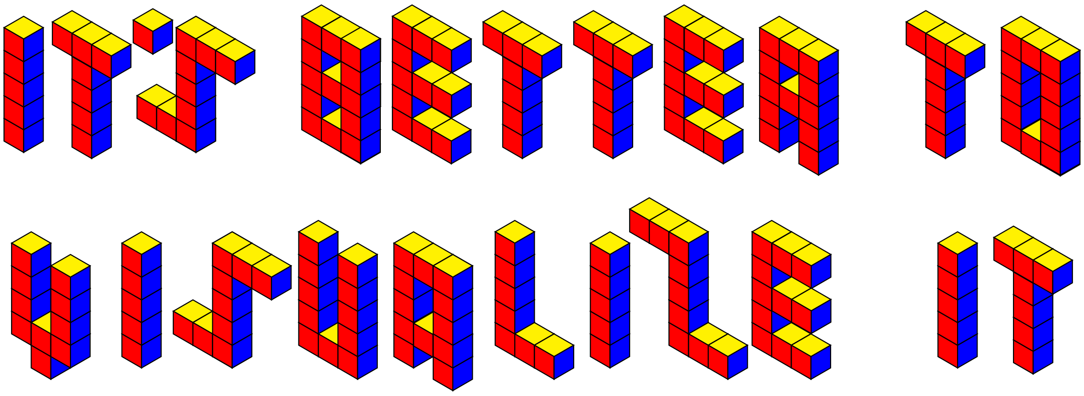

USER MANUAL

MICHELE GRAFFEO

It's Better To Visualize It (IBTVI) is a software designed to allow the visualisation of plane partitions. In addition to the possibility of viewing the plane partition associated with a monomial ideal and the related minimal system of generators, it is possible to perform some basic operations between monomial ideals. Currently the operations of sum product and intersection between monomial ideals have been implemented.

IBTVI gives the possibility to edit a monomial ideal in three variables, directly from its associated plane partition, via simple clicks.

A save/load function is also implemented. This allows to save up to 30 plane partitions and reload them at any time.

CONTENTS

0. INTRODUCTION & BASIC THEORY

IBTVI works with plane partitions. In this section we give their definition and we state their main properties.

Definition 0.1. A $(n-1)$-*partition*, or simply a partition, is a finite subset $\Sigma \subset \mathbb{N}^n$ with the property that, for all $v \in \mathbb{N}^n \setminus \Sigma$, we have

$$(v+\mathbb{N}^n)\cap\Sigma = \left\{\, v+w \;\middle|\; w\in\mathbb{N}^n \,\right\}\cap\Sigma = \emptyset.$$

When $n=3$, we will call Σ a *plane partition* (PP).

Definition 0.2. Let Σ be a $(n-1)$-partition and let $S \subset \mathbb{N}^n$ be a subset. We say that S is a *generating set* for Σ (or that the elements of S generate Σ) if

$$\mathbb{N}^n \smallsetminus \Sigma = S + \mathbb{N}^n = \left\{ s + v \,\middle|\, s \in S,\ v \in \mathbb{N}^n \right\}.$$

The generating set S is said to be *minimal* if, $\forall S' \subset S$, S' is not a generating set for Σ.

Remark 0.3. Let $e_i \in \mathbb{N}^n$, for $i = 1, \ldots, n$, be the vector having 1 at the i-th position and zeroes elsewhere. Then, any generating set of a $(n-1)$-partition has the property that, $\forall i = 1, \ldots, n$, there exists an $n_i \in \mathbb{N} \smallsetminus \{0\}$ such that $n_i e_i \in S$.

Proposition 0.4 ([1, Chapter 1 §4])**.** *Let $\Sigma \subset \mathbb{N}^n$ be a $(n-1)$-partition. Then, Σ has a unique minimal generating set S_Σ. Moreover, S_Σ is finite.*

Let $S \subset \mathbb{N}^n$ be a subset with the property that, $\forall i = 1, \ldots, n$, there exists an $n_i \in \mathbb{N} \smallsetminus \{0\}$ such that $n_i e_i \in S$. Then, $\Sigma_S = \mathbb{N}^n \smallsetminus (S + \mathbb{N}^n)$ is a $(n-1)$-partition. We will call Σ_S the partition generated by S and we will denote it by $\Sigma_S = \langle S \rangle$.

Example 0.5. One-dimesnional partitions are also known in the literature as *Ferrers diagrams* (see [2]).

Let $\Sigma \subset \mathbb{N}^2$ be the 1-partition defined by

$$\Sigma = \{ (0,0), (1,0), (0,1), (0,2) \}.$$

Then, a set of generators of Σ is

$$\{ (0,3), (2,0), (1,1), (2,1) \},$$

while the minimal generating set of Σ is

$$\{ (0,3), (2,0), (1,1) \}.$$

In order to better understand $(n-1)$-partitions it is useful, expecially for $n = 2,3$, to interpret them as ways to stack a number of n-dimensional boxes in the corner of a n-dimensional room. For instances, we can represent $\mathbb{N}^2$ as in Figure 1.

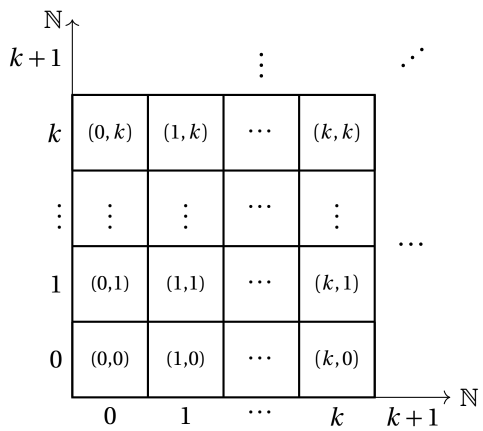

FIGURE 1.

Then, in Example 0.5, one can depict Σ and its minimal set of generators as in Figure 2.

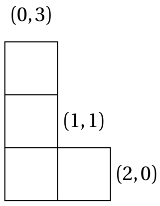

FIGURE 2.

Example 0.6. In dimension 3, again, one can represent $\mathbb{N}^3$ as an infinite set of three-dimensional boxes (cubes) and represent PP as a finite union of them. For instance, the PP

$$\Sigma = \langle \{ (3,0,0); (0,2,0); (0,0,4); (2,1,0); (0,1,1); (1,0,3) \} \rangle$$

corresponds to the configuration of three-dimensional boxes in Figure 3.

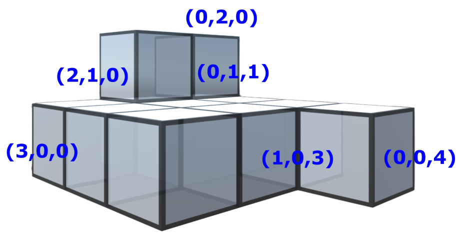

FIGURE 3.

Let us denote by H_{yz}, H_{xz} and H_{xy} the (natural) planes respectively generated by $\{e_2, e_3\}$, $\{e_1, e_3\}$ and $\{e_1, e_2\}$. Let us also denote by π_{yz}, π_{xz} and π_{xy} the corresponding canonical projections. Then, one can also represent Σ via its projections on by H_{yz}, H_{xz} and H_{xy}.

For instance, if Σ is the PP in Figure 3, then $\pi_{yz}(\Sigma)$, $\pi_{xz}(\Sigma)$ and $\pi_{xy}(\Sigma)$ are depicted in Figure 4.

1	
3	
3	
3	2

$\pi_{yz}(\Sigma)$

1		
1	1	1
1	1	1
2	2	1

$\pi_{xz}(\Sigma)$

1	1	
4	3	3

$\pi_{xy}(\Sigma)$

FIGURE 4.

In order to not lose information, we label each square with the number of cubes mapped on it by the corresponding projection.

Definition 0.7. We will call this combinatorical objects the *projected* $\pi_{\clubsuit,\spadesuit}$*-PP*'s (or simply $\pi_{\clubsuit,\spadesuit}$*-PP*'s) where $\clubsuit \neq \spadesuit$ and $\clubsuit, \spadesuit \in \{ x, y, z \}$.

Remark 0.8. Fix $\clubsuit \neq \spadesuit$ and $\clubsuit, \spadesuit \in \{ x, y, z \}$. Given a projected $\pi_{\clubsuit,\spadesuit}$-PP, we can reconstruct the PP uniquely.

0.1. **Partitions and monomial ideals.** Partitions appear in many fields of mathematics. For instance combinatorics, algebra, algebraic gemoetry and enumerative geometry. One of their main feature is that they are in bijection with monomial ideals I such that $\sqrt{I} = (x, y, z) \subset \mathbb{C}[x, y, z]$.

Proposition 0.9 ([1, Chapter 1 §4])**.** *The following correspondence is a bijection.*

$$\begin{array}{ccc} \{\Sigma \subset \mathbb{N}^n \mid \Sigma \textit{ is a } (n-1)\textit{-partitions}\} & \longleftrightarrow & \left\{ I \subset \mathbb{C}[x_1, \ldots, x_n] \,\middle|\, I \textit{ is a monomial ideal and } \sqrt{I} = (x_1, \ldots, x_n) \right\} \\ \langle (v_{1,i}, \ldots, v_{n,i}) \mid i = 1, \ldots, k \rangle & \longleftrightarrow & (x_1^{v_{1,i}} \cdots x_n^{v_{n,i}} \mid i = 1, \ldots, k). \end{array}$$

A useful invariant of a zero-dimensional (monomial) ideal is its Hilbert–Samuel function.

Definition 0.10. Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a zero-dimensional homogeneous ideal. Then, $\mathbb{C}[x_1, \ldots, x_n]/I$ is a graded $\mathbb{C}[x_1, \ldots, x_n]$-module. The *Hilbert–Samuel function* of I (or of $\mathbb{C}[x_1, \ldots, x_n]/I$) is

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\mathrm{HS}_I} & \mathbb{N} \\ d & \longmapsto & \dim_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_n]/I)_d \,. \end{array}$$

0.2. **Organisation of the manual.** This software displays plane partitions trying to be as user friendly as possible. It will be possible to change angle, point of view and position of the plane partition (see Sections 1.2 and 1.4).

IBTVI allows you to work both with the language of n-uple $v \in \mathbb{N}^n$ (see Section 1) and with the language of monomial ideals (see Section 2) . In the latter setting it is also possible to import and export ideals from macaulay2 [3] (see Section 3.4).

Moreover, it will always be possible to have the minimal system of generators, both when working with the language of PPs and when working with the language of monomial ideals.

In Section 0, we reported the definition of PP and its basic properties. In Section 1 we explain how to work with plane partitions without involving monoimal ideals. In Section 2, we explain how to work in the monomial setting and how to visualize the three projected PPs associated with a monomial ideal (see Section 2.7). Furthermore, in Section 2 it is explained how to perform the operations of sum product and intersection of monomial ideals (see Section 2.5). The editor of PP's is also presented in Section 2 (see Section 2.6). Finally in Section 3 we explain how to save and export PP's.

This manual is accessible from the android version clicking the button 

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1. Plane partitions

To work with PPs you must be in *PP mode*. You are in *PP mode* if the icon 

is visible. Click the button 

to enter in *PP mode*.

1.1. **Draw a plane partition.** Follow this instructions to draw the PP generated by

$$\{(a_1, b_1, c_1), \ldots, (a_s, b_s, c_s)\} \subset \mathbb{N}^3.$$

Step 1 Plug in the bottom textbox a generating set for the PP using the following syntax (see Figure 5).

$$(a_1, b_1, c_1); (a_2, b_2, c_2); \ldots; (a_s, b_s, c_s)$$

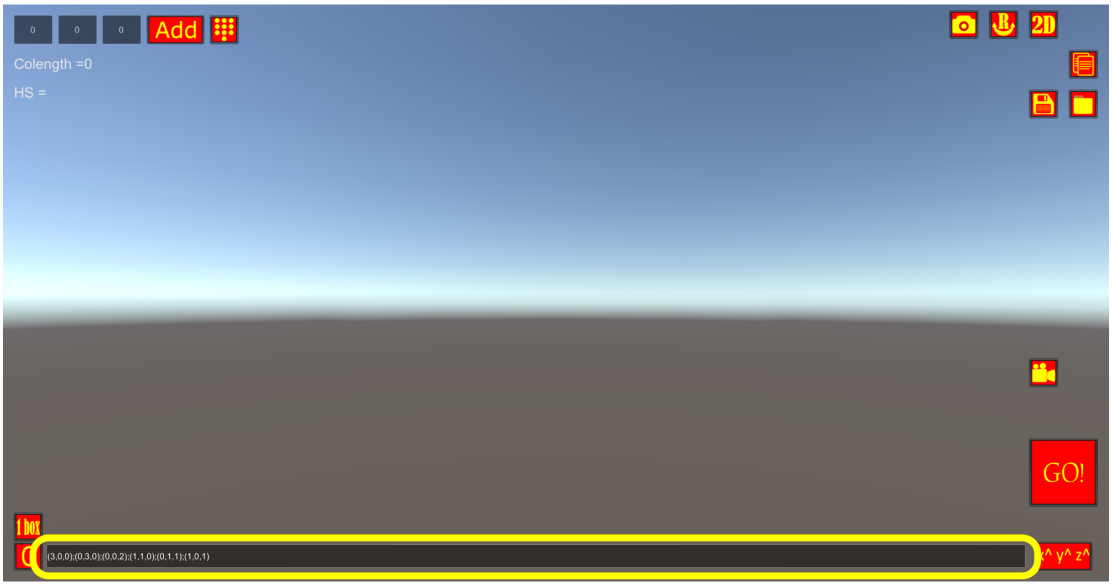

FIGURE 5.

The only allowed characters are

( ) , ; 0 1 2 3 4 5 6 7 8 9

Non-allowed characters will be ignored. Click the button to use the *keyboard* with only the allowed characters (see Figure 6).

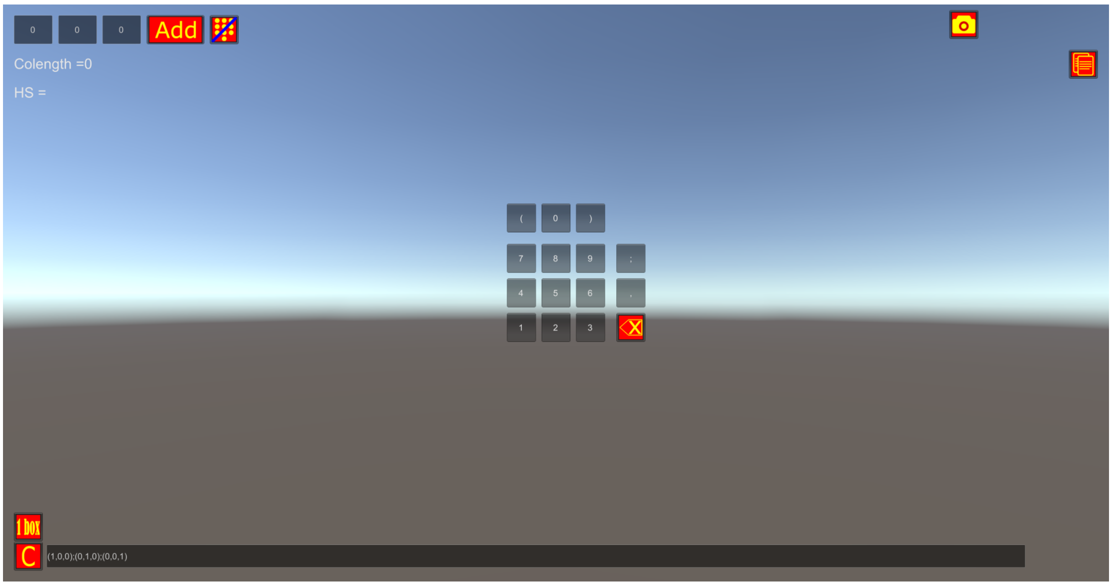

FIGURE 6.

The button cancels the last character.

While using the *keyboard*, you can not enter in *edit mode, camera mode, explorer mode, 2D-mode, monomial mode* and it is not possible to *save*. Click to close the *keyboard*.

Step 2 Click GO! (Figure 7).

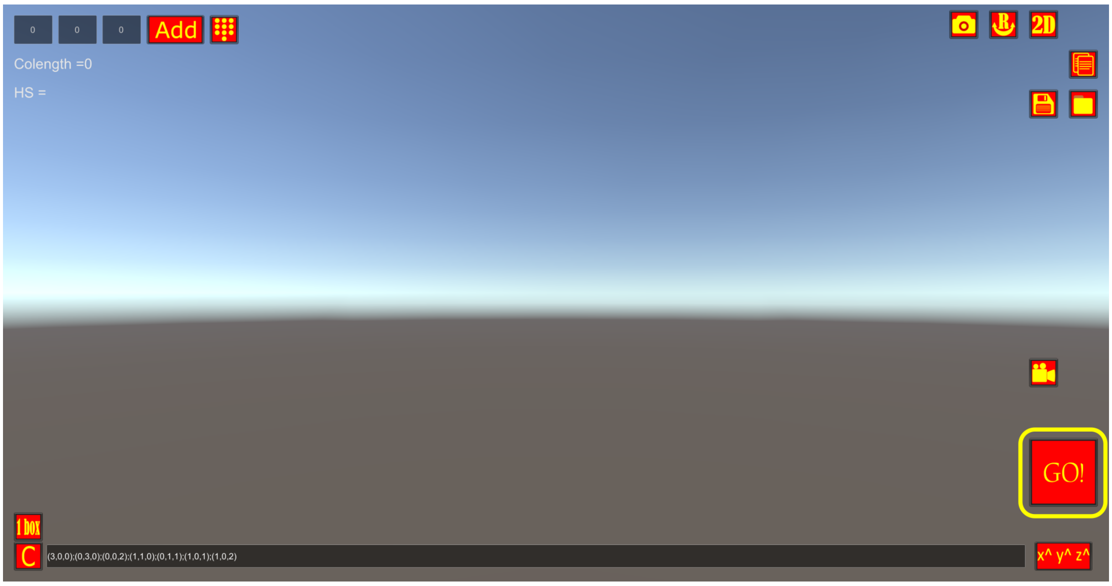

FIGURE 7.

Step 3 The output consists of the PP, its minimal generating set in the textbox, the number of boxes and the Hilbert–Samuel function of the associated monomial ideal (see Figure 8).

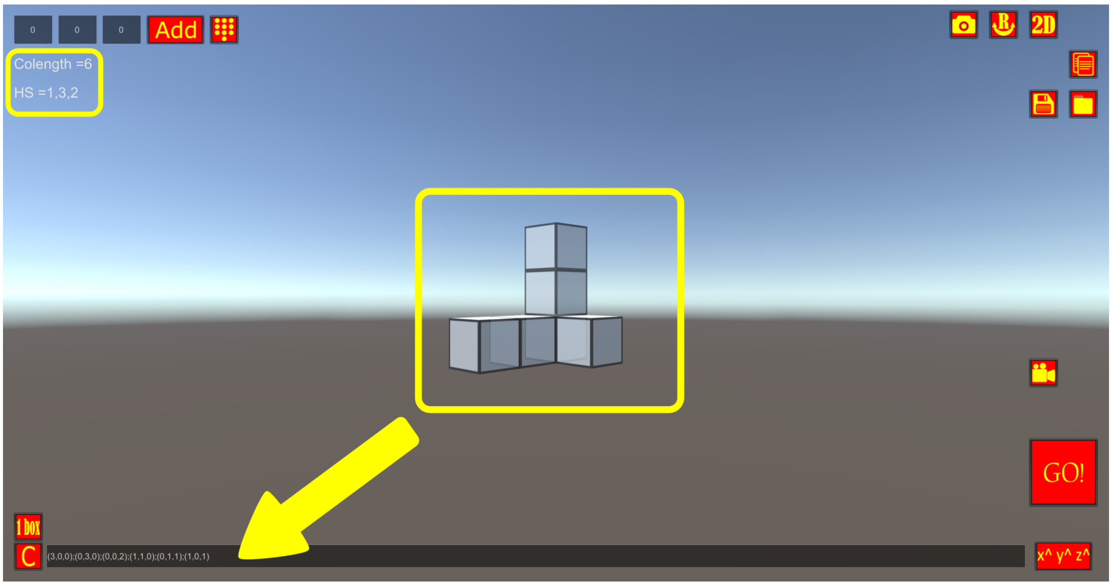

FIGURE 8.

If you want to draw the plane partition with one box, click 1 box to input its generators in the textbox, and then click GO! to generate the PP.

If you want to clean the textbox click C or add the trivial generator $(0,0,0)$ using the button Add (see Section 1.3).

1.2. **Rotate a plane partition.** You can rotate the PP by dragging it. You can not rotate the PP if you are in *edit mode*. Click to reset the position.

1.3. Add a generator to a PP. You can add a generator $(a, b, c) \in \mathbb{N}^3$ to a generating set of a PP Σ using the input area in the top left corner of the screen as follows.

Step 1 Plug the (ordered) entries a, b, c of the new generator in the input area in the top left corner of the screen (see Figure 9).

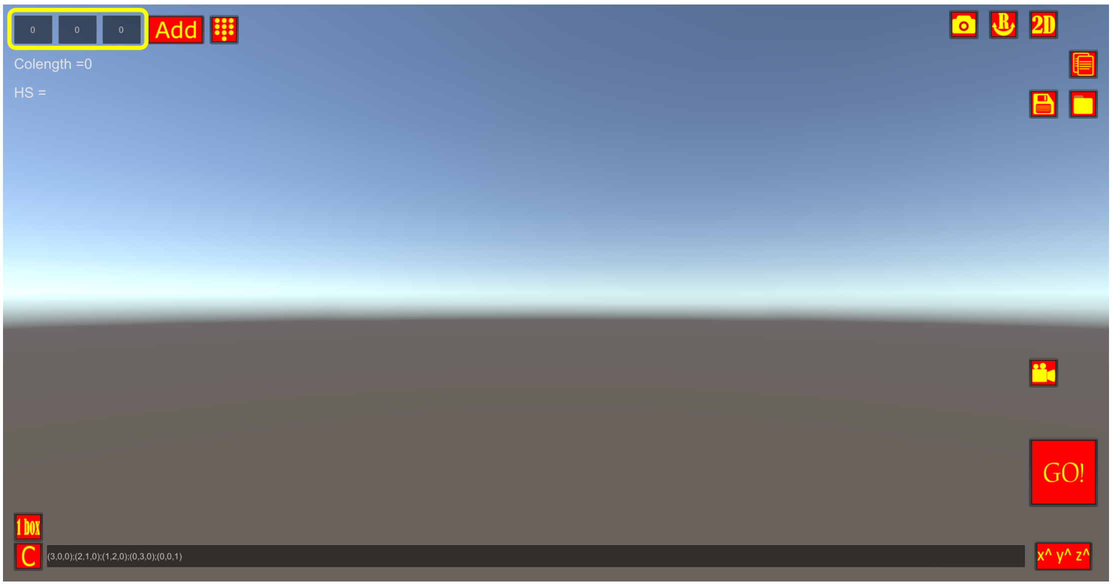

FIGURE 9.

Step 2 Click Add (see Figure 10).

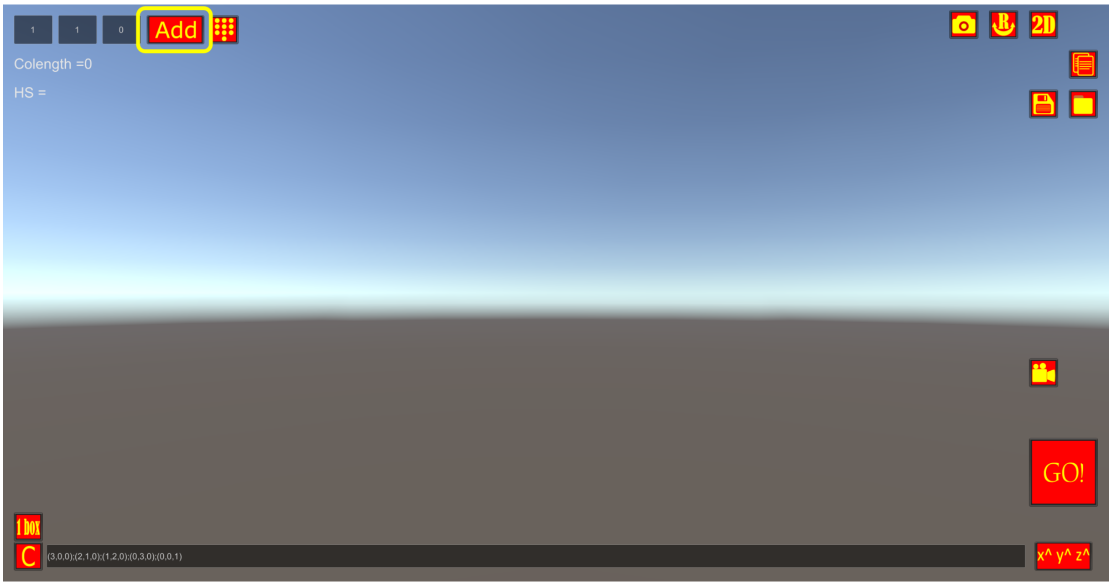

FIGURE 10.

Step 3 The output is in the textbox and it consists in a minimal set of generators for the new PP generated by (a, b, c) and a set of generators for Σ (see Figure 11).

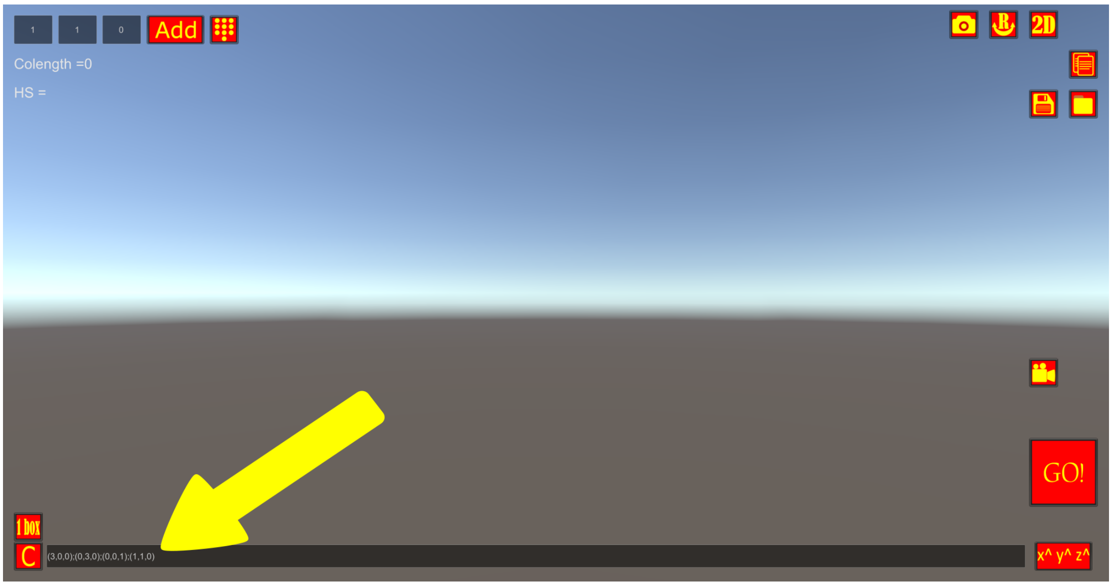

FIGURE 11.

1.4. **Move camera.** You can change the position of camera in *camera mode*. *Camera mode* is active when the icon is visible (see Figure 12).

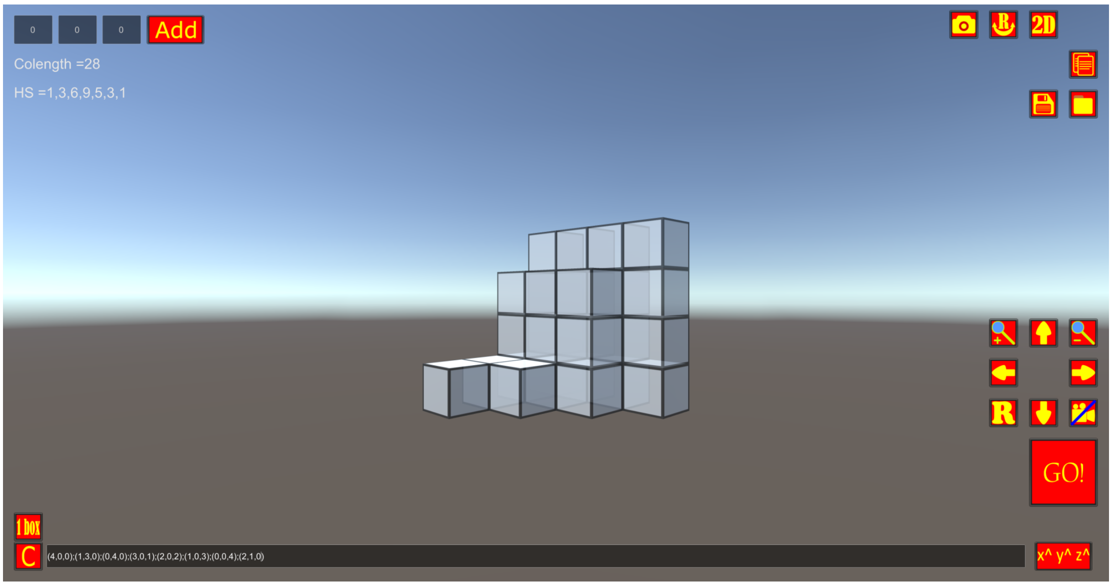

FIGURE 12.

When you are in *camera mode* you can not use the *keyboard* and you can not enter in *edit mode*. Below you find the instructions to move the camera.

Step 1 Click to enter in *camera mode*. You can access to *camera mode*, not only from *PP mode*, but also from *monomial mode* (see Section 2).

Step 2
- Use the arrows to move the camera left, up, right and down.
- Click to zoom-in and to zoom-out.

– Click to reset the position of the camera and the zoom.

Step 3 Click to exit from *camera mode*.

2. MONOMIAL IDEALS

To work with monomial ideals, you must be in *monomial mode*. You are in *monomial mode* if the icon is visible. Click the button to enter in *monomial mode*.

2.1. Draw the plane partition of a monomial ideal.

Follow this instructions to draw the plane partition of a monomial ideal $I = (x^{a_1} y^{b_1} z^{c_1}, \dots, x^{a_s} y^{b_s} z^{c_s})$.

Step 1 Plug, in the bottom textbox, a set of generators for I using the following syntax (see Figure 13).

$$x\hat{}a_1 y\hat{}b_1 z\hat{}c_1, x\hat{}a_2 y\hat{}b_2 z\hat{}c_2, \dots, x\hat{}a_s y\hat{}b_s z\hat{}c_s.$$

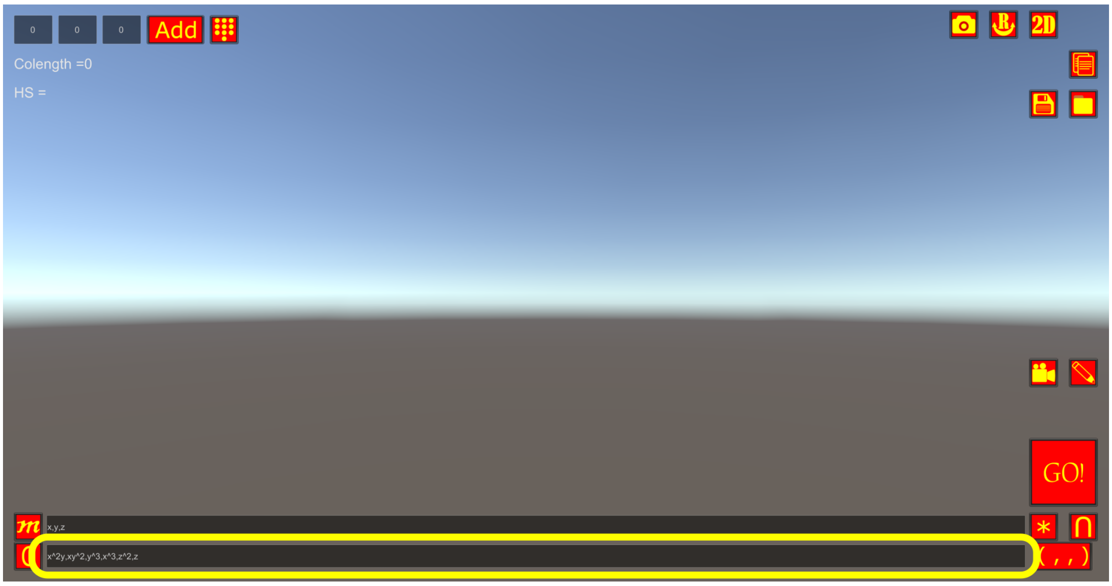

FIGURE 13.

The only allowed characters are

$$x \quad y \quad z \quad , \quad \hat{} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

Non-allowed characters will be ignored. Click the button to use the *keyboard* with only the allowed characters (see Figure 14).

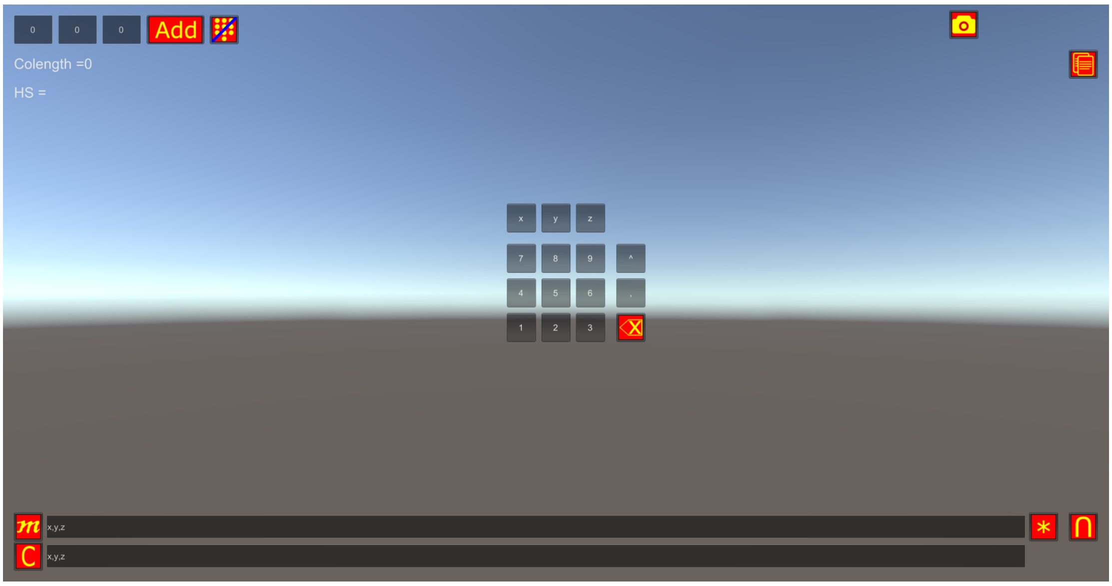

FIGURE 14.

The button cancels last character.

While using the *keyboard*, you can not enter in *edit mode, camera mode, explorer mode, 2D-mode, PP mode* and it is not possible to *save*. Click to close the *keyboard*.

Step 2 Click GO!.

Step 3 The output consists of the minimal set of generators for the ideal I in the bottom textbox, the PP of I decorated with the minimal generators positioned in $\mathbb{N}^3$ coherently with the bijection in Proposition 0.9, the number of boxes and the Hilbert–Samuel function of the monomial ideal I (see Figure 15).

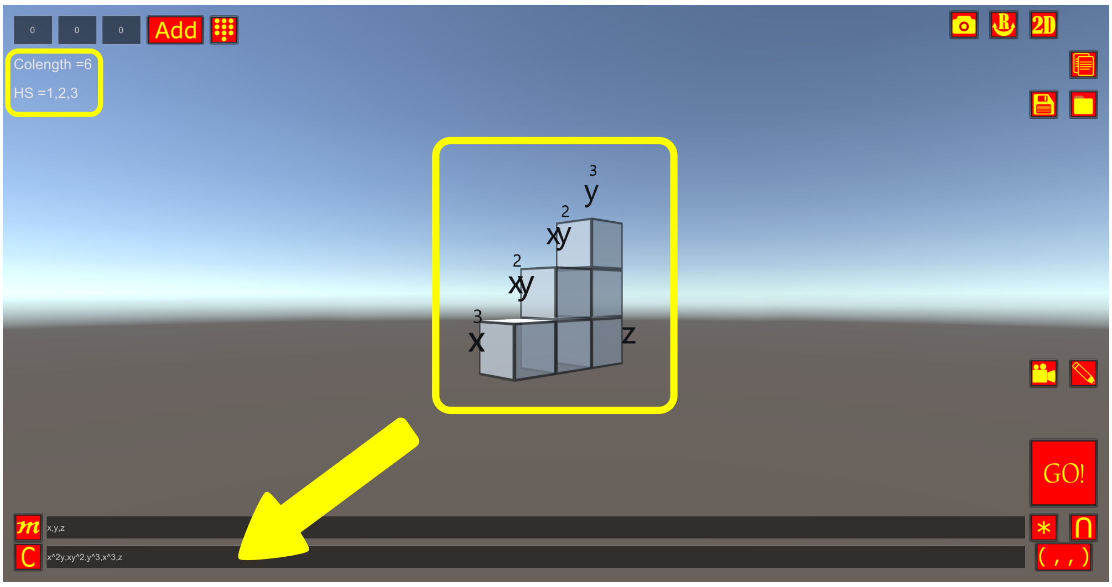

FIGURE 15.

If you want to draw the plane partition of the maximal monomial ideal, click to input its generators in both the textboxes, and then click GO! to generate the PP.

2.2. **Clean the textboxes.** You may want to clean one or both the textboxes.

- If you want to clean both the textboxes click C.
- If you want to clean the bottom textbox add the trivial generator $(0,0,0)$ by using the button Add (see Section 2.3).

2.3. **Add a generator to a monomial ideal.** Given a monomial ideal I, you can build a new monomial ideal by adding a (monomial) generator $x^a y^b z^c$ to I. You can do this by using the input area positioned at the top left corner of the screen (see Section 1.3) as follows.

Step 1 Plug a set of monomial generators of I in the bottom textbox (see Figure 16).

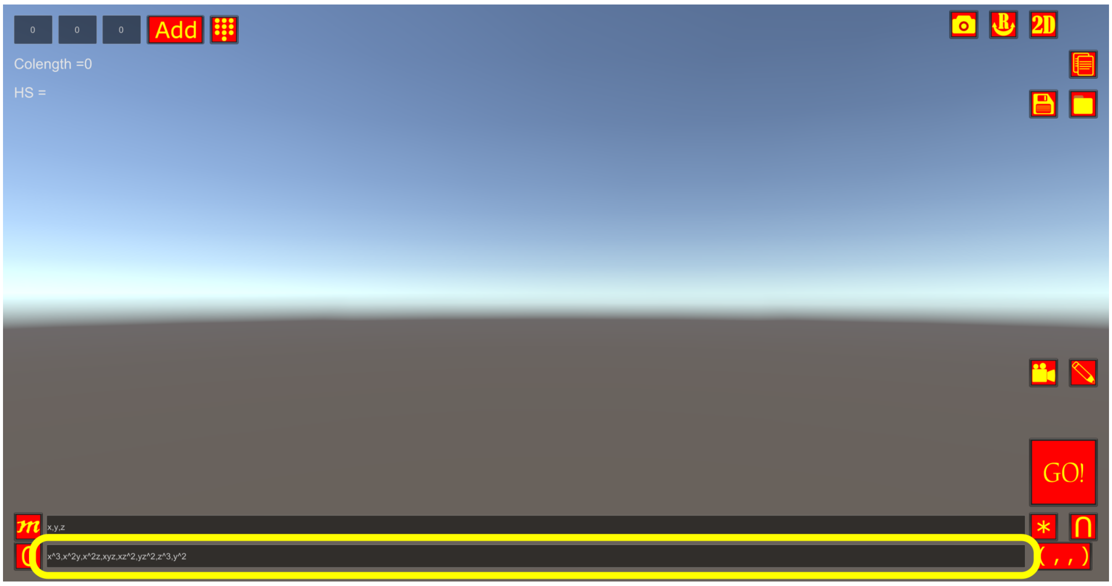

FIGURE 16.

Step 2 Plug the exponents of the new monomial generator in the top left input area (see Section 1.3) first the exponent of the x variable, then the exponent of y and, finally, the exponent of z.

Step 3 Click Add.

Step 4 The output consists of a minimal set of monomial generators for the ideal generated by I and $x^a y^b z^c$ (see Figure 17).

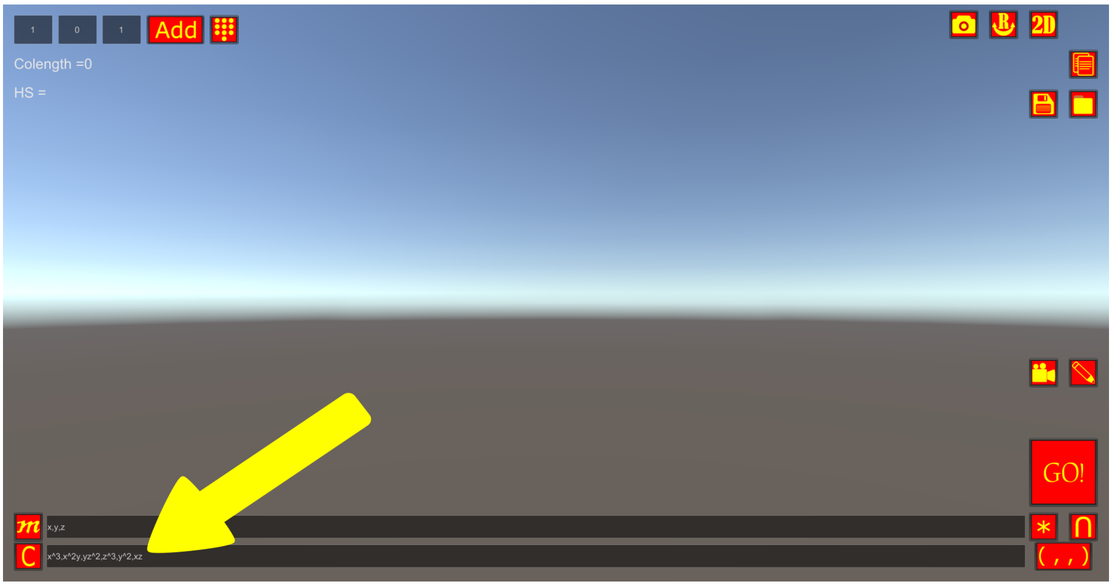

FIGURE 17.

2.4. **Move camera.** See Section 1.4.

2.5. **Operations.** You can also perform *intersection* and *product* of two ideals I_1, I_2 as follows.

Step 1 Pulg a set of monomial generators for I_1 in the bottom textbox and a set of monomial generators for I_2 in the top textbox (see Figure 18).

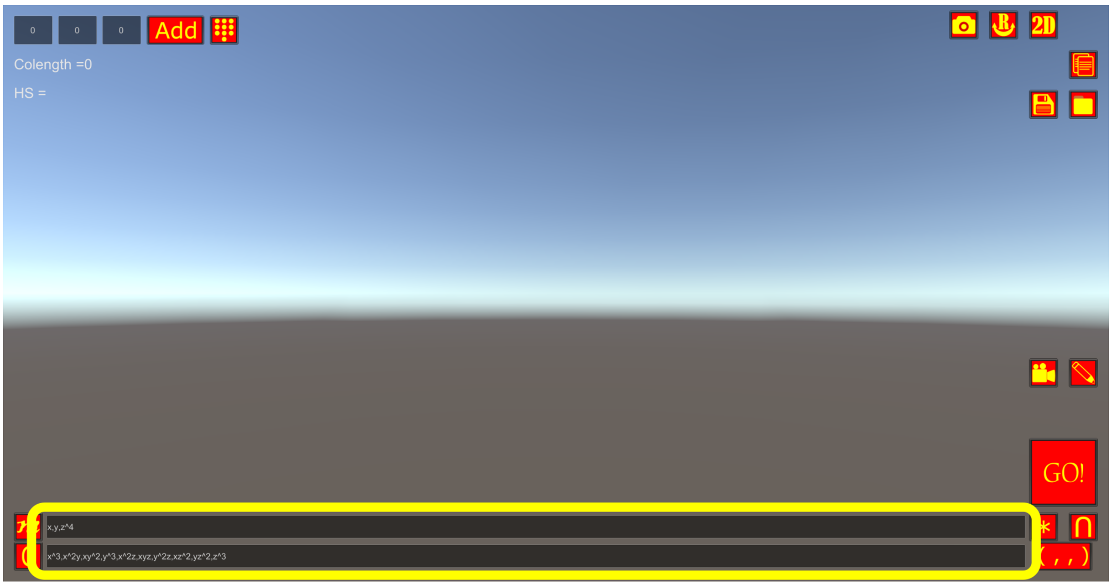

FIGURE 18.

Step 2 Click $*$ for the product and $\cap$ for the intersection.

Step 3 You find the output ideal in terms of its minimal generators in the bottom textbox (see Figure 19 for the product and Figure 20 for the intersection).

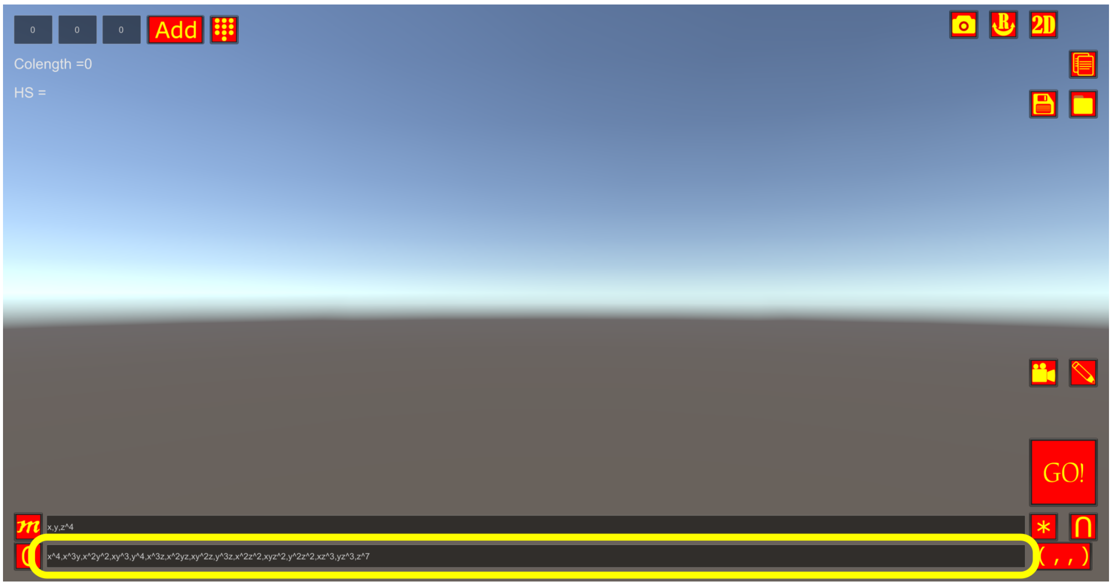

FIGURE 19.

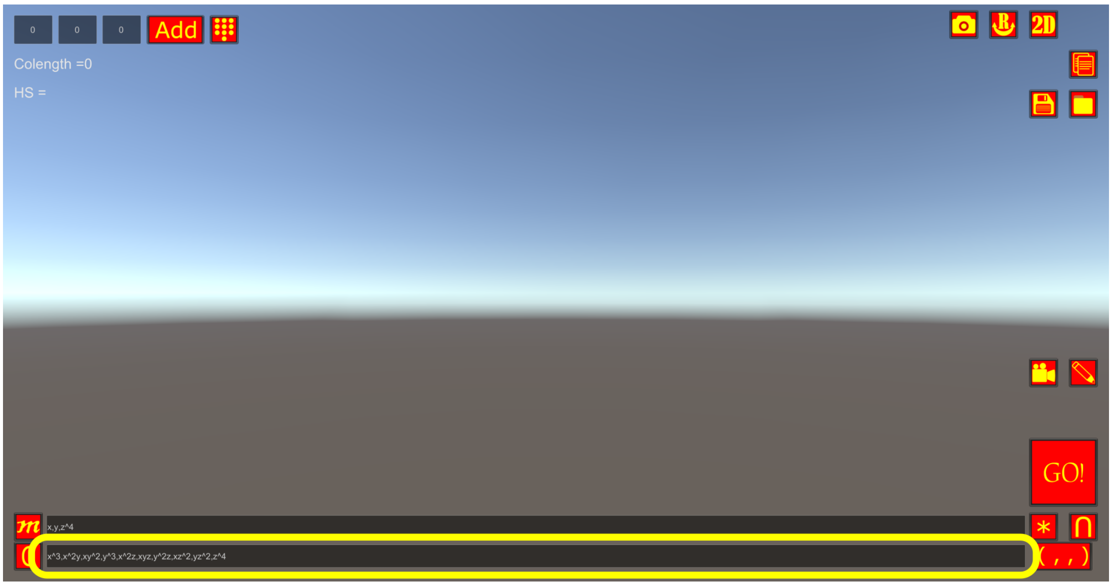

FIGURE 20.

The ideal I_2 in the top textbox remains unchanged.

2.6. **Editor.** When you are in *monomial mode*, you can edit the PP by adding or removing generators via simple clicks. You can do this when you are in *edit mode*. The *edit mode* is active if the icon is visible.

From *edit mode*, you can not enter in *camera mode* or *PP mode* and you can not use the *keyboard*

Step 1 Click to enter in *edit mode*.

- To add a generator you need to enter in *add mode*, you are in *add mode* if the icon is visible. Click to enter in *add mode*, and click to exit from *add mode*.
- Once you are in *add mode*, click on a box to add the generator corresponding to that box.

- To remove a generator you need to enter in *remove mode*, you are in *remove mode* if the icon is visible. Click to enter in *remove mode*, and click to exit from *remove mode*.
- Once you are in *remove mode* click on a generator to remove it. You can not remove power of variables, i.e. generators of the form x^i, y^j, z^k. This would make the ideal not zero-dimensional.

Step 1 Click to exit from *edit mode*.

2.7. **2D-Mode.** ASAP.

3. Save & Export

3.1. **Save a PP.**

- If you are in *PP mode*, click to save the set of generators present in the textbox.
- If you are in *monomial mode*, click to save the generators present in the bottom textbox.

3.2. **Load a saved PP.** If you want to load a saved PP enter in *explorer mode*. You are in *explorer mode* if the icon is visible. When you are in *explorer mode*, you can not enter in *edit mode*, *camera mode*, and *2D-mode*.

In *explorer mode*, you can load one of the last 30 saved PP. They are listed in the gray/white list (see Figure 21).

FIGURE 21.

When you are in *explorer mode*, you can pass form *PP mode* to *monomial mode* and viceversa clicking the buttons (, ,)/x^ y^ z^. If you are in *PP/monomial mode* the gray/white list shows the generators of the saved PP's/ideals. If there are to many generators, then, instead of them, the string *"too long to be showed"* is displayed.

- Click to enter in *explorer mode*.
- Click the button near the saved PP you want to load.
- The generators of the PP appear in the textbox.
- Click to exit from *explorer mode*. You find the loaded ideal in the bottom textbox if you are in *monomial mode* or in the textbox if you are in *PP mode*.

3.3. **Delete a saved PP.** If you want to delete a saved PP enter in *delete mode* form *explorer mode*. You are in *delete mode* if the icon is visible.

- When you are in *explorer mode*, click to enter in *delete mode*.
- Click the button near the saved PP you want to load.
- Click to exit from *delete mode*.

Click Clear to delete all the saved PP/ideals.

3.4. **Export generators.**

- When you are in *PP mode*, you can export a minimal set of generators of the PP generated by the 3-uples in the textbox by clicking and paste it elsewhere.
- When you are in *monomial mode*, you can paste the generators of a monomial ideal copied from Macaulay2 [3] directly on the lower text box. IBTVI will ignore the non-allowed characters. If you want to export to Macalulay2 the minimal set of generators of the ideal generated by the monomial in the bottom textbox, click the button and paste directly on Macaulay2.

3.5. **Screenshot.** Using the PC version, it is possible to take a screenshot by clicking . You will find a png file named *"Screenshot IBTVI.png"* containing the screenshot in your local Image folder.

You can not take screenshots while you are in *explorer mode*.

Acknowledgements. We thank Dr. Teresa Spataro for funding this project.

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