MATH 70095 - Applicable Maths

Autumn 2025 - Assessment 2

Deadline: 13 November 2025, 09:00 (UK time)

You should submit a PDF document containing your answers to these questions, via the Blackboard VLE, by the deadline stated above. Your submission may be a scanned copy of handwritten answers, or a typed document; all submissions should be clear and legible, and should also ***show all of your working***. This coursework should involve approximately **2.5 hours** of effort. The available marks are indicated in square brackets for each question.

This coursework counts for 25% of your total mark for Applicable Maths.

This assignment must be attempted individually; your submission must be your own, unaided work. Candidates are prohibited from discussing assessed coursework, and must abide by Imperial College's rules regarding academic integrity and plagiarism. Unless specifically authorised within the assignment instructions, the submission of output from generative AI tools (e.g., ChatGPT) for assessed coursework is prohibited. Violations will be treated as an examination offence. Enabling other candidates to plagiarise your work constitutes an examination offence. To ensure quality assurance is maintained, departments may choose to invite a random selection of students to an 'authenticity interview' on their submitted assessments.

Q1) Let $X_1, \dots, X_n$ be independent and identically distributed random variables taking values in $[0, 1]$ with mean $\mu := \mathbb{E}[X_1]$. Define the sample mean $\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$.

(a) Using an appropriate concentration inequality, obtain a bound on $\Pr(|\bar{X}_n - \mu| > \epsilon)$ for a given $\epsilon > 0$.

(b) For a given $\delta \in (0, 1)$ and $n \geq 1$, derive a closed form expression for $\epsilon_n(\delta)$ in terms of n and δ, such that

$$\Pr\left(|\bar{X}_n - \mu| > \epsilon_n(\delta)\right) \leq \delta.$$

(c) Use your bound to find a value of n which guarantees that

$$\Pr(|\bar{X}_n - \mu| > 0.05) \leq 0.01.$$

You may use a calculator to evaluate any elementary expressions for this part of the question.

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Q2) *For this question, you may use without proof the standard result that*

$$I(a) := \int_{-\infty}^{\infty} e^{-ay^2}\, \mathrm{d}y = \sqrt{\pi/a}$$

for $a > 0$. For $a \leq 0$, the integral does not converge.

(a) Let $Z \sim \mathcal{N}(0, \sigma^2)$ be a normal random variable with probability density function

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{z^2}{2\sigma^2}\right\}, \qquad z \in \mathbb{R}.$$

Prove that $\mathbb{E}[Z] = 0$ and $\mathrm{Var}[Z] = \sigma^2$. *Hint: it may be useful to consider* $I'(a) \equiv \frac{d}{da} I(a)$.

(b) Derive the moment-generating function $M_Y(t) = \mathbb{E}[e^{tY}]$ of the random variable $Y = Z^2$. Use this to derive $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$.

(c) Let $Z_1, Z_2, \ldots, Z_n$ be n independent and identically distributed normal random variables with mean zero and variance 1, and let $Y_n = \sum_{i=1}^n Z_i^2$. Derive the moment generating function $M_{Y_n}(t) = \mathbb{E}[e^{tY_n}]$ of Y_n.

(d) Use the result from part (c) to prove that Y_n has a $\mathrm{Gamma}(n/2, 1/2)$ distribution.

Note: a random variable $X \sim \mathrm{Gamma}(\alpha, \beta)$ *with shape* α *and rate* β *has probability density function*

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t, \qquad z \in \mathbb{C}, \mathrm{Re}(z) > 0,$$

is the Gamma function.

(e) Characterise the limiting distribution of the random variable $V_n = \frac{Y_n - n}{\sqrt{n}}$ as $n \to \infty$.

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