Real-World Pendulum Experiment

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Abstract

An overview of the upgraded real-world pendulum experiment is given.

1 Introduction

The pendulum is a physical system whose realizations from the linear simple pendulum to more realistic ones such as the non-linear physical pendulum with drag encompass a wealth of fascinating physics. An in-depth discussion is Baker and Blackburn's book [1] with discussions of systems including, "pendulums somewhat simple", "pendulums less simple", the Foucault pendulum, and the torsion pendulum. All four and the spherical pendulum may have some relevance to this lab.

Historically the pendulum was a mainstay of time-keeping - an example is the grand-father clock. This lab builds on the modern ability to make high precision electronic measurements of time intervals to explore the physics involved in the pendulum system. The experimental design allows for precise relative period measurements and precise relative maximum angular speed measurements. Putting these together with the real-world effect of drag, one can:

- Measure the dependence of the period on amplitude.
- Measure the angular speed loss (and so energy loss) per oscillation due to drag.
- Characterize the energy losses of different pendulums as a function of the fluid-dynamics scaled speed (Reynolds number).

This writeup focuses on a top-down discussion of topics that can be investigated with the experiment, presents some illustrative data, and discusses various mathematical models for incorporating the corrections needed to give an adequate description of the observations. The experimental apparatus, its associated programs and procedures for data acquisition, and basic measurement capabilities is documented separately [2]. The previous PHSX516 experiment description [3] can also be consulted. It has extensive discussion of fluid dynamics considerations, and two detailed papers, Nelson-Olsson [4] and McInerney [5].

Two other papers that are helpful are Squire [6] and Kostov [7]. The latter is perhaps the most relevant paper of all as it refers to a similar setup to the current experiment. Finally, [8], shows that drag from the suspended string may be important.

2 Models

The governing differential equation is of the general form:

$$I\frac{d^2\theta}{dt^2} + f(\frac{d\theta}{dt}) + mgl\sin\theta = 0$$

where we have written $f(\frac{d\theta}{dt})$ as the generic form of any drag torque.

Ignoring drag for now, we have the usual form for a simple or compound pendulum,

$$\frac{d^2\theta}{dt^2} + \frac{mgl}{I}\sin\theta = 0 \tag{1}$$

2.1 Linear approximation no drag

Making the small angle approximation of $\sin\theta \approx \theta$, this linearizes to simple harmonic motion

$$\frac{d^2\theta}{dt^2} + \frac{mgl}{I}\,\theta = 0$$

with solution

$$\theta(t) = \theta_0\cos(\omega_0 t - \alpha)$$

where θ_0 is the angular amplitude, α is a phase, and ω_0, the angular frequency, is defined by

$$\omega_0^2 = \frac{mgl}{I}$$

The small-angle period is therefore,

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{I}{mgl}}$$

For a simple pendulum, this simplifies to

$$\omega_0^2 = \frac{mgl}{ml^2} = \frac{g}{l};\quad T_0 = 2\pi\sqrt{\frac{l}{g}}$$

2.2 Non-linear no drag

Returning to equation 1, the exact analytical solution for the period, that makes no approximation for $\sin\theta$ is well known. It gives a result that explicitly shows that the pendulum period does depend on the amplitude.

It is best derived [9] using energy conservation, and using an auxiliary variable, ϕ, which changes from 0 to 2π during one full oscillation of the pendulum, and is defined by

$$\sin\phi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)}$$

This leads to the period being increased with respect to the small-angle period, T_0, as follows,

$$T = T_0\,\frac{2}{\pi}\int_0^{\pi/2}(1-k^2\sin^2\phi)^{-\frac{1}{2}}\,\mathrm{d}\phi \tag{2}$$

where $k=\sin(\theta_0/2)$.

This involves a standard special function, an incomplete elliptic integral of the first kind of Legendre form, defined as

$$F(\phi,k) = \int_0^{\phi}(1-k^2\sin^2 t)^{-\frac{1}{2}}\,\mathrm{d}t$$

For our particular case, with the upper integration limit as $\phi = \pi/2$, this is referred to as a complete elliptic integral, and is sometimes denoted, $K(k)$. Computer implementations are readily available on many platforms. For k values of interest ($\ll 1$) it is also straightforward to evaluate by numerical integration.

There are many articles that attempt to find good closed form approximations to the above exact equation. In practice in this day and age this is probably an exercise in futility. A useful power series expansion, for $k<1$ is

$$\frac{T}{T_0} = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \ldots$$

Note that the coefficients do not decrease much from one term to the next, so sufficiency of for example only four terms depends on k^2 being small.

One can also write this in terms of θ_0 by using a small angle series expansion for k.

$$\frac{T}{T_0} = 1 + \frac{1}{16}\,\theta_0^2 + \frac{11}{3072}\,\theta_0^4 + \frac{173}{737280}\,\theta_0^6 + \ldots \tag{3}$$

The leading correction was known to Bernoulli. This shows that for a small angular amplitude of 40 mrad (2.29°), the increase in period is only 0.01%. One needs a much larger angular amplitude of 400 mrad (22.9°) to see a 1% effect. Fortunately, with the upgraded apparatus, it is not too difficult to see such an effect.

2.3 Drag effects

Various models for including drag are discussed in the cited references.

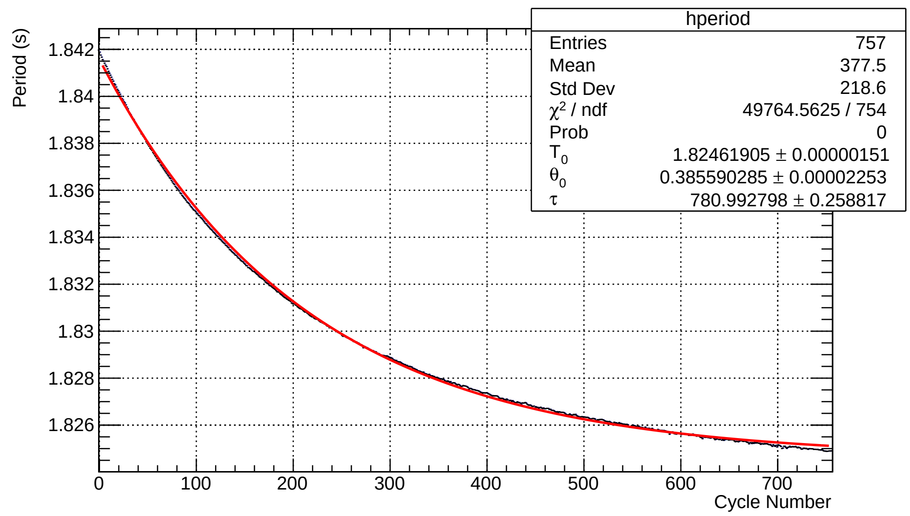

Figure 1: Measured period (s) vs cycle number. Super-imposed is a least-squares fit with the three-parameter function described in the text.

3 Measurements with the Pendulum Experiment

We illustrate here some of the basic measurements that can be done with the pendulum experiment. Shown in Figure 1 is the measured period for each oscillation for a data-taking run lasting 23 minutes using the nominal micro-controller clock time-scale calibration (you can and should do better). This does not use a direct measurement of the angular amplitude, nor even any knowledge of the initial angular amplitude. We do see that the period has decreased by about 0.9% over the course of the run. The uncertainties on the individual period measurements are small, in the range of 10-20 μs.

The variation of the period with cycle number is fitted with a simple three-parameter function, with n as cycle number, and Δt the average period,

$$T(t) = T(n\Delta t) = T_0(1 + \frac{1}{16}\,\theta^2)$$

where

$$\theta(t) = \theta_0 \exp\left(-t/\tau\right)$$

is a reasonable first guess for the variation of the amplitude with time due to drag.

The fit is qualitatively very reasonable, but quantitatively atrocious with a completely off the charts goodness-of-fit. Unfortunately the price to pay for high precision is often the need for understanding sometimes small and subtle effects. We should note that inclusion of the quartic term in the expansion

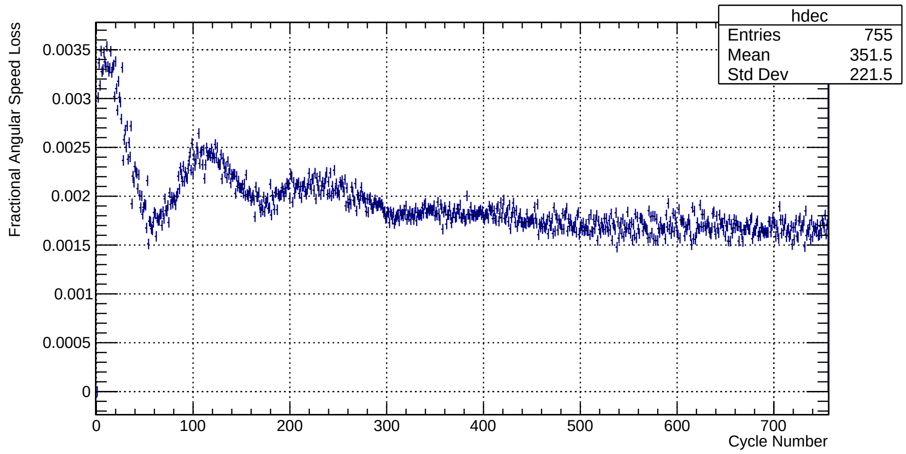

Figure 2: Measured fractional angular speed loss vs cycle number. Uncertainties have been set as 5.4×10^{-5} independent of cycle number.

does improve the fit significantly, but the underlying main cause of the still statistically completely unacceptable fit must be something else.

The fitted curve undershoots the data early, then overshoots until about cycle 250, then undershoots again until about cycle 600 where it starts to seriously overshoot. The reasons for this are probably mainly related to the drag model being naive, and one could do much better by inferring the angular amplitude from the measured angular speed. We will also see that there may be some transient effects associated with the launch of the pendulum. Taking a close look at the data you can see that the measurements are very regular and of high quality. This fit was performed assuming that the time measurement uncertainties are independent of cycle number. So this basically assumes that the time measurement uncertainty is independent of the speed of the bob. You should examine the data, and think about the experimental setup, and see if you think this is a good assumption.

You should be able to figure out reasonable "local" error estimates using low-order polynomial fits to restricted ranges of the data. Finally instead of over-analyzing data that has already been collected you would probably be wise to repeat one or two data-taking runs to establish reproducibility.

Shown in figure 2 is the measured fractional decrease in the average shadow-time angular speed, γ_ω, for cycle n compared to cycle $n-1$ for the same run where this is defined as:

$$\gamma_\omega = \frac{-(|\omega_n| - |\omega_{n-1}|)}{\frac{1}{2}(|\omega_n| + |\omega_{n-1}|)} = 2 \; \frac{(\Delta t)_n - (\Delta_t)_{n-1}}{(\Delta t)_n + (\Delta_t)_{n-1}}$$

where $(\Delta t)_i$ is the average shadow time for cycle i, defined as the average of the two shadow-times denoted, Δt_1 and Δt_3, in the apparatus writeup. While one

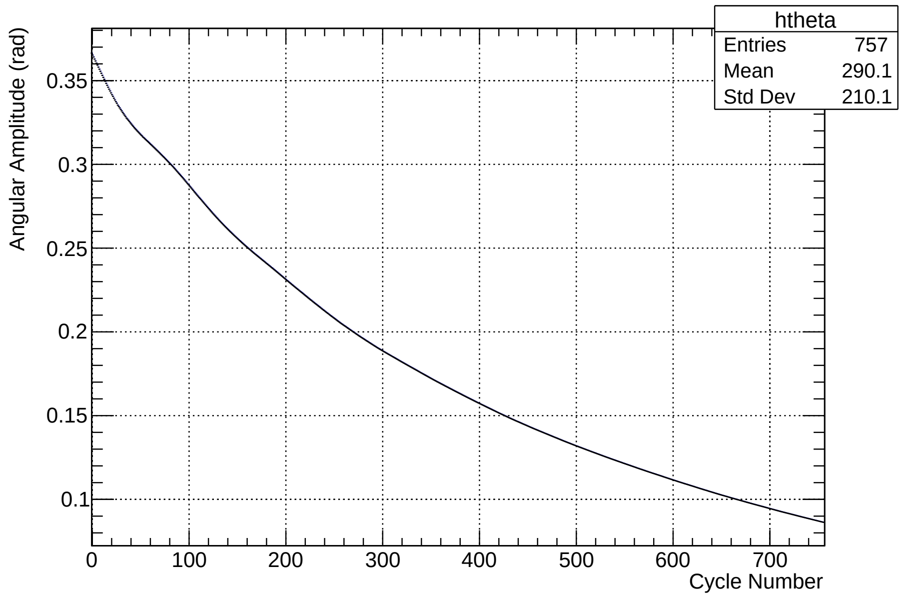

Figure 3: Estimated angular amplitude vs cycle number. Uncertainties have been set as 0.2 μrad independent of cycle number.

needs to know $\delta = D/L$ to compute the angular speeds, $|\omega_i|$, δ cancels in the ratio and one is left with a quantity which depends only on the well-measured time measurements. This is assuming that the effective angle subtended by the bob is independent of bob speed. One sees that for high cycle numbers the speed loss approaches a constant value of around 0.17%, but that at small cycle numbers there is some unexplained oscillatory behavior, and a tendency for the angular speed loss to be higher. The value of γ_ω is directly related to the damping from drag. This particular pendulum has a relatively high Q, so the speed loss is small. I have about four hypotheses for what may be causing the initial oscillations. What do you think could be behind this? This is also maybe a good example of when it may pay dividends to not just focus on taking data, but actually observe the data carefully visually during the pendulum motion. Is the bob moving along the expected path?, or are there some subtle or less subtle deviations?

The angular amplitude can be estimated using a variety of estimators. One method that is fairly simple and elegant, but involves some approximations is to assume simple harmonic motion over one cycle. Defining the start of shadow as

$$t_1 = T/4 - \Delta t$$

and the end of shadow as

$$t_2 = T/4 + \Delta t$$

One then has

$$\theta(t_1) = \theta_0 \cos(\omega t_1) = \theta_0 \cos[(\omega(T/4 - \Delta t)] = \theta_0 \sin(\omega \Delta t)$$

Now by geometry,

$$\theta(t_1) = 2 \arcsin\left(\frac{D}{4L}\right)$$

and so

$$\theta_0 = \frac{2 \arcsin\left(\frac{D}{4L}\right)}{\sin\left(\pi(\Delta t)_{\mathrm{shadow}}/T\right)}$$

Here the overall amplitude scale depends on the angle of first shadow (numerator) and a quantity that depends on the measured fraction of the period that is in shadow. This angular amplitude estimator is shown in Figure 3 where we have used the shadow-time averaged over both half-cycles. This method is approximate in that it neglects the non-linear contributions that are important for large angle, and it ignores drag effects. Other approaches are feasible and preferred. One can start from the expected shadow time fraction in the non-linear model. One could also relate the measured kinetic energy averaged over the shadow to the potential energy at the top taking into account the fact that the measured angular speed is not the maximum angular speed.

One approach would be to simply plot the shadow fraction vs cycle number. Another would use the measured scaled kinetic energy vs cycle number. In these ways one would account for the decrease in period with cycle number but still have a very well defined experimental quantity.

With the more direct estimates of amplitude it will now be feasible to now go back to the first period vs cycle number measurement and turn this data into period vs amplitude measurements that can then be fitted directly to the known forms, or use ideas hinted at by the references.

The data-set used for illustration here has some issues especially in the first 300 cycles. All three graphs show similar issues. Which graph shows the most striking evidence of untoward effects? Maybe you should have a similar graph at hand when taking data.

4 Drag

You should be able to estimate the value of

$$\frac{d^2\theta}{dt^2}$$

the angular acceleration at $\theta = 0$ from cycle-to-cycle by using the rate of change of the angular speed with time. This is tightly related to the γ_ω quantity above. By studying whether these quantities are linear or quadratic in angular speed

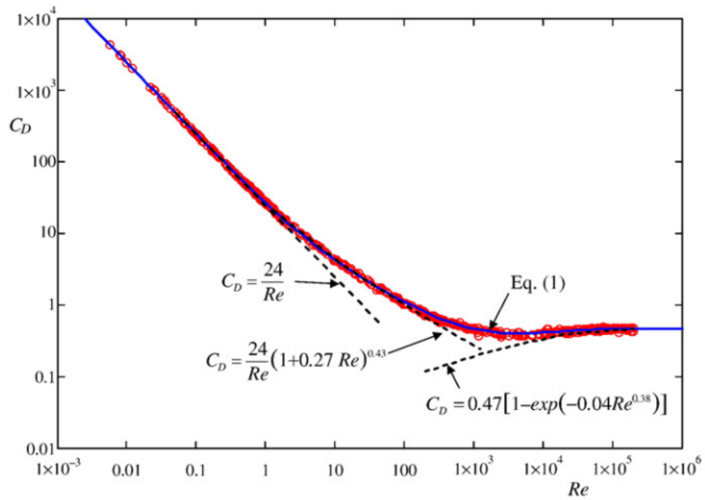

Fig. 1. Standard drag curve represented by Eq. (1) in comparison with experimental data [2].

Figure 4: Drag coefficient for smooth sphere in different fluid dynamic regimes characterized by different Reynolds numbers from Cheng paper [10].

one can determine the form of the drag at the bottom of the swing. You should see very different values as the drag and rotational inertia are varied. See old writeup for more detail. Also there can be effects from drag on the string.

The drag coefficient, C_D, is defined from

$$F = \frac{1}{2}\,\rho\,u^2\,C_D\,A$$

where F is the drag force, ρ is the density of the fluid, u is the speed of the fluid, A is the frontal area. The drag coefficient for a smooth sphere depends on the Reynolds number. See graph from Cheng with data on smooth sphere drag and a six-parameter relatively parsimonious fitting function that describes smooth sphere drag over eight orders of magnitude in Reynolds number.

At high Reynolds number we expect the drag force to be proportional to speed squared. At low Reynolds number, the drag force is linear with speed. In between there is some cross-over. Typical maximum Reynolds number for the pendulum is 2000. So as the pendulum swings the Reynolds number is changing ....

The old writeup claims that this can be averaged over OK using the maximum Reynolds number as the appropriate Reynolds number. This could be verified with a numerical simulation.

Nelson-Olsson and McInerney give suggestions on how to fit the data with a mixture of linear and quadratic drag, but assuming that the coefficients are constant over a cycle. So your mileage may vary.

References

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