MATH 70095 - Applicable Maths

Autumn 2025 - Unassessed Coursework

Deadline: 9 October 2025, 09:00 (UK time)

You should submit a PDF document containing your answers to these questions, via the Blackboard VLE, by the deadline stated above. Your submission may be a scanned copy of handwritten answers, or a typed document; all submissions should be clear and legible, and should also ***show all of your working***. This coursework should involve approximately **3 hours** of effort.

Your submission will be marked and you will be provided with written feedback. The marks available for each question are indicated in square brackets. This mark ***will not*** count towards your overall grade for Applicable Maths.

Plagiarism: Your submission should be *your own work*. Note that software tools are used for plagiarism detection.

Q1) The following derivations are incorrect. For each one, clearly describe any errors that have been made, and give the correct answer. **[5]**

a)

$$f(x) = \frac{x}{x+e^{-x}} \implies f'(x) = \frac{(1-e^{-x})x - (x+e^{-x})}{(x+e^{-x})^2} = \frac{-e^{-x}(x+1)}{(x+e^{-x})^2}.$$

b)

$$\begin{aligned}\int_0^{\pi/4} x^2 \sin 2x \, dx &= \left[2x \cdot \left(-\frac{1}{2}\cos 2x\right)\right]_0^{\pi/4} + \int_0^{\pi/4} x \cos 2x \, dx \\ &= \Big[-x\cos 2x\Big]_0^{\pi/4} + \left[x \cdot \left(\frac{1}{2}\sin 2x\right)\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sin 2x \, dx \\ &= \frac{\pi}{8} + \frac{1}{4}.\end{aligned}$$

Q2) Consider the multivariate function $f(x, y, z) = \log g(x, y) + \log h(y, z)$, where $x, y, z \in \mathbb{R}$ and g, h are both positive-valued bivariate functions. **[9]**

a) Derive a vector-valued expression for the gradient, $\nabla f(x, y, z)$.

b) Derive the Hessian matrix, $H_f = \nabla^2 f(x, y, z)$.

You should provide your answers in terms of the functions g and h and their derivatives.

Q3) Consider the 4-dimensional Euclidean vector space $E = (\mathbb{R}^4, +, \cdot)$, equipped with the usual Euclidean inner product $\langle \cdot, \cdot \rangle$. **[6]**

a) Let $U \subseteq \mathbb{R}^4$ be a subspace spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix}.$$

Using a projection matrix P, calculate the orthogonal projection of $\underline{v} = [1, 2, 1, 3]^T$ onto U.

b) Using the Gram matrix, show that the following vectors are linearly independent:

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

Q1) The following derivations are incorrect. For each one, clearly describe any errors that have been made, and give the correct answer. **[5]**

a)

$$f(x) = \frac{x}{x+e^{-x}} \implies f'(x) = \frac{(1-e^{-x})x - (x+e^{-x})}{(x+e^{-x})^2} = \frac{-e^{-x}(x+1)}{(x+e^{-x})^2}.$$

quotient rule : let $f(x) = \frac{h(x)}{g(x)}$

$$f'(x) = \frac{h'(x)g(x) - h(x)g'(x)}{(g(x))^2}$$

$$f(x) = \frac{x}{x+e^{-x}} \Rightarrow h(x) = x \qquad h'(x) = 1$$

$$g(x) = x+e^{-x} \qquad g'(x) = 1-e^{-x}$$

$$f'(x) = \frac{1(x+e^{-x}) - x(1-e^{-x})}{(x+e^{-x})^2} = \frac{(x+e^{-x}) - (x - xe^{-x})}{(x+e^{-x})^2}$$

$$= \frac{e^{-x} + xe^{-x}}{(x+e^{-x})^2} = \boxed{\frac{e^{-x}(x+1)}{(x+e^{-x})^2}}$$

$$f'(x) = \frac{(1-e^{-x})x - (x+e^{-x})}{(x+e^{-x})^2}$$

The question applies the quotient rule incorrectly, the (incorrect) rule used is :

$$f'(x) = \frac{h(x)g'(x) - h'(x)g(x)}{(g(x))^2} \quad \text{for} \quad f(x) = \frac{h(x)}{g(x)}$$

as a result, the simplified answer ends up being negative, due to the mistake.

b)

Incorrect term in by parts, should be x^2

$$\int_0^{\pi/4} x^2 \sin 2x \, dx = \left[2x \left(-\frac{1}{2}\cos 2x\right)\right]_0^{\pi/4} + \int_0^{\pi/4} x\cos 2x \, dx$$

negation missed on by parts formula

incorrect from by parts formula

$$= \left[-x\cos 2x\right]_0^{\pi/4} + \left[x \cdot \left(\frac{1}{2}\sin 2x\right)\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sin 2x \, dx$$

$$= \frac{\pi}{8} + \frac{1}{4}.$$

missed negative in by parts on previous line, should be $\frac{\pi}{8} - \frac{1}{4}$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

$u = x^2 \rightarrow du = 2x$

$dv = \sin 2x \rightarrow v = -\frac{1}{2}\cos 2x$

$$\int_0^{\frac{\pi}{4}} x^2 \sin 2x \, dx = \left[x^2 \cdot -\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\frac{1}{2}\cos 2x \cdot 2x \, dx$$

$$= \left[-\frac{x^2}{2}\cos 2x\right]_0^{\frac{\pi}{4}} + \int x\cos 2x \, dx$$

$$= \left[-\frac{x^2}{2}\cos 2x\right]_0^{\frac{\pi}{4}} + \left[\frac{x}{2}\sin 2x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin 2x \, dx$$

$$= \left[-\frac{x^2}{2}\cos 2x\right]_0^{\frac{\pi}{4}} + \left[\frac{x}{2}\sin 2x\right]_0^{\frac{\pi}{4}} + \left[\frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{4}}$$

$$= [0 - 0] + \left[\frac{\pi}{8} - 0\right] \quad \left[0 - \frac{1}{4}\right]$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

Q2) Consider the multivariate function $f(x, y, z) = \log g(x, y) + \log h(y, z)$, where $x, y, z \in \mathbb{R}$ and g, h are both positive-valued bivariate functions. **[9]**

a) Derive a vector-valued expression for the gradient, $\nabla f(x, y, z)$.

b) Derive the Hessian matrix, $H_f = \nabla^2 f(x, y, z)$.

You should provide your answers in terms of the functions g and h and their derivatives.

a) Recall gradient $\nabla f(x,y,z) =$

$$\left[\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}\right]^T$$

$$f(x,y,z) = \log g(x,y) + \log h(y,z)$$

$$\frac{df}{dx} = \frac{1}{g(x,y)} \cdot \frac{dg}{dx}(x,y)$$

$$\frac{df}{dy} = \frac{1}{g(x,y)} \cdot \frac{dg}{dy}(x,y) + \frac{1}{h(y,z)} \cdot \frac{dh}{dy}(y,z)$$

$$\frac{df}{dz} = \frac{1}{h(y,z)} \cdot \frac{dh}{dz}(y,z)$$

So $\nabla f(x,y,z) = \left[\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}\right]^T =$

$$\begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \\ \frac{df}{dz} \end{bmatrix} = \begin{bmatrix} \frac{1}{g(x,y)} \cdot \frac{dg}{dx}(x,y) \\ \frac{1}{g(x,y)} \cdot \frac{dg}{dy}(x,y) + \frac{1}{h(y,z)} \cdot \frac{dh}{dy}(y,z) \\ \frac{1}{h(y,z)} \cdot \frac{dh}{dz}(y,z) \end{bmatrix}$$

b) Recall hessian matrix is a matrix of all second order partial derivatives, i.e:

$$\nabla^2 = \begin{bmatrix} \frac{d^2 f}{dx_1^2} & \frac{d^2 f}{dx_1 dx_2} & \cdots & \frac{d^2 f}{dx_1 dx_n} \\ \frac{d^2 f}{dx_2 dx_1} & \frac{d^2 f}{dx_2^2} & \cdots & \frac{d^2 f}{dx_2 dx_n} \\ \vdots & & & \\ \frac{d^2 f}{dx_n dx_1} & \frac{d^2 f}{dx_n dx_2} & \cdots & \frac{d^2 f}{dx_n} \end{bmatrix}$$

so $\nabla^2 f(x,y,z) =$

$$\begin{bmatrix} \frac{d^2 f}{dx^2} & \frac{d^2 f}{dx dy} & \frac{d^2 f}{dx dz} \\ \frac{d^2 f}{dy dx} & \frac{d^2 f}{dy^2} & \frac{d^2 f}{dy dz} \\ \frac{d^2 f}{dz dx} & \frac{d^2 f}{dz dy} & \frac{d^2 f}{dz^2} \end{bmatrix}$$

define shorthand $g = g(x,y)$
$h = h(y,z)$

$g_x = \frac{dg}{dx}$ $g_{xx} = \frac{dg}{dx^2}$ etc

$$\frac{df}{dx} = \frac{1}{g(x,y)} \cdot \frac{dg}{dx}(x,y) = \frac{g_x}{g}$$

$$\frac{df}{dy} = \frac{1}{g(x,y)} \cdot \frac{dg}{dy}(x,y) + \frac{1}{h(y,z)} \cdot \frac{dh}{dy}(y,z) = \frac{g_y}{g} + \frac{h_y}{h}$$

$$\frac{df}{dz} = \frac{1}{h(y,z)} \cdot \frac{dh}{dz}(y,z) = \frac{h_z}{h}$$

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left[\frac{g_x}{g}\right] = \frac{g \cdot g_x - g_x^2}{g^2}$$ (quotient rule)

$$\frac{d^2f}{d_x\, dy} = \frac{d}{dy}\left[\frac{df}{dx}\right] = \frac{d}{dy}\left[\frac{g_x}{g}\right] = \frac{g \cdot g_{xy} - g_x \cdot g_y}{g^2}$$

$$\frac{d^2f}{dx\, dz} = \frac{d}{dz}\left[\frac{df}{dx}\right] = \frac{d}{df}\left[\frac{g_x}{g}\right] = 0$$

$$\frac{d^2f}{dy^2} = \frac{d}{dy}\left[\frac{df}{dy}\right] = \frac{d}{dy}\left[\frac{g_y}{g} + \frac{h_y}{h}\right] = \frac{g \cdot g_{yy} - g_y^2}{g_y^2} + \frac{h \cdot h_{yy} - h_y^2}{h^2}$$

$$\frac{d^2f}{dz\, dy} = \frac{d}{dz}\left[\frac{df}{dy}\right] = \frac{d}{dz}\left[\frac{g_y}{g} + \frac{h_y}{h}\right] = \frac{d}{dz}\left[\frac{h_y}{h}\right] = \frac{h \cdot h_{yz} - h_y h_z}{h^2}$$

$$\frac{d^2f}{dz^2} = \frac{d}{dz}\left[\frac{df}{dz}\right] = \frac{d}{dz}\left[\frac{h_z}{h}\right] = \frac{h \cdot h_{zz} - h_z^2}{h^2}$$

so

$$\nabla^2 f(x,y,z) =$$

$$\begin{bmatrix} \frac{g g_{xx} - g_x^2}{g^2} & \frac{g g_{xy} - g_x g_y}{g^2} & 0 \\ \frac{g g_{xy} - g_x g_y}{g^2} & \frac{g g_{yy} - g_y^2}{g^2} + \frac{h h_{yy} - h_y^2}{h^2} & \frac{h h_{yz} - h_y h_z}{h^2} \\ 0 & \frac{h h_{yz} - h_y h_z}{h^2} & \frac{h h_{zz} - h_z^2}{h^2} \end{bmatrix}$$

Q3) Consider the 4-dimensional Euclidean vector space $E = (\mathbb{R}^4, +, \cdot)$, equipped with the usual Euclidean inner product $\langle \cdot, \cdot \rangle$. **[6]**

a) Let $U \subseteq \mathbb{R}^4$ be a subspace spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix}.$$

Using a projection matrix P, calculate the orthogonal projection of $\underline{v} = [1, 2, 1, 3]^T$ onto U.

b) Using the Gram matrix, show that the following vectors are linearly independent:

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

$$\text{Proj}_U(v) = Pv = A(A^TA)^{-1}A^T$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 7 \\ 4 & 6 & 3 \\ 7 & 3 & 14 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} \frac{29}{126} & -\frac{23}{126} & -\frac{5}{126} \\ -\frac{23}{126} & \frac{37}{126} & -\frac{1}{126} \\ -\frac{5}{126} & -\frac{1}{126} & \frac{11}{126} \end{bmatrix}$$ via calculator.

$$A(A^TA)^{-1}A^Tv =$$

$$A(A^TA^{-1}) \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} =$$

$$= \quad A\left(A^T A^{-1}\right)\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

$$= \quad A \begin{bmatrix} \frac{29}{126} & -\frac{23}{126} & -\frac{5}{126} \\ -\frac{23}{126} & \frac{37}{126} & -\frac{1}{126} \\ -\frac{5}{126} & -\frac{1}{126} & \frac{11}{126} \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

$$= \quad A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 7 \end{bmatrix}$$

$$\text{Proj}_U(v) = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 7 \end{bmatrix}$$

b) Gram matrix $G = \left(\langle x_i, x_j \rangle\right)_{i,j=1}^{4}$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} \quad v_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$G = \begin{bmatrix} 6 & 4 & 5 & 6 \\ 4 & 6 & 3 & 6 \\ 5 & 3 & 14 & 9 \\ 6 & 6 & 9 & 10 \end{bmatrix}$$

v_1, v_2, v_3, v_4 independant $\iff$ G is nonsingular $\iff$ $\det G \neq 0$

Recall modulo-p invertibility trick

If $\det(A) \not\equiv 0 \bmod p \implies \det(A) \neq 0$ over $\mathbb{Z}$ or $\mathbb{R}$
and $\det G \bmod x = \det(G \bmod x)$

$$G \bmod 2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

after row reduction, this will have det $= 0$, so try with mod 3

$$G \bmod 3 = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Gaussian elimination

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ $R_1 \Leftrightarrow R_2$ (Row swap multiplies det by $-1 \equiv 2 \bmod 3$)

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ $R_3 - 2R_1$ (Does not affect det)

swap factor $-1 \equiv 2 \bmod 3$

upper singular, so $\det(G \bmod 3) \equiv (1 \cdot 1 \cdot 2 \cdot 1) \cdot 2 \equiv 2 \cdot 2 \equiv 4 \equiv 1 \bmod 3$

so $\det(G \bmod 3) \equiv 1 \bmod 3 \Rightarrow \det G \equiv 1 \bmod 3$.

$\therefore$ As $\det G \not\equiv 0 \bmod 3$, G is nonsingular over $\mathbb{Z}$ & $\mathbb{R}$, so the vectors are linearly independant.