# Double exponential synaptic implementation note

Double-exponential implementation for synapse with rise time constant as $\tau_r$ and decay time constant as $\tau_d$, and the equation for synaptic current is $I = g(E - V)$ where the equilibrium potential $E$ is also constant. $x$ sort of represents an "activation" variable of the synapse. The problem with this is: if the network is connected more than once with the synapses of the same dynamics and time constant, the number of equations would scale by a factor of $O(n^2)$, in which $n$ is the number of neurons. And the $x$ term would also be sort of complicated to integrate into the equation as well.

$$
\begin{aligned}
\frac{dg}{dt} &= z & \alpha &= k_n\left(\frac{1}{\tau_r} - \frac{1}{\tau_d}\right) & k_n &= \frac{1}{h(t_{peak})} \quad \text{(normz factor)} \\
\frac{dz}{dt} &= \alpha x + \beta z + \gamma g & \beta &= -\left(\frac{1}{\tau_r} + \frac{1}{\tau_d}\right) & h(t) &= e^{-t/\tau_d} - e^{-t/\tau_r} \\
x &= g_{max}\sum_k \delta(t - t_k) & \gamma &= -\frac{1}{\tau_r\tau_d} & t_{peak} &= \tau_r\tau_d \cdot \frac{\ln\tau_d - \ln\tau_r}{\tau_d - \tau_r}
\end{aligned}
$$

So during updating the ODE of the synaptic conductance for the **same type of synapse** (meaning the same time constants and equilibrium potential, hence same constants in $dz/dt$), here's what we could do that can result in only keeping exactly $n$ variables (actually $2n$ counting $g$ and $z$) and hence can generalize for any connectivity pattern of this type of synapse in the network.

Consider knowing what the effects of the network to neuron $i$, we just need to know the summation of each conductance (since $E - V$ is constant). And we consider all conductances sum linearly, in other words, $I_{syn}(t, i) = \sum_j g_{ij}(t) \times (E - V_i)$, where $g_{ij}(t)$ is the conductance of the synapse from neuron $j$ to neuron $i$ at time $t$, and $g_{ii} = 0$ (doesn't actually matter). The equations for each synapse become:

$$
\begin{aligned}
\frac{dg_{ij}}{dt} &= z_{ij} & G_{ij} &: \text{max conductance of synapse from j to i} \\
\frac{dz_{ij}}{dt} &= \alpha x_{ij} + \beta z_{ij} + \gamma g_{ij} & t^{(j)} &: \text{spike (or delayed activation) from neuron j} \\
x_{ij} &= G_{ij}\sum_k \delta(t - t_k^{(j)})
\end{aligned}
$$

However, for simplicity and for the sake of implementation, we wouldn't need to keep track of all the spikes, just whether it spikes (or activates the synapse with a delay) of the last time step, so for the current time step:

$$
x_{ij} = G_{ij}\delta_j \qquad \delta_j : \text{whether j sends out activation in the last time step}
$$

Again, since (we assume) the conductances add linearly, we can define:

$$
\begin{aligned}
g_i &= \sum_j g_{ij} & I_{syn}(t, i) &= \sum_j g_{ij}(t) \times (E - V_i) \\
z_i &= \sum_j z_{ij} & &= g_i(t) \times (E - V_i)
\end{aligned}
$$

Effectively that makes:
(the derivative notation from here on assumses for numerical integration like $a[m] = a[m-1] + da \times dt$, instead of "continuous" derivative)

$$
\begin{aligned}
\frac{dg_i}{dt} &= z_i \\
\frac{dz_i}{dt} &= \alpha X_i + \beta z_i + \gamma g_i \\
X_i &= \sum_j G_{ij}\delta_j
\end{aligned}
$$

Now we consider for a vector of received synaptic conductance for the network (below for implementation in MATLAB using matrix/vector notation):

$$
\begin{aligned}
&\frac{d\vec{g}}{dt} = \vec{z} && \vec{g}_{n\times 1} = [g_1, g_2, ..., g_n]^T \\
&\frac{d\vec{z}}{dt} = \alpha \mathbf{X} + \beta \vec{z} + \gamma \vec{g} && \vec{z}_{n\times 1} = [z_1, z_2, ..., z_n]^T \\
&\mathbf{X} = \mathbf{G} \times \vec{\delta} \quad \text{(matrix multiplication)} && \mathbf{G}_{n\times n} : \text{conductance matrix, (i,j): from j to i} \\
& && \vec{\delta}_{n\times 1} : \text{column vector of network activation in last time bin}
\end{aligned}
$$

Below are notes for the equations describing double exponential synapse. The impulse function should take a form of $h(t) = k_n * (e^{-t/\tau_d} - e^{-t/\tau_r})$, where $\tau_d > \tau_r$ with some normalization factor $k_n$. To find $k_n$, we can just simply set $\frac{dh}{dt} = 0$, in which $t_{peak}$ could be found (above), hence $k_n$.
The Laplace transform of $h(t)$ hence is

$$H(s) = k_n \left( \frac{1}{s + \frac{1}{\tau_d}} - \frac{1}{s + \frac{1}{\tau_r}} \right)$$

And for the activation term $x(t), X(s)$ and synaptic conductance term $g(t), G(s)$ we have $G(s) = H(s)X(s)$. That makes:

$$\tau_d \tau_r \ddot{g} + (\tau_d + \tau_r)\dot{g} + g = k_n(\tau_d + \tau_r)x$$

When we decouple the 2nd derivative with $z = \dot{g}$, we have the first set of equations.