COMPUTATIONS IN ALGEBRAIC GEOMETRY

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ABSTRACT. These notes follow the content of the course *"Computations in algebraic geometry"* held at *Unicamp IMECC* in Spring (Brazilian fall) 2025. The final goal of the course is to learn how to parameterise locally closed subset of the Hilbert schemes of points. During the course, four main topics have been covered in four lectures. These are classical birational geometry, deformations of monomial ideals, deformations of zero-dimensional ideals, Hilbert schemes of points. Each lecture contains explicit examples of computations via Macaulay2.

CONTENTS

1. LECTURE 1

In this lecture we recall the definition of classical maps in algebraic geometry such as Veronese/Segre embeddings, projections and the standard Cremona transformation. In addition, there are examples in the text of how to perform calculations related to this topic using the software *Macaulay2*, [3].

1.1. **Notation.** We work over the field of complex numbers $\mathbb{C}$.

- We adopt throughout the Zariski topology.
- We denote by R a polynomial ring with complex coefficients. If not specified, the set of variables will be
  - $x_0,\dots,x_n$ in the projective setting,
  - $x_1,\dots,x_n$ in the affine setting,
  - x, y in the affine two-dimensional setting.
- The ring R is endowed with the standard grading $\deg x_i = 1$, for all $i = 1,\dots,n$. This gives $R = \oplus_{k\geq 0} R_k$ where
$$R_k = \left\{ f \in R \;\middle|\; f \text{ homogeneous, and } \deg f = k \right\} \cup \{0\}.$$
- We denote by $\mathbb{A}^n = \mathrm{Spec}(R)$ the n-dimensional affine space with coordinates (generators of R) $x_1,\dots,x_n$, if not specified otherwise.
- Similarly, $\mathbb{P}^n = \mathrm{Proj}(R)$ denotes the n-dimensional projective space with homogeneous coordinates (generators of R) $x_0,\dots,x_n$, if not specified otherwise.

- We denote by $H_i = \{ x_i = 0 \} \subset \mathbb{P}^n$, for $i = 0, \ldots, n$, the i-th coordinate hyperplane. Moreover, we denote by $e_i \in \mathbb{P}^n$, for $i = 0, \ldots, n$, the i-th coordinate point.
- The symbols $\mathrm{Mat}(n,m,\mathbb{C})$ and $\mathrm{Sym}(n,\mathbb{C}) \subset \mathrm{Mat}(n,n,\mathbb{C})$ denote respectively the vector spaces of matrices and symmetric matrices with complex entries. The symbol $\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{Mat}(n,n,\mathbb{C})$ denotes the general linear group.
- Given any subset $S \subset \mathbb{P}^n$, we denote by $\langle S \rangle \subset \mathbb{P}^n$ the smallest linear subspace containing S.
- $\mathrm{Bir}(X)$ denotes the group of birational transformations of a variety X.

1.2. **Veronese embeddings.** The first non-trivial example of morphism between projective spaces is provided by the n-th Veronese embedding of degree d.

Definition 1.1. The *n-th Veronese embedding of degree d* is the morphism defined by

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\mathrm{v}_{n,d}} & \mathbb{P}^{\binom{n+d}{d}-1} \\ [x_0 : \cdots : x_n] & \longmapsto & \left[x_0^{\alpha_0} \cdots x_n^{\alpha_n} \mid \sum_{i=0}^n \alpha_i = n \right]. \end{array}$$

Proposition 1.2. *The morphism $\mathrm{v}_{n,d}$ is a closed immersion.*

Exercise 1.3. Prove Proposition 1.2. **Hint:** see Example 1.4.

Example 1.4. Let us explain in details the case $n = 1$. Fix homogeneous coordinates x_0, x_1 on $\mathbb{P}^1$ and y_i, for $i = 0, \ldots, d$ on $\mathbb{P}^d$. The first-Veronese embedding of degree d reads then as

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\mathrm{v}_{1,d}} & \mathbb{P}^d \\ [x_0 : x_1] & \longmapsto & \left[x_0^d : x_0^{d-1} x_1 : \cdots : x_1^d \right]. \end{array}$$

Consider the charts $U_i = \{ x_i \neq 0 \} \cong \mathbb{A}^1$, for $i = 0, 1$ and $V_j = \left\{ y_j \neq 0 \right\} \cong \mathbb{A}^d$, for $j = 0, \ldots, d$. Then, on U_i we have coordinates

$$t_i = \left(\frac{x_1}{x_0} \right)^{(-1)^i},$$

for $i = 0, 1$. The restrictions of $\mathrm{v}_{1,d}$ to U_0 and U_1 have the form

$$(1.1) \qquad \begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ t_0 & \longmapsto & (t_0, t_0^2, \ldots, t_0^d), \end{array} \quad \text{and} \quad \begin{array}{ccc} U_1 & \longrightarrow & V_d \\ t_1 & \longmapsto & (t_1^d, t_1^{d-1}, \ldots, t_1). \end{array}$$

In particular, the image of $\mathrm{v}_{1,d}$ is entirely contained $V_1 \cup V_d$. Notice also that the image is a smooth curve as both the maps in (1.1) are parametrisations of smooth curves.

Let us compute the image of the morphism $\mathrm{v}_{1,d}$. In order to do this, we first observe that any two consecutive entries of the map $\mathrm{v}_{1,d}$ have the same ratio, namely x_0/x_1. Therefore, the equality

$$(1.2) \qquad [y_0 : \cdots : y_d] = \mathrm{v}_{1,d}([x_0 : x_1]) = \left[x_0^d : x_0^{d-1} x_1 : \cdots : x_1^d \right]$$

implies

$$\mathrm{rk} \begin{bmatrix} y_0 & \cdots & y_{d-1} \\ y_1 & \cdots & y_d \end{bmatrix} \leq 1.$$

Define $Y_{1,d} \subset \mathbb{P}^d$ to be the closed subset given by

$$Y_{1,d} = \left\{ [y_0 : \cdots : y_d] \in \mathbb{P}^d \;\middle|\; \mathrm{rk} \begin{bmatrix} y_0 & \cdots & y_{d-1} \\ y_1 & \cdots & y_d \end{bmatrix} \leq 1 \right\}.$$

Clearly $\mathrm{v}_{1,n}(\mathbb{P}^1) \subset Y_{1,d}$. We prove that $\mathrm{v}_{1,d} : \mathbb{P}^1 \to Y_{1,d}$ is invertible and this implies that $\mathrm{v}_{1,d}$ is a closed immersion. Its inverse is

$$\begin{aligned} Y_{1,d} &\xrightarrow{\ \rho_{1,d}\ } \mathbb{P}^1 \\ [y_0 : \cdots : y_d] &\longmapsto [y_0 : y_1]. \end{aligned}$$

To see this we only have to show that the map $\rho_{1,d}$ is well defined. This is true because it can be extended to points of the form $[0:0:y_2:\cdots:y_d] \in Y_{1,d}$ via

$$[y_0 : \cdots : y_d] \mapsto [y_{d-1} : y_d].$$

Indeed, the conditions $y_0 = y_1 = 0$ and $y_{d-1} = y_d = 0$ are incompatible on $Y_{1,d}$ and $[y_0 : y_1] = [y_{d-1} : y_d]$, by (1.2).

Definition 1.5. The rational normal curve of degree d is the image of the morphism $\mathrm{v}_{1,d}$. If $d = 2$, the rational normal curve is called conic, and for $d = 3$, it is called twisted cubic.

Example 1.6. We describe now the degree-2 Veronese embeddings. In this setting the projective space $\mathbb{P}^{\binom{n+2}{2}-1}$ identifies with $\mathbb{P}\,\mathrm{Sym}(n+1,\mathbb{C})$ and we get the following expression for the morphism $\mathrm{v}_{n,2}$.

$$\begin{aligned} \mathbb{P}^n &\xrightarrow{\ \mathrm{v}_{n,2}\ } \mathbb{P}\,\mathrm{Sym}(n+1,\mathbb{C}) \\ [x_0 : \cdots : x_n] &\longmapsto \begin{bmatrix} x_0^2 & \cdots & \cdots & x_0 x_n \\ x_1 x_0 & x_1^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_0 x_n & x_1 x_n & \cdots & x_n^2 \end{bmatrix}. \end{aligned}$$

Exercise 1.7. Prove that

$$\mathrm{v}_{n,2}(\mathbb{P}^n) = \left\{ [M] \in \mathbb{P}\,\mathrm{Sym}(n+1,\mathbb{C}) \,\middle|\, \mathrm{rk}\, M \le 1 \right\}. \tag{1.3}$$

Hint: Clearly $\mathrm{v}_{n,2}(\mathbb{P}^n)$ is contained in the right hand side of Equation (1.3). For the opposite inclusion consider the projections on the i-th row (or the i-th column), for $i = 0, \ldots, n$.

Example 1.8. The easiest example is the second Veronese surface, i.e. the image of the morphism

$$\begin{aligned} \mathbb{P}^2 &\xrightarrow{\ \mathrm{v}_{2,2}\ } \mathbb{P}\,\mathrm{Sym}(3,\mathbb{C}) \cong \mathbb{P}^5 \\ [x_0 : x_1 : x_2] &\longmapsto \begin{bmatrix} x_0^2 & x_0 x_1 & x_0 x_2 \\ x_1 x_0 & x_1^2 & x_1 x_2 \\ x_2 x_0 & x_2 x_1 & x_2^2 \end{bmatrix}. \end{aligned}$$

One of the many possible instances in which Veronese embeddings turn out to be useful is the description of loci parametrising homogeneous ideals.

Exercise 1.9. Fix some integer $d > 1$. Describe the locus

$$V_d^{\circ} = \left\{ [f] \in \mathbb{P}R_d \,\middle|\, \dim_{\mathbb{C}} \mathrm{Span}\left(\frac{\partial}{\partial x_i} f \right)_{i=0}^{n} > 1 \right\}.$$

We get

$$V_d^{\circ} = \mathbb{P}R_d \setminus \mathrm{v}_{n,d}(\mathbb{P}^n).$$

Exercise 1.10. Two important aspects of the Veronese embedding concern its degree and its projective normality.

- Let $L \subset \mathbb{P}^n$ be a line. Then, we have $\mathrm{v}_{n,d}|_L \equiv \mathrm{v}_{1,d}$. Moreover, the generic hyperplane intersects $\mathrm{v}_{n,d}(L)$ in d distinct points. We say that the morphism has topological degree d.
- Let $Y_d \subset \mathbb{P}^n$ be a hypersurface of degree d, i.e. $Y_d = V(f)$ for some $f \in R_d$. Then there exists a hyperplane $H \subset \mathbb{P}^{\binom{n+d}{d}-1}$ such that $Y_d = \mathrm{v}_{n,d}(\mathbb{P}^n) \cap H$. We say that $\mathrm{v}_{n,d}$ is projectively normal.

Remark 1.11. It is worth mentioning that topological degree and projective normality are defined in a much more general context. See [2] for more details on the topological degree and [4, §I.3, §II.5] for projective normality.

The following exercise shows that not all degree d closed immersions are projectively normal.

Exercise 1.12. Show that the composition

$$\begin{array}{ccccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^4 & \dashrightarrow & \mathbb{P}^3 \\ [x_0 : x_1] & \longmapsto & [x_0^4 : x_0^3 x_1 : x_0^2 x_1^2 : x_0 x_1^3 : x_1^4] & \longmapsto & [x_0^4 : x_0^3 x_1 : x_0 x_1^3 : x_1^4] \end{array}$$

is a closed immersion of topological degree 4. Show that it is not projective normal.

1.2.1. *Example of computation.*

```
-- Define a function that remove the common factors from the entries of a map
cleanFactors = f -> (
    L=(entries matrix f)#0;
    G=gcd L;
    Laux= for l in L list sub(l/G,source f );
    return map(target f, source f, Laux);
)
-- Declear ambient spaces
n = 3; d = 3; N = binomial(n+d,d)-1; R = QQ[x_0 .. x_n]; S = QQ[y_0 .. y_N];
-- Construct the embedding and compute the equations
vnd1 = ( ideal R_* )^d_*;                                  -- 1st possibility
vnd2 = (entries(monomials (sum R_*)^d))#0;                 -- 2nd possibility
set vnd1 == set vnd2                                       -- Check the two are the same
Verond = map(R,S,vnd1);                                    -- Define the map
Equations = trim ker Verond;                               -- Compute equations
-- Check of the smoothness of the image of e_0={1,0,0,..,0}.
-- WLOG we restrict to { x_0 != 0 } and { y_0 != 0 }.
-- For simplicity we keep the same symbols for the variables
R0 = QQ[ x_1 .. x_n ]; S0 = QQ[ y_1 .. y_N ];
vnd10 = drop(for v in vnd1 list sub(sub( v , x_0=>1),R0),1);
Verond0 = map(R0,S0,vnd10  );
Eqs0 = trim ker Verond0;
Eqs0 == sub(sub( Equations,y_0=>1),S0)                   -- It agrees with dehomogenising
J = sub( jacobian Eqs0, for i from 1 to N list y_i =>0 ); -- Jacobian at e_0
0 == (rank J - N + n)                                    -- Check it has rank N - n
-- Computation of the inverse map
use R; use S;
Inverse = map(S,R,for i from 0 to n list y_i);
cleanFactors ( Verond * Inverse )           -- Check that the composition is the identity
```

1.3. **Segre embeddings.** Segre embeddings provide a way to realise products of projective spaces as closed subsets of an ambient projective space.

Definition 1.13. Given two integers $n,m \in \mathbb{Z}_{\geq 0}$, the Segre (n,m)-embedding is the morphism

$$\begin{array}{ccc} \mathbb{P}^n \times \mathbb{P}^m & \xrightarrow{\mathrm{s}_{n,m}} & \mathbb{P}\mathrm{Mat}(n+1,m+1,\mathbb{C}) \cong \mathbb{P}^{(n+1)(m+1)-1} \\ ([x_0:\cdots:x_n],[y_0:\cdots:y_m]) & \longmapsto & \begin{bmatrix} x_0y_0 & \cdots & x_0y_m \\ \vdots & \ddots & \vdots \\ x_ny_0 & \cdots & x_ny_m \end{bmatrix} \end{array}$$

Proposition 1.14. *The morphism* $\mathrm{s}_{n,m}$ *is a closed immersion.*

Exercise 1.15. Prove Proposition 1.14.
Hint: Define

$$S_{n,m} = \left\{ [M] \in \mathbb{P}^{(n+1)(m+1)-1} \;\middle|\; \mathrm{rk}\, M \leq 1 \right\}.$$

Clearly, we have

$$\mathrm{s}_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subset S_{n,m}.$$

To conclude consider the morphism

$$\begin{array}{ccc} S_{n,m} & \longrightarrow & \mathbb{P}^n \times \mathbb{P}^m \\ \begin{bmatrix} z_{0,0} & \cdots & z_{0,m} \\ \vdots & \ddots & \vdots \\ z_{n,0} & \cdots & z_{n,m} \end{bmatrix} & \longmapsto & ([z_{0,0}:\cdots:z_{n,0}],[z_{0,0}:\cdots:z_{0,m}]). \end{array}$$

Example 1.16. For $n=m=1$, we get

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\mathrm{s}_{1,1}} & \mathbb{P}^3 \\ ([x_0:x_1],[y_0:y_1]) & \longmapsto & \begin{bmatrix} x_0y_0 & x_0y_1 \\ x_1y_0 & x_1y_1 \end{bmatrix} \end{array}$$

and

$$\mathrm{s}_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) = \left\{ z_{0,0}z_{1,1} - z_{1,0}z_{0,1} = 0 \right\}.$$

Since all smooth quadrics of $\mathbb{P}^n$ differ by projectivities, we deduce that all smooth quadrics of $\mathbb{P}^3$ are isomorphic to $(\mathbb{P}^1)^{\times 2}$.

Example 1.17. Consider the diagonal

$$\Delta = \left\{ (p,q) \in \mathbb{P}^n \times \mathbb{P}^n \;\middle|\; p = q \right\} \cong \mathbb{P}^n.$$

We then restrict the Segre embedding to $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ and we get the second Veronese embedding $\mathrm{v}_{n,2} : \Delta \to \mathbb{P}\,\mathrm{Sym}(n+1,\mathbb{C})$. Precisely, the following diagram

$$\begin{tikzcd} \Delta \arrow[rr, "\mathrm{s}_{n,n}|_\Delta"] & & \mathbb{P}^{(n+1)^2-1} \\ {[x_0:\cdots:x_n]} \arrow[rr, mapsto] \arrow[dr, "\mathrm{v}_{n,2}"'] & & \begin{bmatrix} x_0^2 & \cdots & \cdots & x_0x_n \\ x_1x_0 & x_1^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_0x_n & x_1x_n & \cdots & x_n^2 \end{bmatrix} \\ & \mathbb{P}\,\mathrm{Sym}(n+1,\mathbb{C}), \arrow[ur, hook] & \end{tikzcd}$$

commutes.

1.4. **Projections and blowups of linear subspaces.**

Definition 1.18. Let X and Y be quasi-projective varieties. A rational map $\varphi : X \dashrightarrow Y$ is an equivalence class of pairs (U, f), where U is a dense open subset of X and f is a morphism from U to Y, and where two pairs (U_1, f_1) and (U_2, f_2) are equivalent if there exists a dense open subset $V \subset U_1 \cap U_2$ such that $f_1|_V = f_2|_V$.

Projections are the first examples of rational maps.

Let $H, K \subset \mathbb{P}^n$ be linear subspaces such that $H \cong \mathbb{P}^k$, $K \cong \mathbb{P}^{n-k-1}$ and $H \cap K = \varnothing$. Without loss of generality we put

$$K = \{ x_0 = \cdots = x_k = 0 \},\ H = \{ x_{k+1} = \cdots = x_n = 0 \} \subset \mathbb{P}^n.$$

The projection onto H with centre K is the rational map

$$\begin{array}{ccc} \mathbb{P}^n & \overset{\pi_K}{\dashrightarrow} & H \\ [x_0 : \cdots : x_n] & \longmapsto & [x_0 : \cdots : x_k]. \end{array} \tag{1.4}$$

Notice that we do not report the dependence on H as it is not useful for our applications and we prefer to keep the notation as simple as possible.

Remark 1.19. Geometrically, for all points $p \in \mathbb{P}^n \setminus K$ we consider $W_p = \langle p, K \rangle$. Then we have

$$\pi_K(p) = W_p \cap H.$$

Exercise 1.20. Prove the following basic properties of π_K.

- Any linear subspace $W \subset \mathbb{P}^n$ such that $K \subset W$ and $W \cong \mathbb{P}^{n-k}$ is contracted to a point, i.e. $\pi_K(W) = p \in H$.
- If $L \subset \mathbb{P}^n$ is a line then $\pi_K(L) = p$ is a point if and only if $L \cap K \neq \emptyset$.

Definition 1.21. Given a rational map

$$\begin{array}{ccc} \mathbb{P}^n & \overset{\varphi}{\dashrightarrow} & \mathbb{P}^m \\ [x_0 : \cdots : x_n] & \longmapsto & [f_i([x_0 : \cdots : x_n]) \mid i = 0, \ldots, m], \end{array}$$

where $f_i \in R$, for $i = 0, \ldots, m$, denote by $\operatorname{dom}(\varphi)$ its maximal domain of definition and by $\operatorname{Ind}(\varphi)$ its indeterminacy locus. In symbols[1]

$$\operatorname{dom}(\varphi) = \bigcup_{(U,f)\in\varphi} U, \text{ and } \operatorname{Ind}\varphi = \operatorname{Proj}(R/(f_0, \ldots, f_m)).$$

Then, the graph of φ is

$$\operatorname{graph}\varphi = \left\{ (p,q) \in \operatorname{dom}(\varphi) \times \mathbb{P}^m \,\middle|\, q = \varphi(p) \right\} \subset \mathbb{P}^n \times \mathbb{P}^m.$$

Exercise 1.22. Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map, and denote by $Z = \operatorname{Ind}\varphi$ its indeterminacy locus. Consider the following diagram

$$\begin{tikzcd} & \overline{\operatorname{graph}\varphi} \arrow[dl, "\varepsilon_Z"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^m, \end{tikzcd}$$

where the closure $\overline{\operatorname{graph}\varphi}$ is taken in $\mathbb{P}^n \times \mathbb{P}^m$, the morphisms ε_Z and π are the restrictions of the canonical projections to $\overline{\operatorname{graph}\varphi} \subset \mathbb{P}^n \times \mathbb{P}^m$.

[1] Recall that by definition a rational map is an equivalence class. Thus it is a set.

Prove the following properties.

- The first projection ε_Z is birational. (**Hint:** Look at its restriction to graph φ.)
- Suppose that φ is the projection π_K in (1.4). In particular $Z = K$ is a linear subspace. Show that
  - if $\dim K = n-1$, then ε_K is an isomorphism,
  - the morphism π is a fibration with fibres isomorphic to $\mathbb{P}^{n-k}$,
  - the preimage $E_K = \varepsilon_K^{-1} K$ is a $\mathbb{P}^{n-k-1}$-bundle over K.

Definition 1.23. We say that ε_Z is the blowup of $\mathbb{P}^n$ with centre Z. We will also call blowup the variety $\overline{\operatorname{graph}\varphi}$ ad we will denote it by $\operatorname{Bl}_Z \mathbb{P}^n$. Finally we say that E_Z is the exceptional locus.

Exercise 1.24. Definition 1.23 is well posed, i.e. it does not depend on φ but only on $\operatorname{Ind}\varphi$.

Precisely, show that given two rational maps $\varphi_i : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n_i}$, for $i = 1, 2$ having the same indeterminacy locus $\operatorname{Ind}\varphi_1 = \operatorname{Ind}\varphi_2$ the two closures $\overline{\operatorname{graph}\varphi_i}$, for $i = 1, 2$, are canonically isomorphic.

Exercise 1.25. Extend Definition 1.21, Exercise 1.22 and Definition 1.23 to the following cases.

- Products of projective spaces.
- Arbitrary quasi-projective varieties.

Example 1.26 (Blowup at a point)**.** We describe now the projection π_K in the case $n = 2$ and $\dim K = 0$. Without loss of generality we put $K = \{ e_2 \}$ and we consider the projection

$$\begin{aligned} \mathbb{P}^2 &\overset{\pi_{e_2}}{\dashrightarrow} \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto [x_0 : x_1], \end{aligned}$$

from $\mathbb{P}^2$ with centre the coordinate point $e_2 = [0 : 0 : 1]$. Then, the blowup with centre e_2 is

$$\begin{aligned} \operatorname{Bl}_{e_2} \mathbb{P}^2 &= \overline{\operatorname{graph}(\pi_{e_2})} \\ &= \overline{\left\{ (p,q) \in \operatorname{dom}(\pi_{e_2}) \times \mathbb{P}^1 \,\middle|\, \pi_{e_2}(p) = q \right\}} \subset \mathbb{P}^2 \times \mathbb{P}^1 \\ &= \left\{ ([x_0 : x_1 : x_2], [y_0 : y_1]) \in \operatorname{dom}(\pi_{e_2}) \times \mathbb{P}^1 \,\middle|\, \det \begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} = 0 \right\} \subset \mathbb{P}^2 \times \mathbb{P}^1 \\ &= \left\{ ([x_0 : x_1 : x_2], [y_0 : y_1]) \in \mathbb{P}^2 \times \mathbb{P}^1 \,\middle|\, x_0 y_1 - x_1 y_0 = 0 \right\} \subset \mathbb{P}^2 \times \mathbb{P}^1. \end{aligned}$$

We stress that there is a commutative diagram

$$\begin{tikzcd} & \operatorname{Bl}_{e_2} \mathbb{P}^2 \subset \mathbb{P}^2 \times \mathbb{P}^1 \arrow[dl, "\pi_2|_{\operatorname{Bl}\mathbb{P}^2}"'] \arrow[dr, "\pi_1|_{\operatorname{Bl}_{e_2}\mathbb{P}^2}"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\pi_{e_2}"'] & & \mathbb{P}^1, \end{tikzcd}$$

where $\pi_2|_{\operatorname{Bl}\mathbb{P}^2}$ is the blowup morphism and $\pi_1|_{\operatorname{Bl}_{e_2}\mathbb{P}^2}$ is a $\mathbb{P}^1$-fibration as described in Exercise 1.22.

Exercise 1.27. Compute the equations of some embeddings of $\operatorname{Bl}_{\mathbb{P}^k} \mathbb{P}^n$.

Example 1.28 (Blowup at two points)**.** We construct now the blowup of the projective plane at two points as the closure of a birational map given by a pair of projections. Consider the rational map

$$\begin{aligned} \mathbb{P}^2 &\overset{\varphi}{\dashrightarrow} \mathbb{P}^1 \times \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto ([x_0 : x_1], [x_1 : x_2]). \end{aligned}$$

It is a birational map with inverse

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \overset{\psi}{\dashrightarrow} & \mathbb{P}^2 \\ ([y_0 : y_1],[z_0 : z_1]) & \longmapsto & [y_0 z_0 : y_1 z_0 : y_1 z_1]. \end{array}$$

The two maps have the following indeterminacy loci

$$\mathrm{Ind}(\varphi) = \{\, e_0, e_2 \,\} \quad \text{and} \quad \mathrm{Ind}(\psi) = \{\, ([1:0],[0:1]) \,\}.$$

Figure 1 depicts the construction.

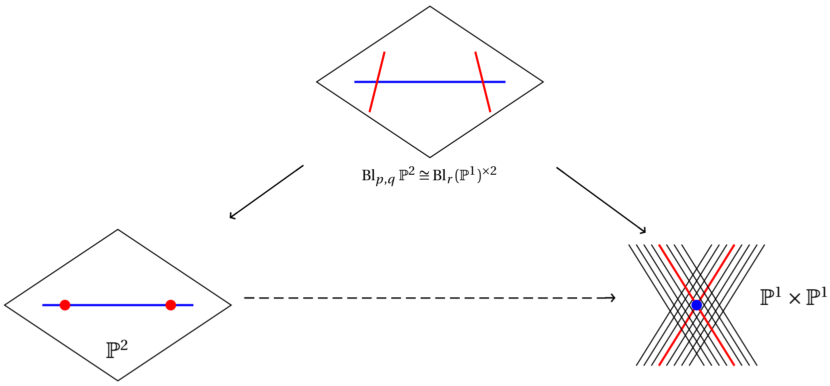

FIGURE 1. Pictorial description of the construction.

As a consequence, we get the isomorphism

$$\mathrm{Bl}_{p,q}\,\mathbb{P}^2 \cong \mathrm{Bl}_r\,\mathbb{P}^1 \times \mathbb{P}^1,$$

where $p, q \in \mathbb{P}^2$ and $r \in \mathbb{P}^1$.

Exercise 1.29. Study the blowup of the projective plane at two points.

- Check the details in Example 1.28.
- Compute the equations of $\mathrm{Bl}_{e_0,e_2}\,\mathbb{P}^2 \subset \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$.
- Show that φ contracts the line trough e_0 and e_2 on the unique point of $\mathrm{Ind}\,\psi$.
- Show that ψ contracts the two lines trough the unique point of $\mathrm{Ind}\,\psi$ on e_0 and e_2.

The following exercise is a direct generalisation of the construction in Example 1.28.

Exercise 1.30. Let $n = 2k+1$ be an odd nonnegative integer and let $Q_n \subset \mathbb{P}^n$ be a smooth quadric. Let also $Q_{n-2} \subset \mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$ be a smooth quadric. Prove that there is an isomorphism

$$\mathrm{Bl}_p\, Q_n \cong \mathrm{Bl}_{Q_{n-2}}\,\mathbb{P}^{n-1}.$$

1.4.1. *Example of computation: use the command* `normalCone` *for the blowup.*

```
    cleanList = L -> (
    G=gcd L;
    if G!= 0 then return for l in L list sub(l/G,ring L#0 );
    return L
)
-- Declear ambient space
R = QQ[x_0 .. x_2];
S = R[w_0 , w_1];
```

```
projection = {x_1,x_0};
I = minors (2,matrix {projection ,S_* });
N = normalCone ideal projection ;
K =sub(ker map(ambient N , ambient ambient N ),S)
I==K                                                    -- Check the construction is the same
O = QQ [R_*|S_*]
point={for r in R_* list sub(r,O),for s in S_* list sub(s,O)};
-- Function to pass from ideal to parametrisation
evalIdeal = I ->  for u in point list cleanList  apply( u , v ->  sub(sub(v,O/I),O) ) ;
E = sub(ideal projection , O ) + sub(I,O)                           -- Preimage of the origin
evalIdeal E
F = saturate(sub(ideal(3*w_0 -w_1),O)+sub(I,O),sub(ideal S_*,O))                  -- Fibration
evalIdeal F
```

1.5. **Standard Cremona transformation.** The standard Cremona transformation $\mathrm{c}_n \in \mathrm{Bir}(\mathbb{P}^n)$, for $n \geq 1$, is a birational transformation of the projective space $\mathbb{P}^n$ and it is the first non trivial example of birational map. We recall now its definition.

Definition 1.31. The standard Cremona transformation $\mathrm{c}_n \in \mathrm{Bir}(\mathbb{P}^n)$ is

$$\begin{array}{ccc} \mathbb{P}^n & \overset{\mathrm{c}_n}{\dashrightarrow} & \mathbb{P}^n \\ [x_0 : \cdots : x_n] & \longmapsto & \left[\frac{1}{x_0} : \cdots : \frac{1}{x_n}\right] = [x_1 \cdots x_n : \cdots : x_0 \cdots x_{n-1}]. \end{array}$$

In dimension $n = 2$ the standard Cremona transformation plays a special role as the following classical result explains.

Theorem 1.32 (Noether-Castelnuovo, [1, 5]). *The following equality of groups holds true,*

$$\mathrm{Bir}(\mathbb{P}^2) = \langle \mathbb{P}\,\mathrm{GL}(3, \mathbb{C}), \mathrm{c}_2 \rangle.$$

Exercise 1.33. Prove the main properties of the standard Cremona transformation listed below.

- $\mathrm{c}_n^2 \equiv \mathrm{id}_{\mathbb{P}^n}$
- $\mathrm{Ind}\,\mathrm{c}_n = \coprod_{0 \leq i < j \leq n} \{ x_i = x_j = 0 \}$
- $\mathrm{c}_n(\{ x_i = 0 \}) = e_i$
- $\mathrm{Fix}\,\mathrm{c}_n = \{ [1 : \pm 1 : \pm 1 : \cdots : \pm 1] \}$
- $|\,\mathrm{Fix}\,\mathrm{c}_n\,| = 2^n$

1.5.1. *Example of computation.*

```
-- Define a function that remove common factor from lists
cleanList = L -> (
    G=gcd L;
    if G!= 0 then return for l in L list sub(l/G,ring L#0 );
    return L
)
-- Function to pass from ideal to parametrisation
evalIdeal = I -> cleanList for l in R_* list sub(sub(l,R/I),R);
-- Declear ambient space
n = 3;   R = QQ[x_0 .. x_n];
Crent = for i from 0 to n list product for j in delete (i, 0..n ) list x_j;
Cremona = map(R , R , Crent );                            -- Define the map
cleanFactorsMap ( Cremona * Cremona )                     -- Check it is an involution
```

```
DDD= set primaryDecomposition ideal Crent                  -- Compute the indeterminacy locus
-- Check that the i-th coord. hyperplane is contracted to the i-th coord. point
for i from  0 to n do print cleanList for a in Crent list  sub( a  , x_i=>0)
DDD1 = set primaryDecomposition  minors(2,matrix{R_*,Crent});
DDD1=DDD1-DDD                                              -- Fixed points
for d in toList DDD1 list evalIdeal d
```

Exercise 1.34 (Blowup at three points)**.** Consider the standard Cremona transformation $\mathrm{c}_2 \in \mathrm{Bir}(\mathbb{P}^2)$. We have a commutative diagram

$$\begin{tikzcd} & B \arrow[dl, "\varepsilon"'] \arrow[dr, "\theta"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\mathrm{c}_2"] & & \mathbb{P}^2 \\ {[x_1 : x_2 : x_3]} \arrow[rr, mapsto] & & \left[\frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}\right], \end{tikzcd} \tag{1.5}$$

where $B = \overline{\mathrm{graph}(\mathrm{c}_2)}$ is the closure of the graph.

- Realise the blowup $\mathrm{Bl}_{e_0,e_1,e_2}\mathbb{P}^2$ as a closed subset of $\mathbb{P}^2 \times \mathbb{P}^2$, see Figure 2.
- Realise the blowup $\mathrm{Bl}_{e_0,e_1,e_2}\mathbb{P}^2$ as a closed subset of $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
- Show that the restriction of the canonical projection
$$\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$
to $\mathrm{Bl}_{e_0,e_1,e_2}\mathbb{P}^2$ is a closed immersion.

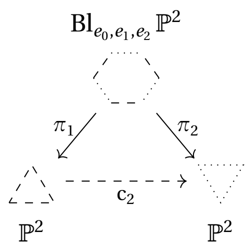

FIGURE 2. The resolution of the indeterminacies of the standard Cremona transformation in dimension 2.

Exercise 1.35. Let $T = \bigcup_{0\le i<j\le 3} \left\{ x_i = x_j = 0 \right\} \subset \mathbb{P}^3$ be the coordinate tetrahedron, i.e. the union of the coordinate lines of $\mathbb{P}^3$. Denote by X the blowup $X = \mathrm{Bl}_T\mathbb{P}^3$.

- Realise X as a closed subset of $\mathbb{P}^3 \times \mathbb{P}^3$.
- Show that X has 12 singular points.
- Identify all the irreducible components of the exceptional locus E_T. (**Hint:** there are 10 of them.)

```
-- Declear ambient space
R = QQ[x_0 .. x_3];
S = R[w_0 .. w_3];

Crent = for i from 0 to 3 list product for j in delete (i, 0..3 ) list x_j;
D = minimalPrimes minors (2,matrix {Crent ,S_* });
I = (select(D , d-> dim d<6 ))#0;
```

```
E= normalCone ideal Crent;
K=sub(ker map(ambient E , ambient ambient E),S)
I==K                                                    -- Check the construction is the same
O = QQ [R_*|S_*]
point={for r in R_* list sub(r,O),for s in S_* list sub(s,O)};
-- Function to pass from ideal to parametrisation
evalIdeal = I ->  for u in point list cleanList  apply( u , v ->  sub(sub(v,O/I),O) ) ;
I = sub(I,O);
J = jacobian I;
U = saturate(saturate(trim(minors (3,J)+I),ideal(x_0..x_3)), ideal(w_0..w_3));
for u in minimalPrimes U list evalIdeal u
```

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