Heat Diffusion – Exercise Set

Project for the course on Model Reduction

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Specifications

As explained in the project description, we consider the model described by the partial differential equation

$$\rho(x,y)c(x,y)\frac{\partial T}{\partial t}(x,y,t) = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} K(x,y) \begin{bmatrix} \frac{\partial T(x,y)}{\partial x} \\ \frac{\partial T(x,y)}{\partial y} \end{bmatrix} + u(x,y,t) \tag{1}$$

Throughout, it is assumed that the model is isotropic, i.e., $K(x,y) = \kappa(x,y)I$. Furthermore, the model is assumed to be homogeneous in the yellow and homogeneous in the blue area separately, so that the values of (ρ, c, κ) are uniform in these two areas. The physical parameters of a specific geometry and material densities and capacities (of silicon) are given in Table 1. The dimensions ℓ_x and ℓ_y of the blue area will vary.

parameter	symbol	quantity	unit
length	L_x	0.2	[m]
width	L_y	0.3	[m]
material densities (yellow, blue area)	ρ	2328, 2300	[kg/m^3]
heat capacities (yellow, blue area)	c	700, 680	[J/(kg K)]
thermal conductivities (yellow, blue area)	κ	148, 148	[W/(m K)]
location heat flux 1	(X_1, Y_1)	$(\frac{L_x}{4}, \frac{L_y}{2})$	([m],[m])
location heat flux 2	(X_2, Y_2)	$(\frac{3L_x}{4}, \frac{L_y}{2})$	([m],[m])
actuator width	W	0.05	[m]
ambient temperature	T_{amb}	309	[K]

Table 1: Physical specifications.

Questions

Properties of the model

Consider the model (1) and assume that it is isotropic, i.e., $K(x,y) = \kappa(x,y)I$.

1. Prove whether this system is linear or nonlinear and prove whether the system is time-variant or time-invariant. In doing so, distinguish between the cases where the model is homogeneous or non-homogeneous.

The homogeneous model

First assume that the model is homogeneous. For this, let $\ell_x = \ell_y = 0$ and let $\rho(x,y) = \rho$, $c(x,y) = c$ and $\kappa(x,y) = \kappa$ be positive constants.

2. Show that under these conditions, (1) admits a solution of the form $T(x,y,t) = a(t)\varphi^{(x)}(x)\varphi^{(y)}(y)$ whenever $u(x,y,t) = 0$. Here, a, $\varphi^{(x)}$ and $\varphi^{(y)}$ are scalar-valued functions on $\mathbb{R}$, $[0, L_x]$ and $[0, L_y]$, respectively, that satisfy the separated differential equations

$$\ddot{\varphi}^{(x)} - \lambda_x \varphi^{(x)} = 0, \quad \ddot{\varphi}^{(y)} - \lambda_y \varphi^{(y)} = 0, \quad \dot{a} - \lambda a = 0$$

for suitable constants λ_x, λ_y and λ.

To numerically simulate the temperature evolution of (1), a spectral decomposition of the temperature evolution is proposed according to

$$T(x,y,t)=\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}a_{k,\ell}(t)\varphi_k^{(x)}(x)\varphi_\ell^{(y)}(y) \tag{2}$$

where we define, for non-negative integers k and ℓ,

$$\varphi_k^{(x)}(x)=\begin{cases}\frac{1}{\sqrt{L_x}} & \text{if } k=0\\ \sqrt{\frac{2}{L_x}}\cos\left(\frac{k\pi x}{L_x}\right) & \text{if } k>0\end{cases};\qquad \varphi_\ell^{(y)}(y)=\begin{cases}\frac{1}{\sqrt{L_y}} & \text{if } \ell=0\\ \sqrt{\frac{2}{L_y}}\cos\left(\frac{\ell\pi y}{L_y}\right) & \text{if } \ell>0\end{cases}$$

and where $a_{k,\ell}(t)$ is a double-indexed time-varying coefficient. Define

$$\varphi_{k,\ell}(x,y):=\varphi_k^{(x)}(x)\varphi_\ell^{(y)}(y) \tag{3}$$

where $x\in[0,L_x]$, $y\in[0,L_y]$. Then $\{\varphi_{k,l}\mid k,\ell=0,1,2,\ldots\}$ denotes an infinite collection of functions that are square integrable on $[0,L_x]\times[0,L_y]$. Let $\mathcal{L}_2$ denote the inner product space of all such functions with the standard inner product

$$\langle\varphi_i,\varphi_j\rangle:=\int_0^{L_x}\int_0^{L_y}\varphi_i(x,y)\varphi_j(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

3. Show that for any $K>0$ and $L>0$ the set $\{\varphi_{k,\ell}\mid 0\le k\le K, 0\le \ell\le L\}$ is an orthonormal set of functions in $\mathcal{L}_2$.

4. Use the Galerkin projection to derive, for arbitrary k and ℓ, an explicit ordinary differential equation for the coefficient function $a_{k,\ell}(t)$ in the spectral expansion (2). Determine the equilibrium solution of this model if $u_1=0$ and $u_2=0$.

5. Consider the given physical specifications of the model in Table 1. Define an arbitrary, but physically realistic initial temperature profile $T(x,y,0)=T_0(x,y)$ and simulate for various values of $K>0$ and $L>0$ the truncated expansion

$$T_{K,L}(x,y,t)=\sum_{k=0}^{K}\sum_{\ell=0}^{L}a_{k,\ell}(t)\varphi_{k,\ell}(x,y) \tag{4}$$

   of temperatures over a time period of about 10 minutes, first with inputs $u_1(t)=u_2(t)=0$ (no external thermal load).

6. Once you have this running, experiment with

   - different approximation orders K and L in (2)
   - different initial temperature profiles
   - realistic and time-varying inputs for the heat fluxes $u_1(t)$ and $u_2(t)$.

   Compare the results that you get with these variations and report your conclusions about the quality of low order truncations (4).

7. Generate the output $T_{K,L}(x,y,t)$ from one representative experiment that you performed in the previous item *with large values of* K *and* L and use this data to compute a POD basis $\{\varphi_i\}_{i=1,\ldots,R}$ in $\mathcal{L}_2$ of order R.

8. Implement a reduced order POD model that takes the heat fluxes (u_1,u_2) as inputs and outputs the coefficients $a_i(t)$ in the truncated expansion

$$T_R(x,y,t)=\sum_{i=1}^{R}a_i(t)\varphi_i(x,y) \tag{5}$$

   where φ_i is the POD basis from the previous item.

9. Validate the quality of the reduced order POD model by looking at different orders of R in (5), different initial temperature profiles T_0 and different heat fluxes u_1 and u_2. What are your conclusions when comparing the quality of the POD basis with the basis (3)?

The non-homogeneous model

Next, we consider the non-homogeneous model where the blue area has dimensions $\ell_x = 0.04$ [m] and $\ell_y = 0.08$ [m]. The specification of the thermal density, heat capacity and conductivity in the blue area are given in Table 1. When viewed over the spatial geometry, the thermal properties in the composite material are discontinuous. This substantially complicates the modeling. We will smoothen this discontinuity by approximating the product $\rho(x, y)c(x, y)$ in (1) by a function of the form

$$e(x, y) := e_{0,0} + \sum_{k=1}^{K} \sum_{\ell=1}^{L} e_{k,\ell}\varphi_{k,\ell}(x, y) \tag{6}$$

where $\varphi_{k,\ell}$ are the basis functions defined in (3) and (K, L) defines the approximation order. Since $K(x, y) = \kappa(x, y)I$ is constant, approximations of $K(x, y)$ in (1) are not necessary, so that the smoothened non-homogeneous model is defined by

$$e(x, y)\frac{\partial T}{\partial t}(x, y, t) = \kappa\left(\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2}\right) + u(x, y, t) \tag{7}$$

10. Derive an expression for the coefficient functions $a_{k,\ell}(t)$ in the expansion (4) so that (4) is a solution of the Galerkin projected non-homogeneous model (7). Make a plot of the approximation (6) of $\rho(x, y)c(x, y)$ and show the temperature profile (4) of the approximate model for a realistic initial condition $T(x, y, 0) = T_0(x, y)$ and realistic inputs $u_1(t)$ and $u_2(t)$ (cf. item 6).