Terminating sequences of Bunny Trainer Transforms

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In this note, we prove a necessary and sufficient condition for the termination of Bunny Trainer Transform sequences.

Definition. A *Bunny Trainer Transform* (BTT) is the following function on pairs of nonnegative integers (m, n):

$$BTT(m,n) \quad = \quad \begin{cases} BTT(n,m) & m > n \\ (2m, n-m) & 3m \leq n \\ (n-m, 2m) & m \leq n < 3m \end{cases}$$

BTT represents the outcome of a match where the bunny trainer corresponding to the smaller integer (e.g. m in (m, n)) bets the entire position value m and wins. BTT is defined in such a way that if $(x, y) = BTT(m, n)$ then $x \leq y$.

Definition. A BTT sequence *terminates* or is *terminating* when $BTT^k(m, n) = (0, p)$ for some nonnegative integers k and p. If no such k and p exist, then the BTT sequence is *nonterminating*.

That is to say, the BTT sequence terminates when successive matches lead to one of the bunny trainers losing everything. We have $BTT(0, p) = (0, p)$, so the BTT sequence terminates if it eventually reaches a fixed point where the first integer in the pair is zero.

Definition. A pair of nonnegative integers (m, n) satisfies the *terminating condition with exponent* k (*TC-k*), if for nonnegative integer k we have

$$m + n = 2^k \gcd(m, n).$$

(We take $\gcd(0, 0) = 0$ by definition.) More generally the pair (m, n) is said to satisfy the *terminating condition* (TC) if for some nonnegative integer k it satisfies TC-k.

Proposition 1. *If (m, n) satisfies TC-k for some positive integer k, then $BTT(m, n)$ satisfies TC-$(k-1)$.*

Proof. $(0,0)$ satisfies TC-k for all nonnegative integers k, and $BTT(0,0) = (0,0)$, so the proposition is true trivially in this case.

$(0,n)$ for a positive integer n only satisfies TC-0 and hence does not meet the condition of the proposition.

So without loss of generality, we can assume that $0 < m \leq n$. Let $d = \gcd(m,n)$, and let $d' = \gcd(2m, n-m)$.

We claim $d' = 2d$. Certainly $2d|2m$ since $d|m$, and since $d|n$ also then $d|(n-m)$. But

$$\frac{m}{d} + \frac{n}{d} = 2^k$$

is even as $k > 0$, and so both integers m/d, n/d are odd. (They can't both be even integers because then d would not be the *greatest* common divisor of m and n.) It follows that

$$\frac{n-m}{d} = \frac{n}{d} - \frac{m}{d}$$

is an even integer, and so $2d|(n-m)$. Thus $2d|d'$, being a divisor both of $2m$ and $n-m$.

Now suppose $d' = 2dq$. Then

$$\frac{2m}{d'} = \frac{m}{dq}$$

is an integer, which contradicts $d = \gcd(m,n)$ unless $q = 1$.

Finally since $BTT(m,n) = (2m, n-m)$ or $(n-m, 2m)$, and

$$2m + (n-m) = m + n = 2^k d = 2^{k-1} d',$$

the proposition is established. □

Proposition 2. *If (m,n) satisfies TC-k for some nonnegative integer k, then every point in the preimage $BTT^{-1}(m,n)$ satisfies*

1. *TC-0 or TC-1, if $k = 0$,*
   - *and if the preimage $BTT^{-1}(m,n)$ is nonempty, it contains points satisfying TC-0;*
2. *TC-$(k+1)$, if $k > 0$.*

Proof. $BTT^{-1}(0,0) = \{(0,0)\}$, so both claims are trivially true for $(0,0)$ as it satisfies TC-k for all nonnegative integers k.

Also $BTT^{-1}(m,n)$ is the empty set when $m > n$, so both claims are trivially true in this case also.

Case 1. TC-0:

So suppose (m,n) satisfies TC-0. That is, $m+n = \gcd(m,n)$. If both m and n are positive this is impossible, since then $m+n > \max(m,n)$ whereas $\gcd(m,n) \leq \min(m,n)$. Hence $m = 0$ or $n = 0$. Without loss of generality we may assume that $m = 0$, $n > 0$.

If n is odd, then $BTT^{-1}(0,n) = \{(0,n),(n,0)\}$, and the first claim is satisfied.
If $n = 2p$ is even, then $BTT^{-1}(0,n) = \{(0,n),(n,0),(p,p)\}$. Since $p > 0$, (p,p) satisfies TC-1, and again the first claim is established.

Case 2. TC-k for $k > 0$:
Suppose (m,n) satisfies TC-k, and without loss suppose $0 < m \leq n$.
If one of m,n is odd and one is even, then $m+n$ is odd, which is impossible since $2^k \gcd(m,n)$ is even when $k > 0$.
If (m,n) is in the image of BTT then one of m,n must be even and hence both must be even, since (m,n) satisfies TC-k. Therefore if both m,n are odd, $BTT^{-1}(m,n)$ is the empty set and the second claim is satisfied.
So now assume that $m = 2p$ and $n = 2q$ for some positive integers $p \leq q$.
Suppose that $(u,v) \in BTT^{-1}(m,n)$, with $u \leq v$. Then we have $(m,n) \in \{(2u, v-u),(v-u, 2u)\}$. It follows that $(u,v) = (p, 2q+p)$ or $(u,v) = (q, 2p+q)$. (So in fact $u < v$.) Let $d = \gcd(m,n)$. d is certainly even, and we know from the proof of Proposition 1 that $2p/d$ and $2q/d$ are both odd.
Consider when $(u,v) = (p, 2q+p)$. Let $d' = \gcd(p, 2q+p)$. We claim that $d = 2d'$.
Certainly $d'|p$ and $d'|(2q+p)$ together imply that $d'|2p$ and $d'|2q$. Hence $d'|d$. Then we can write $d = sd'$ for some positive integer s.
Since d is even, write $d = 2r$ for some positive integer r. Then $d|2p$ and $d|2q$ together imply that $r|p$ and $r|q$ and so $r|(2q+p)$. From this it follows that $r|d'$. So write $d' = tr$ for some positive integer t.
Putting the previous statements together, we see $2r = d = str$, and so $st = 2$. If $s = 1$, $t = 2$ then $d = d'$. But then $d'|p$ means that $d|p$ and so $2p/d$ must be even, a contradiction. The only remaining possibility is that $s = 2$, $t = 1$ and so $d = 2d'$.
Now we can conclude that

$$p + (2q+p) = m + n = 2^k d = 2^{k+1} d',$$

and so (u,v) satisfies TC-$(k+1)$, which was to be shown.
We have not used the fact that $p \leq q$, and so symmetry allows us to make a completely analogous argument to show that $(u,v) = (q, 2p+q)$ satisfies TC-$(k+1)$ also.
If instead $u > v$, we can make the same arguments above with u and v interchanged, since $BTT(u,v) = BTT(v,u)$ by definition. We've shown, by construction, that when $m \leq n$ are both even, positive and (m,n) satisfies TC-k, then $BTT^{-1}(m,n)$ is nonempty and every point in it satisfies TC-$(k+1)$, completing the proof of the second claim.

$\square$

Theorem. *A pair of nonnegative integers (m,n) has a terminating BTT sequence if and only if the pair satisfies the terminating condition.*

Proof. If $m = 0$ then $(0,n) = BTT^k(0,n)$ for any nonnegative k. If $n = 0$ and $m > 0$ then $BTT^k(m,0) = (0,m)$ for any positive k. Thus (m,n) has a terminating BTT sequence when $m = 0$ or $n = 0$, and we can see that such (m,n) also satisfy TC-0. This establishes the theorem when $m = 0$ or $n = 0$.

Now suppose that m,n are positive integers such that $BTT(m,n) = (0,p)$. p must be positive and even, since $BTT^{\text{-1}}(0,0) = \{(0,0)\}$ and $BTT^{\text{-1}}(0,p) = \{(0,p),(p,0)\}$ when p is positive and odd, as we've seen in the proof of Proposition 2, in Case 1. So also from there we know that we must have $m = n = p/2$. $(p/2,p/2)$ satisfies TC-1.

If m,n are positive integers such that $BTT^k(m,n) = (0,p)$ for some $k > 1$, with k the smallest positive integer for which the equation is true, then we will write $BTT(u,v) = (0,p)$ where $(u,v) = BTT^{k-1}(m,n)$. Now u,v are both positive integers (otherwise k is not the smallest positive integer for which $BTT^k(m,n) = (0,p)$ holds), and from the previous paragraph we know that (u,v) satisfies TC-1. But (m,n) is in the $(k-1)$-fold BTT preimage of (u,v), i.e. $(m,n) \in BTT^{\text{-(k-1)}}(u,v)$. Applying Proposition 2(2), inductively $k-1$ times, we find that (m,n) satisfies TC-k.

We've shown the *only if* part of the equivalence, and now must show the *if* case.

Suppose (m,n) are positive integers that satisfy TC-k for some, necessarily positive, integer k. Applying Proposition 1 k times shows that $BTT^k(m,n)$ satisfies TC-0. But from Proposition 2 in Case 1, this means that $BTT^k(m,n) = (0,p)$ for some positive integer p. □