

The Econometrics of DSGE Models

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Neoclassical Growth Model

$$U = E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{(C_t^\lambda (1 - H_t)^{1-\lambda})^{1-\tau}}{1 - \tau}$$

$$Y_t = C_t + I_t$$

$$Y_t = e^{z_t} K_t^\alpha H_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1 - \delta) K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t, |\rho| < 1$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

- C_t : consumption,
- H_t : hours
- Y_t : product
- K_t : capital
- I_t : investment
- z_t : technology shocks
- u_t : exogenous shock
- $\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_u^2)$
structural parameters

Neoclassical Growth Model

- Both welfare theorems hold in this economy.
- Thus, we can solve directly for the social planner's problem:

$$\max_{\{C_t, H_t\}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{(C_t^\lambda (1 - H_t)^{1-\lambda})^{1-\tau}}{1 - \tau}$$

subject to

$$C_t + I_t = e^{z_t} K_t^\alpha H_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1 - \delta)K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t, \quad |\rho| < 1, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

$$z_0, K_0$$

- ▶ maximize the utility of the household subject to the production function, the evolution of technology, the law of motion for capital, the resource constraint, and some initial k_0 and z_0 .

Neoclassical Growth Model

First Order Conditions

The model is **fully** characterized by the first order conditions:

$$1 = E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{\lambda(1-\tau)} \left(\frac{1-H_{t+1}}{1-H_t} \right)^{(1-\lambda)(1-\tau)} R_{t+1} \right]$$

$$R_{t+1} \equiv (1-\delta) + e^{z_{t+1}} K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha}$$

$$(1-\lambda) \frac{1}{(1-H_t)} = \frac{\lambda(1-\alpha) e^{z_t} K_t^\alpha H_t^{-\alpha}}{C_t}$$

$$C_t + K_{t+1} = e^{z_t} K_t^\alpha H_t^{1-\alpha} + (1-\delta) K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t.$$

GMM Estimation of DSGE Models

- Only possible for “some” models
- Main problems
 - ▶ shocks are unobservable
 - ▶ some of the variables in the model are not directly observable
- What we would need is GMM when moment restrictions include latent variables
 - ▶ Giacomini, Gallant, Ragusa (2013)

GMM Estimation: Example

Neoclassical Growth Model

Consider the NGM:

$$E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{(1-\tau)} \left(\frac{1-H_{t+1}}{1-H_t} \right)^{(1-\lambda)(1-\tau)} R_{t+1} - 1 \right] = 0 \quad (1)$$

$$R_{t+1} \equiv (1-\delta) + e^{z_{t+1}} K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha}$$
$$(1-\lambda) \frac{1}{(1-H_t)} = \frac{\lambda(1-\alpha)e^{z_t} K_t^\alpha H_t^{-\alpha}}{C_t} \quad (2)$$

$$C_t + K_{t+1} = e^{z_t} K_t^\alpha H_t^{1-\alpha} + (1-\delta)K_t \quad (3)$$

$$z_t = \rho z_{t-1} + \varepsilon_t \quad (4)$$

$$Y_t = e^{z_t} K_t^\alpha H_t^{1-\alpha} \quad (5)$$

There are 7 parameters to estimate $\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_u^2)$, so we need at least 7 moment conditions.

GMM Estimation: Example

Neoclassical Growth Model

We can 3 conditions from the Euler equations

$$E \left\{ \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{\lambda(1-\tau)} \left(\frac{1-H_{t+1}}{1-H_t} \right)^{(1-\lambda)(1-\tau)} R_{t+1} - 1 \right] \right\} = 0$$

$$E \left\{ \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{\lambda(1-\tau)} \left(\frac{1-H_{t+1}}{1-H_t} \right)^{(1-\lambda)(1-\tau)} R_{t+1} - 1 \right] \frac{C_t}{C_{t-1}} \right\} = 0$$

$$E \left\{ \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{\lambda(1-\tau)} \left(\frac{1-H_{t+1}}{1-H_t} \right)^{(1-\lambda)(1-\tau)} R_{t+1} - 1 \right] R_t \right\} = 0$$

Next, we take expectation of the capital equation:

$$0 = E \left[(1-\lambda) \frac{1}{(1-H_t)} - \frac{\lambda(1-\alpha)e^{z_t} K_t^\alpha H_t^{-\alpha}}{C_t} \right]$$

$$0 = E [C_t + K_{t+1} - e^{z_t} K_t^\alpha H_t^{1-\alpha} - (1-\delta)K_t]$$

GMM Estimation: Example

Neoclassical Growth Model

Three moment restrictions are

$$0 = E(z_{t+1} - \rho z_t)$$

$$0 = E[(z_{t+1} - \rho z_t)z_t] = 0$$

$$0 = E[(z_{t+1} - \rho z_t)^2] - \sigma_z^2 = 0$$

In this case we have 8 moment restrictions a 7 parameters to estimate.

z_t is not observable....

However,

$$Y_t = e^{z_t} K_t^\alpha H_t^{1-\alpha} \implies \log Y_t = z_t + \alpha \log K_t + (1 - \alpha) \log H_t,$$

thus

$$z_t = \log Y_t - \alpha \log K_t + (1 - \alpha) \log H_t.$$

GMM Estimation: Caveat

- Not always possible to obtain expressions for the unobserved shocks, e.g.,

$$E \left\{ \left[\beta \left(\frac{C_{t+1} v_{t+1}}{C_t v_t} \right)^{(1-\tau)} R_{t+1} - 1 \right] \right\} = 0$$

- All variables need to be observables
 - ▶ We do not have a good measure of capital stock
- Identification difficult to show
- Other technical problems
- Unreasonable estimates of the parameters

Simulated method of moments

Basic idea

Suppose we can simulate from the model at a given value of the $\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_u^2)$. Let

structural parameters

$$\bar{D}_s(\theta) = \{C_s(\theta), H_s(\theta), K_s(\theta), Y_s(\theta), R_s(\theta)\}, s = 1, \dots, S \rightarrow \infty,$$

the simulated data and D_t , $t = 1, \dots, T$ the actual data. Then, at the true value of the parameter vector θ_0 , under regularity conditions,

$$\lim_{S \rightarrow \infty} \frac{1}{S} \sum_{t=1}^S f(\bar{D}_s(\theta_0)) = E[f(D_t)],$$

for a function $f(\cdot)$. This suggest the following estimator

$$\min_{\theta \in \Theta} \left[\frac{1}{T} \sum_{t=1}^T f(D_t) - \frac{1}{S} \sum_{t=1}^S f(\bar{D}_s(\theta)) \right]' W \left[\frac{1}{T} \sum_{t=1}^T f(D_t) - \frac{1}{S} \sum_{t=1}^S f(\bar{D}_s(\theta)) \right]$$

Estimation of DSGE Model

Likelihood approach

Recall the steps to obtaining a *state space* representation a DSGE model

- 1 Obtain first order conditions of the model
- 2 (log) linearize the system of equation, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1 x_t(\theta) + C + \Psi(\theta)z_t$$

- 3 Solve the linear rational expectation system, to obtain

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

- 4 Measurement equation (linking data to model variables)

$$\underbrace{\underbrace{y_t}_{\text{observables}} = H(\theta)x_t + \underbrace{m(\theta)\eta_t}_{\text{meas. error}}}_{\text{observation equation}}$$

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observation equation

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State space models

Set aside for the moment the structural parameter θ , and write:

$$\begin{aligned} y_t &= \underset{(k \times 1)}{H} \underset{(k \times k)}{x_t} + \underset{(k \times r)}{m} \eta_t, & \eta_t &\sim N(0, R) \\ x_t &= \underset{(s \times 1)}{G} x_{t-1} + \underset{(s \times q)}{M} \varepsilon_t, & \varepsilon_t &\sim N(0, Q) \end{aligned}$$

The objective is to estimate $\{x_t\}_{t=1}^T$ given the observables $\{y\}_{t=1}^T$.

Smoothed and filtered distribution

Two objects of interest

1 Filtered distribution

$$p(x_t | y_{t-1}, y_{t-2}, \dots, y_1) \equiv p(x_t | y^{t-1})$$

2 Smoothed distribution

$$p(x_t | y_T, y_{T-1}, \dots, y_1) \equiv p(x_t | y^T)$$

Statistical properties

- η_t is independent of x_t
- ε_t is independent of x_0, x_1, \dots, x_t and y_1, \dots, y_t
- *Markov property*

$$x_t | x_0, x_1, \dots, x_{t-1} \sim x_t | x_{t-1}$$

roughly speaking: if we know x_{t-1} , *then knowledge of* x_{t-2}, \dots, x_1, x_0 does not give any more information about x_t

Statistical properties, ctd.

- Let

$$\Sigma_t = E(x_t - E[x_t])(x_t - E[x_t])',$$

denote the variance of $\{x_t\}$ at time t

- The variance, by independence of ε_t and $\{x_t\}$, follows recursion

$$\Sigma_{t+1} = G\Sigma_t G' + MQM'$$

- If G is *stable*, the variance converges Σ which solves the Lyapunov equation

$$\Sigma = G\Sigma G' + MQM'.$$

Intuition

y^T and x^T are **jointly normal**, as they are linear combination of $x_0, \varepsilon_t, \eta_t$:

$$x_t = G^T x_0 + \sum_{j=0}^t G^j M \varepsilon_{t-j},$$

$$y_t = H \left(G^T x_0 + \sum_{j=0}^t G^j \varepsilon_{t-j} \right) + m \eta_t.$$

Thus,

$$E[x_t | y^t] = E[x_t] + \Sigma_{x_t y^t} \Sigma_{y^t}^{-1} (y^t - E[y^t])$$

$$\text{Var}(x_t | y^t) = \Sigma_t + \Sigma_{x_t y^t} \Sigma_{y^t}^{-1} \Sigma'_{x_t y^t}$$

Problem

The inverse in the formula $\Sigma_{y^t}^{-1}$ is of size $(kt \times kt)$ which grows with t

The Kalman Filter

The Kalman filter is a clever method for computing

- 1 $E[x_t|y^t]$
- 2 $E[x_{t+1}|y^t]$
- 3 $\text{Var}(x_t|y^t)$
- 4 $\text{Var}(x_{t+1}|y^t)$

Notation

We use the notation

$$x_{t|s} = E[x_t|y^s]$$

$$\Sigma_{t|s} = E(x_t - x_{t|s})(x_t - x_{t|s})'$$

The Kalman Filter

Suppose we have (from previous recursion)

$$x_{t|t-1} \text{ and } \Sigma_{t|t-1}$$

Two steps:

- 1 [Updating] Obtain

$$x_{t|t} \text{ and } \Sigma_{t|t}$$

in terms of $x_{t|t-1}$ and $\Sigma_{t|t-1}$.

- 2 [Prediction] Obtain

$$x_{t+1|t} \text{ and } \Sigma_{t+1|t}$$

in terms of $x_{t|t}$ and $\Sigma_{t|t}$.

The Kalman Filter

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in terms of $x_{t|t}$ and $\Sigma_{t|t}$.

The Kalman Filter

The Updating step

Notice that

$$\begin{aligned}y_t &= Hx_t + m\eta_t \\x_t &= Gx_{t-1} + M\varepsilon_t\end{aligned}$$

implies that the conditionals (w.r.t. y^{t-1})

$$\begin{aligned}y_t|y^{t-1} &= Hx_t|y^{t-1} + m\eta_t|y^{t-1} \\x_t|y^{t-1} &= Gx_{t-1}|y^{t-1} + M\varepsilon_t\end{aligned}$$

are **jointly normal**

$$\begin{pmatrix} x_t|y^{t-1} \\ y_t|y^{t-1} \end{pmatrix} \sim N \left[\begin{pmatrix} Gx_{t|t-1} \\ Hx_{t|t-1} \end{pmatrix}, \begin{pmatrix} \Sigma_{t|t-1} & \Sigma_{t|t-1}H' \\ H\Sigma_{t|t-1} & H\Sigma_{t|t-1}H' + mRm' \end{pmatrix} \right]$$

The Kalman Filter

The Updating step, ctd.

Now, notice that

$$x_t|y^t \sim (x_t|y^{t-1})|(y_t|y^{t-1}).$$

Using the formula for conditional normals

$$\begin{aligned}x_{t|t} &= x_{t|t-1} + \Sigma_{t|t-1} H' (H \Sigma_{t|t-1} H' + m R m')^{-1} (y_t - H x_{t|t-1}) \\ \Sigma_{t|t} &= \Sigma_{t|t-1} - \Sigma_{t|t-1} H' (H \Sigma_{t|t-1} H' + m R m')^{-1} H \Sigma_{t|t-1},\end{aligned}$$

which can be calculated (given $x_{t|t-1}$ and $\Sigma_{t|t-1}$).

The Kalman Filter

The Prediction step

From

$$x_t = Gx_{t-1} + M\varepsilon_t$$

follows that

$$\begin{aligned} x_{t+1|t} &= Gx_{t|t} + M \underbrace{\varepsilon_{t|t}}_{= E[\varepsilon_t|y^t] = E[\varepsilon_t] = 0} \\ &\quad \underbrace{\hspace{10em}}_{\text{by independence of } \varepsilon_t \text{ and } y^t} \\ &= Gx_{t|t}, \end{aligned}$$

and similarly

$$\begin{aligned} \Sigma_{t+1|t} &= E(x_{t+1|t} - x_{t+1})(x_{t+1|t} - x_{t+1})' \\ &= E(Gx_{t|t} - Gx_t - M\varepsilon_t)(Gx_{t|t} - Gx_t - M\varepsilon_t)' \\ &= G\Sigma_{t|t}G' + MQM' \end{aligned}$$

The Kalman filter

Updating step

$$\begin{aligned}x_{t|t} &= x_{t|t-1} + \Sigma_{t|t-1} H' (H \Sigma_{t|t-1} H' + m R m')^{-1} (y_t - H x_{t|t-1}) \\ \Sigma_{t|t} &= \Sigma_{t|t-1} - \Sigma_{t|t-1} H' (H \Sigma_{t|t-1} H' + m R m')^{-1} H \Sigma_{t|t-1},\end{aligned}$$

Prediction step

$$\begin{aligned}x_{t+1|t} &= G x_{t|t} \\ \Sigma_{t+1|t} &= E(x_{t+1|t} - x_{t+1}) E(x_{t+1|t} - x_{t+1})' \\ &= E(G x_{t|t} - G x_t - M \varepsilon_t) (G x_{t|t} - G x_t - M \varepsilon_t)' \\ &= G \Sigma_{t|t} G' + M Q M'\end{aligned}$$

The Kalman filter

Initial value

- To run the filter we need initial values of $x_{t-1|t-1}$ and $\Sigma_{t-1|t-1}$
- Common practice:

x_0 , and Σ_0 from the unconditional distribution

e.g,

$$x_0 \sim \sum_{j=0}^{\infty} G^j M \varepsilon_{t-j}$$

$$\Sigma_0 = \Sigma \implies (\text{solve } \Sigma = G\Sigma G' + MQM')$$

- Other approaches available (we will see them later)
 - ▶ Diffuse Kalman filter (Rosenberg, 1973; Ansley and Kohn, 1985)
 - ▶ Estimate x_0 and Σ_0

The Kalman Filter

Filtered and smoothed distributions

- Now we now how to recover the following quantities

$$\{x_{t|t-1}\}_{t=1}^T \text{ and } \{x_{t|t}\}_{t=1}^T$$
$$\{\Sigma_{t|t-1}\}_{t=1}^T \text{ and } \{\Sigma_{t|t}\}_{t=1}^T$$

which gives us

- 1 Filtered distribution

$$x_t|y^{t-1} \sim N(x_{t|t-1}, \Sigma_{t|t-1}),$$

- 2 Smoothed distribution

$$x_t|y^T \sim N(x_{t|T}, \Sigma_{t|T}).$$

We can also get something else.....

The Kalman Filter

The likelihood function

- The Kalman Filter also gives the likelihood, that is,

$$p(y^T) = \prod_{t=1}^T p(y_t | y^{t-1})$$

- From the updating step

$$\begin{pmatrix} x_t | y^{t-1} \\ y_t | y^{t-1} \end{pmatrix} \sim N \left[\begin{pmatrix} Gx_{t|t-1} \\ Hx_{t|t-1} \end{pmatrix}, \begin{pmatrix} \Sigma_{t|t-1} & \Sigma_{t|t-1}H' \\ H\Sigma_{t|t-1} & H\Sigma_{t|t-1}H' + mRm' \end{pmatrix} \right]$$

thus

$$p(y^T) = \prod_{t=1}^T p(y_t | y_{t-1}),$$

where

$$p(y_t | y_{t-1}) = \frac{1}{(2\pi)^{k/2} |\Gamma|^{1/2}} \exp \left(-\frac{1}{2} (u - \mu)' \Omega^{-1} (u - \mu) \right)$$

with

$$\mu = Hx_{t|t-1},$$

DSGE Model and Kalman Filter

Let's put back in the parameters

$$\underset{(k \times 1)}{y_t} = \underset{(k \times s)}{H(\theta)} \underset{(s \times 1)}{x_t} + \underset{(k \times r)}{m(\theta)} \underset{(r \times 1)}{\eta_t}, \quad \eta_t \sim N(0, R(\theta))$$

$$\underset{(s \times 1)}{x_t} = \underset{(s \times s)}{G(\theta)} \underset{(s \times 1)}{x_{t-1}} + \underset{(s \times q)}{M(\theta)} \underset{(q \times 1)}{\varepsilon_t}, \quad \varepsilon_t \sim N(0, Q(\theta))$$

- For a given θ
 - ▶ Run the Kalman filter
 - ▶ Calculate the likelihood function

$$p(y^T; \theta) = \prod_{t=1}^T p(y_t | y_{t-1}, \theta),$$

where

$$\mu(\theta) = H(\theta) x_{t|t-1}(\theta)$$

$$\Omega(\theta) = H(\theta) \Sigma_{t|t-1}(\theta) H(\theta)' + m(\theta) R(\theta) m(\theta)'$$

DSGE Model Estimation

Now we can finally see how we estimate θ :

- Maximum Likelihood

$$\max_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^T \log p(y_t | y_{t-1}, \theta),$$

- Bayesian approach

$$p(\theta | y^T) = \frac{p(y^T; \theta)p(\theta)}{p(y^T)}$$

DSGE Model Steps

For each θ

- 1 (Log) linearize the system of equation from FOC, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1(\theta)x_t(\theta) + C(\theta) + \Psi(\theta)z_t$$

- 2 Solve the linear rational expectation system, to obtain

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

- 3 Measurement equation (linking data to model variables)

$$\underbrace{\underbrace{y_t}_{\text{observables}} = H(\theta)x_t + \underbrace{m(\theta)\eta_t}_{\text{meas. error}}}_{\text{observation equation}}$$

- 4 Run the Kalman filter to obtain

$$\ell(y^T, \theta) = \frac{1}{T} \sum_{t=1}^T \log p(y_t | y_{t-1}, \theta)$$

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Maximum Likelihood estimation

$$\hat{\theta}_T \equiv \arg \max_{\theta \in \Theta} \sum_{t=1}^T \log p(y_t | y_{t-1}, \theta)$$

Under **regularity conditions**

$$\hat{\theta}_T \xrightarrow{P} \theta, \text{ and } \sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{d} N(0, H^{-1} \mathcal{I} H^{-1}),$$

where

$$H = -E \left[\underbrace{\frac{\partial \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta \partial \theta'}}_{\text{Hessian of log-likelihood}} \right]$$

and

$$\mathcal{I} = E \left[\underbrace{\frac{\partial \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta'} \frac{\partial \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta'}}_{\text{Fischer information matrix}} \right]$$

Maximum Likelihood estimation

Regularity conditions

- Identification of the model:

$$\theta \neq \tilde{\theta}, \text{ then } p(y^T; \tilde{\theta}) \neq p(y^T; \theta),$$

different value of the parameter must correspond to different values of the likelihood function.

This assumption is difficult to assess in DSGE model — that although linear are highly non-linear in the structural parameters

- ▶ Canova, Fabio, and Luca Sala. "Back to square one: identification issues in DSGE models." *Journal of Monetary Economics* 56.4 (2009): 431-449.
- ▶ Iskrev, Nikolay. "Local identification in DSGE models." *Journal of Monetary Economics* 57.2 (2010): 189-202.
- ▶ Komunjer, Ivana, and Serena Ng. "Dynamic identification of dynamic stochastic general equilibrium models." *Econometrica* 79.6 (2011): 1995-2032.

Maximum Likelihood estimation

Regularity conditions, ctd

- **Model is correctly specified:** if it is not, the MLE estimator is still asymptotically consistent, but for a pseudo-true value

$$\hat{\theta}_T \xrightarrow{P} \theta^{PT}$$

The pseudo true value maximize the Kullback Leibler

- ▶ *White, Halbert. "Maximum likelihood estimation of misspecified models." *Econometrica: Journal of the Econometric Society* (1982): 1-25.*

Maximum Likelihood estimation

Misspecified likelihood estimation

- Let $g(y^T)$ the “true” density of the data. The model is misspecified if

$$g(y^T) \notin \{p(y^T; \theta), \quad \theta \in \Theta\}$$

- In this case, it can be shown that

$$\begin{aligned}\theta^{PT} &= \arg \max_{\theta \in \Theta} \int \log \left(\frac{p(y^T; \theta)}{g(y^T)} \right) g(y^T) \\ &= \arg \min_{\theta \in \Theta} \underbrace{\int \log \left(\frac{g(y^T)}{p(y^T; \theta)} \right) g(y^T)}_{\text{Kullback Leibler "distance"}},\end{aligned}$$

minimize the Kullback–Leibler distance (see White, 1982)

- Remember: “all models are false, but some are useful”

Maximum Likelihood estimation

Regularity conditions, ctd

- Identification

$$\theta \neq \theta_0 \implies \log p(y^T, \theta) \neq \log p(y^T, \theta_0)$$

- Compactness of parameter space

$\theta \in \Theta$, where Θ is a compact set

- Continuity of $p(y^T, \theta)$

$$\Pr \left[\log p(y^T, \theta) \in C^0(\Theta) \right] = 1$$

- Stochastic Dominance

$$|\log p(y_t | y_{t-1}; \theta)| < D(y_t), \quad \text{all } \theta \in \Theta, t \leq T$$

where

$$\int D(y^T) p(y^T, \theta_0) dy^T < \infty$$