The Econometrics of DSGE Models

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Neoclassical Growth Model

$$U = E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{\left(C_t^{\lambda}(1-H_t)^{1-\lambda}\right)^{1-\tau}}{1-\tau}$$

$$Y_t = C_t + I_t$$

$$Y_t = e^{z_t} K_t^{\alpha} H_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1-\delta) K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t, \ |\rho| < 1$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

- C_t : consumption,
- H_t : hours
- Y_t : product
- K_t : capital
- I_t : investment
- z_t: technology shocks
- u_t: exogenous shock
- $\underbrace{\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_u^2)}_{\text{structural parameters}}$

Neoclassical Growth Model

- Both welfare theorems hold in this economy.
- Thus, we can solve directly for the social planner's problem:

$$\max_{\{C_t, H_t\}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{\left(C_t^{\lambda}(1-H_t)^{1-\lambda}\right)^{1-\tau}}{1-\tau}$$

subject to

$$C_t + I_t = e^{z_t} K_t^{\alpha} H_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1-\delta) K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t, \ |\rho| < 1, \ \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

$$z_0, K_0$$

  - maximize the utility of the household subject to the production function, the evolution of technology, the law of motion for capital, the resource constraint, and some initial k_0 and z_0.

Neoclassical Growth Model

First Order Conditions

The model is fully characterized by the first order conditions:

$$
\begin{aligned}
1 &= E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{\lambda(1-\tau)}\left(\frac{1-H_{t+1}}{1-H_t}\right)^{(1-\lambda)(1-\tau)} R_{t+1}\right] \\
R_{t+1} &\equiv (1-\delta)+e^{z_{t+1}}K_{t+1}^{\alpha-1}H_{t+1}^{1-\alpha} \\
(1-\lambda)\frac{1}{(1-H_t)} &= \frac{\lambda(1-\alpha)e^{z_t}K_t^{\alpha}H_t^{-\alpha}}{C_t} \\
C_t+K_{t+1} &= e^{z_t}K_t^{\alpha}H_t^{1-\alpha}+(1-\delta)K_t \\
z_t &= \rho z_{t-1}+\varepsilon_t.
\end{aligned}
$$

GMM Estimation of DSGE Models

- Only possible for “some” models
- Main problems
  - shocks are unobservable
  - some of the variables in the model are not directly observable
- What we would need is GMM when moment restrictions include latent variables
  - Giacomini, Gallant, Ragusa (2013)

GMM Estimation: Example

Neoclassical Growth Model

Consider the NGM:

$$E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{(1-\tau)}\left(\frac{1-H_{t+1}}{1-H_t}\right)^{(1-\lambda)(1-\tau)} R_{t+1}-1\right]=0 \tag{1}$$

$$R_{t+1}\equiv(1-\delta)+e^{z_{t+1}}K_{t+1}^{\alpha-1}H_{t+1}^{1-\alpha}$$

$$(1-\lambda)\frac{1}{(1-H_t)}=\frac{\lambda(1-\alpha)e^{z_t}K_t^{\alpha}H_t^{-\alpha}}{C_t} \tag{2}$$

$$C_t+K_{t+1}=e^{z_t}K_t^{\alpha}H_t^{1-\alpha}+(1-\delta)K_t \tag{3}$$

$$z_t=\rho z_{t-1}+\varepsilon_t \tag{4}$$

$$Y_t=e^{z_t}K_t^{\alpha}H_t^{1-\alpha} \tag{5}$$

There are 7 parameters to estimate $\theta=(\beta,\lambda,\tau,\alpha,\delta,\rho,\sigma_u^2)$, so we need at least 7 moment conditions.

GMM Estimation: Example

Neoclassical Growth Model

We can 3 conditions from the Euler equations

$$E\left\{\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{\lambda(1-\tau)}\left(\frac{1-H_{t+1}}{1-H_t}\right)^{(1-\lambda)(1-\tau)}R_{t+1}-1\right]\right\}=0$$

$$E\left\{\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{\lambda(1-\tau)}\left(\frac{1-H_{t+1}}{1-H_t}\right)^{(1-\lambda)(1-\tau)}R_{t+1}-1\right]\frac{C_t}{C_{t-1}}\right\}=0$$

$$E\left\{\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{\lambda(1-\tau)}\left(\frac{1-H_{t+1}}{1-H_t}\right)^{(1-\lambda)(1-\tau)}R_{t+1}-1\right]R_t\right\}=0$$

Next, we take expectation of the capital equation:

$$0=E\left[(1-\lambda)\frac{1}{(1-H_t)}-\frac{\lambda(1-\alpha)e^{z_t}K_t^{\alpha}H_t^{-\alpha}}{C_t}\right]$$

$$0=E\left[C_t+K_{t+1}-e^{z_t}K_t^{\alpha}H_t^{1-\alpha}-(1-\delta)K_t\right]$$

GMM Estimation: Example

Neoclassical Growth Model

Three moment restrictions are

$$
\begin{aligned}
0 =& E(z_{t+1} - \rho z_t) \\
0 =& E\left[(z_{t+1} - \rho z_t) z_t\right] = 0 \\
0 =& E\left[(z_{t+1} - \rho z_t)^2\right] - \sigma_z^2 = 0
\end{aligned}
$$

In this case we have 8 moment restrictions a 7 parameters to estimate.

z_t is not observable....

However,

$$Y_t = e^{z_t} K_t^{\alpha} H_t^{1-\alpha} \implies \log Y_t = z_t + \alpha \log K_t + (1-\alpha) \log H_t,$$

thus

$$z_t = \log Y_t - \alpha \log K_t + (1-\alpha) \log H_t.$$

GMM Estimation: Caveat

- Not always possible to obtain expressions for the unobserved shocks, e.g.,

$$E\left\{\left[\beta\left(\frac{C_{t+1}\upsilon_{t+1}}{C_t\upsilon_t}\right)^{(1-\tau)}R_{t+1}-1\right]\right\}=0$$

- All variables need to be observables
  - We do not have a good measure of capital stock
- Identification difficult to show
- Other technical problems
- Unreasonable estimates of the parameters

Simulated method of moments

Basic idea

Suppose we can simulate from the model at a given value of the $\underbrace{\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_u^2)}_{\text{structural parameters}}$. Let

$$\bar{D}_s(\theta) = \{C_s(\theta), H_s(\theta), K_s(\theta), Y_s(\theta), R_s(\theta)\}, s = 1, \ldots, S \to \infty,$$

the simulated data and D_t, $t = 1, \ldots, T$ the actual data. Then, at the true value of the parameter vector θ_0, under regularity conditions,

$$\lim_{S \to \infty} \frac{1}{S} \sum_{t=1}^{S} f(\bar{D}_s(\theta_0)) = E[f(D_t)],$$

for a function $f(\cdot)$. This suggest the following estimator

$$\min_{\theta \in \Theta} \left[\frac{1}{T} \sum_{t=1}^{T} f(D_t) - \frac{1}{S} \sum_{t=1}^{S} f(\bar{D}_s(\theta)) \right]' W \left[\frac{1}{T} \sum_{t=1}^{T} f(D_t) - \frac{1}{S} \sum_{t=1}^{S} f(\bar{D}_s(\theta)) \right]$$

Estimation of DSGE Model

Likelihood approach

Recall the steps to obtaining a *state space* representation a DSGE model

1. Obtain first order conditions of the model
2. (log) linearize the system of equation, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1 x_t(\theta) + C + \Psi(\theta)z_t$$

3. Solve the linear rational expectation system, to obtain

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

4. Measurement equation (linking data to model variables)

$$\underbrace{\underbrace{y_t}_{\text{observables}} = H(\theta)x_t \underbrace{(+m(\theta)\eta_t)}_{\text{meas. error}}}_{\text{observation equation}}$$

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State space models

Set aside for the moment the structural parameter θ, and write:

$$
\begin{aligned}
\underset{(k\times 1)}{y_t} &= \underset{(k\times k)}{H} x_t + \underset{(k\times r)}{m} \eta_t, \quad \eta_t \sim N(0,R) \\
\underset{(s\times 1)}{x_t} &= \underset{(s\times k)}{G} x_{t-1} + \underset{(s\times q)}{M} \varepsilon_t, \quad \varepsilon_t \sim N(0,Q)
\end{aligned}
$$

The objective is to estimate $\{x_t\}_{t=1}^T$ given the observables $\{y\}_{t=1}^T$.

Smoothed and filtered distribution

Two objects of interest

1. Filtered distribution

$$p(x_t|y_{t-1}, y_{t-2}, \ldots y_1) \equiv p(x_t|y^{t-1})$$

2. Smoothed distribution

$$p(x_t|y_T, y_{T-1}, \ldots, y_1) \equiv p(x_t|y^T)$$

Statistical properties

- η_t is independent of x_t
- ε_t is independent of $x_0, x_1, \ldots, x_t$ and $y_1, \ldots, y_t$
- *Markov property*

$$x_t | x_0, x_1, \ldots, x_{t-1} \sim x_t | x_{t-1}$$

  roughly speaking: if we know x_{t-1}, *then knowledge of* $x_{t-2}, \ldots, x_1, x_0$ does not give any more information about x_t

Statistical properties, ctd.

- Let
$$\Sigma_t = E(x_t - E[x_t])(x_t - E[x_t])',$$
denote the variance of $\{x_t\}$ at time t
- The variance, by independence of ε_t and $\{x_t\}$, follows recursion
$$\Sigma_{t+1} = G\Sigma_t G' + MQM'$$
- If G is *stable*, the variance converges Σ which solves the Lyapunov equation
$$\Sigma = G\Sigma G' + MQM'.$$

Intuition

y^T and x^T are jointly normal, as they are linear combination of $x_0, \varepsilon_t, \eta_t$:

$$x_t = G^T x_0 + \sum_{j=0}^{t} G^t M \varepsilon_{t-j},$$

$$y_t = H\left(G^T x_0 + \sum_{j=0}^{t} G^t \varepsilon_{t-j}\right) + m\eta_t.$$

Thus,

$$E[x_t|y^t] = E[x_t] + \Sigma_{x_t y^t}\Sigma_{y^t}^{-1}(y^t - E[y^t])$$
$$Var(x_t|y^t) = \Sigma_t + \Sigma_{x_t y^t}\Sigma_{y^t}^{-1}\Sigma'_{x_t y^t}$$

Problem

The inverse in the formula $\Sigma_{y^t}^{-1}$ is of size $(kt \times kt)$ which grows with t........

The Kalman Filter

The Kalman filter is a clever method for computing

1. $E[x_t|y^t]$
2. $E[x_{t+1}|y^t]$
3. $Var(x_t|y^t)$
4. $Var(x_{t+1}|y^t)$

Notation

We use the notation

$$x_{t|s} = E[x_t|y^s]$$
$$\Sigma_{t|s} = E(x_t - x_{t|s})(x_t - x_{t|s})'$$

The Kalman Filter

Suppose we have (from previous recursion)

$$x_{t|t-1} \text{ and } \Sigma_{t|t-1}$$

Two steps:

1. [Updading] Obtain
$$x_{t|t} \text{ and } \Sigma_{t|t}$$
in terms of $x_{t|t-1}$ and $\Sigma_{t|t-1}$.
2. [Prediction] Obtain
$$x_{t+1|t} \text{ and } \Sigma_{t+1|t}$$
in terms of $x_{t|t}$ and $\Sigma_{t|t}$.

The Kalman Filter

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$$x_{t|t-1} \text{ and } \Sigma_{t|t-1}$$

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2. [Prediction] Obtain

$$x_{t+1|t} \text{ and } \Sigma_{t+1|t}$$

   in terms of $x_{t|t}$ and $\Sigma_{t|t}$.

The Kalman Filter

The Updating step

Notice that

$$
\begin{aligned}
y_t &= Hx_t + m\eta_t \\
x_t &= Gx_{t-1} + M\varepsilon_t
\end{aligned}
$$

implies that the conditionals (w.r.t. y^{t-1})

$$
\begin{aligned}
y_t|y^{t-1} &= Hx_t|y^{t-1} + m\eta_t|y^{t-1} \\
x_t|y^{t-1} &= Gx_{t-1}|y^{t-1} + M\varepsilon_t
\end{aligned}
$$

are jointly normal

$$
\begin{pmatrix} x_t|y^{t-1} \\ y_t|y^{t-1} \end{pmatrix} \sim N\left[\begin{matrix} Gx_{t|t-1} \\ Hx_{t|t-1} \end{matrix}, \begin{pmatrix} \Sigma_{t|t-1} & \Sigma_{t|t-1}H' \\ H\Sigma_{t|t-1} & H\Sigma_{t|t-1}H' + mRm' \end{pmatrix}\right]
$$

The Kalman Filter

The Updating step, ctd.

Now, notice that

$$x_t|y^t \sim \left(x_t|y^{t-1}\right)|(y_t|y^{t-1}).$$

Using the formula for conditional normals

$$\begin{aligned} x_{t|t} &= x_{t|t-1} + \Sigma_{t|t-1}H'(H\Sigma_{t|t-1}H' + mRm')^{-1}(y_t - Hx_{t|t-1}) \\ \Sigma_{t|t} &= \Sigma_{t|t-1} + \Sigma_{t|t-1}H'(H\Sigma_{t|t-1}H' + mRm')^{-1}H\Sigma_{t|t-1}, \end{aligned}$$

which can be calculated (given $x_{t|t-1}$ and $\Sigma_{t|t-1}$).

The Kalman Filter

The Prediction step

From

$$x_t = Gx_{t-1} + M\varepsilon_t$$

follows that

$$\begin{aligned}
x_{t+1|t} &= Gx_{t|t} + M \underbrace{\varepsilon_{t|t}}_{\underbrace{= E[\varepsilon_t|y^t] = E[\varepsilon_t] = 0}_{\text{by independence of } \varepsilon_t \text{ and } y^t}} \\
&= Gx_{t|t},
\end{aligned}$$

and similarly

$$\begin{aligned}
\Sigma_{t+1|t} &= E(x_{t+1|t} - x_{t+1})E(x_{t+1|t} - x_{t+1})' \\
&= E(Gx_{t|t} - Gx_t - M\varepsilon_t)(Gx_{t|t} - Gx_t - M\varepsilon_t)' \\
&= G\Sigma_{t|t}G' + MQM'
\end{aligned}$$

The Kalman filter

Updating step

$$x_{t|t} = x_{t|t-1} + \Sigma_{t|t-1} H' (H \Sigma_{t|t-1} H' + mRm')^{-1} (y_t - H x_{t|t-1})$$
$$\Sigma_{t|t} = \Sigma_{t|t-1} + \Sigma_{t|t-1} H' (H \Sigma_{t|t-1} H' + mRm')^{-1} H \Sigma_{t|t-1},$$

Prediction step

$$\begin{aligned}
x_{t+1|t} &= G x_{t|t} \\
\Sigma_{t+1|t} &= E(x_{t+1|t} - x_{t+1}) E(x_{t+1|t} - x_{t+1})' \\
&= E(G x_{t|t} - G x_t - M\varepsilon_t)(G x_{t|t} - G x_t - M\varepsilon_t)' \\
&= G \Sigma_{t|t} G' + MQM'
\end{aligned}$$

The Kalman filter

Initial value

- To run the filter we need initial values of $x_{t-1|t-1}$ and $\Sigma_{t-1|t-1}$
- Common practice:

$$x_0, \text{ and } \Sigma_0 \text{ from the unconditional distribution}$$

e.g,

$$x_0 \sim \sum_{j=0}^{\infty} G^t M \varepsilon_{t-j}$$

$$\Sigma_0 = \Sigma \implies (\text{solve } \Sigma = G \Sigma G' + MQM')$$

- Other approaches available (we will see them later)
  - Diffuse Kalman filter (Rosenberg, 1973; Ansley and Kohn, 1985)
  - Estimate x_0 and Σ_0

The Kalman Filter

Filtered and smoothed distributions

- Now we now how to recover the following quantities

$$\{x_{t|t-1}\}_{t=1}^{T} \text{ and } \{x_{t|t}\}_{t=1}^{T}$$
$$\{\Sigma_{t|t-1}\}_{t=1}^{T} \text{ and } \{\Sigma_{t|t}\}_{t=1}^{T}$$

which gives us

1. Filtered distribution

$$x_t|y^{t-1} \sim N(x_{t|t-1}, \Sigma_{t|t-1}),$$

2. Smoothed distribution

$$x_t|y^{T} \sim N(x_{t|T}, \Sigma_{t|T}).$$

We can also get something else.....

The Kalman Filter

The likelihood function

- The Kalman Filter also gives the likelihood, that is,

$$p(y^T) = \prod_{t=1}^{T} p(y_t | y^{t-1})$$

- From the updating step

$$\begin{pmatrix} x_t | y^{t-1} \\ y_t | y^{t-1} \end{pmatrix} \sim N \left[\begin{matrix} Gx_{t|t-1} \\ Hx_{t|t-1} \end{matrix}, \begin{pmatrix} \Sigma_{t|t-1} & \Sigma_{t|t-1} H' \\ H\Sigma_{t|t-1} & H\Sigma_{t|t-1} H' + mRm' \end{pmatrix} \right]$$

thus

$$p(y^T) = \prod_{t=1}^{T} p(y_t | y_{t-1}),$$

where

$$p(y_t | y_{t-1}) = \frac{1}{(2\pi)^{k/2} |\Gamma|^{1/2}} \exp\left(-\frac{1}{2} (u - \mu)' \Omega^{-1} (u - \mu) \right)$$

with

$$\mu = Hx_{t|t-1},$$

DSGE Model and Kalman Filter

Let's put back in the parameters

$$\underset{(k\times 1)}{y_t} = \underset{(k\times s)}{H(\theta)}x_t + \underset{(k\times r)}{m(\theta)}\eta_t, \quad \eta_t \sim N(0, R(\theta))$$
$$\underset{(s\times 1)}{x_t} = \underset{(s\times s)}{G(\theta)}x_{t-1} + \underset{(s\times q)}{M(\theta)}\varepsilon_t, \quad \varepsilon_t \sim N(0, Q(\theta))$$

- For a given θ
  - Run the Kalman filter
  - Calculate the likelihood function

$$p(y^T;\theta) = \prod_{t=1}^{T} p(y_t | y_{t-1}, \theta),$$

where

$$\mu(\theta) = H(\theta)x_{t|t-1}(\theta)$$
$$\Omega(\theta) = H(\theta)\Sigma_{t|t-1}(\theta)H(\theta)' + m(\theta)R(\theta)m(\theta)'$$

DSGE Model Estimation

Now we can finally see how we estimate θ:

- Maximum Likelihood

$$\max_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \log p(y_t | y_{t-1}, \theta),$$

- Bayesian approach

$$p(\theta | y^t) = \frac{p(y^T; \theta) p(\theta)}{p(y^T)}$$

DSGE Model Steps

For each θ

1. (Log) linearize the system of equation from FOC, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1(\theta)x_t(\theta) + C(\theta) + \Psi(\theta)z_t$$

2. Solve the linear rational expectation system, to obtain

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

3. Measurement equation (linking data to model variables)

$$\underbrace{\underbrace{y_t}_{\text{observables}} = H(\theta)x_t \underbrace{(+m(\theta)\eta_t)}_{\text{meas. error}}}_{\text{observation equation}}$$

4. Run the Kalman filter to obtain

$$\ell(y^T, \theta) = \frac{1}{T}\sum_{t=1}^{T} \log p(y_t|y_{t-1}, \theta)$$

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Maximum Likelihood estimation

$$\hat{\theta}_T \equiv \arg\max_{\theta\in\Theta} \sum_{t=1}^{T} \log p(y_t|y_{t-1},\theta)$$

Under regularity conditions

$$\hat{\theta}_T \xrightarrow{p} \theta, \text{ and } \sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{d} N(0, H^{-1}\mathscr{I}H^{-1}),$$

where

$$H = \underbrace{-E\left[\frac{\partial \log\phi(y_t;\mu(\theta),\Omega(\theta))}{\partial\theta\partial\theta'}\right]}_{\text{Hessian of log-likelihood}}$$

and

$$\mathscr{I} = \underbrace{E\left[\frac{\partial \log\phi(y_t;\mu(\theta),\Omega(\theta))}{\partial\theta'}\frac{\partial \log\phi(y_t;\mu(\theta),\Omega(\theta))}{\partial\theta'}\right]}_{\text{Fischer information matrix}}$$

Maximum Likelihood estimation

Regularity conditions

- Identification of the model:

$$\theta \neq \tilde{\theta}, \text{ then } p(y^T;\tilde{\theta}) \neq p(y^T;\theta),$$

  different value of the parameter must correspond to different values of the likelihood function.
  This assumption is difficult to assess in DSGE model — that although linear are highly non-linear in the structural parameters

  - *Canova, Fabio, and Luca Sala. "Back to square one: identification issues in DSGE models." Journal of Monetary Economics 56.4 (2009): 431-449.*
  - *Iskrev, Nikolay. "Local identification in DSGE models." Journal of Monetary Economics 57.2 (2010): 189-202.*
  - *Komunjer, Ivana, and Serena Ng. "Dynamic identification of dynamic stochastic general equilibrium models." Econometrica 79.6 (2011): 1995-2032.*

Maximum Likelihood estimation

Regularity conditions, ctd

- Model is correctly specified: if it is not, the MLE estimator is still asymptotically consistent, but for a pseudo-true value

$$\hat{\theta}_T \xrightarrow{p} \theta^{PT}$$

The pseudo true value maximize the Kullback Leibler

- *White, Halbert. "Maximum likelihood estimation of misspecified models." Econometrica: Journal of the Econometric Society (1982): 1-25.*

Maximum Likelihood estimation

Misspecified likelihood estimation

- Let $g(y^T)$ the "true" density of the data. The model is misspecified if

$$g(y^T) \notin \{p(y^T;\theta), \quad \theta \in \Theta\}$$

- In this case, it can be shown that

$$\begin{aligned}\theta^{PT} &= \arg\max_{\theta\in\Theta} \int \log\left(\frac{p(y^T;\theta)}{g(y^T)}\right) g(y^T) \\ &= \arg\min_{\theta\in\Theta} \underbrace{\int \log\left(\frac{g(y^T)}{p(y^T;\theta)}\right) g(y^T)}_{\text{Kullback Leibler "distance"}},\end{aligned}$$

  minimize the Kullback–Leibler distance (see White, 1982)

- Remember: "all models are false, but some are useful"

Maximum Likelihood estimation

Regularity conditions, ctd

- Identification

$$\theta \neq \theta_0 \implies \log p(y^T, \theta) \neq \log p(y^T, \theta)$$

- Compactness of parameter space

$$\theta \in \Theta, \text{ where } \Theta \text{ is a compact set}$$

- Continuity of $p(y^T, \theta)$

$$\Pr\left[\log p(y^T, \theta) \in C^0(\Theta)\right] = 1$$

- Stochastic Dominance

$$|\log p(y_t | y_{t-1}; \theta)| < D(y_t), \quad \text{all } \theta \in \Theta, t \leq T$$

where

$$\int D(y^T) p(y^T, \theta_0) dy^T < \infty$$