Lecture 16: Applications of inversive geometry

22 March, 2019

Overview

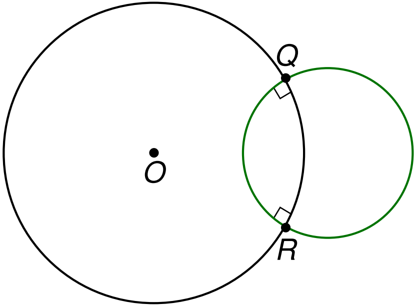
Last time.

- The inverse of a point through a circle
- Inverting lines and circles through a fixed circle
- Properties of inversion
- Inversion preserves angles
- What happens to the centre of a circle under inversion?

Today.

- Circles fixed under inversion
- Inversion preserves the cross-ratio
- The problem of Apollonius
- Steiner's porism
- Peaucellier's linkage
- Constructing the inverse using a ruler and compass

Exercise. Given a circle with centre O and points Q, R on the circle, show how to construct a new circle which intersects the original circle orthogonally at Q and R.

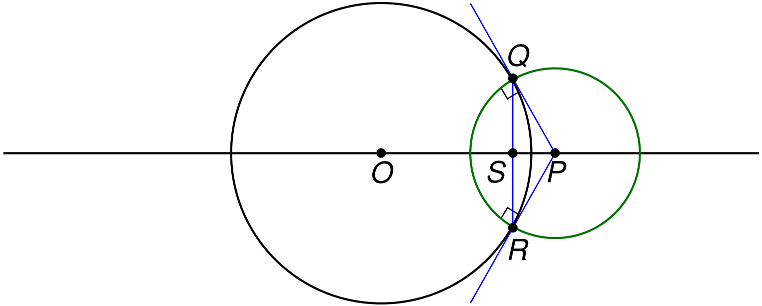


Solution.

Solution. Let S be the midpoint of QR. We know that the centre of the new circle must lie on the perpendicular bisector OS of QR.

Construct a tangent to the original circle at Q. This intersects OS at some point P.

Then P is the centre of the new circle. Draw a circle of radius $|PQ|$ with centre P.



The cross ratio is preserved by inversion

We have shown that angle is preserved by inversion, but length is not. Are there any other quantities that are preserved by inversion?

Definition. Given four points A, B, C, D in the plane, the *absolute value of the cross ratio* is

$$|[A, B; C, D]| = \frac{|AC| \cdot |BD|}{|BC| \cdot |AD|}$$

(We can remove the absolute value sign if we take directed lengths; see the 2014 lecture notes on IVLE for more details)

In inversive geometry we can take this as our definition of cross-ratio for any four distinct points in the extended plane.

If D is the point at infinity, then define $|[A, B; C, D]| = \frac{|AC|}{|BC|}$.

Proposition. The cross ratio is preserved by inversion through any circle in the plane.

We will prove this using our previous result on distances between points under inversion.

The cross ratio is preserved by inversion

Proof. Let A, B, C, D be four distinct points in the plane and let A', B', C', D' be the respective inverses. Then (by definition) we have

$$\big|[A', B'; C', D']\big| = \frac{|A'C'| \cdot |B'D'|}{|B'C'| \cdot |A'D'|}$$

Our previous result on distances shows that

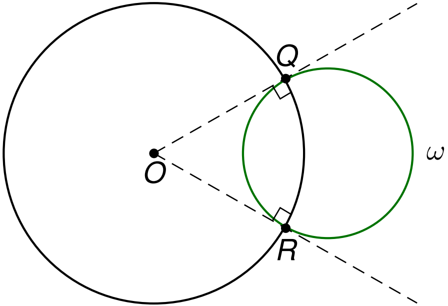
$$\frac{|A'C'| \cdot |B'D'|}{|B'C'| \cdot |A'D'|} = \frac{\frac{r^2|AC|}{|OA| \cdot |OC|} \cdot \frac{r^2|BD|}{|OB| \cdot |OD|}}{\frac{r^2|BC|}{|OB| \cdot |OC|} \cdot \frac{r^2|AD|}{|OA| \cdot |OD|}} = \frac{|AC| \cdot |BD|}{|BC| \cdot |AD|}$$

and so we have $|[A', B'; C', D']| = |[A, B; C, D]|$.

We see that length is not a meaningful quantity in inversive geometry, however the cross ratio is.

Circles fixed by an inversion

Proposition. Consider a circle of radius r and centre O. Let ω be another circle in the plane. Then inversion through the first circle maps $\omega \to \omega$ if and only if ω intersects the first circle orthogonally.



Remark. The points on the circle ω are not fixed by the inversion (except for Q and R). The proposition says that their image is on the circle ω.

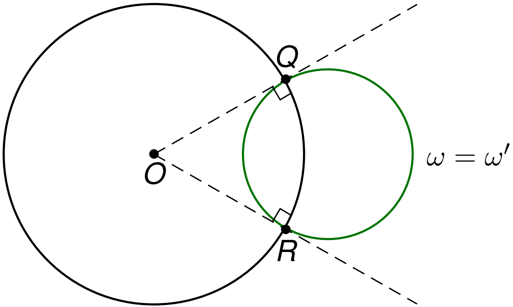
Click here to see an interactive picture of this.

Proof. Last lecture we proved that the circle ω maps to another circle ω'.

We know that Q and R are fixed by the inversion, since they are on the original circle. Therefore ω' passes through Q and R.

Since angles are preserved by inversion then ω' must be orthogonal to the original circle at Q and R.

These properties uniquely define the circle, and so $\omega' = \omega$. ∎



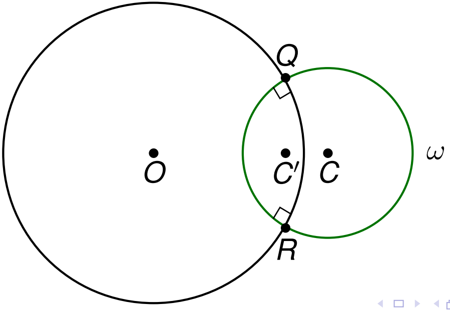
Circles fixed by an inversion

Remark. The centre C of the circle ω is not fixed by the inversion (unless it lies on the original circle).

In the diagram below we see that the centre C is mapped from outside the circle to a point C' inside the circle.

The reverse of the tangent construction from the exercises at the end of today's lecture shows that C' is the midpoint of QR.

Using this idea, we can map the centre C to any point inside ω (see **Tutorial 9** for more details).



Application to the problem of Apollonius

Recall the Problem of Apollonius. Given three circles ω_1, ω_2 and ω_3, we want to find a fourth circle which is tangent to all three.

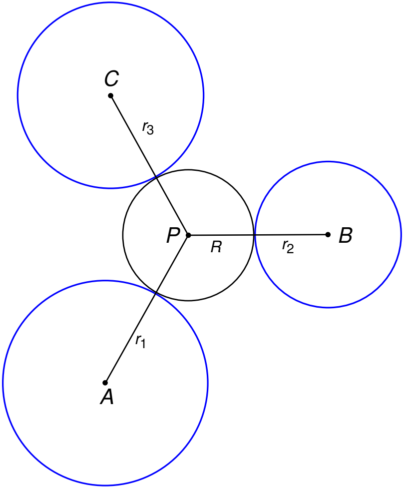
This has many applications, for example hyperbolic navigation and Apollonian circle packing.

In the chapter on conic sections we showed how to solve this problem using intersecting hyperbolas.

This requires being able to accurately construct a hyperbola.

Here we will show how to use inversive geometry to solve the problem.

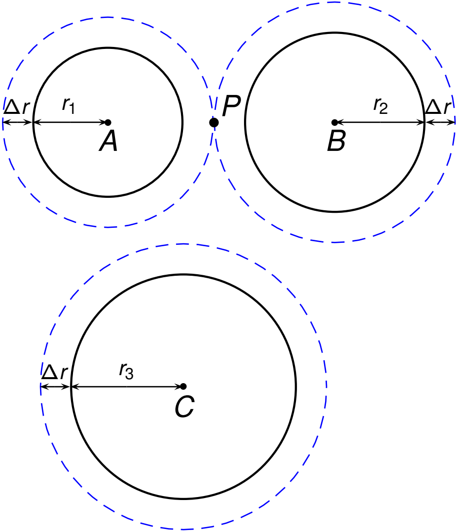
Geometers use symmetries to simplify a problem.



A first simplification is to increase the radius of all of the circles by the same amount Δr. This has the effect of changing the radius of the solution circle by Δr.

We can choose Δr so that two of the circles are now tangent.

Let P be the point of tangency.



Application to the problem of Apollonius

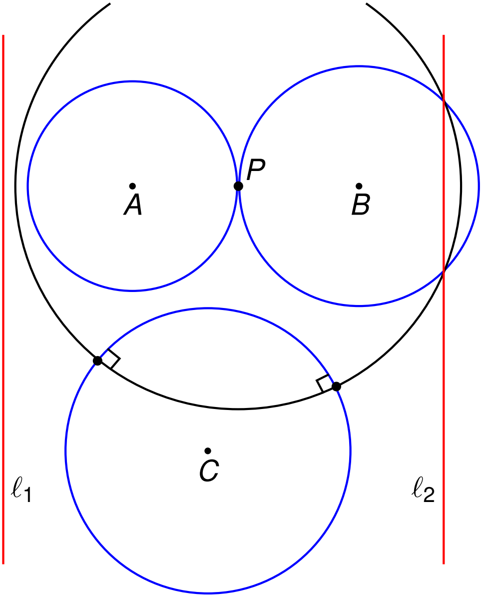
Now apply an inversion through a circle with centre P.

Since two of the circles pass through P then they are transformed into straight lines parallel to the tangent line at P (the two red lines in the picture).

These two circles have the same tangent line, and so the image lines ℓ_1 and ℓ_2 are parallel.

To simplify the inversion, we choose the circle centred at P so that it is orthogonal to the third circle.

Therefore the third circle (centred at C) is preserved by the inversion.

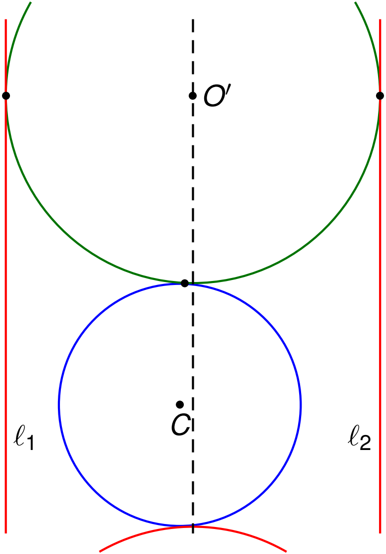


Application to the problem of Apollonius

Now the problem reduces to finding a circle which is tangent to the two parallel lines and the remaining circle.

We know the radius of this circle (it is half the distance between the parallel lines) and we know that the centre lies halfway between the two parallel lines.

Therefore we can find the centre O' of the circle.



Application to the problem of Apollonius

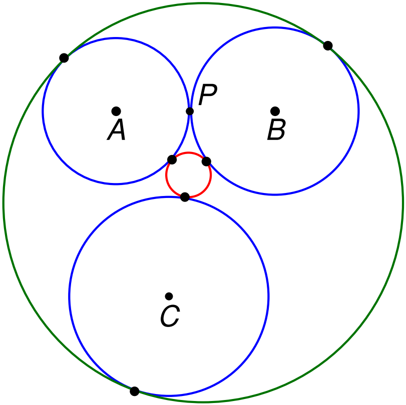
To finish the problem, we just reverse the inversion that we did before.

The circle centred at C remains the same and the two lines ℓ_1 and ℓ_2 invert back to the original circles centred at A and B.

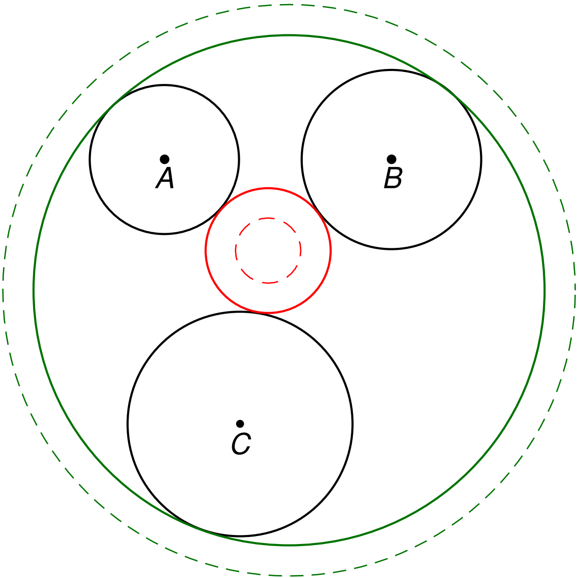
The circle centred at O' inverts to a new circle which is tangent to all three circles.

We can then decrease the radius by the same amount Δr as before to obtain the solution to the problem.

All of the constructions in our solution can be done with a ruler and compass.



Application to the problem of Apollonius



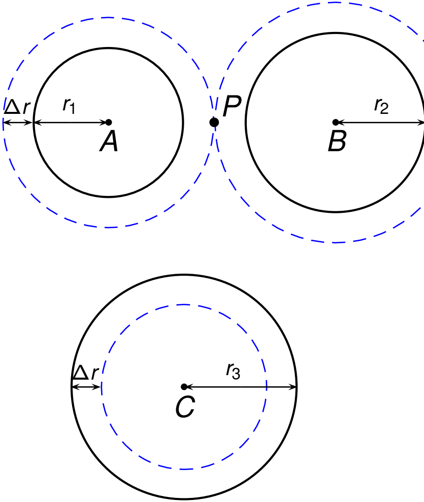
How to construct the other tangent circles?

Note that there are a number of other circles tangent to the two parallel lines and the circle centred at C.

These will also invert back to a circle tangent to all three circles of radius $r_i + \Delta r$, but we cannot decrease or increase the radius to obtain a circle tangent to all three original circles.

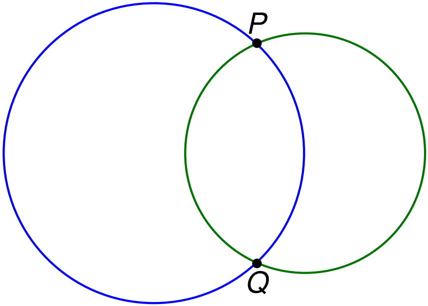
Therefore to find the other circles tangent to all three original circles, we have to increase the radius of some circles and decrease the radius of others.

For example, we could change the radii by $r_1 + \Delta r$, $r_2 + \Delta r$, $r_3 - \Delta r$.



Exercise

Exercise. Consider two circles that intersect at points P and Q. What happens to the circles after we apply an inversion through a circle with centre P?

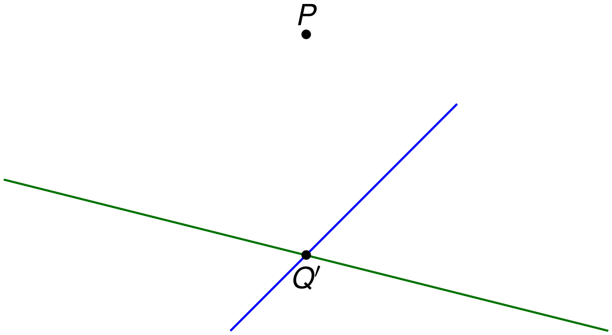


Solution.

Solution

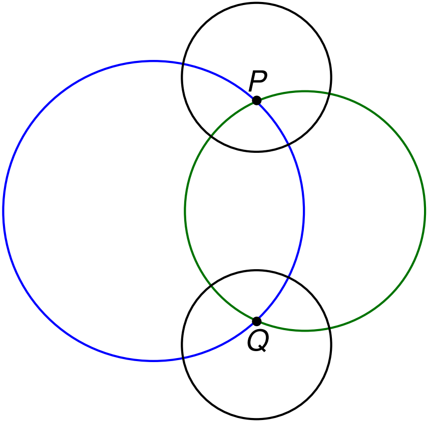
Solution. Since the circles pass through P, then any inversion through a circle centred at P will map these circles to straight lines parallel to the tangent at P.

We also know that both circles intersect at Q. Therefore their images under the inversion must intersect at the image Q' of Q.



Exercise. Suppose now that we have two more circles that intersect the first pair orthogonally. What happens to them under the inversion?

Hint. Angles are preserved by inversion.



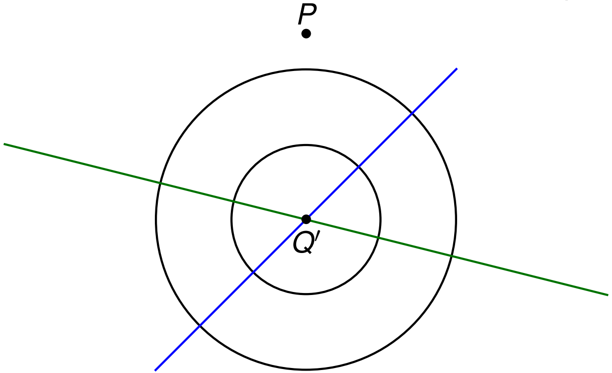
Solution

Solution.

Solution

Solution. We know that the original pair of circles become straight lines after the inversion. We also know that inversion preserves angles, and so the image of each of the two new circles must intersect the lines orthogonally.

Therefore the images of the two circles must be concentric circles centred at the intersection of the two lines (the point Q').



Inverting a pair of circles to a pair of concentric circles

Proposition. Let ω_1 and ω_2 be any pair of non-intersecting circles in the plane. Then there exists an inversion which maps ω_1 and ω_2 to a pair of concentric circles.

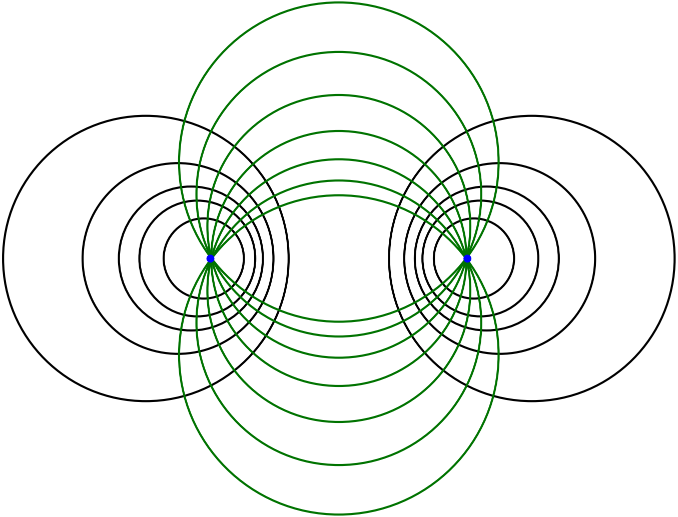
Proof. Any pair of non-intersecting circles defines a hyperbolic pencil of circles. Let P and Q be the degenerate points of this pencil.

The orthogonal elliptic pencil is defined as all of the circles passing through P and Q.

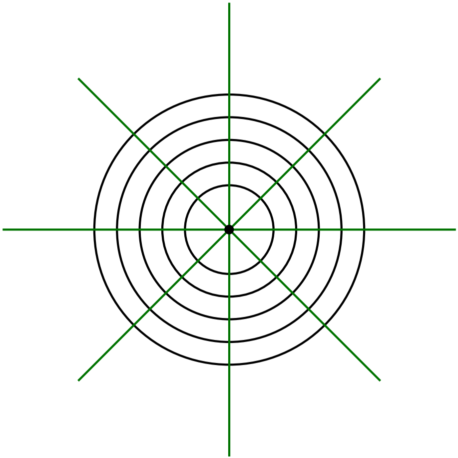
The same idea as the previous two exercises shows that inversion through any circle centred at P sends the circles in the elliptic pencil to straight lines passing through Q, and the circles in the hyperbolic pencil are mapped to concentric circles centred at Q.

In particular, the circles ω_1 and ω_2 are mapped to concentric circles centred at Q. ∎

A hyperbolic pencil and the orthogonal elliptic pencil



The image is a collection of concentric circles



The original hyperbolic pencil is mapped to a new hyperbolic pencil with one degenerate point at infinity.

Here is another example where we use inversion to prove a result that would be difficult to prove directly.

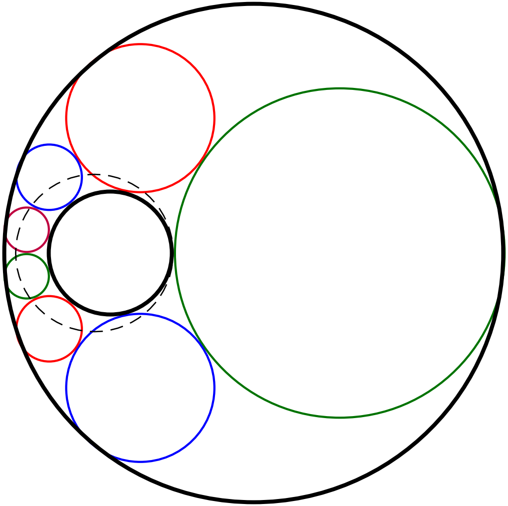
Definition. (Steiner chain) Consider two non-intersecting circles C_1 and C_2, one inside the other. Let ω_1 be a circle tangent to C_1 and C_2, such that ω_1 is inside C_1 and outside C_2. Let ω_2 be a circle tangent to C_1, C_2 and ω_1 such that ω_2 is inside C_1 and outside C_2. Continue such that ω_k is a circle tangent to C_1, C_2 and ω_{k-1} such that ω_k is inside C_1 and outside C_2.

The chain of circles is called a Steiner chain if there exists n such that ω_n is tangent to ω_1.

Theorem. (Steiner's porism) If the circles $\omega_1, \ldots, \omega_n$ form a Steiner chain then the same is true for any initial position of ω_1.

Click here for an animation.

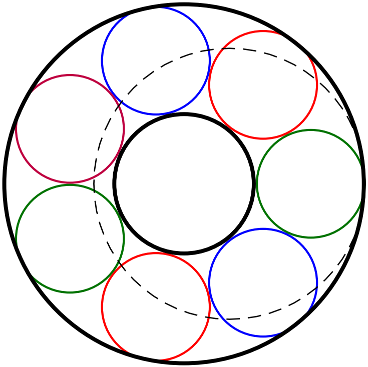
Steiner 's porism (idea of proof)



In the Steiner chain shown here, the inner and outer circle are coloured black.

Given such a chain of circles, we apply an inversion through the dashed circle to make the inner and outer circle concentric.

Steiner 's porism (idea of proof)



Here is the resulting figure. The circle of inversion is the dashed circle.

The inner and outer circles are now concentric, and so the circles in the chain must have the same radius.

Therefore, if we know the radius of the inner and outer circle and the number of circles in the chain, then determining whether the circles form a Steiner chain is now a matter of trigonometry.

Moreover, since the inner and outer circles are concentric then we see that *it does not matter where we place the first circle in the chain*. Either: (a) the chain always closes up (and forms a Steiner chain) or (b) the chain never closes up.

The Peaucellier linkage

One final application of inversive geometry is the following engineering problem.

It is often important to convert linear motion (such as that from a piston in an engine) to circular motion (to drive a wheel).

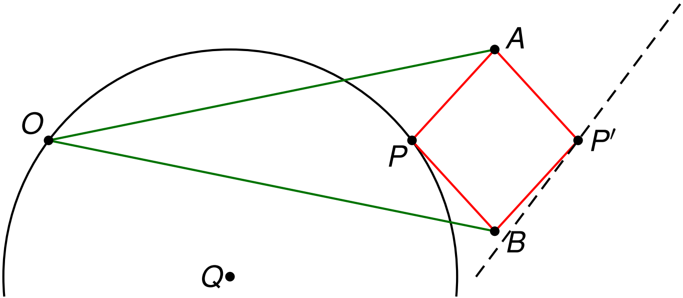
In car engines this is done by using movable hinges and lubricant (since the piston does not move in a perfectly straight line, so it will rub against the sides of the engine).

On the next slide is a picture of the Peaucellier linkage, which uses inversion to convert the rotational motion of the point P into the linear motion of the point P'.

The linkage applies an inversion to the point P, converting the circle through P into a straight line. Can you figure out how it works?

Exercise. (for interest) Try to construct a physical version of the Peaucellier linkage.

Peaucellier linkage



The red lines AP, AP', PB and $P'B'$ represent rigid bars of equal length. The green lines OA and OB also represent rigid bars of equal length.

Exercise. Prove that P' is the inverse of P with respect to a circle centred at O.

We allow the point P to move along the circle centred at Q. Since this circle passes through O then the point P' moves along a straight line.

Peaucellier linkage (solution)

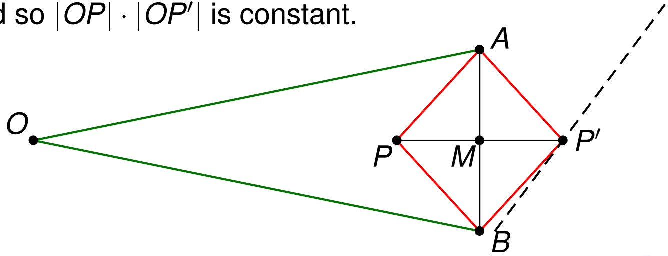
We want to show that $|OP| \cdot |OP'|$ is constant, and then P' will be the inverse of P with respect to a circle centred at O of radius $\sqrt{|OP| \cdot |OP'|}$. Let M be the midpoint of PP'. Then

$$|OP| \cdot |OP'| = (|OM| - |PM|)(|OM| + |P'M|) = |OM|^2 - |PM|^2$$

Pythagoras then shows that $|OA|^2 = |OM|^2 + |AM|^2$ and so we have

$$|OP| \cdot |OP'| = |OA|^2 - |AM|^2 - |PM|^2 = |OA|^2 - |AP|^2$$

Since $|OA|$ and $|AP|$ are rigid bars then their length is constant and so $|OP| \cdot |OP'|$ is constant.



Next time

Monday's class will be a review of all of the constructions from the second half of the semester.

On Friday we will begin hyperbolic and spherical geometry.

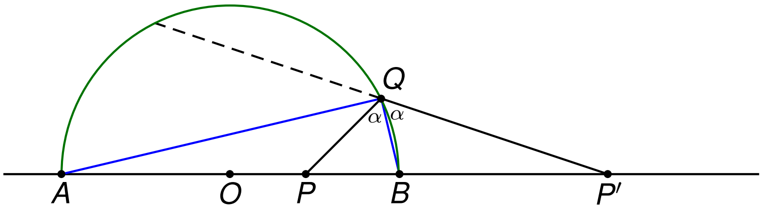
- Basic definitions
- Models for hyperbolic and spherical geometry
- Euclid Book I revisited
- The angle sum of a triangle in hyperbolic and spherical geometry
- Spherical trigonometry

Construction of the inverse using harmonic conjugates

Consider a circle with centre O and let P be a point inside the circle. Let AB be a diameter of the circle such that A, O, P, B are collinear. W.l.o.g. assume that P is between O and B.

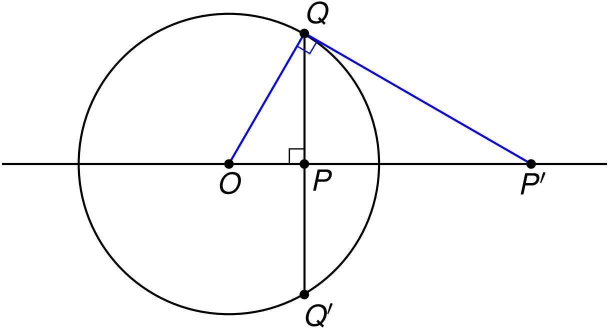
Let Q be any point on the circle and construct P' on $\overrightarrow{OP}$ such that QB is the bisector of $\angle PQP'$.

The earlier exercise shows that P' is the inverse of P with respect to the circle.



Consider a circle with centre O and let P be a point inside the circle. Draw a perpendicular to OP through P and let Q, Q' be the points of intersection with the circle. Draw the tangent to the circle at Q and let P' be the point of intersection with OP.

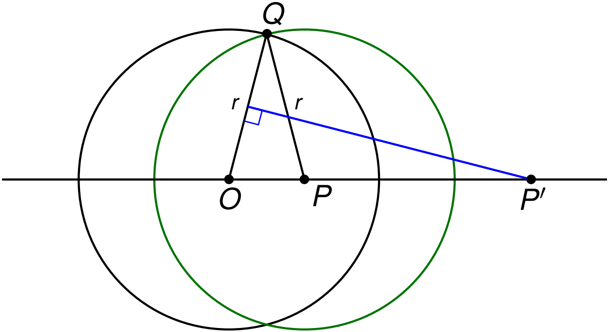
Prove that P' is the inverse of P.



Construction of the inverse using isosceles triangles

Consider a circle with centre O and radius r and let P be a point inside the circle. Transfer the radius r to P and draw a circle of radius r and centre P. Let Q be one of the intersection points of this circle with the original circle.

Let P' be the intersection of the perpendicular bisector of OQ with OP. Prove that P' is the inverse of P.



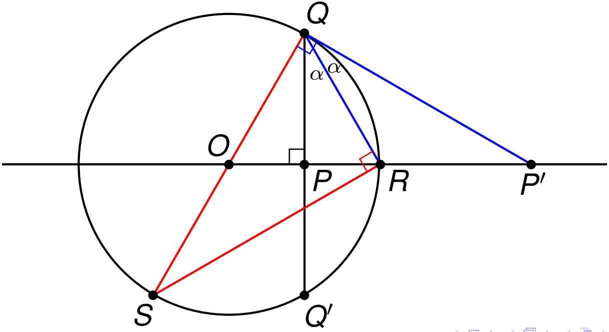
Construction of the inverse using tangents (solution)

Construction of the inverse using tangents (solution)

Let R be the intersection of $\overrightarrow{OP}$ with the circle. Draw the diameter QS.

We have $\angle QSR = 90 - \angle SQR = \angle RQP'$. Since $\angle QSR$ is subtended by the arc QR and $\angle PQR$ is subtended by the arc $Q'R$ of equal length then $\angle PQR = \angle QSR = \angle RQP'$.

Therefore QR is the angle bisector of $\angle PQP'$ and the exercise from Tuesday's lecture shows that P' is the inverse of P.



Construction of the inverse using isosceles triangles (solution)

Construction of the inverse using isosceles triangles (solution)

Since P' is the intersection of the perpendicular bisector of OQ with OP, then $\Delta OP'Q$ is isosceles. Since $|OQ| = |QP| = r$ then ΔOQP is also isosceles.

We also have $\angle QOP = \angle P'OQ$ and so $\Delta OP'Q \sim \Delta OQP$. Therefore

$$\frac{|OP|}{|OQ|} = \frac{|OQ|}{|OP'|} \Leftrightarrow |OP| \cdot |OP'| = |OQ|^2 = r^2$$

and so P' is the inverse of P.