Continuity

Metric Equivalence: Assume (X, d_X) and (Y, d_Y) are metric spaces. Show that a map $f : X \to Y$ is continuous at $x \in X$ if and only if for each $\varepsilon > 0$, we can find a $\delta > 0$ such that $d_X(x', x) < \delta$ for $x' \in X$ implies $d_Y(f(x'), f(x)) < \varepsilon$.

($\Rightarrow$) Let $x \in X$, $\varepsilon > 0$. Suppose $f : X \to Y$ is continuous. Then we know $B(f(x), \varepsilon)$ is open so $f^{-1}(B(f(x), \varepsilon)$ is open. Since $x \in f^{-1}(B(f(x), \varepsilon)$ there exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon)$. Thus, we have $f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$ and thus if $d(x, y) < \delta$ then we have $d(f(x), f(y)) < \varepsilon$.
($\Leftarrow$) Suppose for each $\varepsilon > 0$ we can find a $\delta > 0$ such that $d_X(x', x) < \delta$ for $x' \in X$ implies $d_Y(f(x'), f(x)) < \varepsilon$. Let $x \in X$ and consider $V(f(x))$. Let $\varepsilon > 0$ be such that $B(f(x), \varepsilon) \subseteq V(x)$. Then we know that there exists $\delta > 0$ such that if $y \in B(x, \delta)$ then $f(y) \in B(f(x), \varepsilon)$. Thus, $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)) \subset f^{-1}(V(f(x)))$. Then for every neighborhood $V(f(x))$ we know there exists a neighborhood $V(x) \subseteq f^{-1}(V(f(x)))$ so f is continuous.