Terminating sequences of Bunny Trainer Transforms

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2023-07-03

In this note, we prove a necessary and sufficient condition for the termination of Bunny Trainer Transform sequences.

Definition. A *Bunny Trainer Transform* (BTT) is the following function on pairs of nonnegative integers (m, n):

$$BTT(m, n) \quad = \quad \begin{cases} BTT(n, m) & m > n \\ (2m, n - m) & 3m \leq n \\ (n - m, 2m) & m \leq n < 3m \end{cases}$$

The BTT represents the outcome of a match where the bunny trainer corresponding to the smaller integer (e.g. m in (m, n)) bets the entire position value m and wins. BTT is defined in such a way that if $(x, y) = BTT(m, n)$ then $x \leq y$.

Definition. A BTT sequence *terminates* or is *terminating* when $BTT^k(m, n) = (0, p)$ for some nonnegative integers k and p. If no such k and p exist, then the BTT sequence is *nonterminating*.

That is to say, the BTT sequence terminates when successive matches lead to one of the bunny trainers losing everything. We have $BTT(0, p) = (0, p)$, so the BTT sequence terminates if it eventually reaches a fixed point where the first integer in the pair is zero.

Definition. A pair of nonnegative integers (m, n) satisfies the *terminating condition with exponent* k (*TC-k*), if for nonnegative integer k we have

$$m + n = 2^k \gcd(m, n).$$

(We take $\gcd(0, 0) = 0$ by definition.) More generally the pair (m, n) is said to satisfy the *terminating condition* (TC) if for some nonnegative integer k it satisfies TC-k.

Proposition. *If (m, n) satisfies TC-k for some positive integer k, then $BTT(m, n)$ satisfies TC-$(k - 1)$.*

Proof. $(0,0)$ satisfies TC-k for all nonnegative integers k, and $BTT(0,0) = (0,0)$, so the proposition is true trivially in this case.

$(0,n)$ for a positive integer n only satisfies TC-0 and hence does not meet the condition of the proposition.

So without loss of generality, we can assume that $0 < m \leq n$. Let $d = \gcd(m,n)$, and let $d' = \gcd(2m, n-m)$.

We claim $d' = 2d$. Certainly $2d|2m$ since $d|m$, and since $d|n$ also then $d|(n-m)$. But

$$\frac{m}{d} + \frac{n}{d} = 2^k$$

is even as $k > 0$, and so both integers m/d, n/d are odd. (They can't both be even integers because then d would not be the *greatest* common divisor of m and n.) It follows that

$$\frac{n-m}{d} = \frac{n}{d} - \frac{m}{d}$$

is an even integer, and so $2d|(n-m)$. Thus $2d|d'$, being a divisor both of $2m$ and $n-m$.

Now suppose $d' = 2dq$. Then

$$\frac{2m}{d'} = \frac{m}{dq}$$

is an integer, which contradicts $d = \gcd(m,n)$ unless $q = 1$.

Finally since $BTT(m,n) = (2m, n-m)$ or $(n-m, 2m)$, and

$$2m + (n-m) = m + n = 2^k d = 2^{k-1} d',$$

the proposition is established. □

Proposition. *If (m,n) satisfies TC-k for some positive integer k, then every point in the preimage $BTT^{-1}(m,n)$ satisfies TC-$(k+1)$.*

Proof. $BTT^{-1}(0,0) = \{(0,0)\}$, so the claim is trivially true for $(0,0)$ as it satisfies TC-k for all nonnegative integers k. □

Theorem. *A pair of nonnegative integers (m,n) has a terminating BTT sequence if and only if the pair satisfies the terminating condition.*

Proof. □