COMPUTATIONS IN ALGEBRAIC GEOMETRY

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ABSTRACT. These notes follow the content of the course *"Computations in algebraic geometry"* held at *Unicamp IMECC* in Spring (Brazilian fall) 2025. The final goal of the course is to learn how to parameterise locally closed subset of the Hilbert schemes of points. During the course, four main topics have been covered in four lectures. These are classical birational geometry, deformations of monomial ideals, deformations of zero-dimensional ideals, Hilbert schemes of points. Each lecture contains explicit examples of computations via Macaulay2.

CONTENTS

1. LECTURE 1

In this lecture we recall the definition of classical maps in algebraic geometry such as Veronese/Segre embeddings, projections and the standard Cremona transformation. In addition, there are examples in the text of how to perform calculations related to this topic using the software *Macaulay2*, [8].

1.1. Notation.

We work over the field of complex numbers $\mathbb{C}$.

- We adopt throughout the Zariski topology.
- We denote by R a polynomial ring with complex coefficients. If not specified, the set of variables will be
  - $x_0, \dots, x_n$ in the projective setting,
  - $x_1, \dots, x_n$ in the affine setting,

  - x, y in the affine two-dimensional setting.
- The ring R is endowed with the standard grading $\deg x_i = 1$, for all $i = 1, \dots, n$. This gives $R = \oplus_{k\geq 0} R_k$ where

$$R_k = \left\{ f \in R \,\middle|\, f \text{ homogeneous, and } \deg f = k \right\} \cup \{0\}.$$

- We denote by $\mathbb{A}^n = \operatorname{Spec}(R)$ the n-dimensional affine space with coordinates (generators of R) $x_1, \dots, x_n$, if not specified otherwise.
- Similarly, $\mathbb{P}^n = \operatorname{Proj}(R)$ denotes the n-dimensional projective space with homogeneous coordinates (generators of R) $x_0, \dots, x_n$, if not specified otherwise.
- We denote by $H_i = \{ x_i = 0 \} \subset \mathbb{P}^n$, for $i = 0, \dots, n$, the i-th coordinate hyperplane. Moreover, we denote by $e_i \in \mathbb{P}^n$, for $i = 0, \dots, n$, the i-th coordinate point.
- The symbols $\operatorname{Mat}(n, m, \mathbb{C})$ and $\operatorname{Sym}(n, \mathbb{C}) \subset \operatorname{Mat}(n, n, \mathbb{C})$ denote respectively the vector spaces of matrices and symmetric matrices with complex entries. The symbol $\operatorname{GL}(n, \mathbb{C}) \subset \operatorname{Mat}(n, n, \mathbb{C})$ denotes the general linear group.
- Given any subset $S \subset \mathbb{P}^n$, we denote by $\langle S \rangle \subset \mathbb{P}^n$ the smallest linear subspace containing S.
- $\operatorname{Bir}(X)$ denotes the group of birational transformations of a variety X.

1.2. **Veronese embeddings.** The first non-trivial example of morphism between projective spaces is provided by the n-th Veronese embedding of degree d.

Definition 1.1. The *n-th Veronese embedding of degree d* is the morphism defined by

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\quad \mathrm{v}_{n,d} \quad} & \mathbb{P}^{\binom{n+d}{d}-1} \\ [x_0 : \cdots : x_n] & \longmapsto & \left[x_0^{\alpha_0} \cdots x_n^{\alpha_n} \mid \sum_{i=0}^n \alpha_i = n \right]. \end{array}$$

Proposition 1.2. *The morphism $\mathrm{v}_{n,d}$ is a closed immersion.*

Exercise 1.3. Prove Proposition 1.2. **Hint:** see Example 1.4.

Example 1.4. Let us explain in details the case $n = 1$. Fix homogeneous coordinates x_0, x_1 on $\mathbb{P}^1$ and y_i, for $i = 0, \dots, d$ on $\mathbb{P}^d$. The first-Veronese embedding of degree d reads then as

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\quad \mathrm{v}_{1,d} \quad} & \mathbb{P}^d \\ [x_0 : x_1] & \longmapsto & \left[x_0^d : x_0^{d-1} x_1 : \cdots : x_1^d \right]. \end{array}$$

Consider the charts $U_i = \{ x_i \neq 0 \} \cong \mathbb{A}^1$, for $i = 0, 1$ and $V_j = \left\{ y_j \neq 0 \right\} \cong \mathbb{A}^d$, for $j = 0, \dots, d$. Then, on U_i we have coordinates

$$t_i = \left(\frac{x_1}{x_0} \right)^{(-1)^i},$$

for $i = 0, 1$. The restrictions of $\mathrm{v}_{1,d}$ to U_0 and U_1 have the form

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ t_0 & \longmapsto & (t_0, t_0^2, \dots, t_0^d), \end{array} \quad \text{and} \quad \begin{array}{ccc} U_1 & \longrightarrow & V_d \\ t_1 & \longmapsto & (t_1^d, t_1^{d-1}, \dots, t_1). \end{array} \tag{1.1}$$

In particular, the image of $\mathrm{v}_{1,d}$ is entirely contained $V_1 \cup V_d$. Notice also that the image is a smooth curve as both the maps in (1.1) are parametrisations of smooth curves.

Let us compute the image of the morphism $\mathrm{v}_{1,d}$. In order to do this, we first observe that any two consecutive entries of the map $\mathrm{v}_{1,d}$ have the same ratio, namely x_0/x_1. Therefore, the equality

$$[y_0:\cdots:y_d]=\mathrm{v}_{1,d}([x_0:x_1])=\left[x_0^d:x_0^{d-1}x_1:\cdots:x_1^d\right] \tag{1.2}$$

implies

$$\operatorname{rk}\begin{bmatrix} y_0 & \cdots & y_{d-1} \\ y_1 & \cdots & y_d \end{bmatrix}\leq 1.$$

Define $Y_{1,d}\subset\mathbb{P}^d$ to be the closed subset given by

$$Y_{1,d}=\left\{[y_0:\cdots:y_d]\in\mathbb{P}^d \;\middle|\; \operatorname{rk}\begin{bmatrix} y_0 & \cdots & y_{d-1} \\ y_1 & \cdots & y_d \end{bmatrix}\leq 1\right\}.$$

Clearly $\mathrm{v}_{1,n}(\mathbb{P}^1)\subset Y_{1,d}$. We prove that $\mathrm{v}_{1,d}:\mathbb{P}^1\longrightarrow Y_{1,d}$ is invertible and this implies that $\mathrm{v}_{1,d}$ is a closed immersion. Its inverse is

$$\begin{aligned} Y_{1,d} &\xrightarrow{\ \rho_{1,d}\ } \mathbb{P}^1 \\ [y_0:\cdots:y_d] &\longmapsto [y_0:y_1]. \end{aligned}$$

To see this we only have to show that the map $\rho_{1,d}$ is well defined. This is true because it can be extended to points of the form $[0:0:y_2:\cdots:y_d]\in Y_{1,d}$ via

$$[y_0:\cdots:y_d]\mapsto[y_{d-1}:y_d].$$

Indeed, the conditions $y_0=y_1=0$ and $y_{d-1}=y_d=0$ are incompatible on $Y_{1,d}$ and $[y_0:y_1]=[y_{d-1}:y_d]$, by (1.2).

Definition 1.5. The rational normal curve of degree d is the image of the morphism $\mathrm{v}_{1,d}$. If $d=2$, the rational normal curve is called conic, and for $d=3$, it is called twisted cubic.

Example 1.6. We describe now the degree-2 Veronese embeddings. In this setting the projective space $\mathbb{P}^{\binom{n+2}{2}-1}$ identifies with $\mathbb{P}\operatorname{Sym}(n+1,\mathbb{C})$ and we get the following expression for the morphism $\mathrm{v}_{n,2}$.

$$\begin{aligned} \mathbb{P}^n &\xrightarrow{\ \mathrm{v}_{n,2}\ } \mathbb{P}\operatorname{Sym}(n+1,\mathbb{C}) \\ [x_0:\cdots:x_n] &\longmapsto \begin{bmatrix} x_0^2 & \cdots & \cdots & x_0x_n \\ x_1x_0 & x_1^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_0x_n & x_1x_n & \cdots & x_n^2 \end{bmatrix}. \end{aligned}$$

Exercise 1.7. Prove that

$$\mathrm{v}_{n,2}(\mathbb{P}^n)=\left\{[M]\in\mathbb{P}\operatorname{Sym}(n+1,\mathbb{C}) \;\middle|\; \operatorname{rk}M\leq 1\right\}. \tag{1.3}$$

Hint: Clearly $\mathrm{v}_{n,2}(\mathbb{P}^n)$ is contained in the right hand side of Equation (1.3). For the opposite inclusion consider the projections on the i-th row (or the i-th column), for $i=0,\dots,n$.

Example 1.8. The easiest example is the second Veronese surface, i.e. the image of the morphism

$$\begin{aligned} \mathbb{P}^2 &\xrightarrow{\ \mathrm{v}_{2,2}\ } \mathbb{P}\operatorname{Sym}(3,\mathbb{C})\cong\mathbb{P}^5 \\ [x_0:x_1:x_2] &\longmapsto \begin{bmatrix} x_0^2 & x_0x_1 & x_0x_2 \\ x_1x_0 & x_1^2 & x_1x_2 \\ x_2x_0 & x_2x_1 & x_2^2 \end{bmatrix}. \end{aligned}$$

One of the many possible instances in which Veronese embeddings turn out to be useful is the description of loci parametrising homogeneous ideals.

Exercise 1.9. Fix some integer $d > 1$. Describe the locus

$$V_d^{\mathrm{o}} = \left\{ [f] \in \mathbb{P}R_d \;\middle|\; \dim_{\mathbb{C}} \operatorname{Span}\left(\frac{\partial}{\partial x_i} f \right)_{i=0}^{n} > 1 \right\}.$$

We get

$$V_d^{\mathrm{o}} = \mathbb{P}R_d \setminus \mathrm{v}_{n,d}(\mathbb{P}^n).$$

Exercise 1.10. Two important aspects of the Veronese embedding concern its degree and its projective normality.

- Let $L \subset \mathbb{P}^n$ be a line. Then, we have $\mathrm{v}_{n,d}|_L \equiv \mathrm{v}_{1,d}$. Moreover, the generic hyperplane intersects $\mathrm{v}_{n,d}(L)$ in d distinct points. We say that the morphism has topological degree d.
- Let $Y_d \subset \mathbb{P}^n$ be a hypersurface of degree d, i.e. $Y_d = V(f)$ for some $f \in R_d$. Then there exists a hyperplane $H \subset \mathbb{P}^{\binom{n+d}{d}-1}$ such that $Y_d = \mathrm{v}_{n,d}(\mathbb{P}^n) \cap H$. We say that $\mathrm{v}_{n,d}$ is projectively normal.

Remark 1.11. It is worth mentioning that topological degree and projective normality are defined in a much more general context. See [7] for more details on the topological degree and [9, §I.3, §II.5] for projective normality.

The following exercise shows that not all degree d closed immersions are projectively normal.

Exercise 1.12. Show that the composition

$$\begin{array}{ccccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^4 & \dashrightarrow & \mathbb{P}^3 \\ [x_0 : x_1] & \longmapsto & [x_0^4 : x_0^3 x_1 : x_0^2 x_1^2 : x_0 x_1^3 : x_1^4] & \longmapsto & [x_0^4 : x_0^3 x_1 : x_0 x_1^3 : x_1^4] \end{array}$$

is a closed immersion of topological degree 4. Show that it is not projective normal.

1.2.1. *Example of computation.*

```
-- Define a function that remove the common factors from the entries of a map
cleanFactors = f -> (
    L=(entries matrix f)#0;
    G=gcd L;
    Laux= for l in L list sub(l/G,source f );
    return map(target f, source f, Laux);
)
-- Declear ambient spaces
n = 3; d = 3; N = binomial(n+d,d)-1; R = QQ[x_0 .. x_n]; S = QQ[y_0 .. y_N];
-- Construct the embedding and compute the equations
vnd1 = ( ideal R_* )^d_*;                                   -- 1st possibility
vnd2 = (entries(monomials (sum R_*)^d))#0;                  -- 2nd possibility
set vnd1 == set vnd2                                        -- Check the two are the same
Verond = map(R,S,vnd1);                                     -- Define the map
Equations = trim ker Verond;                                -- Compute equations
-- Check of the smoothness of the image of e_0={1,0,0,..,0}.
-- WLOG we restrict to { x_0 != 0 } and { y_0 != 0 }.
-- For simplicity we keep the same symbols for the variables
R0 = QQ[ x_1 .. x_n ]; S0 = QQ[ y_1 .. y_N ];
vnd10 = drop(for v in vnd1 list sub(sub( v , x_0=>1),R0),1);
Verond0 = map(R0,S0,vnd10  );
```

```
Eqs0 = trim ker Verond0;
Eqs0 == sub(sub( Equations,y_0=>1),S0)              -- It agrees with dehomogenising
J = sub( jacobian Eqs0, for i from 1 to N list y_i =>0 ); -- Jacobian at e_0
0 == (rank J - N + n)                               -- Check it has rank N - n
-- Computation of the inverse map
use R; use S;
Inverse = map(S,R,for i from 0 to n list y_i);
cleanFactors ( Verond * Inverse )          -- Check that the composition is the identity
```

1.3. **Segre embeddings.** Segre embeddings provide a way to realise products of projective spaces as closed subsets of an ambient projective space.

Definition 1.13. Given two integers $n, m \in \mathbb{Z}_{\geq 0}$, the Segre (n, m)-embedding is the morphism

$$\begin{array}{ccc}\mathbb{P}^n \times \mathbb{P}^m & \xrightarrow{\mathrm{s}_{n,m}} & \mathbb{P}\mathrm{Mat}(n+1, m+1, \mathbb{C}) \cong \mathbb{P}^{(n+1)(m+1)-1} \\ ([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) & \longmapsto & \begin{bmatrix} x_0 y_0 & \cdots & x_0 y_m \\ \vdots & \ddots & \vdots \\ x_n y_0 & \cdots & x_n y_m \end{bmatrix}\end{array}$$

Proposition 1.14. *The morphism* $\mathrm{s}_{n,m}$ *is a closed immersion.*

Exercise 1.15. Prove Proposition 1.14.
Hint: Define

$$S_{n,m} = \left\{ [M] \in \mathbb{P}^{(n+1)(m+1)-1} \,\middle|\, \mathrm{rk}\, M \leq 1 \right\}.$$

Clearly, we have

$$\mathrm{s}_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subset S_{n,m}.$$

To conclude consider the morphism

$$\begin{array}{ccc}S_{n,m} & \longrightarrow & \mathbb{P}^n \times \mathbb{P}^m \\ \begin{bmatrix} z_{0,0} & \cdots & z_{0,m} \\ \vdots & \ddots & \vdots \\ z_{n,0} & \cdots & z_{n,m} \end{bmatrix} & \longmapsto & ([z_{0,0} : \cdots : z_{n,0}], [z_{0,0} : \cdots : z_{0,m}]).\end{array}$$

Example 1.16. For $n = m = 1$, we get

$$\begin{array}{ccc}\mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\mathrm{s}_{1,1}} & \mathbb{P}^3 \\ ([x_0 : x_1], [y_0 : y_1]) & \longmapsto & \begin{bmatrix} x_0 y_0 & x_0 y_1 \\ x_1 y_0 & x_1 y_1 \end{bmatrix}\end{array}$$

and

$$\mathrm{s}_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) = \left\{ z_{0,0} z_{1,1} - z_{1,0} z_{0,1} = 0 \right\}.$$

Since all smooth quadrics of $\mathbb{P}^n$ differ by projectivities, we deduce that all smooth quadrics of $\mathbb{P}^3$ are isomorphic to $(\mathbb{P}^1)^{\times 2}$.

Example 1.17. Consider the diagonal

$$\Delta = \left\{ (p, q) \in \mathbb{P}^n \times \mathbb{P}^n \,\middle|\, p = q \right\} \cong \mathbb{P}^n.$$

We then restrict the Segre embedding to $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ and we get the second Veronese embedding $\mathrm{v}_{n,2} : \Delta \to \mathbb{P}\,\mathrm{Sym}(n+1,\mathbb{C})$. Precisely, the following diagram

$$\begin{tikzcd}
\Delta \arrow[rr, "\mathrm{s}_{n,n}|_{\Delta}"] & & \mathbb{P}^{(n+1)^2-1} \\
{[x_0:\cdots:x_n]} \arrow[rr, mapsto] \arrow[dr, "\mathrm{v}_{n,2}"'] & & \begin{bmatrix} x_0^2 & \cdots & \cdots & x_0x_n \\ x_1x_0 & x_1^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_0x_n & x_1x_n & \cdots & x_n^2 \end{bmatrix} \\
& \mathbb{P}\,\mathrm{Sym}(n+1,\mathbb{C}), \arrow[ur, hook] &
\end{tikzcd}$$

commutes.

1.4. **Projections and blowups of linear subspaces.**

Definition 1.18. Let X and Y be quasi-projective varieties. A rational map $\varphi : X \dashrightarrow Y$ is an equivalence class of pairs (U, f), where U is a dense open subset of X and f is a morphism from U to Y, and where two pairs (U_1, f_1) and (U_2, f_2) are equivalent if there exists a dense open subset $V \subset U_1 \cap U_2$ such that $f_1|_V = f_2|_V$.

Projections are the first examples of rational maps.

Let $H, K \subset \mathbb{P}^n$ be linear subspaces such that $H \cong \mathbb{P}^k$, $K \cong \mathbb{P}^{n-k-1}$ and $H \cap K = \emptyset$. Without loss of generality we put

$$K = \{\, x_0 = \cdots = x_k = 0 \,\},\ H = \{\, x_{k+1} = \cdots = x_n = 0 \,\} \subset \mathbb{P}^n.$$

The projection onto H with centre K is the rational map

$$\begin{aligned} \mathbb{P}^n & \overset{\pi_K}{\dashrightarrow} H \\ [x_0 : \cdots : x_n] & \longmapsto [x_0 : \cdots : x_k]. \end{aligned} \tag{1.4}$$

Notice that we do not report the dependence on H as it is not useful for our applications and we prefer to keep the notation as simple as possible.

Remark 1.19. Geometrically, for all points $p \in \mathbb{P}^n \setminus K$ we consider $W_p = \langle p, K \rangle$. Then we have

$$\pi_K(p) = W_p \cap H.$$

Exercise 1.20. Prove the following basic properties of π_K.

- Any linear subspace $W \subset \mathbb{P}^n$ such that $K \subset W$ and $W \cong \mathbb{P}^{n-k}$ is contracted to a point, i.e. $\pi_K(W) = p \in H$.
- If $L \subset \mathbb{P}^n$ is a line then $\pi_K(L) = p$ is a point if and only if $L \cap K \neq \emptyset$.[1]

Definition 1.21. Given a rational map

$$\begin{aligned} \mathbb{P}^n & \overset{\varphi}{\dashrightarrow} \mathbb{P}^m \\ [x_0 : \cdots : x_n] & \longmapsto [f_i([x_0 : \cdots : x_n]) \mid i = 0, \ldots, m], \end{aligned}$$

where $f_i \in R$, for $i = 0, \ldots, m$, denote by $\mathrm{dom}(\varphi)$ its maximal domain of definition and by $\mathrm{Ind}(\varphi)$ its indeterminacy locus. In symbols[2]

$$\mathrm{dom}(\varphi) = \bigcup_{(U,f)\in\varphi} U, \text{ and } \mathrm{Ind}\,\varphi = \mathrm{Proj}(R/(f_0, \ldots, f_m)).$$

[1] In particular, we require $L \not\subset K$ so that it makes sense to consider $\pi_K(L)$.

[2] Recall that by definition a rational map is an equivalence class. Thus it is a set.

Then, the graph of φ is

$$\operatorname{graph}\varphi = \left\{ (p,q) \in \operatorname{dom}(\varphi) \times \mathbb{P}^m \;\middle|\; q = \varphi(p) \right\} \subset \mathbb{P}^n \times \mathbb{P}^m.$$

Exercise 1.22. Let $\varphi \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map, and denote by $Z = \operatorname{Ind}\varphi$ its indeterminacy locus. Consider the following diagram

$$\begin{tikzcd} & \overline{\operatorname{graph}\varphi} \arrow[dl, "\varepsilon_Z"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^m, \end{tikzcd}$$

where the closure $\overline{\operatorname{graph}\varphi}$ is taken in $\mathbb{P}^n \times \mathbb{P}^m$, the morphisms ε_Z and π are the restrictions of the canonical projections to $\overline{\operatorname{graph}\varphi} \subset \mathbb{P}^n \times \mathbb{P}^m$.

Prove the following properties.

- The first projection ε_Z is birational. (**Hint:** Look at its restriction to $\operatorname{graph}\varphi$.)
- Suppose that φ is the projection π_K in (1.4). In particular $Z = K$ is a linear subspace. Show that
  - if $\dim K = n-1$, then ε_K is an isomorphism,
  - the morphism π is a fibration with fibres isomorphic to $\mathbb{P}^{n-k}$,
  - the preimage $E_K = \varepsilon_K^{-1} K$ is a $\mathbb{P}^{n-k-1}$-bundle over K.

Definition 1.23. We say that ε_Z is the blowup of $\mathbb{P}^n$ with centre Z. We will also call blowup the variety $\overline{\operatorname{graph}\varphi}$ ad we will denote it by $\operatorname{Bl}_Z \mathbb{P}^n$. Finally we say that E_Z is the exceptional locus.

Exercise 1.24. Definition 1.23 is well posed, i.e. it does not depend on φ but only on $\operatorname{Ind}\varphi$.

Precisely, show that given two rational maps $\varphi_i \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n_i}$, for $i = 1,2$ having the same indeterminacy locus $\operatorname{Ind}\varphi_1 = \operatorname{Ind}\varphi_2$ the two closures $\overline{\operatorname{graph}\varphi_i}$, for $i = 1,2$, are canonically isomorphic.

Exercise 1.25. Extend Definition 1.21, Exercise 1.22 and Definition 1.23 to the following cases.

- Products of projective spaces.
- Arbitrary quasi-projective varieties.

Example 1.26 (Blowup at a point)**.** We describe now the projection π_K in the case $n = 2$ and $\dim K = 0$. Without loss of generality we put $K = \{ e_2 \}$ and we consider the projection

$$\begin{aligned} \mathbb{P}^2 &\overset{\pi_{e_2}}{\dashrightarrow} \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto [x_0 : x_1], \end{aligned}$$

from $\mathbb{P}^2$ with centre the coordinate point $e_2 = [0:0:1]$. Then, the blowup with centre e_2 is

$$\begin{aligned} \operatorname{Bl}_{e_2} \mathbb{P}^2 &= \overline{\operatorname{graph}(\pi_{e_2})} \\ &= \overline{\left\{ (p,q) \in \operatorname{dom}(\pi_{e_2}) \times \mathbb{P}^1 \;\middle|\; \pi_{e_2}(p) = q \right\}} \subset \mathbb{P}^2 \times \mathbb{P}^1 \\ &= \overline{\left\{ ([x_0 : x_1 : x_2], [y_0 : y_1]) \in \operatorname{dom}(\pi_{e_2}) \times \mathbb{P}^1 \;\middle|\; \det \begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} = 0 \right\}} \subset \mathbb{P}^2 \times \mathbb{P}^1 \\ &= \left\{ ([x_0 : x_1 : x_2], [y_0 : y_1]) \in \mathbb{P}^2 \times \mathbb{P}^1 \;\middle|\; x_0 y_1 - x_1 y_0 = 0 \right\} \subset \mathbb{P}^2 \times \mathbb{P}^1. \end{aligned}$$

We stress that there is a commutative diagram

$$\begin{tikzcd} & \mathrm{Bl}_{e_2}\mathbb{P}^2 \subset \mathbb{P}^2 \times \mathbb{P}^1 \arrow[dl, "\pi_2|_{\mathrm{Bl}\,\mathbb{P}^2}"'] \arrow[dr, "\pi_1|_{\mathrm{Bl}_{e_2}\mathbb{P}^2}"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\pi_{e_2}"'] & & \mathbb{P}^1, \end{tikzcd}$$

where $\pi_2|_{\mathrm{Bl}\,\mathbb{P}^2}$ is the blowup morphism and $\pi_1|_{\mathrm{Bl}_{e_2}\mathbb{P}^2}$ is a $\mathbb{P}^1$-fibration as described in Exercise 1.22.

Exercise 1.27. Compute the equations of some embeddings of $\mathrm{Bl}_{\mathbb{P}^k}\mathbb{P}^n$.

Example 1.28 (Blowup at two points)**.** We construct now the blowup of the projective plane at two points as the closure of a birational map given by a pair of projections. Consider the rational map

$$\begin{aligned} \mathbb{P}^2 &\overset{\varphi}{\dashrightarrow} \mathbb{P}^1 \times \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto ([x_0 : x_1], [x_1 : x_2]). \end{aligned}$$

It is a birational map with inverse

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\overset{\psi}{\dashrightarrow} \mathbb{P}^2 \\ ([y_0 : y_1], [z_0 : z_1]) &\longmapsto [y_0 z_0 : y_1 z_0 : y_1 z_1]. \end{aligned}$$

The two maps have the following indeterminacy loci

$$\mathrm{Ind}(\varphi) = \{ e_0, e_2 \} \quad \text{and} \quad \mathrm{Ind}(\psi) = \{ ([1:0],[0:1]) \}.$$

Figure 1 depicts the construction.

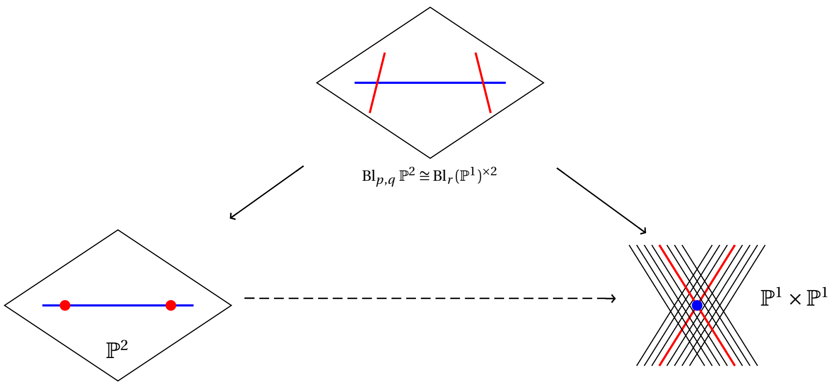

FIGURE 1. Pictorial description of the construction.

As a consequence, we get the isomorphism

$$\mathrm{Bl}_{p,q}\mathbb{P}^2 \cong \mathrm{Bl}_r(\mathbb{P}^1 \times \mathbb{P}^1),$$

where $p, q \in \mathbb{P}^2$ and $r \in \mathbb{P}^1 \times \mathbb{P}^1$.

Exercise 1.29. Study the blowup of the projective plane at two points.

- Check the details in Example 1.28.
- Compute the equations of $\mathrm{Bl}_{e_0,e_2}\mathbb{P}^2 \subset \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$.

- Show that φ contracts the line trough e_0 and e_2 on the unique point of $\operatorname{Ind}\psi$.
- Show that ψ contracts the two lines trough the unique point of $\operatorname{Ind}\psi$ on e_0 and e_2.

The following exercise is a direct generalisation of the construction in Example 1.28.

Exercise 1.30. Let $n = 2k+1$ be an odd nonnegative integer and let $Q_n \subset \mathbb{P}^n$ be a smooth quadric. Let also $Q_{n-2} \subset \mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$ be a smooth quadric. Prove that there is an isomorphism

$$\operatorname{Bl}_p Q_n \cong \operatorname{Bl}_{Q_{n-2}} \mathbb{P}^{n-1}.$$

1.4.1. *Example of computation: use the command* `normalCone` *for the blowup.*

```
    cleanList = L -> (
    G=gcd L;
    if G!= 0 then return for l in L list sub(l/G,ring L#0 );
    return L
)
-- Declear ambient space
R = QQ[x_0 .. x_2];
S = R[w_0 , w_1];
projection = {x_1,x_0};
I = minors (2,matrix {projection ,S_* });
N = normalCone ideal projection ;
K =sub(ker map(ambient N , ambient ambient N ),S)
I==K                                                    -- Check the construction is the same
O = QQ [R_*|S_*]
point={for r in R_* list sub(r,O),for s in S_* list sub(s,O)};
-- Function to pass from ideal to parametrisation
evalIdeal = I ->  for u in point list cleanList  apply( u , v ->  sub(sub(v,O/I),O) ) ;
E = sub(ideal projection , O ) + sub(I,O)                           -- Preimage of the origin
evalIdeal E
F = saturate(sub(ideal(3*w_0 -w_1),O)+sub(I,O),sub(ideal S_*,O))                 -- Fibration
evalIdeal F
```

1.5. **Standard Cremona transformation.** The standard Cremona transformation $c_n \in \operatorname{Bir}(\mathbb{P}^n)$, for $n \geq 1$, is a birational transformation of the projective space $\mathbb{P}^n$ and it is the first non trivial example of birational map. We recall now its definition.

Definition 1.31. The standard Cremona transformation $c_n \in \operatorname{Bir}(\mathbb{P}^n)$ is

$$\begin{array}{ccc} \mathbb{P}^n & \overset{c_n}{\dashrightarrow} & \mathbb{P}^n \\ [x_0 : \cdots : x_n] & \longmapsto & \left[\frac{1}{x_0} : \cdots : \frac{1}{x_n}\right] = [x_1 \cdots x_n : \cdots : x_0 \cdots x_{n-1}]. \end{array}$$

In dimension $n = 2$ the standard Cremona transformation plays a special role as the following classical result explains.

Theorem 1.32 (Noether-Castelnuovo, [4, 11])**.** *The following equality of groups holds true,*

$$\operatorname{Bir}(\mathbb{P}^2) = \langle \mathbb{P}\mathrm{GL}(3,\mathbb{C}), c_2 \rangle.$$

Exercise 1.33. Prove the main properties of the standard Cremona transformation listed below.

- $c_n^2 \equiv \mathrm{id}_{\mathbb{P}^n}$
- $\operatorname{Ind} c_n = \coprod_{0 \leq i < j \leq n} \{ x_i = x_j = 0 \}$

- $\mathrm{c}_n(\{x_i = 0\}) = e_i$
- $\operatorname{Fix} \mathrm{c}_n = \{[1 : \pm 1 : \pm 1 : \cdots : \pm 1]\}$
- $|\operatorname{Fix} \mathrm{c}_n| = 2^n$

1.5.1. *Example of computation.*

```
-- Define a function that remove common factor from lists
cleanList = L -> (
    G=gcd L;
    if G!= 0 then return for l in L list sub(l/G,ring L#0 );
    return L
)
-- Function to pass from ideal to parametrisation
evalIdeal = I -> cleanList for l in R_* list sub(sub(l,R/I),R);
-- Declear ambient space
n = 3;   R = QQ[x_0 .. x_n];
Crent = for i from 0 to n list product for j in delete (i, 0..n ) list x_j;
Cremona = map(R , R , Crent );                              -- Define the map
cleanFactorsMap ( Cremona * Cremona )                       -- Check it is an involution
DDD= set primaryDecomposition ideal Crent                   -- Compute the indeterminacy locus
-- Check that the i-th coord. hyperplane is contracted to the i-th coord. point
for i from  0 to n do print cleanList for a in Crent list  sub( a  , x_i=>0)
DDD1 = set primaryDecomposition  minors(2,matrix{R_*,Crent});
DDD1=DDD1-DDD                                               -- Fixed points
for d in toList DDD1 list evalIdeal d
```

Exercise 1.34 (Blowup at three points)**.** Consider the standard Cremona transformation $\mathrm{c}_2 \in \operatorname{Bir}(\mathbb{P}^2)$. We have a commutative diagram

$$\begin{tikzcd} & B \arrow[dl, "\varepsilon"'] \arrow[dr, "\theta"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\mathrm{c}_2"] & & \mathbb{P}^2 \\ {[x_1 : x_2 : x_3]} \arrow[rr, mapsto] & & \left[\frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}\right], \end{tikzcd} \tag{1.5}$$

where $B = \overline{\operatorname{graph}(\mathrm{c}_2)}$ is the closure of the graph.

- Realise the blowup $\operatorname{Bl}_{e_0,e_1,e_2} \mathbb{P}^2$ as a closed subset of $\mathbb{P}^2 \times \mathbb{P}^2$, see Figure 2.
- Realise the blowup $\operatorname{Bl}_{e_0,e_1,e_2} \mathbb{P}^2$ as a closed subset of $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
- Show that the restriction of the canonical projection
$$\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$
to $\operatorname{Bl}_{e_0,e_1,e_2} \mathbb{P}^2$ is a closed immersion.

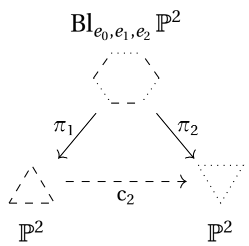

FIGURE 2. The resolution of the indeterminacies of the standard Cremona transformation in dimension 2.

Exercise 1.35. Let $T = \bigcup_{0\le i<j\le 3} \left\{ x_i = x_j = 0 \right\} \subset \mathbb{P}^3$ be the coordinate tetrahedron, i.e. the union of the coordinate lines of $\mathbb{P}^3$. Denote by X the blowup $X = \mathrm{Bl}_T \mathbb{P}^3$.

- Realise X as a closed subset of $\mathbb{P}^3 \times \mathbb{P}^3$.
- Show that X has 12 singular points.
- Identify all the irreducible components of the exceptional locus E_T. (**Hint:** there are 10 of them.)

```
-- Declear ambient space
R = QQ[x_0 .. x_3];
S = R[w_0 .. w_3];

Crent = for i from 0 to 3 list product for j in delete (i, 0..3 ) list x_j;
D = minimalPrimes minors (2,matrix {Crent ,S_* });
I = (select(D , d-> dim d<6 ))#0;
E= normalCone ideal Crent;
K=sub(ker map(ambient E , ambient ambient E),S)
I==K                                                    -- Check the construction is the same
O = QQ [R_*|S_*]
point={for r in R_* list sub(r,O),for s in S_* list sub(s,O)};
-- Function to pass from ideal to parametrisation
evalIdeal = I ->  for u in point list cleanList  apply( u , v ->  sub(sub(v,O/I),O) ) ;
I = sub(I,O);
J = jacobian I;
U = saturate(saturate(trim(minors (3,J)+I),ideal(x_0..x_3)), ideal(w_0..w_3));
for u in minimalPrimes U list evalIdeal u
```

2. Lecture 2

In this second lecture, we introduce some useful tools for dealing with ideals of the polynomial ring, such as the notion of socle and Macaulay duality. Along the way, we will see many examples concerning monomial ideals. As a computational example, we treat the combinatorial problem of counting higher-dimensional partitions. Finally, we present the first examples of deformation theory in terms of the module $\mathrm{Hom}_R(I, R/I)$.

2.1. Notation.

We work over the field of complex numbers $\mathbb{C}$.

- We denote by R a polynomial ring with complex coefficients and by $\mathfrak{m} \subset R$ the maximal ideal generated by the variables. If not specified, the set of variables will be
  - $x_1, \dots, x_n$ in the n-dimensional setting,
  - x, y in the two-dimensional setting.
- The ring R is endowed with the standard grading $\deg x_i = 1$, for all $i = 1, \dots, n$. This gives $R = \oplus_{k\ge 0} R_k$ where

$$R_k = \left\{ f \in R \,\middle|\, f \text{ homogeneous, and } \deg f = k \right\} \cup \{0\}.$$

- We will denote by the same symbols the variables in R and their image in the quotient R/I.
- Given a $\mathfrak{m}$-primary ideal I, we denote by $\operatorname{len} R/I$ its colength, i.e.

$$\operatorname{len} R/I = \dim_{\mathbb{C}} R/I.$$

- The semigroup $\mathbb{N}^n$ is endowed with the poset structure given by componentwise comparison. All its subsets will be considered as poset with the restricted structure.

2.2. **Graded modules and Hilbert–Samuel function.**

Definition 2.1. An algebra A is graded if there exists a direct sum decomposition $A=\bigoplus_{k\in\mathbb{Z}} A_k$ such that the multiplication on A reads as follows in each degree $k\in\mathbb{Z}$,

$$\begin{aligned} A_h \times A_k &\longrightarrow A_{h+k} \\ (a,b) &\longmapsto ab. \end{aligned}$$

Given a graded algebra $A=\bigoplus_{k\in\mathbb{Z}} A_k$, an A-module M is graded if $M=\bigoplus_{k\in\mathbb{Z}} M_k$ and the action of A on M restricts as follows

$$A_h \times M_k \longrightarrow M_{h+k},$$

for all $h,k\in\mathbb{Z}$.

Definition 2.2. Let $(A,\mathfrak{m}_A)$ be a local algebra. Its associated graded algebra $\mathscr{G}\mathrm{r}_{\mathfrak{m}_A} A$ is the algebra

$$\mathscr{G}\mathrm{r}_{\mathfrak{m}_A} A=\bigoplus_{i\geq 0}\mathfrak{m}_A^i/\mathfrak{m}_A^{i+1}.$$

Exercise 2.3. Prove the following properties.

- The associated graded algebra $\mathscr{G}\mathrm{r}_{\mathfrak{m}_A} A$ is graded.
- Recall that a homogeneous ideal $I\subset R$ is an ideal which admits a generating set containing only homogeneous elements. Let I be a homogeneous ideal. Show that
  - I is a graded R-module,
  - R/I is a graded R-module,
  - $\mathrm{Hom}_R(I,R/I)$ is a graded module with

    $$\mathrm{Hom}_R(I,R/I)_d=\left\{\varphi\in\mathrm{Hom}_R(I,R/I)\,\middle|\,\varphi(I_i)\subset(R/I)_{i+d},\text{ for all } i\in\mathbb{Z}\right\}\cup\{0\}.$$

    Notice that, even if $I_d\cong(R/I)_d\cong(0)$ for all $d<0$, the same is not necessarily true for the module $\mathrm{Hom}_R(I,R/I)$.

Definition 2.4. Consider an element $f\in R$ and write it as

$$f=\sum_{i=k_f}^{\deg(f)} f_i,$$

where $f_i\in R_i$, for $i=k_f,\dots,\deg(f)$. Then, the initial form of f is $\mathrm{In} f=f_{k_f}$. Moreover, if $I\subset R$ is any ideal, its initial ideal is

$$\mathrm{In} I=\left\{\mathrm{In} f\,\middle|\,f\in I\right\}.$$

Exercise 2.5. Let $I\subset R$ be any ideal. Then,

- the initial ideal $\mathrm{In} I$ is homogeneous,
- if I is $\mathfrak{m}$-primary, $\mathscr{G}\mathrm{r}_{\mathfrak{m}/I} R/I\cong R/\mathrm{In} I$.

Example 2.6. Let $f=y^2-x^2(x+1)$ be the polynomial defining the nodal cubic. Then, we have $\mathrm{In} f=y^2-x^2$. While, for the ideal $I=(x+y^2+y^3+y^4,y^{10})$, we have $\mathrm{In}(I)=(x,y^{10})$.

Exercise 2.7. If the scheme $X = \operatorname{Spec}(R/I)$ is smooth at the origin $0 \in X \subset \mathbb{A}^n$, then we have an isomorphism of $\mathbb{A}^n$-schemes

$$\operatorname{Spec} R/\operatorname{In} I \cong \operatorname{Spec} \operatorname{Sym}(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2),$$

where $\overline{\mathfrak{m}} = \mathfrak{m}/I$.

Remark 2.8. In general, the initial ideal defines the so-called tangent cone to a variety $X \subset \mathbb{A}^n$ at the origin $0 \in \mathbb{A}^n$. Roughly speaking it is the union of the lines having the maximum possible multiplicity intersection with X at the origin, see [5, §5.4] for more details.

Definition 2.9. Let A be a graded $\mathbb{C}$-algebra of finite type, and let $M = \bigoplus_{k\in\mathbb{Z}} M_k$ be a finitely generated graded A-module. The Hilbert–Samuel function associated to M is

$$\begin{aligned} \mathbb{Z} &\xrightarrow{h_M} \mathbb{N} \\ i &\longmapsto \dim_{\mathbb{C}} M_i. \end{aligned}$$

Now, let $(A, \mathfrak{m}_A)$ be an Artinian local $\mathbb{C}$-algebra of finite type. The Hilbert–Samuel function associated to A is

$$\begin{aligned} \mathbb{Z} &\xrightarrow{h_A} \mathbb{N} \\ i &\longmapsto \dim_{\mathbb{C}} {\mathfrak{m}_A}^i/{\mathfrak{m}_A}^{i+1}. \end{aligned}$$

Exercise 2.10. Definition 2.9 is well posed, i.e. the vector spaces M_k have finite dimension over $\mathbb{C}$, for all $k \in \mathbb{Z}$.

Remark 2.11. Notice that

$$h_A \equiv h_{\mathscr{G}\mathrm{r}_{\mathfrak{m}_A} A}.$$

Recall that a $\mathbb{C}$-algebra of finite type A is of the form $A = \mathbb{C}[x_1,\dots,x_n]/I$, for some $n \in \mathbb{N}$ and some ideal $I \subset \mathbb{C}[x_1,\dots,x_n]$. Equivalently, there is a closed immersion $\operatorname{Spec}(A) \hookrightarrow \mathbb{A}^n$ for some n.

Exercise 2.12. Compute h_R and show that if I is homogeneous, then

$$h_R = h_I + h_{R/I},$$

see Exercise 2.3.

Exercise 2.13. Let $I \subset R$ be a $\mathfrak{m}$-primary ideal. Then

- $h_{R/I}(0) = 1$
- $|h_{R/I}| = \sum_{i\geq 0} h_{R/I}(i) = \operatorname{len} R/I$.

Notice that the two together imply $0 \leq h_{R/I}(1) \leq (\operatorname{len} R/I) - 1$. Moreover, since R/I is a finite dimensional vector space, the function $h_{R/I}$ must vanish definitively. Therefore, we represent it as a vector, implicitly assuming that the values not displayed are zero.

Example 2.14. Consider the $\mathfrak{m}$-primary ideal $I = (x^2, xy, y^4)$. Then, as a vector space, R/I has the following direct sum decomposition

$$\mathbb{C}[x,y]/I = \underbrace{1\cdot\mathbb{C}}_{(R/I)_0} \oplus \underbrace{x\cdot\mathbb{C} \oplus y\cdot\mathbb{C}}_{(R/I)_1} \oplus \underbrace{y^2\cdot\mathbb{C}}_{(R/I)_2} \oplus \underbrace{y^3\cdot\mathbb{C}}_{(R/I)_3}.$$

And we get

$$h_A = (1,2,1,1).$$

Notice also that the $\mathbb{C}[x,y]$-module structure of the quotient $\mathbb{C}[x,y]/I$ is encoded in the diagram in Figure 3.

Precisely, moving right (resp. down) corresponds to the multiplication by x (resp. y).

$$\begin{array}{|c|c|}\hline 1 & x \\ \hline y & \multicolumn{1}{c}{} \\ \cline{1-1} y^2 & \multicolumn{1}{c}{} \\ \cline{1-1} y^3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

FIGURE 3. Graphical description of the R-module structure of R/I.

2.2.1. *Computation of the Hilbert–Samuel function.*

```
R = QQ[x,y]; mm=ideal(x,y);
I=ideal(x^2+y^3+5*y^4,y^5,x*y)
--The function tangentCone computes the initial ideal.
J=tangentCone I
-- M2 prints the value of the H--S function of R/J computed at i asking hilbertFunction(i,J)
for i from 0 to 10 list hilbertFunction (i,J)
for i from 0 to 10 list hilbertFunction (i,R/J)
for i from 0 to 10 list hilbertFunction (i,R)
```

Definition 2.15. The number $h_{R/I}(1)$ is called the embedding dimension of R/I. In symbols, we write $\mathrm{emb}_{R/I} = h_{R/I}(1)$.

Exercise 2.16. Let I be a $\mathfrak{m}$-primary ideal then $\mathrm{emb}_{R/I}$ is the smallest integer k such that there exists a closed immersion $\mathrm{Spec}(R/I) \hookrightarrow \mathbb{A}^k$ sending the unique closed point of the support of $\mathrm{Spec}\, R/I$ to the origin $0 \in \mathbb{A}^k$.

Hint: See Exercise 2.7 and Remark 2.8.

2.3. **Apolarity.** Apolarity (or Macaulay duality) is a powerful tool in commutative algebra. This technique has many application in algebraic geometry. For instance it helps in many cases to construct families of homogeneous ideals. The moral behind apolarity is:

construct the quotient instead of the ideal.

Working in characteristic 0 is crucial in this section.

Let us set

$$R = \mathbb{C}[x_1,\dots,x_n], \quad R^* = \mathbb{C}[y_1,\dots,y_n].$$

We view R^* as a R-module via the action

$$\begin{array}{ccc} R \times R^* & \longrightarrow & R^* \\ (x_1^{\alpha_1}\cdots x_n^{\alpha_n}, p(y_1,\dots,y_n)) & \longmapsto & \dfrac{\partial^{\sum_{i=1}^n \alpha_i}}{\partial^{\alpha_1} y_1 \cdots \partial^{\alpha_n} y_n} p, \end{array}$$

where $\alpha_i \in \mathbb{Z}_{\geq 0}$ for $i = 1,\dots,n$. This induces, for every $k \geq 0$, a perfect pairing

$$R_k \times R_k^* \longrightarrow R_0^* = \mathbb{C}$$

and, consequently, a notion of orthogonality.

We say that a vector subspace $T \subset R^*$ is graded if

$$T = \bigoplus_{k\in\mathbb{Z}} T \cap R_k^*.$$

Definition 2.17. An *inverse system* is a graded vector subspace $T \subset R^*$ closed under differentiation. If $S \subset R^*$ is a finite subset containing only homogeneous elements, then the inverse system generated by S is the smallest graded subspace $\langle S\rangle \subset R^*$ containing S and closed under differentiation. The *apolar ideal* attached to T is

$$T^\perp = \{ r \in R \mid r \cdot T = 0 \} \subset R.$$

If $I \subset R$ is a homogeneous ideal, its associated inverse system is

$$I^\perp = \{ r^* \in R^* \mid I \cdot r^* = 0 \} \subset R^*.$$

Example 2.18. Consider the subset $S = \left\{ y_2^3, y_1 \right\} \subset R^*$. Then we have

$$\langle S\rangle = \left\{ 1, y_1, y_2, y_2^2, y_2^3 \right\}.$$

Notice[3] that, if $I \subset \mathbb{C}[x, y]$ is the ideal in Example 2.14, then the classes in $\mathbb{C}[x, y]/I$ of the elements in $\langle S\rangle$ provide, after an appropriate relabelling of the variables, a basis for $\mathbb{C}[x, y]/I$.

Then we compute the apolar ideal to $\langle S\rangle$ and we get

$$\langle S\rangle^\perp = (x_1^2, x_1x_2, x_2^4) \subset \mathbb{C}[x_1, x_2].$$

In particular, we get $R^*/\langle S\rangle^\perp \cong \mathbb{C}[x, y]/I$.

```
--Check for previous example
R = QQ[x,y]
ideal(y^4 , x*y,x^2) == inverseSystem matrix {{y^3,x}}
```

Example 2.18 is a special instance of a more general behaviour that we explain in Remark 2.19.

Remark 2.19. Notice that if $I \subset R$ is a homogeneous ideal, then $I^\perp \subset R^*$ is a graded subspace closed under differentiation. Conversely, every graded vector subspace $T \subset R^*$ closed under differentiation is orthogonal to the homogeneous ideal $T^\perp \subset R$. Moreover, if $V \subset R_j^*$ is a vector subspace, then

$$\dim_{\mathbb{C}}(V^\perp)_j = \dim_{\mathbb{C}} R_j^*/V,$$

which yields an isomorphism of graded vector spaces $R/I \cong I^\perp$, see [3, Sec. 2].

2.4. **Monomial ideals and Partitions.** From now on we focus on monomial ideals, i.e. ideals admitting a generating set consisting only of monomials. In particular these ideals are homogeneous.

Definition 2.20. Fix $n, d \in \mathbb{Z}_{\geqslant 0}$. An $(n-1)$*-dimensional partition of size* d is a collection of d points $\lambda = \{\mathbf{a}_1, \dots, \mathbf{a}_d\} \subset \mathbb{N}^n$ such that if $\mathbf{y} \in \mathbb{N}^n$ satisfies $\mathbf{y} \le \mathbf{a}_i$ for some $i = 1, \dots, d$, then $\mathbf{y} \in \lambda$. We call $|\lambda| = d$ the *size* of λ and we denote by P_d^n the set of $(n-1)$-dimensional partitions of size d, and by p_d^n the cardinality $|\mathrm{P}_d^n|$.

Exercise 2.21. Prove that the following associations are bijections

$$\left\{ \mathfrak{m}\text{-primary monomial ideals} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Subsets of } \mathbb{N}^n \text{ closed by translations} \\ \text{with finite complement} \end{array} \right\} \longleftrightarrow \left\{ \text{Partitions} \right\}$$

$$I \xmapsto{\quad\varphi\quad} \left\{ (\alpha_1, \dots, \alpha_n) \,\middle|\, \textstyle\prod_{i=1}^n x_i^{\alpha_i} \in I \right\}$$

$$A \xmapsto{\quad\psi\quad} A^c.$$

[3] Keep in mind the moral.

Definition 2.22. Let $(A, \mathfrak{m}_A)$ be a local Artinian $\mathbb{C}$-algebra of finite type. The socle of A is

$$\operatorname{Soc}(A) = (0_A : \mathfrak{m}_A) = \{\, a \in A \mid \mathfrak{m}_A \cdot a = 0 \,\}.$$

Exercise 2.23. Let $I \subset R$ be a $\mathfrak{m}$-primary ideal. Prove the following properties of $\operatorname{Soc}(R/I)$.

- The socle $\operatorname{Soc}(R/I)$ is a R-submodule of R/I.
- If I is a homogeneous ideal, then the socle $\operatorname{Soc}(R/I)$ is a graded R-module.
- If I is a monomial ideal, then $\operatorname{Soc}(R/I)$ corresponds to $\max \psi(\varphi(I))$, where φ, ψ are defined in Exercise 2.21.

2.4.1. *Computation of the socle of R/I, for $I \subset R$ monomial ideal.*

```
R = QQ[x,y]; mm=ideal(x,y);
-- Function to compute the socle of R/I with M monomial ideal
socleMonomial = I->  for u in  (trim sub(I:mm,R/I))_* list  sub(u,R);
J=ideal(x^2,y^5,x*y)
SOCLE = socleMonomial J
K = inverseSystem matrix {SOCLE}
K==J                                         -- Check that double-perp is the identity
```

Definition 2.24. A subset $S \subset \mathbb{N}^n$ is admissible if every two elements $s_1, s_2 \in S$ are not comparable with respect to the poset structure of $\mathbb{N}^n$.

Exercise 2.25. Prove that any admissible subset $S \subset \mathbb{N}^n$ is finite.
Hint: Induction.

Exercise 2.26. There is a bijection

$$\big\{\, \mathfrak{m}\text{-primary monomial ideals} \,\big\} \longleftrightarrow \{\, S \subset \mathbb{N}^n \text{ admissible} \,\}.$$

Hint: See Exercise 2.23.

2.5. **Special case $n = 2$.** The case $n = 2$ is special since, as we show in what follows, all admissible sets in dimension 2 are totally ordered.

Definition 2.27. Given two points $(n_1, n_2), (m_1, m_2) \in \mathbb{N}^2$ not comparable with respect to the poset structure of $\mathbb{N}^2$ we say that $(n_1, n_2) \prec (m_1, m_2)$ if

$$\begin{cases} m_2 > n_2, \\ n_1 > m_1. \end{cases}$$

Exercise 2.28. Definition 2.27 induces a total order on every admissible subset of $\mathbb{N}^2$.

Exercise 2.29. There is a bijection

$$\big\{\, \mathfrak{m}\text{-primary monomial ideals in 2 variables} \,\big\} \longleftrightarrow \big\{\, \text{totally } (\prec)\text{−ordered subsets of } \mathbb{N}^2 \,\big\}.$$

Exercise 2.30. List all the monomial ideals in two variables of colength smaller or equal to 6.

Exercise 2.31. List all the pairs (I_1, I_2) of monomial ideals in two variables of respective colengths $d_1 \le d_2 \le 6$ such that $\psi(\varphi(I_1)) \subset \psi(\varphi(I_2))$ and depict the respective partitions.

2.5.1. *Pictorial description of the bijiection in Exercise 2.29.*

```
R = QQ[x,y]; mm=ideal(x,y); d=7;
--Function to depicts 1-partitions
drawPartition = d -> for a in d do (  print (for b from 0 to  a -1 list "*");)
--Function to convert partitions to ideals
partToideal = a -> trim(ideal(x^#a)+ideal(for b from 0 to #a-1 list y^(a#b)*x^b));
--Function to compute socle
socleMonomial = I-> for u in  (trim sub(I:mm,R/I))_* list  sub(u,R);
-- Computation of monomial ideals of colength d in 2 variables
P = partitions(d)
M= for a in P list partToideal a;
for p in P do (
    drawPartition p;
    print(" ");
    print(partToideal p);
    print("socle: ", socleMonomial partToideal p);
    print(" "); print(" ");
)
```

2.6. **The computation of** $\mathrm{Hom}_R(I,R/I)$ **for a monomial ideal** I**.** Let us recall the definition of minimal free resolution of a finitely generated R-module M.

Definition 2.32. Let M be a finitely generated R-module. A minimal free resolution is an exact sequence of the form

$$0 \longleftarrow M \longleftarrow F^\bullet, \tag{2.1}$$

where

$$F^\bullet: \qquad \cdots \longleftarrow F_{i-1} \xleftarrow{\delta_i} F_i \longleftarrow \cdots,$$

with all the F_i's free modules, i.e. $F_i \cong R^{n_i}$, for some $n_i \in \mathbb{N}$, and such that $\delta_i(F_i) \subset \mathfrak{m}F_{i-1}$.

Remark 2.33. The elements in F_0 are then called generators of M. The name generators is related to the fact that the first map $F_0 \xrightarrow{\delta_0} M$ in (2.1) is surjective, thus the image of a generating set for F_0 via δ_0 is a generating set for M.

The elements in F_1 are called first syzygies. This term means relation. Since the sequence (2.1) is exact, we have $\ker\delta_0 = \mathrm{Im}\,\delta_1$, i.e. any element in F_1 gives a relation between the generators of M, via the map δ_1.

The following theorem due to Hilbert guarantees that, when working in the polynomial setting, minimal free resolutions have finite length.

Proposition 2.34 ([1, Hilbert syzygy theorem])**.** *Any graded* $\mathbb{C}[x_1,\dots,x_n]$*-module* M *of finite type has a graded free resolution of length at most* n*.*

Exercise 2.35. Compute a minimal free resolution of the R module R/I where I is the ideal of the twisted cubic, see Definition 1.5.

2.6.1. *The minimal free graded resolution of the ideal of twisted cubic.*

```
n=3; R = QQ [x_0 .. x_n];
-- Computation of the ideal I_C of the twisted cubic
M = matrix { apply(n,i->R_*#i),apply(n,i->R_*#(i+1)) }
```

```
C = trim minors(2,M);
D = res C                              -- res C computes the resolution of R/C
for i from 1 to n-1 do (
    print D.dd_i;
    print(" ");
)                                      -- Differential of the resolution
D.dd_1*D.dd_2                          -- Check it is a complex
```

Given an ideal $I \subset R$, the space of homomorphisms $\mathrm{Hom}_R(I, R/I)$ provides very important geometrical informations about the zero locus of I. Precisely it encodes the infinitesimal first order deformations of $\mathrm{Spec}(R/I)$ as we will see in Lecture 4.

We clearly have an injection

$$\mathrm{Hom}_R(I, R/I) \overset{\iota}{\hookrightarrow} \mathrm{Hom}_{\mathbb{C}}(I, R/I).$$

On the other hand a $\mathbb{C}$-linear homomorphism $\varphi \in \mathrm{Hom}_{\mathbb{C}}(I, R/I)$ is also R-linear if and only if it is compatible with the syzygies of I, i.e. given a relation

$$\sum_{j=1}^{s} p_j i_j = 0 \in R,$$

for some $p_j \in R$, $i_j \in I$, for $j = 1, \ldots, s$, we require

$$\sum_{j=1}^{s} p_j \varphi(i_j) = 0 \in R/I.$$

Remark 2.36. These are the basics of deformation theory, see Lectures 3 and 4.

Exercise 2.37. If $I \subset R$ is a monomial ideal and $\{ m_1, \ldots, m_s \}$ is a minimal set of monomial generators, then any R-linear homomorhism is uniquely determined by the images of the m_i's.

Exercise 2.38. Let $I \subset R$ be a $\mathfrak{m}$-primary homogeneous ideal.

- Show that partial derivatives $\frac{\partial}{\partial x_i}$ are well defined and linearly independent elements of $\mathrm{Hom}_R(I, R/I)_{-1}$.
- Show that any $\mathbb{C}$-linear homomorphism in $\mathrm{Hom}_{\mathbb{C}}(I, R/I)$ with target in the socle $\mathrm{Soc}(R/I)$ is naturally a R-homomorphism.

The following theorem is one possible incarnation of the celebrated result by Fogarty [6] about smoothness of the Hilbert scheme of points on a smooth surface.

Theorem 2.39. *Let $I \subset R = \mathbb{C}[x, y]$ be a $\mathfrak{m}$-primary homogeneous ideal. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(I, R/I) = 2 \dim_{\mathbb{C}} R/I.$$

On the other hand the dimension of the non-negative part of the space $\mathrm{Hom}_{\mathbb{C}[x,y]}(I, \mathbb{C}[x, y]/I)$ was computed independently by Iarrobino and Briançon in [10] and [2] respectively.

Theorem 2.40 ([2, Thm. III.3.1] and [10, Thm 1])**.** *Let $I \subset R = \mathbb{C}[x, y]$ be a $\mathfrak{m}$-primary homogeneous ideal and let $h_{R/I}$ be the associated Hilbert–Samuel function. Denote*[4] *by $d, s > 0$ the integers such that*

$$h_{R/I} = (1, 2, \ldots, d, h_d, \ldots, h_{d+s-1}, 0, \ldots),$$

with $h_d < d + 1$ and $h_i \geqslant h_{i+1}$, for $i \geqslant d$. Then, we have

[4] Notice that there always exist such d, s.

$$\dim_{\mathbb{C}} \bigoplus_{k\geq 0} \mathrm{Hom}_R(I,R/I)_k = |h_{R/I}| - d - \sum_{i\geqslant d} \binom{h_{i-1}-h_i}{2}, \tag{2.2}$$

and

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(I,R/I)_0 = \sum_{i\geqslant d} (h_{i-1}-h_i+1)(h_i-h_{i+1}).$$

Example 2.41. Consider the ideal $I=\mathfrak{m}^2=(x^2,xy,y^2)$ and let $\varphi\in\mathrm{Hom}_R(I,R/I)$ be a R-linear homomorphism. Since I is monomial, by Exercise 2.37, the homomorphism φ is uniquely determined by the images of x^2,xy,y^2.

Consider, for $A=(\alpha,\alpha_x,\alpha_y,\beta,\beta_x,\beta_y,\gamma,\gamma_x,\gamma_y)\in\mathbb{C}^6$ the $\mathbb{C}$-linear homomorphism

$$\begin{array}{rcl}
\mathrm{Span}_{\mathbb{C}}(x^2,xy,y^2) & \xrightarrow{\overline{\varphi}_A} & R/I \\
x^2 & \longmapsto & \alpha+\alpha_x x+\alpha_y y \\
xy & \longmapsto & \beta+\beta_x x+\beta_y y \\
y^2 & \longmapsto & \gamma+\gamma_x x+\gamma_y y.
\end{array}$$

Then, the map $\overline{\varphi}_A$ extends to a (unique) R-linear homomorphism $\varphi_A\in\mathrm{Hom}_R(I,R/I)$ if and only in the following conditions are satisfied

$$y\overline{\varphi}_A(x^2)-x\overline{\varphi}_A(xy)=0,\quad x\overline{\varphi}_A(y^2)-y\overline{\varphi}_A(xy)=0,\quad y^2\overline{\varphi}_A(x^2)-x^2\overline{\varphi}_A(y^2)=0.$$

This imposes the conditions

$$\alpha=\beta=\gamma=0. \tag{2.3}$$

Notice that, since the last syzygy is algebraically dependent from the first two, it is enough to consider the first two syzygies to compute the module $\mathrm{Hom}(I,R/I)$.

We also remark that this result can be obtained via a different argument. Indeed, any $\mathbb{C}$-homomorphism taking values in the socle is R-linear by Exercise 2.38. This, together with Fogarty's result, see Theorem 2.39, already implies conditions (2.3). As expected, we get

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(I,R/I)=2\cdot 3=6.$$

Notice that the tangent space is concentrated in degree -1.

2.6.2. *Computations from Example 2.41.*

```
R = QQ[x,y];
m2 =  (ideal R_*)^2;                            -- Square of the maximal ideal
N = Hom(m2,R/m2)                                -- Computation of Hom module
hilbertFunction (-1,N)                          -- Degree -1 H--S function of N
F = res m2;                                     -- Resolution of R/m2
A = F.dd_1                                      -- Generators
B = F.dd_2                                      -- First syzygies
-- Straightforward computation of Hom(m2,R/m2)
S = R[a_1..a_3,b_1..b_3,c_1..c_3]
m2 = sub(m2,S);
Syz_1 = entries (B_0);
Syz_2 = entries (B_1);
```

```
-- Impose syzygies to the image of a morphism
Im = for i from 1 to 3 list (a_i + b_i *x + c_i*y)                       -- Image of a morphism
Check = ideal for j from 1 to 2 list sum for i from 0 to 2 list (Im#i)*(Syz_j#i) -- Imposition
sub(Check , S/m2)                                          -- Recover the conditions a_1=a_2=a_3=0
```

Exercise 2.42. Compute the modules $\operatorname{Hom}_R(I, R/I)$, for I one of the following ideals.

- $I = (x, y) \subset \mathbb{C}[x, y]$,
- $I = (x, y, z)^2 \subset \mathbb{C}[x, y, z]$,
- $I = (x, y)^2 + (z, t)^2 + (xz - yt) \subset \mathbb{C}[x, y, z, t]$.

In the third case show that

$$\dim_{\mathbb{C}} \operatorname{Hom}_R(I, R/I)_{-1} = 4.$$

In particular, by Exercise 2.38, we get

$$\operatorname{Hom}_R(I, R/I)_{-1} = \operatorname{Span}_{\mathbb{C}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right).$$

2.6.3. *Some computations for Exercise 2.42.*

```
-- Ideal I1
R1 = QQ[x,y]; I1 = ideal R1_*; N1 =Hom(I1,R1/I1);
-- Ideal I2
R2 = QQ[x,y,z]; I2 = (ideal R2_*)^2; N2 =Hom(I2,R2/I2);
-- Ideal I3
R3=QQ[x,y,z,t]; I3=(ideal(x,y))^2+(ideal(z,t))^2+det(matrix{{x,z},{y,t}}); N3=Hom(I3,R3/I3);
-- Dimension of tangent spaces
(degree N1, degree N2, degree N3)
-- Degree -1 part of N3
hilbertFunction(-1,N3)
```

Exercise 2.43. Consider the power of the maximal ideal $(x, y)^k \subset \mathbb{C}[x, y]$, for some $k \geq 1$. Then $\operatorname{Hom}_R(I, R/I)$ is concentrated in degree -1.

```
n= 3; k= 4; R = QQ[x_1..x_n]; m=ideal R_*;
N=Hom(m^k,R/m^k);
-- User-friendly output
for i from -2 to 2 do print("h("|toString i |")="|toString hilbertFunction(i,N))
```

3. Lecture 3

4. Lecture 4

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