Lecture 5: Special circles associated to triangles

28 January, 2019

Overview

Last time.

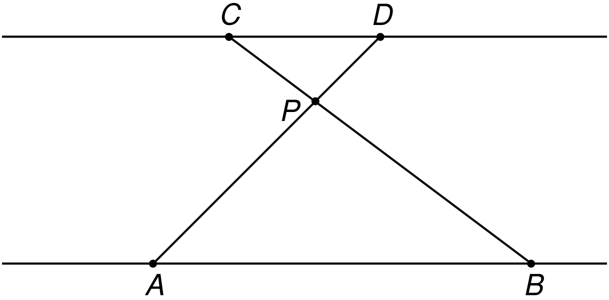
- Similar triangles
- Another proof of Pythagoras' Theorem
- More applications of similar triangles
- Pappus' theorem and Desargues' theorem
- Constructions in Euclidean geometry

Today.

- Ceva's Theorem
- The centroid of a triangle
- The circumcentre of a triangle
- The incentre of a triangle
- The orthocentre of a triangle

Exercise

Exercise. Let AB and CD be parallel lines, and let P be a point not on either line. Prove that $\Delta ABP \sim \Delta DCP$. ("$\sim$" means "similar")

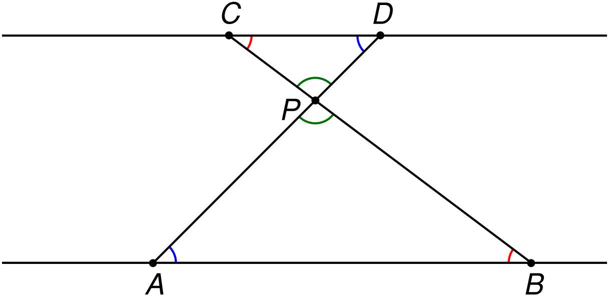


Solution

Solution

Solution.

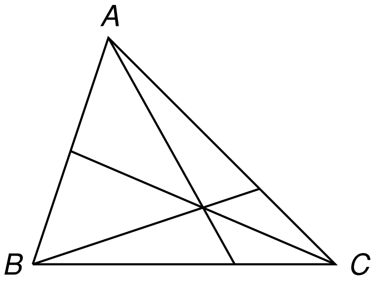
Solution. The angles $\angle APB$ and $\angle DPC$ are opposite angles and therefore equal.
Angles $\angle BAP$ and $\angle CDP$ are alternate and therefore equal.
Angles $\angle ABP$ and $\angle DCP$ are alternate and therefore equal.
Therefore $\Delta ABP \sim \Delta DCP$. (AAA)



Ceva's Theorem

Definition. A cevian is a line segment joining a vertex of a triangle ΔABC to any given point on the opposite side.

Definition. Three lines are concurrent if and only if they meet at a single point.

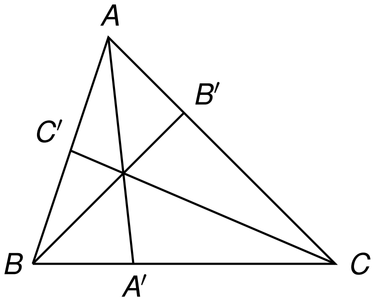


An example of three concurrent cevians.

Ceva's Theorem

Theorem. (Ceva) Let ΔABC be a triangle. Then three cevians AA', BB' and CC' are concurrent if and only if

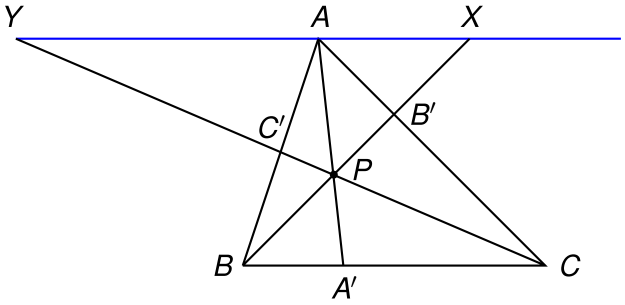
$$\frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} \cdot \frac{|AC'|}{|C'B|} = 1$$



Proof. Draw a line through A which is parallel to BC.

Extend the line segment BB' so that it intersects this parallel line at X.

Similarly, extend the line segment CC' so that it intersects this parallel line at Y.



Proof of Ceva's Theorem (cont.)

Suppose first that the cevians AA', BB' and CC' are concurrent at P.

Since XY is parallel to BC then the result of the previous exercise gives us the following pairs of similar triangles.

$$\Delta AC'Y \sim \Delta BC'C, \quad \Delta AB'X \sim \Delta CB'B, \quad \Delta A'PC \sim \Delta APY, \quad \Delta A'PB \sim \Delta APX$$

These four pairs of similar triangles give us four pairs of equations

$$\frac{|AC'|}{|AY|} = \frac{|C'B|}{|BC|}, \quad \frac{|B'A|}{|AX|} = \frac{|CB'|}{|BC|}, \quad \frac{|AP|}{|AY|} = \frac{|A'P|}{|A'C|},$$

$$\text{and } \frac{|AP|}{|AX|} = \frac{|A'P|}{|BA'|}.$$

Proof of Ceva's Theorem (cont.)

Now we can solve the equations to get

$$|AY| = \frac{|AC'|}{|C'B|} \cdot |BC| = \frac{|AP|}{|A'P|} \cdot |A'C|$$

and

$$|AX| = \frac{|B'A|}{|CB'|} \cdot |BC| = \frac{|AP|}{|A'P|} \cdot |BA'|$$

Dividing the first equation by the second gives us

$$\frac{|AC'|}{|C'B|} \cdot \frac{|CB'|}{|B'A|} = \frac{|A'C|}{|BA'|}$$

which we can then rearrange to obtain

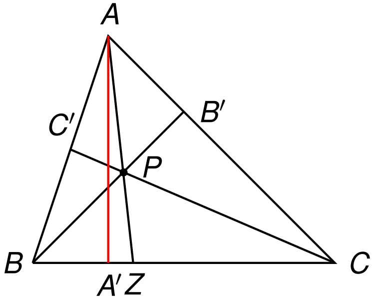
$$\frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} \cdot \frac{|AC'|}{|C'B|} = 1$$

Proof of Ceva's Theorem (cont.)

Now we prove the converse. Suppose that

$$\frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} \cdot \frac{|AC'|}{|C'B|} = 1$$

Let P be the intersection of BB' and CC', and let Z be the intersection of AP with BC.



Proof of Ceva's Theorem (cont.)

Since AZ, BB' and CC' are concurrent, then the previous proof shows that

$$\frac{|BZ|}{|ZC|} \cdot \frac{|CB'|}{|B'A|} \cdot \frac{|AC'|}{|C'B|} = 1$$

Combining this with the equation

$$\frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} \cdot \frac{|AC'|}{|C'B|} = 1$$

shows that $\frac{|BZ|}{|ZC|} = \frac{|BA'|}{|A'C|}$ and so $A' = Z$. Therefore AA', BB' and CC' are all concurrent. ∎

The centroid of a triangle

Now let's apply Ceva's theorem in the following situation.

Definition. A median of a triangle is a cevian from a vertex to the midpoint of the opposite side.

Given a triangle ΔABC, construct A' as the midpoint of BC, B' as the midpoint of AC and C' as the midpoint of AB.

Lemma. The three medians AA', BB' and CC' are concurrent.

Proof. Since A', B' and C' are the respective midpoints of BC, AC and AB, then

$$\frac{|BA'|}{|A'C|} = 1, \quad \frac{|CB'|}{|B'A|} = 1, \quad \frac{|AC'|}{|C'B|} = 1$$

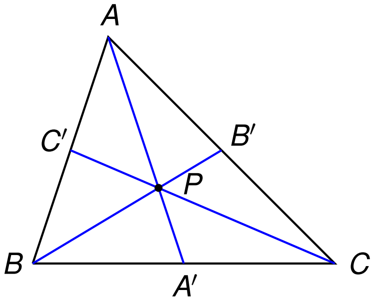
Therefore

$$\frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} \cdot \frac{|AC'|}{|C'B|} = 1$$

and so the medians are concurrent by Ceva's theorem.

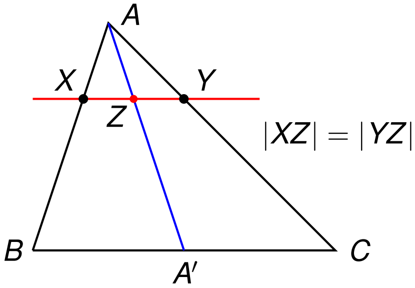
The centroid of a triangle

Definition. The centroid of a triangle ΔABC is the common intersection point of the three medians.



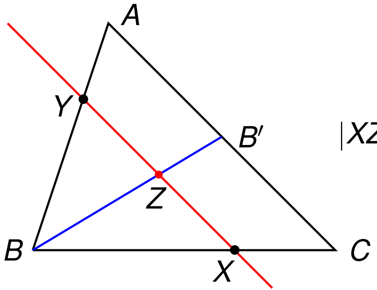
The centroid is the centre of gravity

Consider a triangle ΔABC and draw the median from A to A'. Given any line XY which is parallel to BC, the line AA' will cut XY into two pieces of equal length. (This follows from Thales' theorem (Exercise).)



Imagine the triangle as a thin sheet of rigid metal. The above argument shows that if you balance the triangle on the line AA', the gravitational forces will cancel in the direction parallel to BC. (There may still be gravitational forces perpendicular to BC.)

Now draw the median from B to B'. The same argument shows that if we balance the triangle along the line BB' then the gravitational forces will cancel in the direction parallel to AC.

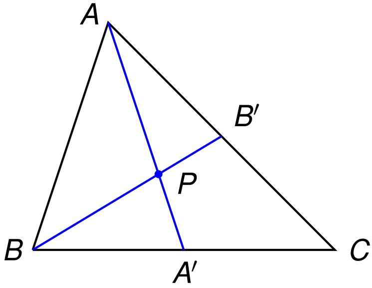


$$|XZ| = |YZ|$$

The centroid is the centre of gravity (cont.)

Therefore if we balance the triangle at the point P where the two medians intersect, then the gravitational forces will cancel

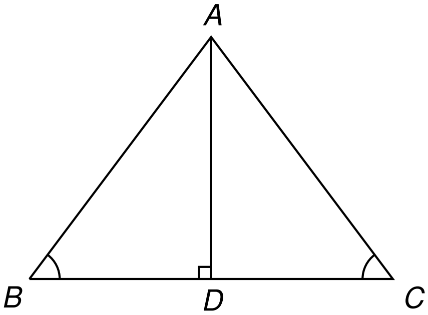
- in the direction parallel to BC, and
- in the direction parallel to AC.



Therefore the triangle will balance perfectly and so the centroid P is the centre of gravity of the triangle.

Exercise. Let ΔABC be a triangle and let D be the midpoint of BC. (Therefore $|BD| = |DC|$)
Prove that $\angle BDA = 90^\circ = \angle CDA$ if and only if $|AB| = |AC|$ (i.e. ΔABC is isosceles).

Hint. Use congruent triangles.



See also Euclid Prop. III.3 for an equivalent statement.

Solution

Solution

Solution.

Solution

Solution. First assume that $\angle BDA = 90^\circ = \angle CDA$. Since D is the midpoint of BC, then $|BD| = |DC|$. The triangles ΔBDA and ΔCDA have the side AD in common, therefore we can use SAS to show that $\Delta BDA \cong \Delta CDA$.

Since the triangles are congruent, then the corresponding sides are equal and so $|AB| = |AC|$.

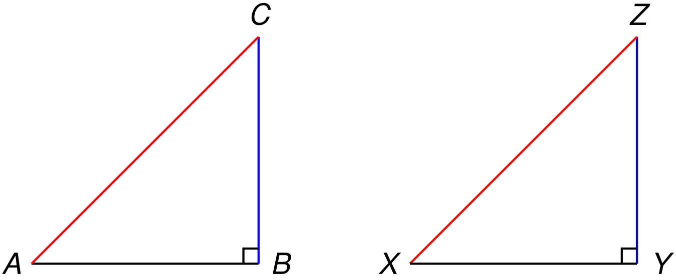
Now assume that $|AB| = |AC|$. We know that $|BD| = |DC|$ and that ΔBDA and ΔCDA have the side AD in common.

Therefore $\Delta BDA \cong \Delta CDA$ by SSS and so $\angle BDA = \angle CDA$. Since $\angle BDA + \angle CDA = 180^\circ$ then $\angle BDA = 90^\circ = \angle CDA$. ■

Exercise

Exercise. Now consider a triangle ΔABC with $|AB| = |AC|$ (see the picture from the previous exercise), and construct D on BC such that AD is perpendicular to BC.
Prove that $|BD| = |DC|$.

Deduce the *Right-angle Side Side* (RASS) congruence theorem: If ΔABC and ΔXYZ have $\angle ABC = 90^\circ = \angle XYZ$, $|AB| = |XY|$ and $|AC| = |XZ|$ then they are congruent.



Exercise Solution

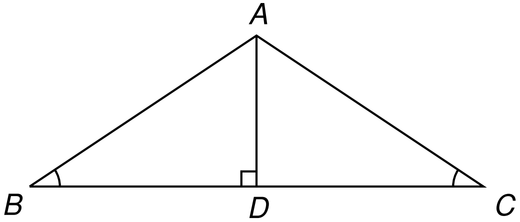
Solution.

Exercise Solution

Solution. Since $|AB| = |AC|$ and $\angle BDA = \angle CDA$ then

$$|BD|^2 = |AB|^2 - |AD|^2 = |AC|^2 - |AD|^2 = |CD|^2 \quad \text{(Pythagoras)}$$

Therefore the triangles ΔABD and ΔACD are congruent by SSS.

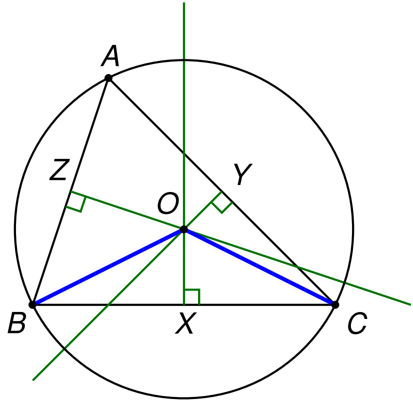


The RASS theorem then follows from the same idea: Use Pythagoras' theorem and then SSS.

Remark. We could also use the fact that in an isosceles triangle, the angles opposite the equal sides are equal (see Euclid Prop. I.5) and then AAS.

The circumcentre of a triangle

Suppose first that a circle intersects the three vertices of a triangle ΔABC. What properties does it have?



Let X, Y and Z be the midpoints of the sides BC, AC and AB respectively.

The circumcentre of a triangle (cont.)

We see that the centre O is equidistant from each vertex A, B and C. Therefore we can form an isosceles triangle with base BC and show that O lies on the perpendicular bisector of BC.

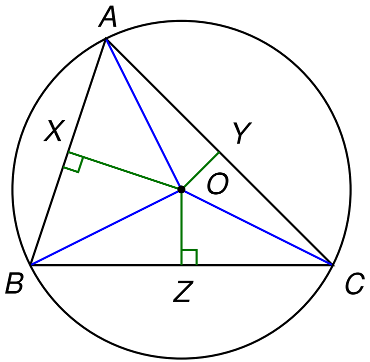
Similarly, O must lie on the perpendicular bisectors of AB and AC.

Therefore, a *necessary* condition for the existence of a circle that intersects each vertex of a triangle ΔABC is that the perpendicular bisectors of each side must be concurrent.

The circumcentre of a triangle (cont.)

Now suppose that you are given a triangle ΔABC. Is there a circle intersecting the three vertices? If so, then is it unique?

Lemma. Given a triangle ΔABC, the perpendicular bisectors of each side are concurrent.



Proof. Let X, Y and Z be the midpoints of AB, AC and BC respectively.

Let O be the intersection point of the perpendicular bisectors of AB and BC (since the angle is less than 180° then they intersect by Euclid's Axiom 5).

First we want to show that $|OA| = |OB|$. We know that $|AX| = |BX|$ (since X is the bisector of AB) and $\angle AXO = 90^\circ = \angle BXO$. Also, the triangles ΔAXO and ΔBXO have the side OX in common.

Therefore by SAS the triangle ΔAXO is congruent to the triangle ΔBXO and so $|OA| = |OB|$.

We can apply the same idea to show that the triangle ΔBZO is congruent to the triangle ΔCZO. Therefore $|OB| = |OC|$.

The circumcentre of a triangle (cont.)

Proof. (cont.)
Finally, we want to show that OY is the *perpendicular* bisector of AC.
(*A priori* we only know that Y is a bisector of AC.)

Note that $|OA| = |OC|$, $|AY| = |YC|$ (since Y is the bisector of AC) and the triangles ΔAYO and ΔCYO have the side OY in common.

Therefore $\Delta AYO \cong \Delta CYO$ by SSS, which implies that $\angle AYO = \angle CYO$. Since $\angle AYO + \angle CYO = 180^\circ$ then $\angle AYO = 90^\circ$ and so OY is the *perpendicular* bisector of AC.

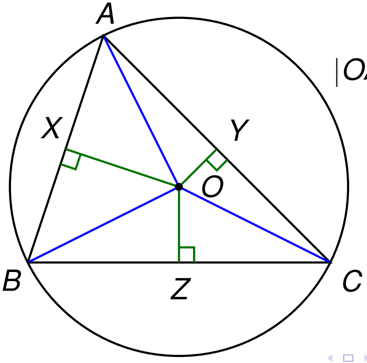
Therefore we have shown that the three perpendicular bisectors intersect at O and are therefore concurrent. ∎

Remark. Euclid does not actually prove this result in (Euclid Prop. IV.5), he only shows that the intersection of two perpendicular bisectors is the circumcentre.

The circumcentre of a triangle (cont.)

Proposition. (Euclid Prop. IV.5) Given a triangle ΔABC, there is a unique circle intersecting each of the vertices. This circle is called the circumcircle of the triangle ΔABC.

The circumcentre of a triangle ΔABC is the centre of the circumcircle. It is the common intersection point of the perpendicular bisectors of the sides AB, BC and AC.

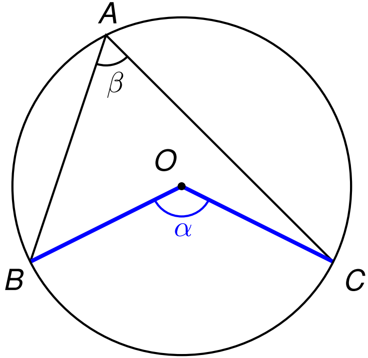


$$|OA| = |OB| = |OC|$$

Angles subtended by the same sector are equal

Choose three points A, B, C on a circle with centre O. How does the angle $\alpha = \angle BOC$ relate to $\beta = \angle BAC$?

The orientations of the angles are chosen so that α and β measure the angle of the sector BC that does not contain A.



Proposition. (Euclid Prop. III.20) $\beta = \frac{1}{2}\alpha$.

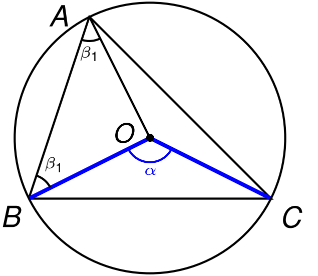
Corollary. (Euclid Prop. III.21) The angle $\angle BAC$ is independent of the point A on the circle.

Angles subtended by the same sector are equal

Proof. Let $\beta_1 = \angle OAB$. Then $\beta - \beta_1 = \angle OAC$.
Since $|AO| = |OB|$ then ΔAOB is isosceles and so $\angle ABO = \beta_1$.
Similarly, $\angle ACO = \beta - \beta_1$.
Since ΔBCO is isosceles then $\angle OBC = \angle OCB = \frac{1}{2}(180 - \alpha)$.
Adding all of these up gives us the internal angles of ΔABC

$$180 = \angle ABC + \angle BCA + \angle CAB = 2\beta_1 + 180 - \alpha + 2(\beta - \beta_1)$$

Solving the equation gives us $180 = 180 - \alpha + 2\beta$ and so we have the relation $2\beta = \alpha$.



Question. What happens if O is not contained in the triangle ΔABC?

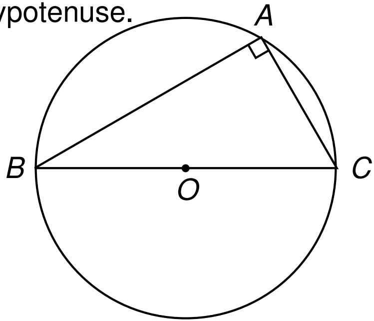
See Euclid Prop. III.20 for a more detailed proof (see also the exercise in Lecture 7).

A special case of the result

Now suppose that $\beta = 90°$. Then $\alpha = 180°$, and so the circumcentre O lies on the side BC. Since $|OB| = |OC|$ then O is the bisector of BC.

We have proven

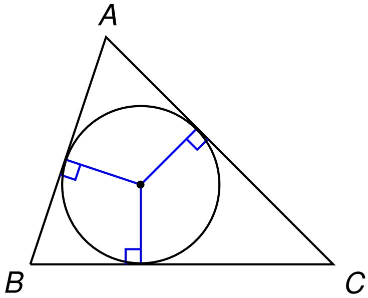
Theorem. In a right-angled triangle, the circumcentre is the midpoint of the hypotenuse.



(There is also a simpler proof given by doubling the triangle to form a rectangle. See Tutorial 3.)

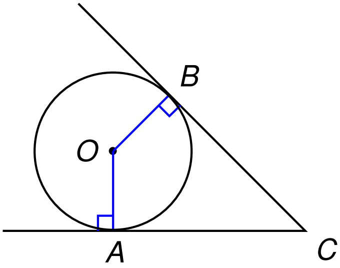
The incentre of a triangle

Suppose now that a circle lies inside a triangle ΔABC and touches each side in exactly one point.



Euclid proves in Prop. III.16 and Prop. III.18 that a circle touches a line in exactly one point if and only if that line is perpendicular to the diameter of the circle which intersects the line at the given point on the circle.

Exercise. Let O and A be two points in the plane and consider a circle with centre O and radius $|OA|$. Let B be another point on the circle. Draw the tangents at A and B and suppose that they intersect at C.
Prove that $\angle ACO = \angle BCO$, and therefore O lies on the angle bisector of $\angle ACB$.



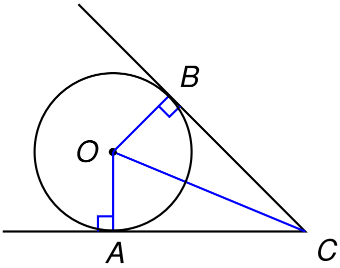
Solution

Solution

Solution.

Solution. Draw the line OC. We claim that the triangles ΔAOC and ΔBOC are congruent. To see this, note that $|OA| = |OB|$ (since they are radii of the same circle), the side OC is common to both triangles, and $\angle OAC = \angle OBC = 90^\circ$. Therefore $\Delta AOC \cong \Delta BOC$ by RASS (Right Angle Side Side) congruence.

Therefore $\angle ACO = \angle BCO$.



The incentre of a triangle (cont.)

Let ΔABC be a triangle in the plane and suppose that a circle inside the triangle touches each side in exactly one point.

Question. What properties does this circle have?

We see that the centre of the circle must lie on the intersection of the angle bisectors.

Therefore, a necessary condition for the existence of such a circle is that the angle bisectors must be concurrent.

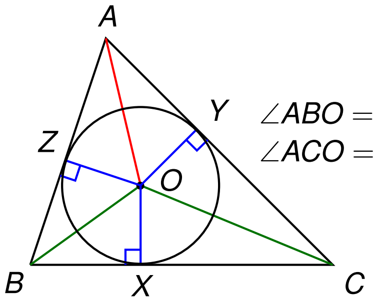
Conversely, given a triangle ΔABC, is it possible to construct a circle tangent to each of the sides?

Equivalently, are the angle bisectors concurrent?
This is called inscribing the circle inside the triangle.
Is such a circle unique?

Proposition. (Euclid Prop. IV.4) Given a triangle ΔABC, there is a unique circle tangent to each of the sides. This circle is called the incircle.

The centre of the incircle is called the incentre.

Strategy of Proof.



$$\angle ABO = \angle CBO$$
$$\angle ACO = \angle BCO$$

The angle bisectors of $\angle ABC$ and $\angle BCA$ intersect at a point O. We want to show that OA is the angle bisector of $\angle BAC$.

The incentre of a triangle (cont.)

Proof. Given a triangle ΔABC, let O be the point of intersection of the two angle bisectors $\angle ABC$ and $\angle BCA$.

Let X be the point on BC such that OX is perpendicular to BC. Similarly, define Y and Z to be the points on AC and AB respectively such that OY is perpendicular to AC and OZ is perpendicular to AB.

Since $\angle OBX = \angle OBZ$, $\angle OXB = 90^\circ = \angle OZB$ and the triangles ΔOXB and ΔOZB have the side OB in common, then they are congruent by AAS.

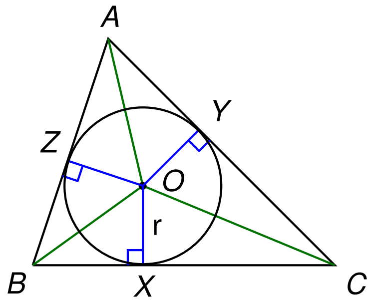
Similarly, $\Delta OXC \cong \Delta OYC$.

Therefore $|OY| = |OZ|$, $\angle OYA = 90^\circ = \angle OZA$ and the triangles ΔOZA and ΔOYA have the side OA in common. Therefore they are congruent by RASS, and hence $\angle OAY = \angle OAZ$.

Therefore OA is the angle bisector of $\angle BAC$ and hence the three angle bisectors are concurrent. ∎

How could we compute the radius of the incircle?

Let r be the radius of the incircle. Consider the following diagram.



Let $a = |BC|$, $b = |AC|$ and $c = |AB|$. Then

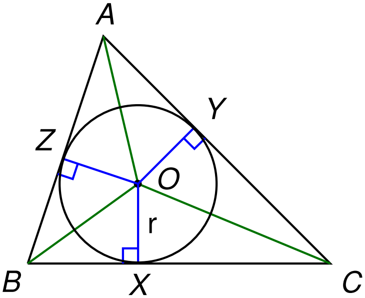
$$\text{Area}(\Delta BOC) = \frac{1}{2}ra, \text{Area}(\Delta AOC) = \frac{1}{2}rb$$

$$\text{and} \;\; \text{Area}(\Delta AOB) = \frac{1}{2}rc$$

The radius of the incircle (cont.)

We also know that

$$\text{Area}(\Delta ABC) = \text{Area}(\Delta BOC) + \text{Area}(\Delta AOC) + \text{Area}(\Delta AOB)$$



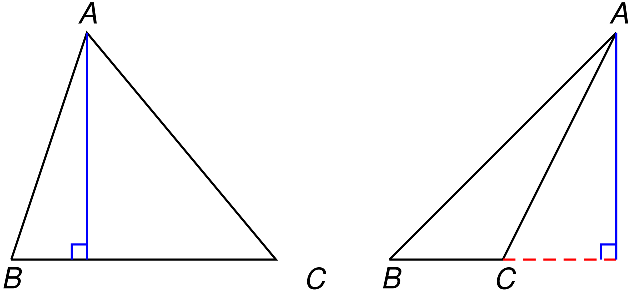
Therefore

$$r = \frac{\text{Area}(\Delta ABC)}{\frac{1}{2}(a+b+c)}$$

and so we can compute the radius of the incircle in terms of the area of the original triangle.

Definition. Let ΔABC be a triangle. The altitude of the vertex A is the unique line perpendicular to BC and passing through A.

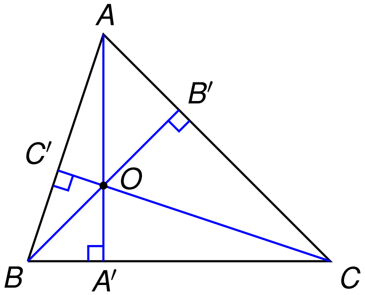
Remark. The altitude may lie outside the triangle.



The orthocentre of a triangle (cont.)

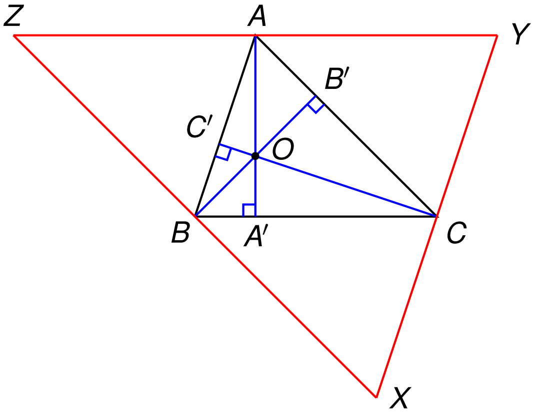
Theorem. The altitudes of ΔABC are concurrent.

Definition. The orthocentre of a triangle ΔABC is the point of intersection of the three altitudes.



The orthocentre of a triangle (cont.)

Strategy of Proof. First, draw a line through A which is parallel to BC, a line through B which is parallel to AC and a line through C which is parallel to AB.



We want to show that the altitudes of ΔABC intersect at the circumcentre of ΔXYZ.

Proof. First note that $AZBC$ and $AYBC$ are parallelograms. Therefore the opposite sides have equal length (see the exercise at the beginning of Lecture 3) and so $|AZ| = |BC| = |AY|$.

Since AA' is perpendicular to BC, then it is also perpendicular to YZ, since BC and YZ are parallel. Therefore AA' is the perpendicular bisector of YZ.

Similarly, we can show that BB' is the perpendicular bisector of XZ and CC' is the perpendicular bisector of XY.

We know that the perpendicular bisectors of the triangle ΔXYZ are concurrent and that they intersect in the circumcentre.

Therefore the altitudes of ΔABC are also concurrent. ∎

Corollary. The orthocentre of ΔABC is the circumcentre of ΔXYZ.

Next time

We will continue studying special circles associated to triangles.

- Properties of these circles
- The nine point circle
- The Euler line
- The orthic triangle
- The Simson line

We will also continue doing constructions in Euclidean geometry.