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GRAHAM DIB

MATH 70095 - Applicable Maths

Autumn 2025 - Assessment 1

Deadline: 23 October 2025, 09:00 (UK time)

You should submit a PDF document containing your answers to these questions, via the Blackboard VLE, by the deadline stated above. Your submission may be a scanned copy of handwritten answers, or a typed document; all submissions should be clear and legible, and should also ***show all of your working***. This coursework should involve approximately **2.5 hours** of effort. The available marks are indicated in square brackets for each question.

This coursework counts for 25% of your total mark for Applicable Maths.

This assignment must be attempted individually; your submission must be your own, unaided work. Candidates are prohibited from discussing assessed coursework, and must abide by Imperial College's rules regarding academic integrity and plagiarism. Unless specifically authorised within the assignment instructions, the submission of output from generative AI tools (e.g., ChatGPT) for assessed coursework is prohibited. Violations will be treated as an examination offence. Enabling other candidates to plagiarise your work constitutes an examination offence. To ensure quality assurance is maintained, departments may choose to invite a random selection of students to an 'authenticity interview' on their submitted assessments.

Q1) Let $\underline{x}_1, \underline{x}_2, \underline{x}_3 \in \mathbb{R}^2$ be the Euclidean vectors

$$\underline{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \underline{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \underline{x}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For each of the following sets of vectors $\underline{y}_1, \underline{y}_2, \underline{y}_3 \in \mathbb{R}^2$, determine whether there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\underline{y}_i = T\underline{x}_i$ for all $i \in \{1, 2, 3\}$. If it exists, write down the matrix $A \in \mathbb{R}^{2\times 2}$ which represents this transformation. Explain your reasoning. This can be a formal derivation, or a written/graphical description of the transformation. If it does not exist, provide a proof.

(a)

$$\underline{y}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \qquad \underline{y}_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \qquad \underline{y}_3 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

(b)

$$\underline{y}_1 = \begin{pmatrix} -3 \\ 1/2 \end{pmatrix}, \qquad \underline{y}_2 = \begin{pmatrix} -3 \\ -1/2 \end{pmatrix}, \qquad \underline{y}_3 = \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

(c)

$$\underline{y}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \underline{y}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \underline{y}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(d)

$$\underline{y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \underline{y}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \underline{y}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

(e)

$$\underline{y}_1 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \qquad \underline{y}_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \qquad \underline{y}_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Hint: what do the equations $A\underline{x}_1 = \underline{y}_1$ and $A\underline{x}_2 = \underline{y}_2$ tell you about A here?

[8]

Q2) Let $A \in \mathbb{C}^{n\times n}$ be a diagonalisable matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Using the properties of the trace function, show that the trace of A is equal to the sum of its eigenvalues, i.e. that

$$\mathrm{tr}(A) = \lambda_1 + \cdots \lambda_n.$$

[4]

Q3) Consider the following matrices:

$$A = \begin{pmatrix} a & b \\ b & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} a & b & 0 \\ b & 2 & c \\ 0 & c & 4 \end{pmatrix}.$$

a) For what values of $a, b \in \mathbb{R}$ is the matrix A positive definite?

b) For what values of $a, b, c \in \mathbb{R}$ is the matrix B positive definite?

[5]

Q4) Consider the dataset

$$\underline{a}_1 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \qquad \underline{a}_2 = \begin{pmatrix} -6 \\ -4 \end{pmatrix}, \qquad \underline{a}_3 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \qquad \underline{a}_4 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}.$$

a) Compute the sample covariance matrix

$$S = \frac{1}{n}\sum_{i=1}^{n} \underline{a}_i \underline{a}_i^\top, \qquad n = 4$$

of $\underline{a}_1, \ldots, \underline{a}_4$.

b) Compute $\tilde{\underline{b}}_1$ and $\tilde{\underline{b}}_2$, the first and second principal components, respectively, of $\underline{a}_1, \ldots, \underline{a}_4$.

c) For each data point $\underline{a}_1, \ldots, \underline{a}_4$, compute its coordinates in the principal component basis $\tilde{\underline{b}}_1, \tilde{\underline{b}}_2$. That is, the coordinates $\{\tilde{x}_{ij} \,:\, i = 1, \ldots, 4;\ j = 1, 2\}$ such that

$$\underline{a}_i = \sum_{j=1}^{2} \tilde{x}_{ij} \tilde{\underline{b}}_j.$$

d) Compute $\underline{\tilde{a}}_1, \ldots, \underline{\tilde{a}}_4$, the (orthogonal) projection of the data $\underline{a}_1, \ldots, \underline{a}_4$ onto the subspace spanned by their first principal component $\tilde{b}_1$.

[8]

Q1) Let $\underline{x}_1, \underline{x}_2, \underline{x}_3 \in \mathbb{R}^2$ be the Euclidean vectors

$$\underline{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \underline{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \underline{x}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For each of the following sets of vectors $\underline{y}_1, \underline{y}_2, \underline{y}_3 \in \mathbb{R}^2$, determine whether there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\underline{y}_i = T\underline{x}_i$ for all $i \in \{1, 2, 3\}$. If it exists, write down the matrix $A \in \mathbb{R}^{2\times 2}$ which represents this transformation. Explain your reasoning. This can be a formal derivation, or a written/graphical description of the transformation. If it does not exist, provide a proof.

Definition 2.3.2. Let $\mathcal{V}, \mathcal{W}$ be vector spaces, defined over the common scalar field F. The univariate vector space mapping $L : V \mapsto W$ is a linear transformation if, for all $\underline{v}_1, \underline{v}_2 \in V$ and for all $s \in F$,

$$L(\underline{v}_1 + \underline{v}_2) = L(\underline{v}_1) + L(\underline{v}_2) \qquad \text{and} \qquad L(s \cdot \underline{v}_1) = s \cdot L(\underline{v}_1)$$

Using def 2.3.2, $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation if

1. $T(\underline{v}_1 + \underline{v}_2) = T(\underline{v}_1) + T(\underline{v}_2)$ and $T(s \cdot \underline{v}_1) = s \cdot L(\underline{v}_1)$ for all $\underline{v}_1, \underline{v}_2 \in V$ and all $s \in F$.

$$\underline{x}_1 + \underline{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 \cdot \underline{x}_3$$

$$\therefore \underline{x}_3 = \tfrac{1}{2}(\underline{x}_1 + \underline{x}_2)$$

We can see that $\underline{x}_3 = \frac{1}{2}(\underline{x}_1 + \underline{x}_2)$, so using 1

$$\underline{y}_3 = T(\underline{x}_3) = T\left(\tfrac{1}{2}(\underline{x}_1 + \underline{x}_2)\right) = \tfrac{1}{2}T(\underline{x}_1) + \tfrac{1}{2}T(\underline{x}_2)$$
$$= \tfrac{1}{2}(\underline{y}_1 + \underline{y}_2)$$

so for T to be a linear transformation:

2. $\underline{y}_3 = \frac{1}{2}(\underline{y}_1 + \underline{y}_2)$ must be valid.

We will use 2 to check if a map is valid, but alternatively we can use derive the matrix A using similtaneous equations with x_1, x_2, y_1, y_2 and check for consistency with x_3 and y_3

This is valid because only x_1 and x_2 are linearly independant, so they form a basis of $\mathbb{R}^2$, so any linear map is determined by $T(x_1)$ and $T(x_2)$

(a)

$$\underline{y}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \qquad \underline{y}_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \qquad \underline{y}_3 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Using 2 :

$$\frac{1}{2}(\underline{y}_1 + \underline{y}_2) = \frac{1}{2}\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \underline{y}_3$$

So linear transformation $T: R^2 \to R^2$ is valid.

Clearly T is scaling $\underline{x}_i$ by a factor of 2, so $T(x) = 2x$ or $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Formal derivation by simultaneous eq:

$\underline{y}_1 = T(\underline{x}_1)$ Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\underline{y}_1 = A\underline{x}_1$

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} \to \begin{matrix} a+b = 2 & (1) \\ c+d = 2 & (2) \end{matrix}$$

$\underline{y}_2 = T(\underline{x}_2)$

$\underline{y}_2 = \underline{A}\,x_2$

$$\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a-b \\ c-d \end{pmatrix} \to \begin{matrix} a-b = 2 & (3) \\ c-d = -2 & (4) \end{matrix}$$

$$+\begin{array}{l} ① \ a+b=2 \\ ③ \ a-b=2 \\ \hline \end{array} \qquad \begin{array}{l} ② \ c+d=2 \\ ④ \ c-d=-2 \\ \hline \end{array}$$

①+③ $2a = 4$ ②+④ $2c = 0$

$a = 2$ $c = 0$

sub into ① $(2)+b = 2$ sub into ② $0+d=2$

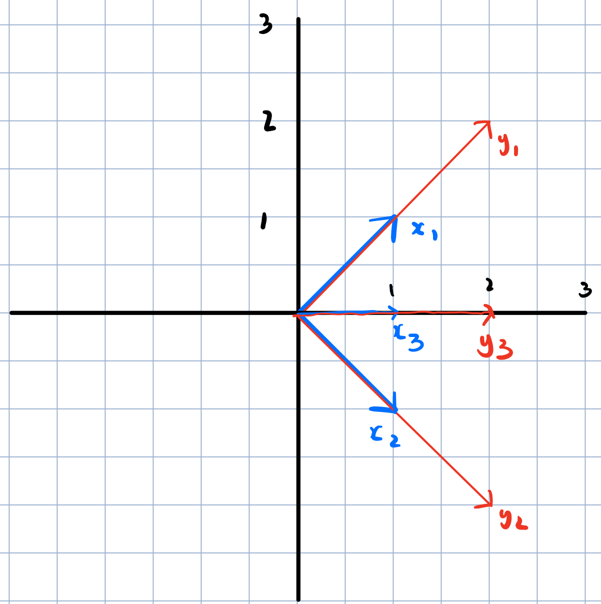
$b = 0$ $d = 2$

so $A \in R^{2\times2}$, A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad T(\underline{x}) = 2\underline{x}$$

or graphically:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$



a) T is a scalar with factor of 2, so

$A = 2\underline{I}$, where I is the identity matrix $I \in R^{2\times2}$

(b)

$$\underline{y}_1 = \begin{pmatrix} -3 \\ 1/2 \end{pmatrix}, \quad \underline{y}_2 = \begin{pmatrix} -3 \\ -1/2 \end{pmatrix}, \quad \underline{y}_3 = \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

$$\underline{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \underline{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \underline{x}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

first check if ② holds, to check if a linear map is valid.

$\frac{1}{2}(\underline{y}_1 + \underline{y}_2) = \frac{1}{2}\left[\begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} -3 \\ -\frac{1}{2} \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} -6 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \underline{y}_3 \rightarrow$ so a linear map T is valid.

Derive using simultaneous equations:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\underline{y}_1 = T(\underline{x}_1)$

$= \underline{y}_1 = A\underline{x}_1$

$= \underline{y}_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix} \rightarrow$ $a+b = -3$ ① ; $c+d = \frac{1}{2}$ ②

$\underline{y}_2 = T(\underline{x}_2)$

$= \underline{y}_2 = A(\underline{x}_2)$

$\underline{y}_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a-b \\ c-d \end{pmatrix} = \begin{pmatrix} -3 \\ -\frac{1}{2} \end{pmatrix} \rightarrow$ $a-b = -3$ ③ ; $c-d = -\frac{1}{2}$ ④

$+$ ① $a+b = -3$

③ $a-b = -3$

①+③ $2a = -6 \rightarrow a = -3$

$b = 0$

② $c+d = \frac{1}{2}$

④ $c-d = -\frac{1}{2}$

②+④ $2c = 0 \rightarrow c = 0$

$d = \frac{1}{2}$

so $A = \begin{pmatrix} -3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ so transformation T scales the x coordinate by -3, and y by $\frac{1}{2}$

(c)

$$\underline{y}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \underline{y}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \underline{y}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Check if 2 holds for linear map to be valid

$$\frac{1}{2}\left(\underline{y}_1 + \underline{y}_2\right) = \frac{1}{2}\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{y}_3$$ so linear map is valid

Derive A by similtaneous equations:

$\underline{y}_1 = T(\underline{x}_1) = A\underline{x}_1$

$$\underline{y}_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rightarrow \begin{array}{l} ① \; a+b = -1 \\ ② \; c+d = 1 \end{array}$$

$\underline{y}_2 = T(\underline{x}_2) = A\underline{x}_2$

$$\underline{y}_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a-b \\ c-d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{array}{l} ① \; a-b = 1 \\ ② \; c-d = 1 \end{array}$$

$a+b = -1$
$a-b = 1$

$a = 0$
$b = -1$

$c+d = 1$
$c-d = 1$

$2c = 2 \quad c = 1$
$d = 0$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ which is a rotation counterclockwise by $90°$

(d)

$$\underline{y}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \underline{y}_2 = \begin{pmatrix}0\\1\end{pmatrix}, \quad \underline{y}_3 = \begin{pmatrix}1\\1\end{pmatrix},$$

$$\underline{x}_1 = \begin{pmatrix}1\\1\end{pmatrix}, \quad \underline{x}_2 = \begin{pmatrix}1\\-1\end{pmatrix}, \quad \underline{x}_3 = \begin{pmatrix}1\\0\end{pmatrix}.$$

First use ① to check validity of linear map

$$\tfrac{1}{2}(\underline{y}_1 + \underline{y}_2) = \tfrac{1}{2}\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) = \tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix} \neq \underline{y}_3$$

$\therefore$ A linear map is not valid, we formalise this in the following proof by contradiction:

For the vectors $\underline{x}_1, \underline{x}_2, \underline{x}_3 \in \mathbb{R}^2$ and $\underline{y}_1, \underline{y}_2, \underline{y}_3 \in \mathbb{R}^2$, assume a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ exists, s.t. $\underline{y}_i = T\underline{x}_i$ for all $i \in \{1,2,3\}$.

Therefore, by the definition of linear transformations (2.3.2 in the lecture notes), T has the properties of additivity and homogenity.

Using def 2.3.2, $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation if

$$T(\underline{v}_1 + \underline{v}_2) = T(\underline{v}_1) + T(\underline{v}_2) \text{ and } T(s \cdot \underline{v}_1) = s \cdot L(\underline{v}_1)$$

for all $\underline{v}_1, \underline{v}_2 \in V$ and all $s \in F$.

or

$$T(\alpha \underline{v}_1 + \beta \underline{v}_2) = \alpha T(\underline{v}_1) + \beta T(\underline{v}_2) \quad \text{for all } \underline{v}_1, \underline{v}_2 \in V \text{ and } \alpha, \beta \in F$$

$$\underline{x}_1 + \underline{x}_2 = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} = 2 \cdot \begin{pmatrix}1\\0\end{pmatrix} = 2 \cdot \underline{x}_3$$

$$\therefore \underline{x}_3 = \tfrac{1}{2}(\underline{x}_1 + \underline{x}_2)$$

We can see that $\underline{x}_3 = \frac{1}{2}(\underline{x}_1 + \underline{x}_2)$, so using ①,

$$\underline{y}_3 = T(\underline{x}_3) = T\left(\tfrac{1}{2}(\underline{x}_1 + \underline{x}_2)\right) = \tfrac{1}{2}T(\underline{x}_1) + \tfrac{1}{2}T(\underline{x}_2)$$
$$= \tfrac{1}{2}(\underline{y}_1 + \underline{y}_2)$$

so for T to be a linear transformation:

2 $\quad \underline{y}_3 = \frac{1}{2}(\underline{y}_1 + \underline{y}_2)$ must be valid.

computing $\frac{1}{2}(\underline{y}_1 + \underline{y}_2)$:

$$\frac{1}{2}(\underline{y}_1 + \underline{y}_2) = \frac{1}{2}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \neq \underline{y}_3$$

however this contradicts our validity claim, as $\frac{1}{2}(\underline{y}_1 + \underline{y}_2) \neq \underline{y}_3$

so linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ for $T(\underline{x}_i) = \underline{y}_i$ does not exist, by proof by contradiction.

(e)

$$\underline{y}_1 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \qquad \underline{y}_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \qquad \underline{y}_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Hint: what do the equations $A\underline{x}_1 = \underline{y}_1$ and $A\underline{x}_2 = \underline{y}_2$ tell you about A here?

First use 1 to check the validity of the linear map.

$$\underline{y}_3 = \frac{1}{2}(\underline{y}_2 + \underline{y}_1)$$

$$\frac{1}{2}(\underline{y}_1 + \underline{y}_2) = \frac{1}{2}\left[\begin{pmatrix} 4 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

so linear map is valid.

Use sim eq to find A.

$$\underline{y}_1 = T\underline{x}_1$$

$$\underline{y}_1 = A\underline{x}_1$$

let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \rightarrow \begin{matrix} a+b = 4 \\ c+d = 4 \end{matrix}$$

$$\underline{y}_2 = T\underline{x}_2$$

$$\underline{y}_2 = A x_2$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a-b \\ c-d \end{pmatrix} = \begin{matrix} -2 \\ 2 \end{matrix} \rightarrow \begin{matrix} a-b = -2 \\ c-d = 2 \end{matrix}$$

$$\begin{array}{r} a+b = 4 \\ + \quad a-b = -2 \\ \hline 2a = 2 \\ a = 1 \\ b = 3 \end{array} \qquad \begin{array}{r} c+d = 4 \\ + \quad c-d = 2 \\ \hline 2c = 6 \\ c = 3 \\ d = 1 \end{array}$$

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Q2) Let $A \in \mathbb{C}^{n\times n}$ be a diagonalisable matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Using the properties of the trace function, show that the trace of A is equal to the sum of its eigenvalues, i.e. that

$$\mathrm{tr}(A) = \lambda_1 + \cdots \lambda_n.$$

[4]

Let $A \in \mathbb{C}^{n\times n}$ be a diagonalisable matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we have the following representation for A:

$A = P\Lambda P^{-1}$ (3.1 from lecture notes)

Where P is an invertible matrix $P \in \mathbb{C}^{n\times n}$ and $\Lambda \in \mathbb{C}^{n\times n}$ is a diagonal matrix with eigenvalue entries, i.e:

$$P = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & \lambda_n \end{pmatrix}$$

Applying the trace function to A:

$\mathrm{tr}(A) = \mathrm{tr}(P\Lambda P^{-1})$

We know from the properties of the trace function: (Section 2.2.2 Matrix algebra in lecture notes)

$\mathrm{tr}(CD) = \mathrm{tr}(DC)$ for any $C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{n\times p}$ (invariance of trace function)

so:

$$\mathrm{tr}(A) = \mathrm{tr}(P\Lambda P^{-1}) = \mathrm{tr}(PP^{-1}\Lambda) = \mathrm{tr}(I\Lambda) = \mathrm{tr}(\Lambda)$$

and as $\mathrm{tr}(B)$ is defined as the sum of the diagonal elements of B:

Suppose $B = [B_{ij}]$ is a square matrix of order $n \times n$ (Definition 2.2.6 in the lecture notes)

$$\mathrm{tr}(B) = \sum_{i=1}^{n} a_{ij}$$

then:

$$\mathrm{tr}(A) = \mathrm{tr}(\Lambda) = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \ldots \lambda_n \qquad \text{QED}$$

Q3) Consider the following matrices:

$$A = \begin{pmatrix} a & b \\ b & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} a & b & 0 \\ b & 2 & c \\ 0 & c & 4 \end{pmatrix}.$$

a) For what values of $a, b \in \mathbb{R}$ is the matrix A positive definite?

b) For what values of $a, b, c \in \mathbb{R}$ is the matrix B positive definite?

[5]

a) Recall Definition of positive definite, from definition 2.2.21:

Consider the Hermitian square matrix $A \in \mathbb{C}^{n \times n}$. A is defined to be positive definite if:

$\underline{u}^* A \underline{u} > 0$ for all $\underline{u} \in \mathbb{C}^{n \times 1}$, $\underline{u} \neq 0$.

In our case, matrix A is defined over the reals, so the property holds if A is symmetric, which it is.

We can use Sylvester's criterion to determine which values ensure A is PD, that is all leading principle minors must be positive

1. Upper left 1×1 corner of A

$\det(a) > 0$

$\underline{a > 0}$

2. Upper left 2×2 corner of A / A itself

$\det(A) > 0$

$2a - b^2 > 0$

$2a > b^2$

$\underline{a > \frac{b^2}{2}}$

So for A to be PD, $a > 0$ and $a > \frac{b^2}{2}$

b) Recall Definition of positive definite, from definition 2.2.21:

Consider the Hermitian square matrix $A \in \mathbb{C}^{n \times n}$. A is defined to be positive definite if:

$\underline{u}^* A \underline{u} > 0$ for all $\underline{u} \in \mathbb{C}^{n \times 1}, \underline{u} \neq 0$.

In our case, matrix A is defined over the reals, so the property holds if A is symmetric, which it is.

We can use Sylvester's Criterion to determine which values ensure A is PD, that is all leading principle minors must be positive

1. Upper left 1x1 corner of B

$\det(a) > 0$

$\underline{a > 0}$

2. Upper left 2x2 corner of B

$$\det\begin{pmatrix} a & b \\ b & 2 \end{pmatrix} > 0$$

$$2a - b^2 > 0$$
$$2a > b^2$$
$$a > \frac{b^2}{2}$$

3. B itself

$\det(B) > 0$

$$a \det\begin{vmatrix} 2 & c \\ c & 4 \end{vmatrix} - b \det\begin{vmatrix} b & 0 \\ c & 4 \end{vmatrix} + 0 \cdot \det\begin{vmatrix} b & 2 \\ 0 & c \end{vmatrix} > 0$$

$$a \cdot (8 - c^2) - b \cdot 4b > 0$$

$$a(8 - c^2) - 4b^2 > 0$$

if $c = \sqrt{8}$, then $-4b^2 < 0$, so $c \neq \sqrt{8}$

if $8 - c^2 < 0$ then $a(8 - c^2) - 4b^2 < 0$ as $a > 0$, so $8 - c^2 < 0$

so $8 - c^2 > 0$ also

so for B to be PD, we have conditions:

$a > 0$, $a > \frac{b^2}{2}$, $a(8-c^2) - 4b^2 > 0$ and $8 - c^2 > 0$

Q4) Consider the dataset

$$\underline{a}_1 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \qquad \underline{a}_2 = \begin{pmatrix} -6 \\ -4 \end{pmatrix}, \qquad \underline{a}_3 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \qquad \underline{a}_4 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}.$$

a) Compute the sample covariance matrix

$$S = \frac{1}{n} \sum_{i=1}^{n} \underline{a}_i \underline{a}_i^\top, \qquad n = 4$$

of $\underline{a}_1, \ldots, \underline{a}_4$.

$$S = \frac{1}{4}\left[a_1 a_1^T + a_2 a_2^T + a_3 a_3^T + a_4 a_4^T\right]$$

$$S = \frac{1}{4}\left[\begin{pmatrix} 6 \\ 4 \end{pmatrix}\begin{pmatrix} 6 & 4 \end{pmatrix} + \begin{pmatrix} -6 \\ -4 \end{pmatrix}\begin{pmatrix} -6 & -4 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}\begin{pmatrix} 4 & 0 \end{pmatrix} + \begin{pmatrix} -4 \\ 0 \end{pmatrix}\begin{bmatrix} -4 & 0 \end{bmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix} 36 & 24 \\ 24 & 16 \end{pmatrix} + \begin{pmatrix} 36 & 24 \\ 24 & 16 \end{pmatrix} + \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix} 2 \cdot 36 + 2 \cdot 16 & 2 \cdot 24 \\ 2 \cdot 24 & 2 \cdot 16 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 104 & 48 \\ 48 & 32 \end{bmatrix} = \begin{bmatrix} 26 & 12 \\ 12 & 8 \end{bmatrix}$$

Q4) Consider the dataset

$$\underline{a}_1 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \quad \underline{a}_2 = \begin{pmatrix} -6 \\ -4 \end{pmatrix}, \quad \underline{a}_3 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad \underline{a}_4 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}.$$

b) Compute $\underline{b}_1$ and $\underline{b}_2$, the first and second principal components, respectively, of $\underline{a}_1, \ldots, \underline{a}_4$.

From 3.4.2 of the lecture notes, the basis vectors $\underline{\tilde{b}}_1, \ldots, \underline{\tilde{b}}_r$ that satisfy the sequential variance maximisation procedure to find the principle components are the eigenvectors of the data covariance matrix calculated in part (a).

We will do this by eigendecomposition (instead of SVD)

So we need to find the eigenvectors of

$$S = \begin{pmatrix} 26 & 12 \\ 12 & 8 \end{pmatrix}$$

To find the eigenvectors, we first need to find the eigenvalues.

Solve characteristic polynomial $|A - \lambda I| = 0$ to find eigenvalues.

$$\rightarrow \quad |S - \lambda I| = 0$$

$$\det \begin{pmatrix} 26-\lambda & 12 \\ 12 & 8-\lambda \end{pmatrix} = (26-\lambda)(8-\lambda) - 12 \cdot 12$$

$$= 208 - 26\lambda - 8\lambda + \lambda^2 - 144$$

$$= \lambda^2 - 34\lambda + 64$$

Quadratic formula

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda_1, \lambda_2 = \frac{34 \pm \sqrt{34^2 - 4(64)}}{2}$$

$$= \frac{34 \pm 30}{2}$$

$$\lambda_1 = \frac{34+30}{2} = 32 \qquad \lambda_2 = \frac{34-30}{2} = 2$$

Now find eigenvectors by solving:

$(S - \lambda I) v = 0$ for non-zero vector v.

$(S - \lambda_1 I) v_1 = 0$

$(S - 32 I) v_1 = 0$

let $v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\left[\begin{pmatrix} 26 & 12 \\ 12 & 8 \end{pmatrix} - \begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -6 & 12 \\ 12 & -24 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -6x_1 + 12x_2 \\ 12x_1 - 24x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-6x_1 + 12x_2 = 0$
$12x_1 - 24x_2 = 0$

so $x_1 = 2x_2$

let $x_1 = 2$ $x_2 = 1$

so $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

now normalise:

$$u_1 = \frac{1}{\sqrt{2^2+1^2}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

for second eigenvector:

$(S - \lambda I)v = 0$ for non-zero vector v.

$(S - \lambda_2 I)v_2 = 0$

$(S - 2I)v_2 = 0$

let $v_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\left[\begin{pmatrix} 26 & 12 \\ 12 & 8 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 24 & 12 \\ 12 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 24x_1 + 12x_2 \\ 12x_1 + 6x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$24x_1 + 12x_2 = 0$
$12x_1 + 6x_2 = 0$

so $x_1 = -\frac{1}{2}x_2$

let $x_1 = 1$ $x_2 = -2$

so $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

now normalise:

$$u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Finally, the k^{th} principle component is equal to the eigenvector with the k^{th} largest eigenvalue, so:

$\underline{b}_1 = \underline{a}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with corresponding eigenvalue $\lambda_1 = 32$

$\underline{b}_2 = \underline{a}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ with corresponding eigenvalue $\lambda_2 = 2$

Q4) Consider the dataset

$$\underline{a}_1 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \quad \underline{a}_2 = \begin{pmatrix} -6 \\ -4 \end{pmatrix}, \quad \underline{a}_3 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad \underline{a}_4 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}.$$

c) For each data point $\underline{a}_1, \ldots, \underline{a}_4$, compute its coordinates in the principal component basis $\underline{\tilde{b}}_1, \underline{\tilde{b}}_2$. That is, the coordinates $\{\tilde{x}_{ij} : i = 1, \ldots, 4;\ j = 1, 2\}$ such that

$$\underline{a}_i = \sum_{j=1}^{2} \tilde{x}_{ij} \underline{\tilde{b}}_j.$$

c)

$$a_i = \sum_{j=1}^{2} \tilde{x}_{ij} \underline{\tilde{b}}_j$$

$$a_i = \tilde{x}_{i1} \underline{\tilde{b}}_j + \tilde{x}_{i2} \underline{\tilde{b}}_j$$

let $\underline{x}_i = \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix}$, a vector of the coordinates

and B a 2x2 matrix with collumns equal to the basis vectors, i.e

$$\underline{b}_j = \begin{pmatrix} b_{j1} \\ b_{j2} \end{pmatrix} \rightarrow B = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$$

then $a_i = B\underline{x}_i$

$$B^T a_i = x_i$$

So calculate $B^T a_i$ to work out coordinates.

$$B = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$B^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$ (makes sense as B is orthogonal & symmetric)

$\underline{x}_i = B^T \underline{a}_i$

$$\begin{aligned} \underline{x}_1 &= B^T a_1 \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} 16 \\ -2 \end{pmatrix} \end{aligned}$$ so $x_{11} = \frac{16}{\sqrt{5}}$ $x_{12} = \frac{-2}{\sqrt{5}}$

$$\begin{aligned} \underline{x}_2 &= B^T a_2 \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -6 \\ -4 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -16 \\ 2 \end{pmatrix} \end{aligned}$$ so $x_{21} = \frac{-16}{\sqrt{5}}$ / $x_{22} = \frac{2}{\sqrt{5}}$

$$\begin{aligned} \underline{x}_3 &= B^T a_3 \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 8 \\ 4 \end{pmatrix} \end{aligned}$$ so $x_{31} = \frac{8}{\sqrt{5}}$ $x_{32} = \frac{4}{\sqrt{5}}$

$$\begin{aligned} x_4 &= B^T a_4 \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -8 \\ -4 \end{pmatrix} \end{aligned}$$ so $x_{41} = \frac{-8}{\sqrt{5}}$ $x_{42} = \frac{-4}{\sqrt{5}}$

Q4) Consider the dataset

$$\underline{a}_1 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \quad \underline{a}_2 = \begin{pmatrix} -6 \\ -4 \end{pmatrix}, \quad \underline{a}_3 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad \underline{a}_4 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}.$$

d) Compute $\tilde{\underline{a}}_1, \ldots, \tilde{\underline{a}}_4$, the (orthogonal) projection of the data $\underline{a}_1, \ldots, \underline{a}_4$ onto the subspace spanned by their first principal component $\tilde{b}_1$.

[8]

Recall the formula for projection from 2.3.2 of the lecture notes:

$$\underline{x}_i' = \frac{bb^T}{\|\underline{b}\|^2} x_i$$

so we need to multiply data points $a_1, \ldots, a_4$ by P, where P is:

$$P = \frac{bb^T}{b^T b}$$

$$P = \frac{\bar{b}_1 \bar{b}_1^T}{\tilde{b}_1^T \bar{b}_1}$$

$$\tilde{b}_1 \tilde{b}_1^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\|b^2\| = \left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 = 1 \quad (\text{as unit vector})$$

now calculate the orthogonal projections of the data

$$\tilde{a}_1 = b_1^T b_1 a_1$$

$$= \frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 32 \\ 16 \end{pmatrix}$$

$$\tilde{a}_2 = b_1^T b_1 a_2$$

$$= \frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -6 \\ -4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -32 \\ -16 \end{pmatrix}$$

$$\tilde{a}_3 = b_1^T b_1 a_3$$

$$= \frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 16 \\ 8 \end{pmatrix}$$

$$\tilde{a}_4 = b_1^T b_1 a_4$$

$$= \frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -16 \\ -8 \end{pmatrix}$$