Optimisation formulations - brainstorming

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1 Introduction

We start with the formulation Φ_0 detailed below (as discussed on 15th Feb 2024). This formulation minimises a deterministic objective function subject to annual baseline building constraints and a total budget constraint.

Let $T \in Z^+$ be a fixed horizon over which to model queue behaviour and to make building decisions.

Let vectors $\boldsymbol{h} = \{h_t \ \forall t \in 1, ..., T\}$ and $\boldsymbol{s} = \{s_t \ \forall t \in 1, ..., T\}$ denote annual house and shelter building rates, respectively. For simplicity we say that housing/shelter building rates are constant within each year. We can therefore consider $h(t) = \boldsymbol{h}$ and $s(t) = \boldsymbol{s}$ to be step functions.

Let $c_h = 1$ be the cost of increasing h_t by one, for any t.

Let c_s be the cost of increasing s_t by one, for any t.

Let C be a total budget for building housing and shelter

Let B be a baseline minimum annual house/shetler building rate

Let $y(\boldsymbol{h}, \boldsymbol{s})$ be a deterministic objective function.

$$
\begin{aligned}
\Phi_0 = \min_{\boldsymbol{h}, \boldsymbol{s}} \quad & y(\boldsymbol{h}, \boldsymbol{s}) \\
\text{s.t.} \quad & \sum_{t=1}^{T} c_h h_t + c_s s_t \leq C \\
& h_t, s_t \geq B \ \ \forall t \in \{1, ..., T\}
\end{aligned}
$$

where the objective function is the average over time of the expected unsheltered queue length.

$$y(\boldsymbol{h}, \boldsymbol{s}) = \frac{1}{T} \int_0^T \mathbb{E}[unsh(t; \boldsymbol{h}, \boldsymbol{s})] dt$$

where $unsh(t; \boldsymbol{h}, \boldsymbol{s})$ is the unsheltered queue at time t, as evaluated with a deterministic model. When we set C to be sufficiently large, this effectively amounts to enforcing the building of B shelters and B houses each year and allowing the surplus budget to spent at any time on either housing or shelter. What we find with this formulation is that we always prefer to spend this surplus in the first year and we always either spend this surplus all on housing, or all on shelter. The preference depends on c_s, which dicates how many shelters we can build with our budget and on μ_0 which is a parameter of the deterministic model indicating the service rate at each housing unit.

This formulation does not yet capture all of the interesting features of the problem - we now introduce four possible extensions to this formulation.

2 Including the length of the sheltered queue

Motivation: we would like to capture the fact that a large sheltered population is undesirable.

Suggested formulation: here we include the length of the sheltered queue in the objective function. As discussed previously, when using the fluid model we are assuming that the unsheltered queue never vanishes, and in this case the length of the sheltered queue at time t will simply be the number of shelters at time t, which we denote $n_{\boldsymbol{s}}(t)$. Remember that:

$$n_{\boldsymbol{s}}(t) = \int_0^t s(t)dt$$

where $s(t) = \boldsymbol{s}$ is a step function giving the rate of building shelters in each year $t \in \{1, ..., T\}$.

We can therefore introduce Φ_1 which is identical to Φ_0 except for an extra term in the objective function $y(\boldsymbol{h}, \boldsymbol{s})$:

$$\begin{aligned}
\Phi_1 = \min_{\boldsymbol{h},\boldsymbol{s}} \quad & y(\boldsymbol{h}, \boldsymbol{s}) \\
\text{s.t.} \quad & \sum_{t=1}^{T} c_h h_t + c_s s_t \leq C \\
& h_t, s_t \geq B \ \ \forall t \in \{1, ..., T\}
\end{aligned}$$

where the y($\boldsymbol{h}$,$\boldsymbol{s}$) is the average over time of the sum of the expected unsheltered and sheltered queue lengths:

$$y(\boldsymbol{h}, \boldsymbol{s}) = \frac{1}{T} \int_0^T \mathbb{E}[unsh(t; \boldsymbol{h}, \boldsymbol{s})]dt + \frac{1}{T} \int_0^T n_{\boldsymbol{s}}(t)dt$$

Comments: the effect of the second term in the objective function is to penalise the building of shelters. We would still expect an optimal solution to build surplus budget either all on shelter, or all on housing, where we have here tipped the balance in favour of housing by introducing a penalty for building shelters.

3 Squared queue lengths

Motivation: we would like to discourage the spending of all surplus budget on one type of accommodation (either housing or shelter) as clearly in practice a balance is sought.

Suggested formulation: here we square the expected length of the unsheltered queue and square the penalty term on the building of shelters. Minimising the sum of these squared terms will encourage a balance between building shelter and building housing.

- **Shelter** quickly reduces unsheltered queue but at the cost of a large **sheltered** population.
- **Housing** gives long-term relief to the system, but with an initial large **unsheltered** population.

We can therefore introduce Φ_2 which is identical to Φ_1 except for squared terms in the objective function $y(\boldsymbol{h}, \boldsymbol{s})$:

$$\begin{aligned}\Phi_2 = \min_{\boldsymbol{h},\boldsymbol{s}} \quad & y(\boldsymbol{h}, \boldsymbol{s}) \\ \text{s.t.} \quad & \sum_{t=1}^{T} c_h h_t + c_s s_t \leq C \\ & h_t, s_t \geq B \ \ \forall t \in \{1, ..., T\}\end{aligned}$$

where the y($\boldsymbol{h}$,$\boldsymbol{s}$) is the average over time of the sum of the expected squared queue lengths:

$$y(\boldsymbol{h}, \boldsymbol{s}) = \frac{1}{T} \int_0^T \mathbb{E}[unsh^2(t; \boldsymbol{h}, \boldsymbol{s})]dt + \frac{1}{T} \int_0^T n_{\boldsymbol{s}}^2(t)dt$$

where we have that:

$$\mathbb{E}[unsh(t; \boldsymbol{h}, \boldsymbol{s})^2] = \mathbb{E}[unsh(t; \boldsymbol{h}, \boldsymbol{s})]^2 + \mathrm{Var}[unsh(t; \boldsymbol{h}, \boldsymbol{s})].$$

Comments: The effect of the first term in the objective function is to penalise an extreme reliance on housing which is slow to reduce the size of the unsheltered queue. The second term penalises the extreme reliance on shelter. We expect that spending the surplus budget on a mixture of housing and shelter will optimise this objective function.

4 Time-dependent weighting of queue lengths

Motivation: we would like to recognise that while an unsheltered population is never acceptable, a sheltered population is acceptable as a short-term measure, but not in the long term.

Suggested formulation: here we introduce a weighting to the squared penalty term on shelter. This weighting is a function of time - representing the tolerance for shelter in the short term. We can therefore introduce Φ_3 which is identical to Φ_2 except for a weighted average over time of the squared penalty term on shelter in $y(\boldsymbol{h}, \boldsymbol{s})$:

$$\begin{aligned}
\Phi_3 = \min_{\boldsymbol{h},\boldsymbol{s}} \quad & y(\boldsymbol{h}, \boldsymbol{s}) \\
\text{s.t.} \quad & \sum_{t=1}^{T} c_h h_t + c_s s_t \leq C \\
& h_t, s_t \geq B \ \ \forall t \in \{1, ..., T\}
\end{aligned}$$

where the y($\boldsymbol{h}$,$\boldsymbol{s}$) is the average over time of the sum of the expected squared queue lengths:

$$y(\boldsymbol{h}, \boldsymbol{s}) = \frac{1}{T} \int_0^T \mathbb{E}[unsh^2(t; \boldsymbol{h}, \boldsymbol{s})]dt + \int_0^T w(t) n_{\boldsymbol{s}}^2(t) dt$$

where $w(t)$ is a linear function of time:

$$w(t) = w_0 + w_1 t$$

and we require:

$$\int_0^T w(t)dt = 1$$

which means:

$$w_0 T + w_1 \frac{T^2}{2} = 1$$

so for a given gradient w_1, we require:

$$w_0 = \frac{1}{T} - \frac{w_1 T}{2}$$

Comments: The effect of the second term in the objective function is to give more tolerance to the extreme reliance on shelter at an early stage, compared to later stages. We expect that spending the surplus budget on a mixture of housing and shelter will optimise this objective function, with a preference for more shelter initially.

5 Shape constraints on the build functions