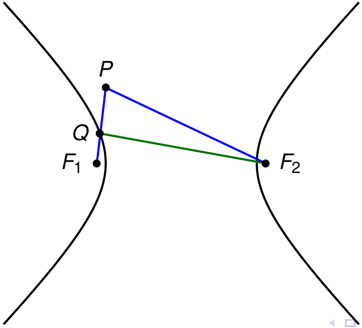
A geometric property of the hyperbola

Proof. Let Q be the intersection of F_1P with the hyperbola. Then $|F_1P| = |F_1Q| + |QP|$. The triangle inequality shows that $|F_2P| < |F_2Q| + |QP| = |F_2Q| + |F_1P| - |F_1Q|$.

Therefore $|F_2P| - |F_1P| < |F_2Q| - |F_1Q| = \ell$.

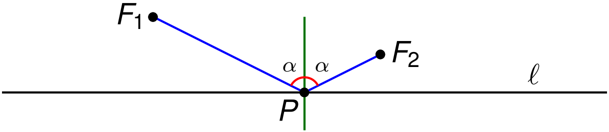
If we switch F_1 and F_2 and repeat the argument (now Q' is on the line from F_2 to P) then we get $|F_1P| - |F_2P| < \ell$.

Therefore the absolute value of $|F_1P| - |F_2P|$ is less than ℓ. ∎



Optical properties (preliminary results)

Lemma 1. Let ℓ be a line and let F_1 and F_2 be two points on one side of ℓ. As we vary a point P on the line, the function $|F_1P| + |F_2P|$ is minimised if and only if the perpendicular to ℓ through P bisects the angle $\angle F_1PF_2$.
Equivalently, the lines F_1P and F_2P make the same angle with the line ℓ.



If you shine a light from F_1 to P then it will reflect off the line ℓ to F_2 if and only if P minimises the length $|PF_1| + |PF_2|$.

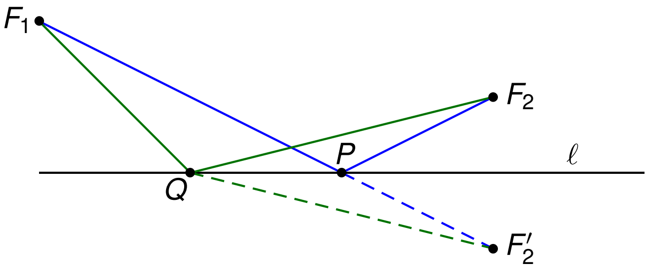
This lemma is the basis for studying the *optical properties* of the ellipse.

Optical properties (preliminary results)

Proof. Reflect F_2 in the line ℓ to get a new point F_2'. Then $|PF_2| = |PF_2'|$ and so $|PF_1| + |PF_2| = |PF_1| + |PF_2'|$.

The point P which minimises this length is clearly the point lying on the straight line F_1F_2'.

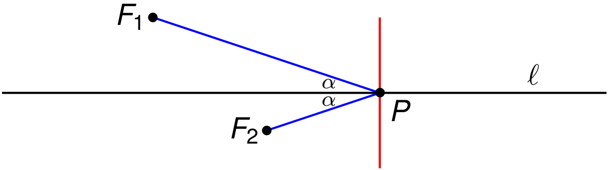
Therefore the lines F_1P and $F_2'P$ make the same angle with the line ℓ, and so F_2P also makes the same angle with ℓ. ∎



$$|F_1P| + |F_2P| = |F_1P| + |F_2'P| < |F_1Q| + |F_2'Q| = |F_1Q| + |F_2Q|$$

Optical properties (preliminary results)

Lemma 2. Let ℓ be a line and let F_1 and F_2 be two points on opposite sides of ℓ. As we vary a point P on the line, the absolute value of $|F_1P| - |F_2P|$ is maximised if and only if ℓ bisects the angle $\angle F_1PF_2$.



If you shine a light from F_1 to P then it will reflect off the perpendicular to the line ℓ to F_2 if and only if P maximises the absolute value of $|PF_1| - |PF_2|$.

This lemma is the basis for studying the *optical properties* of the hyperbola.

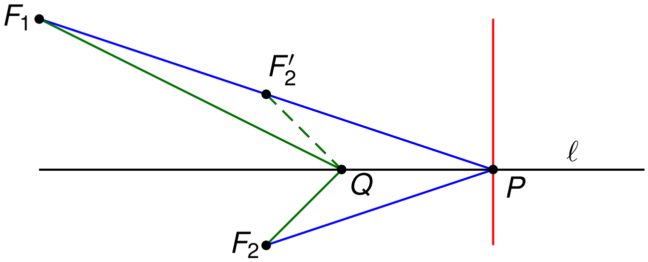
Optical properties (preliminary results)

Proof. Reflect F_2 in the line ℓ to get a new point F_2'. Then $|PF_2| = |PF_2'|$ and so $|PF_1| - |PF_2| = |PF_1| - |PF_2'|$.

The triangle inequality shows that $|F_1P| \leq |F_2'P| + |F_1F_2'|$ and $|F_2'P| \leq |F_1P| + |F_1F_2'|$.

Therefore $\big||F_1P| - |F_2'P|\big| \leq |F_1F_2'|$, with equality if and only if P lies on the line F_1F_2'.

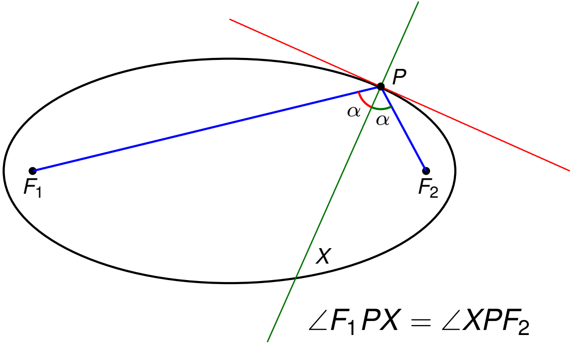
Therefore the lines F_1P and $F_2'P$ make the same angle with the line ℓ, and so F_2P also makes the same angle with ℓ. ∎



Optical property of the ellipse

Theorem. (Optical property of the ellipse) Let P be a point on an ellipse with focal points F_1 and F_2. Then F_1P and F_2P make the same angle with the tangent to the ellipse at P.

The perpendicular to the tangent bisects the angle $\angle F_1PF_2$

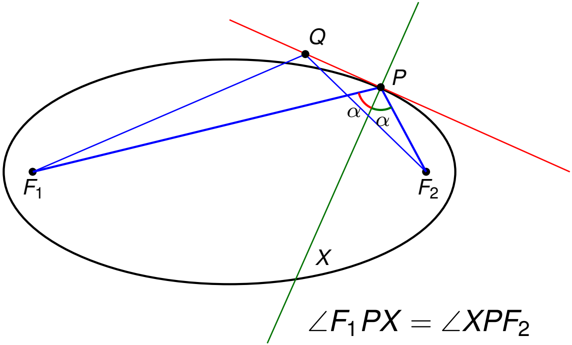


$$\angle F_1PX = \angle XPF_2$$

If you shine a beam of light from F_1 to any point P on the ellipse, then it will reflect back to F_2.

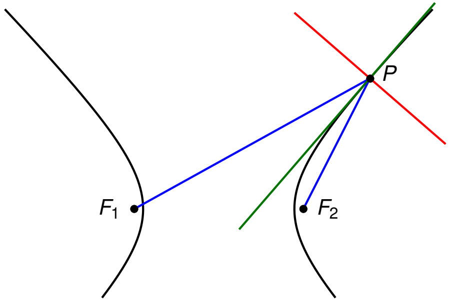
Optical property of the ellipse

Proof. Let Q be any other point on the tangent line. Since Q is outside the ellipse, then $|F_1Q| + |F_2Q| > \ell = |F_1P| + |F_2P|$. Lemma 1 then shows that F_1P and F_2P make the same angle with the tangent to the ellipse at P. Equivalently, PX is the angle bisector of $\angle F_1PF_2$.



$$\angle F_1PX = \angle XPF_2$$

Theorem. (Optical property of the hyperbola) Let P be a point on a hyperbola with focal points F_1 and F_2. Then F_1P and F_2P make the same angle with the tangent to the hyperbola at P. The tangent bisects the angle $\angle F_1PF_2$



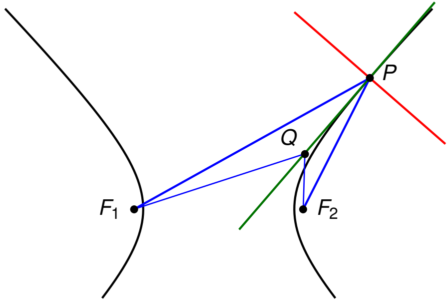
If you shine a beam of light from F_1 to any point P on the hyperbola, then it will reflect off the perpendicular to the tangent back to F_2.

Optical property of the hyperbola

Proof. Suppose w.l.o.g. that P lies on the component of the hyperbola closest to F_2.

Let Q be any other point on the tangent line. Then Lemma 2 shows that the tangent line bisects the angle $\angle F_1QF_2$ if and only if the absolute value of $|F_1Q| - |F_2Q|$ is maximised.

Since all the points on the tangent line lie in between the two components of the hyperbola then $||F_1Q| - |F_2Q||$ is maximised if and only if $P = Q$.



Next time

This Friday we will have the midterm exam. Next week we will continue with conic sections and study the following topics.

- Geometric properties of the parabola
- Optical properties of the parabola
- Properties of triangles passing through the focal points of an ellipse
- Constructing conics

Next Friday we will also learn how to draw in perspective as an introduction to projective geometry.