The Econometrics of DSGE Models

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Neoclassical Growth Model

$$U = E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{\left(C_t^{\lambda}(1-H_t)^{1-\lambda}\right)^{1-\tau}}{1-\tau}$$

$$Y_t = C_t + I_t$$

$$Y_t = e^{z_t} K_t^{\alpha} H_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1-\delta)K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t, \; |\rho| < 1$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

- C_t : consumption,
- H_t : hours
- Y_t : product
- K_t : capital
- I_t : investment
- z_t: technology shocks
- u_t: exogenous shock
- $\underbrace{\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_u^2)}_{\text{structural parameters}}$

Neoclassical Growth Model

- Both welfare theorems hold in this economy.
- Thus, we can solve directly for the social planner's problem:

$$\max_{\{C_t, H_t\}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{\left(C_t^{\lambda}(1-H_t)^{1-\lambda}\right)^{1-\tau}}{1-\tau}$$

subject to

$$C_t + I_t = e^{z_t} K_t^{\alpha} H_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1-\delta) K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t, \; |\rho| < 1, \; \varepsilon_t \sim N(0, \sigma_{\varepsilon}^2)$$

$$z_0, K_0$$

  - maximize the utility of the household subject to the production function, the evolution of technology, the law of motion for capital, the resource constraint, and some initial k_0 and z_0.

Neoclassical Growth Model

First Order Conditions

The model is fully characterized by the first order conditions:

$$
\begin{aligned}
1 &= E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{\lambda(1-\tau)}\left(\frac{1-H_{t+1}}{1-H_t}\right)^{(1-\lambda)(1-\tau)} R_{t+1}\right]\\
R_{t+1} &\equiv (1-\delta)+e^{z_{t+1}}K_{t+1}^{\alpha-1}H_{t+1}^{1-\alpha}\\
(1-\lambda)\frac{1}{(1-H_t)} &= \frac{\lambda(1-\alpha)e^{z_t}K_t^{\alpha}H_t^{-\alpha}}{C_t}\\
C_t+K_{t+1} &= e^{z_t}K_t^{\alpha}H_t^{1-\alpha}+(1-\delta)K_t\\
z_t &= \rho z_{t-1}+\varepsilon_t.
\end{aligned}
$$

Estimation of DSGE Model

Likelihood approach

Recall the steps to obtaining a *state space* representation a DSGE model

1. Obtain first order conditions of the model
2. (log) linearize the system of equation, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1 x_t(\theta) + C + \Psi(\theta)z_t$$

3. Solve the linear rational expectation system, to obtain

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

4. Measurement equation (linking data to model variables)

$$\underbrace{\underbrace{y_t}_{\text{observables}} = H(\theta)x_t \underbrace{(+m(\theta)\eta_t)}_{\text{meas. error}}}_{\text{observation equation}}$$

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MA(∞) representation

Consider the following

$$
\begin{aligned}
y_t &= H(\theta)x_t + m(\theta)\eta_t \\
x_t &= G(\theta)x_{t-1} + M(\theta)\varepsilon_t
\end{aligned}
$$

For a given θ

$$(1 - G(\theta)L)x_t = M(\theta)\varepsilon_t$$

if $(1 - G(\theta)L)$ is invertible

$$x_t = \Psi(L,\theta)M(\theta)\varepsilon_t$$

where $\Psi(L,\theta) = (1 - G(\theta)L)^{-1}$. In practice,

$$x_t = \sum_{j=0}^{\infty} G(\theta)^j M(\theta)\varepsilon_{t-j}$$

MA(∞) representation

Substituting into the equation for y_t, we have

$$\begin{aligned} y_t &= H(\theta)\sum_{j=0}^{\infty} G(\theta)^j M(\theta)\varepsilon_{t-j} + m(\theta)\eta_t \\ &= H(\theta)\Psi(L,\theta)M(\theta)\varepsilon_t + m(\theta)\eta_t \end{aligned}$$

- The impulse response function of the DSGE model is

$$H(\theta)\Psi(L,\theta)M(\theta)$$

more specifically

$$\underbrace{H(\theta)M(\theta)}_{s=1} \quad \underbrace{H(\theta)G(\theta)M(\theta)}_{s=2} \quad \cdots \quad \underbrace{H(\theta)G^{j-1}(\theta)M(\theta)}_{s=j}$$

Estimation

To calculate the impulse response function we need to estimate θ....

- Run the Kalman filter
- Calculate the likelihood function

$$p(y^T;\theta) = \prod_{t=1}^{T} \phi(y_t;\mu(\theta),\Omega(\theta)),$$

where

$$\mu(\theta) = H(\theta)x_{t|t-1}$$
$$\Omega(\theta) = H(\theta)\Sigma_{t|t-1}(\theta)H(\theta)' + m(\theta)R(\theta)m(\theta)'$$

Estimation

Now we can finally see how we estimate θ:

- Maximum Likelihood

$$\max_{\theta \in \Theta} \prod_{t=1}^{T} \phi(y_t; \mu(\theta), \Omega(\theta)),$$

- Bayesian approach

$$p(\theta | y^t) = \frac{p(y^T; \theta) p(\theta)}{p(y^T)}$$

DSGE Model Steps

1. (Log) linearize the system of equation from FOC, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1(\theta)x_t(\theta) + C(\theta) + \Psi(\theta)z_t$$

2. Solve the linear rational expectation system, to obtain

$$x_{t+1} = G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

3. Measurement equation (linking data to model variables)

$$y_t = H(\theta)x_t + m(\theta)\eta_t$$

4. Run the Kalman filter to obtain

$$\prod_{t=1}^{T} \phi(y_t; \mu(\theta), \Omega(\theta))$$

5. Estimate θ either my MLE of Bayesian procedures
6. Construct the IRF

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Maximum Likelihood estimation

$$\hat{\theta}_T \equiv \arg\max_{\theta \in \Theta} \sum_{t=1}^{T} \log \phi(y_t; \mu(\theta), \Omega(\theta))$$

Under regularity conditions

$$\hat{\theta}_T \xrightarrow{p} \theta, \text{ and } \sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{d} N(0, H^{-1} \mathscr{I} H^{-1}),$$

where

$$H = \underbrace{-E\left[\frac{\partial \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta \partial \theta'}\right]}_{\text{Hessian of log-likelihood}}$$

and

$$\mathscr{I} = \underbrace{E\left[\frac{\partial \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta'} \frac{\partial \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta'}\right]}_{\text{Fischer information matrix}}$$