Financial Contagion and CDOs

We value synthetic sCDO spreads using a graph theory parametrization of the Markov default process transition matrix.

by Graham Crowell

Outline

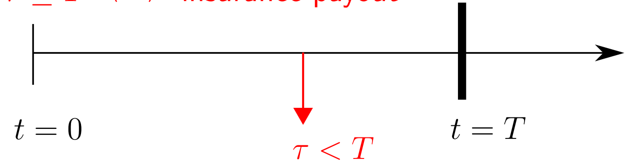
1. Price Model for Credit Default Swap
2. Price Model for Collateralized Debt Obligation
3. Markose Market Graph
4. Herbertsson Loss Distribution
5. Future Study

Insurance and Arrival Time

Insurance is protection against an event with a random **arrival time** τ

$\tau > T \iff$ NO insurance payout

$\tau \leq T \iff$ insurance payout



Poisson Model for Single Default

Define a Poisson process $X_t \sim \mathsf{POI}(\lambda t)$ on $t > 0$ such that

$$\begin{aligned}
\Pr(X_t = 1) :&= \text{default probability} \\
&= \Pr(t \leq \tau) \\
&= \mathsf{e}^{-\lambda t} \\
&= \mathsf{E}\left[\mathbb{1}_{\{t \leq \tau\}}\right]
\end{aligned}$$

where

$$\mathbb{1}_{\{t \leq \tau\}} = \begin{cases} 1 & \text{if } t \leq \tau \\ 0 & \text{if } t < \tau \end{cases}$$

Survival Probability

The probability of survival is given by

$$\Pr(\text{survival}) := 1 - \mathsf{e}^{-\lambda t}$$

Discount Factor

Protection buyer pays periodic fee to protection seller.
Each future payment must be discounted to it's present value:

$$PV = \mathrm{e}^{-r\Delta t} FV$$

where $r :=$ risk-free interest rate,
$PV :=$ present value and
$FV :=$ future value

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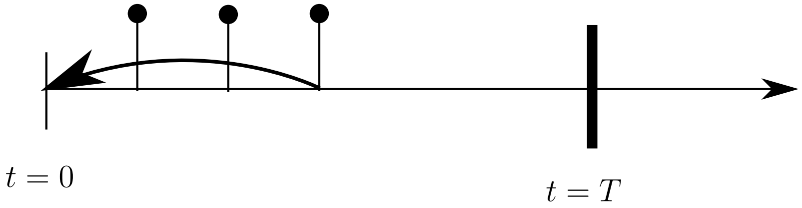

$$PV = \mathsf{e}^{-r3\Delta t}FV$$

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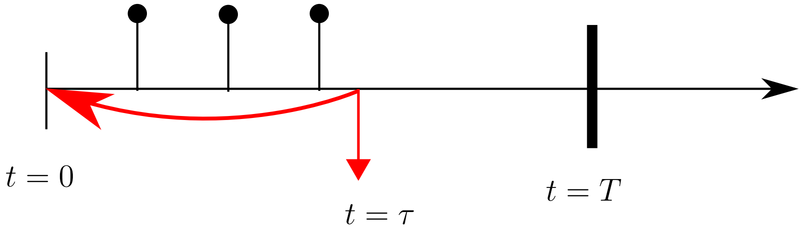

$$PV = \mathrm{e}^{-r\left(3+\frac{1}{2}\right)\Delta t} FV$$

where $r :=$ risk-free interest rate,
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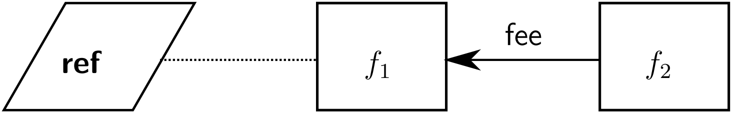
Credit Default Swap (CDS)

A CDS is a bilateral insurance contract between two *counter parties*: f_1 and f_2.

insurance against 1 default

- contract references some defaultable entity/cash-flow **ref**.
- f_1 sells insurance to f_2 against **ref** defaulting.
- f_2 pays insurance premiums to f_1.
- if **ref** defaults during lifetime of CDS then f_1 pays f_2 an insurance payout.

if $0 < t < \tau < T$



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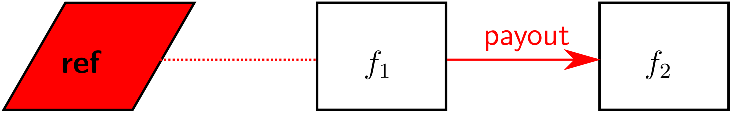

Components of CDS

reference entity **ref** $:=$ risky asset to which protection refers

maturity $T :=$ time length of protection $= 5$ years

notional $N :=$ nominal value of protection $= \$1$

credit event $X_t :=$ random variable that triggers payout

recovery rate $\phi :=$ value of **ref** asset after credit event $= 40\%$

Credit Loss for a CDS

$$
\begin{aligned}
L_t &: = \text{payout} \\
&= \begin{cases} N(1-\phi) & \text{if } \tau \le t \\ 0 & \text{if } \tau > t \end{cases} \\
&= N(1-\phi)\mathbb{1}_{\{\tau \le t\}}
\end{aligned}
$$

CDS Contingent Leg

Contingent Leg

The expected present value of the insurance payout from f_1 to f_2 is given by

$$\begin{aligned} V_{\mathsf{Cont}}(t) &= \mathsf{E}\left[\int_0^t \mathsf{e}^{-rs} N(1-\phi)\mathbb{1}_{\{\tau<s\}}\mathsf{d}s\right] \\ &= \mathsf{e}^{-rt}\mathsf{E}\left[L_t\right] \end{aligned}$$

CDS Par Value

Premium Leg

The expected present value of insurance premiums fee x from f_2 to f_1 subject to $t < \tau$ is given by

$$V_{\mathsf{Fee}}(t) = x\mathsf{E}\left[\sum_{i=1}^{n_T} \mathsf{e}^{-rt_i} N\Delta_i \mathbb{1}_{\{t_i<\tau\}}\right]$$

where

$$\Delta_i = t_i - t_{i-1}$$

Par Value

At initiation time $t = 0$ choose x so that

$$V_{\mathsf{Fee}} + V_{\mathsf{Acc}} = V_{\mathsf{Cont}}$$

Portfolio of CDSs

Consider a basket of $m = 4$ underlying reference assets and associated default times $\tau_1, \tau_2, \ldots, \tau_m$
with ordering $0 < T_1 < T_2 < \ldots < T_k < \infty$.

Accumulated Credit Loss

The accumalated credit loss at time t is given by

$$L_t = \sum_{i=1}^{m} N(1 - \phi_i)\mathbb{1}_{\{\tau_i \leq t\}}$$

CDS
CDS
CDS
CDS

$$t < T_1 \implies k = 0$$

$$\therefore L_t = 0$$

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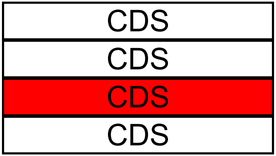

$$T_1 < t < T_2 \implies k = 1$$

$$\therefore L_t = N(1 - \phi)$$

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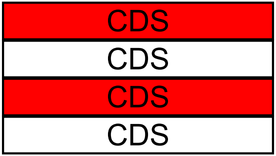

$$T_2 < t < T_3 \implies k = 2$$

$$\therefore L_t = 2N(1 - \phi)$$

Synthetic Collateralized Debt Obligation (sCDO)

A sCDO offers exposure to a portion of a CDS portfolio defined in terms L_t and attachment points $0 = k_0 < k_1 < \ldots < k_\kappa = mN$.

Tranche (slice)

A tranche is an interval of portfolio losses $[k_{\gamma-1}, k_\gamma]$ and the accumulated tranche losses is given by,

$$L_t^{(\gamma)} = \begin{cases} L_t - k_{\gamma-1} & \text{if } L_t \in [k_{\gamma-1}, k_\gamma] \\ k_\gamma - k_{\gamma-1} & \text{if } L_t > k_\gamma \end{cases}$$

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where

$$L_t = \sum_{i=1}^{m} N(1 - \phi_i)\mathbb{1}_{\{\tau_i \le t\}}$$

sCDO Contingent Leg

Contingent Leg

$$
\begin{aligned}
V^{(\gamma)}_{\mathsf{Cont}}(t) &= \mathsf{E}\left[\int_0^t \mathsf{e}^{-rs}\mathsf{d}L^{(\gamma)}_t\right] \\
&= \int_0^t \mathsf{e}^{-rs}\mathsf{d}s\mathsf{E}\left[L^{(\gamma)}_T\right] + \int_0^t r\mathsf{e}^{-rs}\mathsf{E}\left[L^{(\gamma)}_s\right]\mathsf{d}s
\end{aligned}
$$

where

$$
\begin{aligned}
L^{(\gamma)}_t &= (L_t - k_{\gamma-1})\mathbb{1}_{\{L_t\in[k_{\gamma-1},k_\gamma]\}} + (k_\gamma - k_{\gamma-1})\mathbb{1}_{\{L_t>k_\gamma\}} \\
L_t &= \sum_{i=1}^{m} N(1-\phi_i)\mathbb{1}_{\{\tau_i\le t\}}
\end{aligned}
$$

sCDO Fee Leg

Premium Leg

$$V_{\mathsf{Fee}}^{(\gamma)}(t) = x\mathsf{E}\left[\sum_{i=1}^{n_T} \mathsf{e}^{-rt_i}\left(\Delta k_\gamma - \mathsf{E}\left[L_t^{(\gamma)}\right]\right)\Delta_i\right]$$

where

$$L_t^{(\gamma)} = (L_t - k_{\gamma-1})\mathbb{1}_{\{L_t \in [k_{\gamma-1}, k_\gamma]\}} + (k_\gamma - k_{\gamma-1})\mathbb{1}_{\{L_t > k_\gamma\}}$$

$$\Delta k_\gamma = k_{\gamma-1} - k_\gamma$$

$$\Delta_i = t_i - t_{i-1}$$

Contagion

Contagion is the propagation of a disease in a system or population.

Ecology

There is a non-zero probability that a transferable infection will become an epidemic iff the average number of secondary infections $\mathcal{R}_0 > 1$.

Random Graph Theory

A phase transition occurs when the average second neighbours of a randomly chosen vertex equals 1.
If the average secondary neighbours is greater than 1 a graph contains a connected component of infinite size.

Market Graph

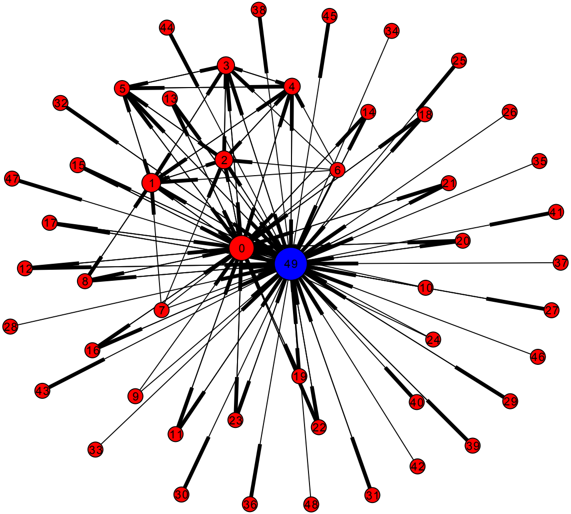


Figure: Markose 2010 type graph of Q4-2004 credit derivatives market

Markose Counter Party Exposures

Let x_{ij} be the amount of gross derivative obligations from counter party f_i to f_j.

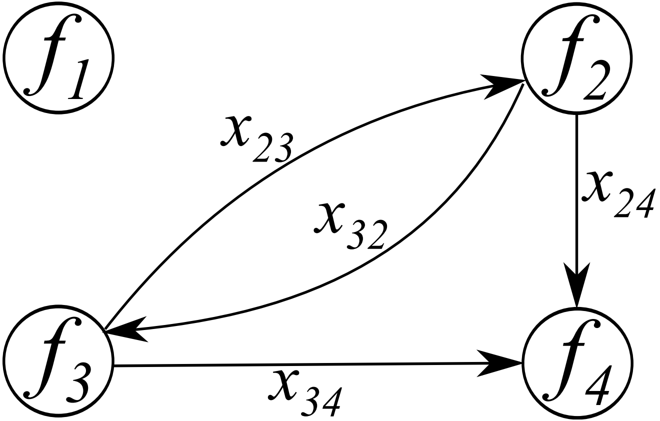


Figure: Markose 2012 bilateral derivatives.

Markose Net Bilateral Exposures

Let $x_{ij} - x_{ji}$ be the net amount of gross derivative obligations between counter parties f_i and f_j.

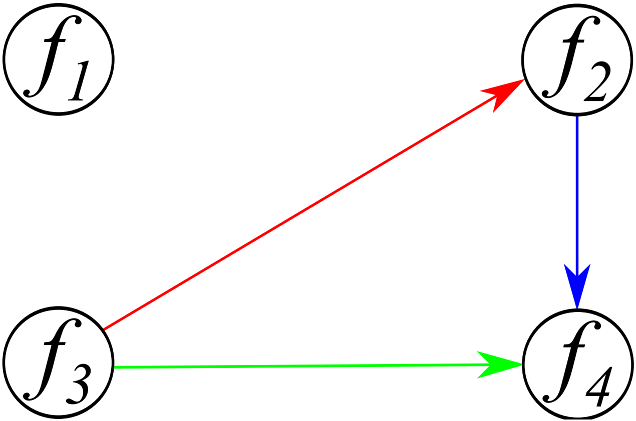


Figure: Markose bilateral derivatives.

Markose Derivative Capital Ratio

Let Θ_{ij} be the initial derivative capital ratio of counter party of f_i due losses from default of f_j.

$$\Theta_{ij} = \frac{(x_{ij} - x_{ji})^+}{C_i}$$

Note that Θ_{ij} is a capital buffer that provides a measure of counter party risk.

Default Condition

We assume that counter party f_i defaults if $\Theta_{ij} > \rho$.
Markose assumes $\rho = 6\%$.

Furfine Iteration

Furfine 2002 provides a frame work to model the affect counter party default.

Furfine trigger default

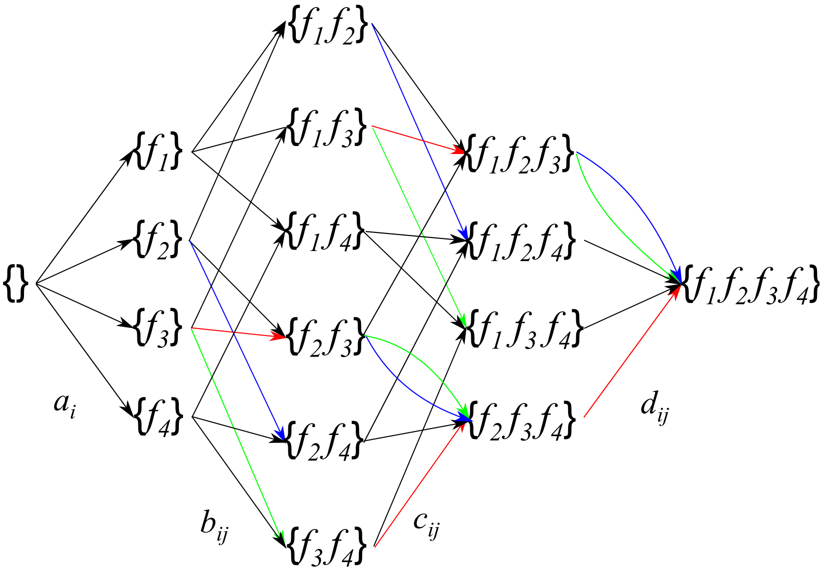
Let $f^* \in \{f_i\}$ be a counter party who has defaulted.

Furfine iteration

Each Furfine iteration measures affect on counter parties due to the default of f^*.
For $q = 1$,

$$\frac{(x_{if^*} - x_{f^*i})^+}{C_i} > \rho \Longrightarrow f_i \text{ defaults}$$

Herbertsson State Space



Herbertsson Point Process

Let $\boldsymbol{f} = \{f_1, f_2, \dots, f_k\}$ be a state with $0 \leq k \leq m$,
let $\boldsymbol{S}_k = \{\boldsymbol{f} = \{f_1, f_2, \dots, f_k\} : 1 \leq j_i \leq m, i = 1, \dots, k\}$,
let $\boldsymbol{S} = \bigcup \boldsymbol{S}_k$ be the state space with $|\boldsymbol{S}| = 2^m$.

Markov Intensity Matrix

The elements of the Markov point process intensity matrix $\boldsymbol{Q}$ specify the transition between states in $\boldsymbol{S}$.
If $\boldsymbol{j} = \{\boldsymbol{f}, f_k\}$ then

$$\boldsymbol{Q}_{\boldsymbol{f}\boldsymbol{j}} = a_k + \sum_{i=1}^{k-1} b_{\boldsymbol{f}_k, \boldsymbol{f}_i}$$

Herbertsson Intensity Matrix Q

$$\begin{bmatrix}
-4a & a & a & a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3b & 0 & 0 & 0 & b & b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3b & 0 & 0 & b & 0 & 0 & b & {\color{blue}b} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3b & 0 & 0 & b & 0 & {\color{red}b} & 0 & {\color{green}b} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3b & 0 & 0 & b & 0 & b & b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2c & 0 & 0 & 0 & 0 & 0 & c & {\color{blue}c} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2c & 0 & 0 & 0 & 0 & {\color{red}c} & 0 & {\color{blue}c} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2c & 0 & 0 & 0 & 0 & c & c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2c & 0 & 0 & c & 0 & 0 & {\color{blue}\mathbf{c}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2c & 0 & 0 & c & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2c & 0 & 0 & c & {\color{red}c} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d & 0 & 0 & 0 & {\color{blue}\mathbf{d}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d & {\color{red}d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Herbertsson Loss Distribution

Let $\boldsymbol{f} = \{f_1, f_2, \dots, f_k\}$ be a state with $0 \le k \le m$,
let $\boldsymbol{S}_k = \{\boldsymbol{f} = \{f_1, f_2, \dots, f_k\} : 1 \le j_i \le m, i = 1, \dots, k\}$,
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Loss Distribution

$$\mathsf{E}\left[L_t^{(\gamma)}\right] = \boldsymbol{\alpha} \mathrm{e}^{\boldsymbol{Q}t} \boldsymbol{l}^{(\gamma)}$$

where

$$\boldsymbol{l_f}^{(\gamma)} = \begin{cases} 0 & \text{if } L(\boldsymbol{f}) < k_{\gamma-1} \\ L(\boldsymbol{f}) - k_{\gamma-1} & \text{if } L(\boldsymbol{f}) \in [k_{\gamma-1}, k_\gamma] \\ k_\gamma - k_{\gamma-1} & \text{if } L(\boldsymbol{f}) > k_\gamma \end{cases}$$

$$L(\boldsymbol{f}) = \sum_{i=1}^{|\boldsymbol{f}|} N(1 - \phi)$$

Idea

Use Furfine iteration and Markose capital ratio as a parametrization for Herbertsson intensity matrix.
If $\boldsymbol{j} = \{\boldsymbol{f}, f_k\}$ consider a trigger default $f^* \in \boldsymbol{f}$ then

$$\boldsymbol{Q}_{\boldsymbol{fj}} \propto \frac{\left(x_{f^*k} - x_{kf^*}\right)^+ + \sum_{i \in \boldsymbol{f}} \left(x_{ik} - x_{ki}\right)^+}{C_k}$$

Conclusion

- Industry standard models for the spread of a sCDO ignore the affect of counter party risk.
- Furfine provides a frame work to model the affect counter party default.
- Markose provides a data based measure of counter party exposures.
- Herbertsson provides a frame work to price the affect counter party default contagion.
- Markose and Furfine provide a parametrization for Herbertsson's model.