The Econometrics of DSGE Models

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Neoclassical Growth Model

$$
\begin{aligned}
&U = E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{\left(C_t^{\lambda}(1-H_t)^{1-\lambda}\right)^{1-\tau}}{1-\tau} \\
&Y_t = C_t + I_t \\
&Y_t = e^{z_t} K_t^{\alpha} H_t^{1-\alpha} \\
&K_{t+1} = I_t + (1-\delta) K_t \\
&z_t = \rho z_{t-1} + \varepsilon_t, \ |\rho| < 1 \\
&\varepsilon_t \sim N(0, \sigma_{\varepsilon}^2)
\end{aligned}
$$

- C_t : consumption,
- H_t : hours
- Y_t : product
- K_t : capital
- I_t : investment
- z_t: technology shocks
- u_t: exogenous shock
- $\underbrace{\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_u^2)}_{\text{structural parameters}}$

Neoclassical Growth Model

- Both welfare theorems hold in this economy.
- Thus, we can solve directly for the social planner's problem:

$$\max_{\{C_t, H_t\}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{\left(C_t^{\lambda}(1-H_t)^{1-\lambda}\right)^{1-\tau}}{1-\tau}$$

subject to

$$C_t + I_t = e^{z_t} K_t^{\alpha} H_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1-\delta) K_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t, \ |\rho| < 1, \ \varepsilon_t \sim N(0, \sigma_{\varepsilon}^2)$$

$$z_0, K_0$$

  - maximize the utility of the household subject to the production function, the evolution of technology, the law of motion for capital, the resource constraint, and some initial k_0 and z_0.

Neoclassical Growth Model

First Order Conditions

The model is fully characterized by the first order conditions:

$$
\begin{aligned}
1 &= E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{\lambda(1-\tau)}\left(\frac{1-H_{t+1}}{1-H_t}\right)^{(1-\lambda)(1-\tau)}R_{t+1}\right]\\
R_{t+1} &\equiv (1-\delta)+e^{z_{t+1}}K_{t+1}^{\alpha-1}H_{t+1}^{1-\alpha}\\
(1-\lambda)\frac{1}{(1-H_t)} &= \frac{\lambda(1-\alpha)e^{z_t}K_t^{\alpha}H_t^{-\alpha}}{C_t}\\
C_t+K_{t+1} &= e^{z_t}K_t^{\alpha}H_t^{1-\alpha}+(1-\delta)K_t\\
z_t &= \rho z_{t-1}+\varepsilon_t.
\end{aligned}
$$

GMM Estimation of DSGE Models

- Only possible for "some" models
- Main problems
  - shocks are unobservable
  - some of the variables in the model are not directly observable
- What we would need is GMM when moment restrictions include latent variables
  - Giacomini, Gallant, Ragusa (2013, 2016)

GMM Estimation: Example

Neoclassical Growth Model

Consider the NGM:

$$E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{(1-\tau)}\left(\frac{1-H_{t+1}}{1-H_t}\right)^{(1-\lambda)(1-\tau)}R_{t+1}-1\right]=0 \tag{1}$$

$$R_{t+1}\equiv(1-\delta)+e^{z_{t+1}}K_{t+1}^{\alpha-1}H_{t+1}^{1-\alpha}$$

$$(1-\lambda)\frac{1}{(1-H_t)}=\frac{\lambda(1-\alpha)e^{z_t}K_t^{\alpha}H_t^{-\alpha}}{C_t} \tag{2}$$

$$C_t+K_{t+1}=e^{z_t}K_t^{\alpha}H_t^{1-\alpha}+(1-\delta)K_t \tag{3}$$

$$z_t=\rho z_{t-1}+\varepsilon_t \tag{4}$$

$$Y_t=e^{z_t}K_t^{\alpha}H_t^{1-\alpha} \tag{5}$$

There are 7 parameters to estimate $\theta=(\beta,\lambda,\tau,\alpha,\delta,\rho,\sigma_\varepsilon^2)$, so we need at least 7 moment conditions.

GMM Estimation: Example

Neoclassical Growth Model

We can 3 conditions from the Euler equations

$$E\left\{\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{\lambda(1-\tau)}\left(\frac{1-H_{t+1}}{1-H_t}\right)^{(1-\lambda)(1-\tau)}R_{t+1}-1\right]\right\}=0$$

$$E\left\{\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{\lambda(1-\tau)}\left(\frac{1-H_{t+1}}{1-H_t}\right)^{(1-\lambda)(1-\tau)}R_{t+1}-1\right]\frac{C_t}{C_{t-1}}\right\}=0$$

$$E\left\{\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{\lambda(1-\tau)}\left(\frac{1-H_{t+1}}{1-H_t}\right)^{(1-\lambda)(1-\tau)}R_{t+1}-1\right]R_t\right\}=0$$

Next, we take expectation of the capital equation:

$$0=E\left[(1-\lambda)\frac{1}{(1-H_t)}-\frac{\lambda(1-\alpha)e^{z_t}K_t^{\alpha}H_t^{-\alpha}}{C_t}\right]$$

$$0=E\left[C_t+K_{t+1}-e^{z_t}K_t^{\alpha}H_t^{1-\alpha}-(1-\delta)K_t\right]$$

GMM Estimation: Example

Neoclassical Growth Model

Three moment restrictions are

$$
\begin{aligned}
0 =& E(z_{t+1} - \rho z_t) \\
0 =& E\left[(z_{t+1} - \rho z_t) z_t\right] = 0 \\
0 =& E\left[(z_{t+1} - \rho z_t)^2\right] - \sigma_z^2 = 0
\end{aligned}
$$

In this case we have 8 moment restrictions a 7 parameters to estimate.

z_t is not observable....

However,

$$Y_t = e^{z_t} K_t^{\alpha} H_t^{1-\alpha} \implies \log Y_t = z_t + \alpha \log K_t + (1-\alpha) \log H_t,$$

thus

$$z_t = \log Y_t - \alpha \log K_t + (1-\alpha) \log H_t.$$

GMM Estimation: Caveat

- Not always possible to obtain expressions for the unobserved shocks, e.g.,

$$E\left\{\left[\beta\left(\frac{C_{t+1}\upsilon_{t+1}}{C_t\upsilon_t}\right)^{(1-\tau)}R_{t+1}-1\right]\right\}=0$$

- All variables need to be observables
  - We do not have a good measure of capital stock
- Identification difficult to show
- Other technical problems
- Unreasonable estimates of the parameters

Simulated method of moments

Basic idea

Suppose we can simulate from the model at a given value of the $\underbrace{\theta = (\beta, \lambda, \tau, \alpha, \delta, \rho, \sigma_u^2)}_{\text{structural parameters}}$. Let

$$\bar{D}_s(\theta) = \{C_s(\theta), H_s(\theta), K_s(\theta), Y_s(\theta), R_s(\theta)\}, s = 1, \dots, S \to \infty,$$

the simulated data and D_t, $t = 1, \dots, T$ the actual data. Then, at the true value of the parameter vector θ_0, under regularity conditions,

$$\lim_{S \to \infty} \frac{1}{S} \sum_{t=1}^{S} f(\bar{D}_s(\theta_0)) = E[f(D_t)],$$

for a function $f(\cdot)$. This suggest the following estimator

$$\min_{\theta \in \Theta} \left[\frac{1}{T} \sum_{t=1}^{T} f(D_t) - \frac{1}{S} \sum_{t=1}^{S} f(\bar{D}_s(\theta)) \right]' W \left[\frac{1}{T} \sum_{t=1}^{T} f(D_t) - \frac{1}{S} \sum_{t=1}^{S} f(\bar{D}_s(\theta)) \right]$$

Estimation of DSGE Model

Likelihood approach

Recall the steps to obtaining a *state space* representation a DSGE model

1. Obtain first order conditions of the model
2. (log) linearize the system of equation, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1(\theta)x_t + C(\theta) + \Psi(\theta)z_t$$

3. Solve the linear rational expectation system, to obtain

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

4. Measurement equation (linking data to model variables)

$$\underbrace{\underbrace{y_t}_{\text{observables}} = H_0(\theta) + H_1(\theta)x_t \underbrace{(+m(\theta)\eta_t)}_{\text{meas. error}}}_{\text{observation equation}}$$

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Linear, gaussian, state space models

Consider the general linear gaussian state space model

$$\alpha_t = c_t + T_t\alpha_{t-1} + R_t\eta_t, \quad \eta_t \sim N(0, Q)$$
$$y_t = d_t + Z_t\alpha_t + \varepsilon_t, \quad \varepsilon \sim N(0, H_t)$$

where

$$\alpha_0 \sim N(a_0, \Sigma_0)$$

- The $p \times 1$ vector y_t is the observation
- The $m \times 1$ vector α_t is the state
- The disturbances ε_t and η_t are independent sequences of independent normal vectors (white noise).

Object of interest

Two objects of interest

1. **[Kalman Filter]** Filtered state: Estimation of $E(\alpha_{t+1}|Y_t)$ and $Var(\alpha_{t+1}|Y_t)$
2. **[State smooter]** Smoothed distribution: Estimation of $\hat{\alpha}_t = E(\alpha_t|Y_T)$ and $\hat{V}_t = Var(\alpha_t|Y_T)$

Statistical properties, ctd.

- Let
$$\Sigma_t = E(\alpha_t - E(\alpha_t))(\alpha_t - E(\alpha_t))',$$
denote the variance of α_t at time t
- The variance, by independence of η_t and α_t, follows recursion
$$\Sigma_{t+1} = T_t \Sigma_t T_t' + R_t Q_t R_t'$$
- If T_t and R_t are *constant and* T *is stable,* the variance converges Σ which solves the Lyapunov equation
$$\Sigma = T \Sigma T' + RQR'.$$

The Kalman Filter

The Kalman filter is a *clever* method for computing

1. $E(\alpha_t | y^t)$
2. $E(\alpha_{t+1} | y^t)$
3. $Var(\alpha_t | y^t)$
4. $Var(\alpha_{t+1} | y^t)$

Notation

We use the notation

$$\alpha_{t|s} = E(\alpha_t | y^s)$$
$$\Sigma_{t|s} = E(\alpha_t - \alpha_{t|s})(\alpha_t - \alpha_{t|s})'$$

The Kalman Filter

Suppose we have (from previous recursion)

$$\alpha_{t|t-1} \text{ and } \Sigma_{t|t-1}$$

Two steps:

1. [Updading] Yields

$$\alpha_{t|t} \text{ and } \Sigma_{t|t}$$

   in terms of $\alpha_{t|t-1}$ and $\Sigma_{t|t-1}$.

2. [Prediction] Yields

$$\alpha_{t+1|t} \text{ and } \Sigma_{t+1|t}$$

   in terms of $\alpha_{t|t}$ and $\Sigma_{t|t}$.

The Kalman Filter

Suppose we have (from previous recursion)

$$\alpha_{t|t-1} \text{ and } \Sigma_{t|t-1}$$

Two steps:

1. [Updading] Yields
$$\alpha_{t|t} \text{ and } \Sigma_{t|t}$$
in terms of $\alpha_{t|t-1}$ and $\Sigma_{t|t-1}$.
2. [Prediction] Yields
$$\alpha_{t+1|t} \text{ and } \Sigma_{t+1|t}$$
in terms of $\alpha_{t|t}$ and $\Sigma_{t|t}$.

Kalman Filter

The key result one needs to keep in mind is the following. If

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Big| w \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix} \right)$$

then

$$z_2 | w, z_1 \sim N(m, S)$$

where

$$\begin{aligned} m &= \mu_2 + \Omega_{21} \Omega_{11}^{-1} (z_1 - \mu_1) \\ S &= \Omega_{22} - \Omega_{21} \Omega_{11}^{-1} \Omega_{12}. \end{aligned}$$

Kalman filter

Suppose we are at $t = 1$ and $\alpha_1 | y^0 \sim N(\alpha_{1|0}, \Sigma_{1|0})$. Let $y^0 = \{\emptyset\}$.

We have

$$E(y_1|y_0) = d_t + Z_t E(\alpha_1|y^0) + E(\varepsilon_t|y^0) = d_t + Z_t \alpha_{1|0}$$

and

$$y_1 - E(y_1|y_0) = Z_t(\alpha_1 - \alpha_{1|0}) + \varepsilon_t$$

From which it follows that

$$Var(y_1|y^0) = E\left[(y_1 - E(y_1|y_0))(y_1 - E(y_1|y_0))'\right] = Z_t \Sigma_{1|0} Z_t + H_t$$

$$\begin{aligned} Cov(y_1, \alpha_1|y^0) &= E\left[(y_1 - E(y_1|y_0))(\alpha_1 - E(\alpha_1|y_0))'\right] \\ &= Z_t' E\left[(\alpha_1 - E(\alpha_1|y_0))(\alpha_1 - E(\alpha_1|y_0))'\right] \\ &= Z_t' \Sigma_{1|0} \end{aligned}$$

Kalman filter

Now, since $\alpha_1|^0 \sim N(\alpha_{1|0}, \Sigma_{1|0})$ and $y_1|\alpha_1, y_0$ is also normal, we have

$$\begin{pmatrix} \alpha_1 \\ y_1 \end{pmatrix} \Bigg| y^0 \sim N\left(\begin{bmatrix} \alpha_{1|0} \\ d_1 + Z_1\alpha_{1|0} \end{bmatrix}, \begin{bmatrix} \Sigma_{1|0} & \Sigma_{1|0}Z_1 \\ Z_1'\Sigma_{1|0} & Z_1\Sigma_{1|0}Z_1 + H_1 \end{bmatrix} \right)$$

Now, noting that $y_1 = (y^0, y_1)$, $\alpha_1|y^1$can be obtained by the normal formula

$$\alpha_{1|1} = E(\alpha_1|y^1) = \alpha_{1|0} + \Sigma_{1|0}Z_1(Z_1\Sigma_{1|0}Z_1 + H_1)^{-1}(y_1 - d_1 - Z_1\alpha_{1|0})$$
$$\Sigma_{1|1} = Var(\alpha_1|y^1) = \Sigma_{1|0} - \Sigma_{1|0}Z_1(Z_1\Sigma_{1|0}Z_1 + H_1)^{-1}Z_1'\Sigma_{1|0}$$

Updating

This is the updating step, since we have updated the distribution of $\alpha_1|y^0$ to $\alpha_1|y^1$.

The Kalman Filter

From

$$\alpha_2 = c_2 + T_2 \alpha_1 + R_2 \eta_2$$

follows that

$$\alpha_{2|1} = c_2 + T_2 \alpha_{1|1} + \underbrace{\overbrace{R_2 \eta_{2|1}}^{} = R_2 E[\eta_2 | y^1] = E[\eta_2] = 0}_{\text{by independence of } \eta_t \text{ and } y^1}$$

$$= c_2 + T_2 \alpha_{1|1},$$

and similarly

$$\begin{aligned} \Sigma_{2|1} &= E(\alpha_{2|1} - \alpha_2) E(\alpha_{2|1} - \alpha_2)' \\ &= T_2 \Sigma_{1|1} T_2' + R_2 Q_2 R_2' \end{aligned}$$

Prediction

This is the prediction step

Kalman filter

We are now at $t = 2$, and we know that

$$\alpha_2 | y^1 \sim N(c_2 + T_2 \alpha_{1|1}, \Sigma_{2|1})$$

thus, we can apply the updating step and the prediction step to obtain the distribution of $\alpha_3 | y^2$ which is given

$$\alpha_3 | y^2 \sim N(c_3 + T_3 \alpha_{2|2}, T_3 \Sigma_{2|2} T_3' + R_3 Q_3 R_3')$$

where

$$\alpha_{2|2} = \alpha_{2|1} + \Sigma_{2|1} Z_2 (Z_2 \Sigma_{2|1} Z_2 + H_2)^{-1} (y_2 - d_2 - Z_2 \alpha_{2|1})$$
$$\Sigma_{2|2} = \Sigma_{2|1} - \Sigma_{2|1} Z_2 (Z_2 \Sigma_{2|1} Z_2 + H_2)^{-1} Z_2' \Sigma_{2|1}$$

The Kalman filter

Thus for a generic t and given the normality of $\alpha_t | y^{t-1}$ we can summarize the algorithm as follows

Updating step

$$\alpha_{t|t} = \alpha_{t|t-1} + \Sigma_{t|t-1} Z_t' (Z_t \Sigma_{t|t-1} Z_t' + H_t)^{-1} (y_t - d_t - Z_t \alpha_{t|t-1})$$
$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1} Z_t' (Z_t \Sigma_{t|t-1} Z_t' + H_t)^{-1} Z_t' \Sigma_{t|t-1},$$

Prediction step

$$\alpha_{t+1|t} = c_t + T_t \alpha_{t|t}$$
$$\Sigma_{t+1|t} = T_t \Sigma_{t-1|t-1} T_t' + R_t Q_t R_t'$$

The Kalman filter

Initial value

- To run the filter we need to *initialize the filter*
  - $\alpha_0 \sim N(a_0, \Sigma_0)$, $\alpha_1|Y^0 \sim N(c_1 + T_1 k, R_1 Q_1 R_1' + \Sigma_0)$
  - $\Sigma_0 = 0$ and $\alpha_0 = a_0$, from which $\alpha_1|Y^0 \sim N(c_1 + T_1 a_0, R_1 Q_1 R_1')$
- When $\{c_t, T_t, R_t, Q_t, d_t, Z_t, H_t\}$ are time invariant, common practice is to set a_0 and Σ_0 equal to the moment of the unconditional distribution of α_t

$$\alpha_t = c + T\alpha_{t-1} + R\eta_t \implies E(\alpha_t) = (I - T)^{-1}c, \quad vec(Var(\alpha_t)) = (I - T \otimes T)^{-1} vec(Q)$$

provided that the system is stable, that is, the root of

$$det(I - Tz) = 0$$

lie outside the complex unit circle.

- Other approaches available (we will see them later)
  - Diffuse Kalman filter (Rosenberg, 1973; Ansley and Kohn, 1985)
  - Estimate a_0 and Σ_0

The Kalman Filter

Filtered and smoothed distributions

- Now we now how to recover the following quantities

$$\{\alpha_{t|t-1}\}_{t=1}^{T} \text{ and } \{\alpha_{t|t}\}_{t=1}^{T}$$
$$\{\Sigma_{t|t-1}\}_{t=1}^{T} \text{ and } \{\Sigma_{t|t}\}_{t=1}^{T}$$

  which gives us

1. Filtered distribution

$$\alpha_t|y^{t-1} \sim N(\alpha_{t|t-1}, \Sigma_{t|t-1}),$$

2. Smoothed distribution

$$x_t|y^T \sim N(x_{t|T}, \Sigma_{t|T}).$$

We can also get something else.....

The Kalman Filter

The likelihood function

- The Kalman Filter also gives the likelihood, that is,

$$p(y^T) = \prod_{t=1}^{T} p(y_t|y^{t-1})$$

- From the updating step

$$\begin{pmatrix} \alpha_t|y^{t-1} \\ y_t|y^{t-1} \end{pmatrix} \sim N\left[\begin{matrix} c_t + T_t\alpha_{t|t-1} \\ d_t + Z_t\alpha_{t|t-1} \end{matrix}, \begin{pmatrix} \Sigma_{t|t-1} & \Sigma_{t|t-1}Z_t' \\ Z_t\Sigma_{t|t-1} & Z_t\Sigma_{t|t-1}Z_t' + H_t \end{pmatrix}\right]$$

Thus the likelihood function $p(y^T)$ is the product of T multivariate normal densities:

$$p(y_t|y^{t-1}) = \frac{1}{(2\pi)^{k/2}|\Omega_t|^{1/2}} \exp\left(-\frac{1}{2}(u-\mu_t)'\Omega_t^{-1}(u-\mu_t)\right)$$

with

$$\Omega_t = Z_t\Sigma_{t|t-1}Z_t' + H_t, \quad \mu_t = d_t + Z_t\alpha_{t|t-1}$$

DSGE Model and Kalman Filter

Let's put back in the parameters

$$\underbrace{\underbrace{y_t}_{\text{observables}} = H_0(\theta) + H_1(\theta)x_t \underbrace{(+m(\theta)\eta_t)}_{\text{meas. error}}}_{\text{observation equation}}$$

$$y_t = H_0(\theta) + H_1(\theta)x_t\,(+m(\theta)\eta_t)\,, \quad \eta_t \sim N(0, R(\theta))$$

$$x_t = G_0(\theta) + G(\theta)x_{t-1} + M(\theta)\varepsilon_t, \quad \varepsilon_t \sim N(0, Q(\theta))$$

- For a given θ
  - Run the Kalman filter
  - Calculate the likelihood function

$$p(y^T;\theta) = \prod_{t=1}^{T} p(y_t|y_{t-1},\theta),$$

where

$$\mu(\theta) = H_0(\theta) + H_1(\theta)x_{t|t-1}$$
$$\Omega(\theta) = H(\theta)\Sigma_{t|t-1}(\theta)H(\theta)' + m(\theta)R(\theta)m(\theta)'$$

DSGE Model Estimation

Now we can finally see how we estimate θ:

- Maximum Likelihood

$$\max_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \log p(y_t | y_{t-1}, \theta),$$

- Bayesian approach

$$p(\theta | y^t) = \frac{p(y^T; \theta) p(\theta)}{p(y^T)}$$

DSGE Model Steps

For each θ

1. (Log) linearize the system of equation from FOC, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1(\theta)x_t + C(\theta) + \Psi(\theta)z_t$$

2. Solve the linear rational expectation system, to obtain

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

3. Measurement equation (linking data to model variables)

$$\underbrace{\underbrace{y_t}_{\text{observables}} = H_0(\theta) + H_1(\theta)x_t \underbrace{(+m(\theta)\eta_t)}_{\text{meas. error}}}_{\text{observation equation}}$$

4. Run the Kalman filter to obtain

$$\ell(y^T, \theta) = \frac{1}{T}\sum_{t=1}^{T} \log p(y_t | y_{t-1}, \theta)$$

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$$\underbrace{\underbrace{y_t}_{\text{observables}} = H_0(\theta) + H_1(\theta)x_t \underbrace{(+m(\theta)\eta_t)}_{\text{meas. error}}}_{\text{observation equation}}$$

4. Run the Kalman filter to obtain

$$\ell(y^T, \theta) = \frac{1}{T}\sum_{t=1}^{T} \log p(y_t|y_{t-1}, \theta)$$

DSGE Model Steps

For each θ

1. (Log) linearize the system of equation from FOC, to obtain

$$\Gamma_0(\theta)E[x_{t+1}] = \Gamma_1(\theta)x_t + C(\theta) + \Psi(\theta)z_t$$

2. Solve the linear rational expectation system, to obtain

$$x_{t+1} = G_0(\theta) + G(\theta)x_t + M(\theta)\varepsilon_{t+1}$$

3. Measurement equation (linking data to model variables)

$$\underbrace{\underbrace{y_t}_{\text{observables}} = H_0(\theta) + H_1(\theta)x_t \underbrace{(+m(\theta)\eta_t)}_{\text{meas. error}}}_{\text{observation equation}}$$

4. Run the Kalman filter to obtain

$$\ell(y^T, \theta) = \frac{1}{T}\sum_{t=1}^{T} \log p(y_t|y_{t-1}, \theta)$$

Maximum Likelihood estimation

$$\hat{\theta}_T \equiv \arg\max_{\theta \in \Theta} \sum_{t=1}^{T} \log p(y_t | y_{t-1}, \theta)$$

Under regularity conditions

$$\hat{\theta}_T \xrightarrow{p} \theta, \text{ and } \sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{d} N(0, H^{-1} \mathscr{I} H^{-1}),$$

where

$$H = \underbrace{-E\left[\frac{\partial \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta \partial \theta'}\right]}_{\text{Hessian of log-likelihood}}$$

and

$$\mathscr{I} = \underbrace{E\left[\frac{\partial \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta'} \frac{\partial \log \phi(y_t; \mu(\theta), \Omega(\theta))}{\partial \theta'}\right]}_{\text{Fischer information matrix}}$$

Maximum Likelihood estimation

Regularity conditions

- Identification of the model:

$$\theta \neq \tilde{\theta}, \text{ then } p(y^T; \tilde{\theta}) \neq p(y^T; \theta),$$

  different value of the parameter must correspond to different values of the likelihood function. This assumption is difficult to assess in DSGE model — that although linear are highly non-linear in the structural parameters

  - *Canova, Fabio, and Luca Sala. "Back to square one: identification issues in DSGE models." Journal of Monetary Economics 56.4 (2009): 431-449.*
  - *Iskrev, Nikolay. "Local identification in DSGE models." Journal of Monetary Economics 57.2 (2010): 189-202.*
  - *Komunjer, Ivana, and Serena Ng. "Dynamic identification of dynamic stochastic general equilibrium models." Econometrica 79.6 (2011): 1995-2032.*

Maximum Likelihood estimation

Regularity conditions, ctd

- Model is correctly specified: if it is not, the MLE estimator is still asymptotically consistent, but for a pseudo-true value

$$\hat{\theta}_T \xrightarrow{p} \theta^{PT}$$

The pseudo true value maximize the Kullback Leibler

  - *White, Halbert. "Maximum likelihood estimation of misspecified models." Econometrica: Journal of the Econometric Society (1982): 1-25.*

Maximum Likelihood estimation

Misspecified likelihood estimation

- Let $g(y^T)$ the "true" density of the data. The model is misspecified if

$$g(y^T) \notin \{p(y^T;\theta), \quad \theta \in \Theta\}$$

- In this case, it can be shown that

$$\begin{aligned}
\theta^{PT} &= \arg\max_{\theta\in\Theta} \int \log\left(\frac{p(y^T;\theta)}{g(y^T)}\right) g(y^T) \\
&= \arg\min_{\theta\in\Theta} \underbrace{\int \log\left(\frac{g(y^T)}{p(y^T;\theta)}\right) g(y^T)}_{\text{Kullback Leibler "distance"}},
\end{aligned}$$

  minimize the Kullback–Leibler distance (see White, 1982)

- Remember: "all models are false, but some are useful"

Maximum Likelihood estimation

Regularity conditions, ctd

- Identification

$$\theta \neq \theta_0 \implies \log p(y^T, \theta) \neq \log p(y^T, \theta)$$

- Compactness of parameter space

$$\theta \in \Theta, \text{ where } \Theta \text{ is a compact set}$$

- Continuity of $p(y^T, \theta)$

$$\Pr\left[\log p(y^T, \theta) \in C^0(\Theta)\right] = 1$$

- Stochastic Dominance

$$|\log p(y_t | y_{t-1}; \theta)| < D(y_t), \quad \text{all } \theta \in \Theta, t \leq T$$

where

$$\int D(y^T) p(y^T, \theta_0) dy^T < \infty$$