Black-Scholes Pricing Model... Ad nauseam

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Presentation Outline

Background and Context

- financial derivative
- underlying asset
- risk free rate of return r
- financial derivative: call option

Derivations of Price Model

- fundamental theorem
- Black-Scholes PDE
- 1D Feymann-Kac (F-K)

Heston Model

- multi-dimensional Itô and F-K
- Heston 2 factor model

Definitions

'underlying' (asset) stock, bond, commodity, mortgage, weather

- $dS_t = \mu(S_t, t)S_t dt + \sigma(S_t, t)S_t dW_t$

financial 'derivative' asset whose **value depends on 'underlying'**

- $C_t(S_t, t) =$??? (stay tuned)

'risk-free rate' of return denoted r

- theorical **interest rate** when
$$\Pr(\text{default or bankruptcy}) = 0$$
- deterministic rate of return of 'risk-less' asset with price X:
$$d\Pi = r\Pi dt \implies \Pi_T = \Pi_t e^{r(T-t)}$$
- investments that grow faster than r must have **risk**
- proxies r used in finance:
  - LIBOR (London interbank overnight rate)
  - interest rate on short-term U.S. Treasury bonds

Financial Derivative: European Call Option

- gives owner **right without obligation** to buy underlying at **expiration** time $t = T$ for **strike price** K
- traded on exchanges (like stocks)

At expiration (boundary time $t = T$) nothing

T	expiration date
$S_T \geq 0$	price of the 'underlying' **S**tock at $t = T$
K	strike price (constant \$)
r	risk-free interest rate (assumed constant %)
$C_T = \max(S_T - K, 0)$	payoff of **C**all option at $t = T$ (\$)

$$\boxed{C_T(S_T, T) = \max(S_T - K, 0)}$$

Payout (ie boundary condition) of call option:

If $S_T > K$: then $C_T = \max(S_T - K, 0) = S_T - K > 0$
$\therefore$ you recieve payout of $S_T - K$

else $S_T < K$: then $C_T = \max(S_T - K, 0) = 0$
$\therefore$ option expires worthless

Fundamental Theorem of Asset Pricing (FTAP)

The following statements are assumed true for all financial assets:

- Law of One Price: at fixed time t the asset must have a unique price
- No Arbitrage: selling price of any asset must equal it's buying price
- Equivalent Martingales: there exists a probability measure that makes the process of it's price a martingale

Martingales and Stochastic Integrals in the Theory of Continuous Trading by Harrison and Pliska (1980)

S_t underlying asset (eg **S**tock)

$C_t(S_t, t)$ derivative (eg **C**all Option)

$C_T(S_T, T)$ boundary condition (eg $C_T = \max(S_T - K, 0)$)

Then FTAP asserts that: $\exists \mathbb{Q}$ such that:

$$\boxed{C_t(S_t, t) = e^{-r(T-t)} E_{\mathbb{Q}} \left[C_T(S_T, T) | \mathcal{F}_t \right]}$$

where $\mathbb{Q}$ is the **equivalent martingale measure** (EMM)

Stochastic Model for Underlying Asset Price S_t

Assume that underlying stock price S_t follows GBM:

$$dS_t = \mu(S_t,t)S_t dt + \sigma(S_t,t)S_t dW_t$$

Let $f(x) = \ln x \implies \boxed{X_t = \ln S_t}$ then Itô gives:

$$\ln S_T = \ln S_t + (\mu - \frac{\sigma^2}{2})(T-t) + \sigma W_{T-t}$$

$$\sim N\left[\ln S_t + (\mu - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}\right]$$

So S_T is a log-normal random variable:

$$\boxed{S_T = e^{X_T} = S_t e^{(\mu - \frac{\sigma^2}{2})(T-t) + \sigma W_{T-t}}}$$

and W_{T-t} is a normal random variable:

$$W_{T-t} \sim N\left[0, \sqrt{T-t}\right]$$

$$E(W_{T-t}) = 0$$

$$\mathrm{Var}\,(W_{T-t}) = T-t$$

Value of Call Option at Datetime $0 \le t < T$

From previous slides:

$$C_t = e^{-r(T-t)} E_{\mathbb{Q}} \left[\max\left(S_T - K, 0\right) | \mathcal{F}_t\right]$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} \max(S_T - K, 0) \frac{e^{-\frac{1}{2}\left(\frac{w-0}{1\sqrt{T-t}}\right)^2}}{\sqrt{2\pi(T-t)}} dw$$

where $\max$ operator restricts bounds of the integral:

$$\int_{-\infty}^{\infty} \left[\max\left(S_T - K, 0\right) | \mathcal{F}_t\right] \frac{e^{-\frac{1}{2}\left(\frac{w}{\sqrt{T-t}}\right)^2}}{\sqrt{2\pi(T-t)}} dw \neq 0 \iff S_T > K$$

$$\iff \ln S_T > \ln K$$

$$\iff \ln S_t + (\mu - \frac{\sigma^2}{2})(T-t) + \sigma W_{T-t} > \ln K$$

solve for normal random variable W_{T-t}:

$$W_{T-t} > \frac{\ln \frac{K}{S_t} - (\mu - \frac{\sigma^2}{2})(T-t)}{\sigma} = a$$

Value of C_t for $0 \le t < T$

From previous slide:

$$W_{T-t} > \frac{\ln \frac{K}{S_t} - (\mu - \frac{\sigma^2}{2})(T-t)}{\sigma} = a$$

and

$$C_t = e^{-r(T-t)} E_{\mathbb{Q}} \left[\max(S_T - K, 0) | \mathcal{F}_t \right]$$

$$= e^{-r(T-t)} \int_a^\infty (S_T - K) \frac{e^{-\frac{1}{2}\left(\frac{w}{\sqrt{T-t}}\right)^2}}{\sqrt{2\pi(T-t)}} dw$$

$$= e^{-r(T-t)} \int_a^\infty \left(S_t e^{(\mu - \frac{\sigma^2}{2})(T-t) + W_{T-t}} - K \right) \frac{e^{-\frac{1}{2}\left(\frac{w}{\sqrt{T-t}}\right)^2}}{\sqrt{2\pi(T-t)}} dw$$

convention is to rewrite above as:

$$\boxed{C_t = S_t \mathcal{N}\left(\frac{-a}{\sqrt{T-t}} + \sigma\sqrt{T-t} \right) + e^{-r(T-t)} K \mathcal{N}\left(\frac{-a}{\sqrt{T-t}} \right)}$$

Black-Scholes-Merton Risk Neutral Argument #1

Risk Neutral Portfolio

Consider a 'portfolio' with total value Π_t:

$$\Pi_t = C_t + nS_t$$

Portfolio consists of:

- 1 call option with price C_t
- $n \in \mathbb{R}$ units of 'underlying' each with price S_t
- S_t is underlying asset of C_t (ie C_t is a function of t and S_t)

If portfolio is 'risk-neutral' it's value grows at 'risk-free' rate r:

$$\frac{d\Pi}{dt} = r\Pi$$

So **'risk-neutral' assets are deterministic.**

Black-Scholes-Merton Risk Neutral Argument #2

$$\boxed{d\Pi_t = dC_t + ndS_t}$$

Main idea:

- n **makes** Π_t **'risk-neutral'** (ie PDE instead of SDE)
- n is unknown
- find expression for n so that:

$$\begin{aligned} d\Pi_t &= dC_t + ndS_t \\ &= r\Pi_t dt \end{aligned}$$

- rational price of C_t is the price that satasfies $\Pi_t = C_t + nS_t$

Assume that rate of return (ie growth) of underlying asset follows GBM:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

Black-Scholes PDE #1

$$\Pi = C_t + nS_t \implies d\Pi = dC_t + ndS_t$$

- assume S_t follows $dS_t = \mu S_t dt + \sigma S_t dW_t$
- by definition the price of any financial derivative depends on price of it's underlying

let $C_t = g(S_t, t)$ apply Itô formula to $g(x, t)$

$$\begin{aligned} dC_t &= \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial x} dS_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial x^2}(dS_t)^2 \\ &= \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial x}\left(\mu S_t dt + \sigma S_t dW_t\right) + \frac{1}{2}\frac{\partial^2 C_t}{\partial x^2} S_t^2 \sigma^2 dt \end{aligned}$$

so portfolio equation becomes:

$$\begin{aligned} d\Pi &= dC_t + ndS_t \\ &= \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial x}\left(\mu S_t dt + \sigma S_t dW_t\right) + \frac{1}{2}\frac{\partial^2 C_t}{\partial x^2} S_t^2 \sigma^2 dt \\ &\quad + n\left(\mu S_t dt + \sigma S_t dW_t\right) \end{aligned}$$

Black-Scholes PDE #2

From previous slide:

$$d\Pi = \frac{\partial C_t}{\partial t}dt + \frac{\partial C_t}{\partial x}\left(\mu S_t dt + \sigma S_t dW_t\right) + \frac{1}{2}\frac{\partial^2 C_t}{\partial x^2}S_t^2\sigma^2 dt$$
$$+ n\left(\mu S_t dt + \sigma S_t dW_t\right)$$

group drift terms and diffusion terms:

$$d\Pi = \left(\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial x}\mu S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial x^2}\sigma^2 S_t^2 + n\mu S_t\right)dt$$
$$+ \left(\frac{\partial C_t}{\partial x}\sigma S_t + n\sigma S_t\right)dW_t$$

to **eliminate diffusion** term dW_t let $n = -\frac{\partial C_t}{\partial x}$

$$d\Pi = \left(\frac{\partial C_t}{\partial t} + \frac{1}{2}\frac{\partial^2 C_t}{\partial x^2}\sigma^2 S_t^2\right)dt$$

so portfolio equation went from being SDE to PDE!

Black-Scholes PDE #3

Since Π_t is deterministic is grows at 'risk-free' rate of return:

$$\begin{aligned} d\Pi_t &= \left(\frac{\partial C_t}{\partial t} + \frac{1}{2} \frac{\partial^2 C_t}{\partial x^2} \sigma^2 S_t^2 \right) dt \\ r\Pi_t dt &= \\ r(C_t + nS_t)dt &= \\ r(C_t + \frac{\partial C_t}{\partial x} S_t)dt &= \end{aligned}$$

Black-Scholes PDE for price of **any** financial derivative:

$$\boxed{\frac{\partial C_t}{\partial t} + \frac{1}{2} \frac{\partial^2 C_t}{\partial x^2} \sigma^2 S_t^2 + \frac{\partial C_t}{\partial x} r S_t - r C_t = 0}$$

For call options apply terminal condition:

$$g(S_T, t) = C_T = \max(S_T - K, 0)$$

to get the same solution as before:

$$C_t = S_t \mathcal{N}\left(\frac{-a}{\sqrt{T-t}} + \sigma\sqrt{T-t} \right) + e^{-r(T-t)} K \mathcal{N}\left(\frac{-a}{\sqrt{T-t}} \right)$$

Feymann-Kac Theorem for 1 dimension

If X_t follows arithmatic Brownian motion

$$dX_t = adt + bdW_t$$

where W_t is under measure $\mathbb{Q}$

and boundary condition $f(X_T, T)$ is known so that:

$$f(X_t, t) = E_{\mathbb{Q}}\left[f(X_T, T)|\mathcal{F}_t\right]$$

then $f(x, t)$ must follow PDE:

$$\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial x^2} - rf = 0$$

where generator of SDE $\mathcal{A}$ is defined as the operator:

$$\mathcal{A} = a\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial^2}{\partial x^2}$$

and the PDE is often rewriten in terms of $\mathcal{A}$:

$$\frac{\partial f}{\partial t} + \mathcal{A}f(x, t) - rf(x, t) = 0$$

Black-Scholes and Feynmann-Kac

Let $X_t = S_t$, $a = S_t\mu$, $b = S_t\sigma$ to recover SDE model of underlying price:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

and let $f(x,t) = C_t = g(S_t, t)$ to recover Black-Scholes PDE:

$$\frac{\partial C_t}{\partial t} + \mu S_t \frac{\partial C_t}{\partial x} + \frac{1}{2}(S_t\sigma)^2 \frac{\partial^2 C_t}{\partial x^2} - rC_t = 0$$

F-K explains connection between Black-Scholes PDE and FTAP expected value.

BC for call option is

$$C_T = g(S_T, T) = \max(S_T - K, 0)$$

F-K says solution is expected value just like in FTAP:

$$\begin{aligned} C_t &= e^{-r(T-t)} E_{\mathbb{Q}}\left[\max(S_T - K, 0) | \mathcal{F}_t\right] \\ &= S_t \mathcal{N}\left(\frac{-a}{\sqrt{T-t}} + \sigma\sqrt{T-t}\right) + e^{-r(T-t)} K \mathcal{N}\left(\frac{-a}{\sqrt{T-t}}\right) \end{aligned}$$

Expected Value

multi-dimensional Itô's Formula

Consider system of n SDEs

$$
\begin{aligned}
d\boldsymbol{X}_t &= \boldsymbol{\mu} dt + \boldsymbol{\sigma} d\boldsymbol{W}_t \\
&= \begin{cases}
dX_1 = \mu_1 dt & + & \sigma_{11} dW_1 & + & \cdots & + & \sigma_{1m} dW_m \\
dX_2 = \mu_2 dt & + & \sigma_{21} dW_1 & + & \cdots & + & \sigma_{2m} dW_m \\
\vdots & & \vdots & & \ddots & & \vdots \\
dX_n = \mu_n dt & + & \sigma_{n1} dW_1 & + & \cdots & + & \sigma_{nm} dW_m
\end{cases}
\end{aligned}
$$

Define smooth function $f(x,t)$

$$
\begin{aligned}
Y_t &= (f_1(x,t), f_2(x,t), \ldots, f_p(x,t)) \\
&= f(x,t) \in C^2\left(\mathbb{R}^n \times [0,\infty), \mathbb{R}^p\right)
\end{aligned}
$$

Itô's formula for k^{th} component of Y_t is:

$$
\boxed{dY_k = \frac{\partial f_k(X,t)}{\partial t} dt + \frac{\partial_k(X,t)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f_k(X,t)}{\partial x^2} (dX_t)^2}
$$

multi-dimensional Feymann-Kac Theorem

Consider system of n SDEs

$$d\boldsymbol{X}_t = \boldsymbol{\mu} dt + \boldsymbol{\sigma} d\boldsymbol{W}_t$$

and smooth function $f(x,t)$

$$Y_t = g(x,t) \in C^2\left(\mathbb{R}^n \times [0,\infty), \mathbb{R}^p\right)$$

such that

$$Y_t = f(\boldsymbol{X}_t, t) = E_{\mathbb{Q}}\left[e^{-\int_t^T r(\boldsymbol{X}_u, u) du} f(\boldsymbol{X}_T, T) | \mathcal{F}_t\right]$$

where boundary condition $f(\boldsymbol{X}_T, T)$ is known

then $Y_t = f(x,t)$ is a solution to the PDE:

$$\frac{\partial f(\boldsymbol{X}_t, t)}{\partial t} + \mathcal{A} f(\boldsymbol{X}_t, t) - r(\boldsymbol{X}_t, t) f(\boldsymbol{X}_t, t) = 0$$

with generator $\mathcal{A}$

$$\mathcal{A} = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

Ornstein-Uhlembeck Process

On the Theory of Brownian Motion
by Ornstein and Uhlembeck (1930)
SDE for process whose velocity that tends to 0 (with speed c)

$$dX_t = -cX_t dt + \beta dB_t$$

Optimum Consumption and Portfolio Rules in a Continuous-Time Model
by Merton (1970)
SDE equilibrium of process $\theta \neq 0$

$$dX_t = c(\theta - X_t)dt + \beta dB_t$$

Heston Model

Heston 2 Factor Model

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_{1,t} \\
dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_{2,t} \\
E\left[dW_{1,t}, dW_{2,t}\right] &= \rho dt
\end{aligned}
$$

Model Parameters:

- μ drift of stock processs
- κ mean reversion speed for variance
- θ equilibrium level of variance
- σ volitility of the variance
- ρ correlation between Brownian motions W_1 and W_2
- v_0 initial $(t = 0)$ variance

Apply Itô Formula then Feymann-Kac

Let $X_t = \ln S_t$ and apply Itô to get:

$$dX_t = (\mu - \frac{1}{2}v_t)dt + \sqrt{v_t}dW_1$$

Let $dZ_1 = dW_1$ and $dZ_2 = \rho dW_1 + \sqrt{1-\rho}dW_2$ where $Z_1 \perp\!\!\!\perp Z_2$ and rewrite as martix:

$$d\begin{pmatrix} X_t \\ v_t \end{pmatrix} = \begin{pmatrix} \mu - \frac{1}{2}v_t \\ \kappa(\theta - v_t) \end{pmatrix} dt + \begin{pmatrix} \sqrt{v_t} & 0 \\ \rho\sigma\sqrt{v_t} & \sigma\sqrt{v_t(1-\rho)} \end{pmatrix} \begin{pmatrix} dZ_1 \\ dZ_2 \end{pmatrix}$$

The generator $\mathcal{A}$ of the process is given by:

$$\mathcal{A} = (\mu - \frac{1}{2}v_t)\frac{\partial}{\partial x} + \kappa(\theta - v_t)\frac{\partial}{\partial y} + \frac{1}{2}\left[v_t\frac{\partial^2}{\partial x^2} + \sigma^2 v_t\frac{\partial^2}{\partial y^2} + 2\rho\sigma v_t\frac{\partial^2}{\partial x \partial y}\right]$$

Consider some function of $C_t = g(x,y,t) \in C^2\left(\mathbb{R}^2 \times [0,\infty)\right)$ with boundary condition $C_T(S_T, v_T, T) = \max(S_T - K, 0)$.

Feyman-Kac then gives PDE

$$\frac{\partial C_t}{\partial t} + \mathcal{A}C_t - rC_t = 0$$

Characteristic Function

Convention to let $\tau = T - t \implies \frac{\partial C_t}{\partial t} = -\frac{\partial C_t}{\partial \tau}$
Heston assumes that solution (ie characteristic equation) of PDE has form:

$$\phi(x, y, u) = e^{\alpha(\tau)+\beta(\tau)y+ux}$$

which produces 2 ODEs with boundary conditions:

$$\phi(X_T, T) = e^{uX_T} = e^{u \ln S_T}$$

$$\alpha(0) = 0$$

$$\beta(0) = 0$$

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