Illuminating the Hidden Structures of Equations: A Examination of Equation Visualization Methods which Explicitly Convey Expression Hierarchy

John Gralyan

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Student Mentor: Mineta Wilde

Capstone Facilitator: Dr. Kimberly Handy

Abstract

This project aims to explore effective methods of explicitly teaching the structure of equations to students learning algebra. Since the traditional method of writing equations utilizes symbols and spacial arrangements to implicitly describe it's underlying structure, students often struggle with parsing and interpreting these classically written equations. The first part of this study allows students and teachers to create and explore alternative graphical representations which explicitly describe the structures of equations. Based on these designs, novel methods of visualizing equations are developed and tested for their ability to appropriately diagram and teach the hidden structures. A lesson plan has been developed and adapted to each of the visualization methods and randomly assigned to students later tested by an algebra assessment.

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Chapter 1

Topic and Problem

Many researchers (e.g. Sleeman, 1984; Ernest, 1987; Kirshner, 1989; Kirshner & Awtry, 2004; Jansen, Marriott, & Yelland, 2007; Landy & Goldstone, 2007a) have noted that a visual dissection of equations, which confers an understanding of their underlying structure, is a vital requirement in utilizing and solving for such equations. This project attempts to further build upon this base of research by using a mixed method approach to qualitatively open to discussion new methods of visualizing equations, followed by a quantitative analysis of their effectiveness in teaching each structural element commonly found in elementary level algebraic equations.

Topic

One of the major challenges algebra students face is the lack of understanding the hidden structure within equations. Anecdotal conjectures by algebra instructors have been backed by research (e.g. Bush, 2011; Lewis, 1981; Matz, 1982; Sleeman, 1984) which shows that misconceptions of an equation's structure are a common cause of errors made by algebra students.

Much like any larger problem, looking at a situation as a whole can seem very daunting. It is only when we take a closer look at its individual components do we see that this large problem is actually composed of many simple pieces. In the case of an equation, students need to see the grouping of individual expressions that they are familiar with nested inside one another. The dissection of an equation into its elementary components is an important skill to master because it allows one to focus on sub expressions in isolation and work through a problem effectively. It is also just as important to see the structure of these interrelated pieces in order to perform algebraic alterations as necessary.

This act of reading through and dissecting an equation into nested sub expressions is known as *parsing* the equation. This can be done consciously with some effort given some basic rules, but is more often done unconsciously by parts of the brain that are well conditioned to do so through practice (Maruyama, Pallier, Jobert, Sigman & Dehaene, 2012). Even though it is a vital component of using equations, parsing is never explicitly taught in the classroom beyond simply relaying the order of operations which provide the rules for appropriately parsing expressions.

The language of math, though very ubiquitous and universally understood internationally, is not designed to be intuitive as much as it is compact and efficient. This is not to say there is a need to change the well grounded syntax, but it does give instructors the challenge of presenting equations in a more intuitive way while still conserving the universally accepted conventions. This analysis aims to explore how alternative visuals can help students better understand the structure of equations by testing their effectiveness.

Topic choice. This topic was chosen in particular for its potential impact as a promising area for improvement in algebra education. Students learning algebra often experience difficulty transforming expressions because they do not see the appropriate nesting of the constituent expressions and therefore perform invalid changes that do not preserve equality. This trouble with the skill in properly parsing expressions can be attributed to the fact that there is a lack of visual cues embedded in the common semantics of equations for its structure. The main goal of this study is to explore potential alternatives to the classical mathematical syntax which lacks an explicit schematic explanation of the hierarchical structure of the relationships it represents. The structure can be parsed by a trained individual, but is not intuitive enough to be done without a strong background of practice. An alternative visualization method which more clearly illustrates the equation's structure could help students more efficiently learn the rote basics of algebra in order to allow more focus on the appropriate usage of it in real world situations.

The goal here is not to replace the well grounded, time tested, compact and efficient syntax of modern mathematics as it has proven to be very functional. It is also not the purpose of this study to find the perfect visualization method to diagram the complex relationships in equations. The true strive of this study is to build upon research on the subject by exploring alternative visualization methods of equations which may have the potential to effectively teach the the structure of equations.

The ability to parse through and understand the structure of an equation is an important part of many basic competencies in elementary algebra. It is possible that there is a more useful learning tool for each of these skills. This study tests the effects of using the visualization methods on each of these competencies individually as well as overall to see if there is any significant difference between using each method in each case.

Topic Importance. It is a commonly held belief that Algebra 1 is a gateway course, the understanding of which is a crucial component to graduating high school and succeeding in college (Asquith, Stephens, Knuth, & Alibali, 2007; Bottoms, 2003; Capraro & Joffrion, 2006; Choike, 2000; Edwards, 2000; Erbas, 2005; House & Telese, 2008; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Kaput, 2000a; Nathan & Koellner, 2007; Spielhagen, 2006a; Stephens, 2005; Welder, 2007; Witzel, 2005). The subject is deemed so because it is an important part of many succeeding math and science classes. It is so vital to these core classes that it is a mandatory prerequisite to many of courses in this area available in college. Without a solid foundation in algebra, students experience a very difficult time going through many math intensive science classes and find later mathematical courses nearly impossible to understand.

Mathematics is a subject most students learn in formal schooling over many years. These ancient teachings build upon one another and branch in many different directions which encompass every aspect of our abstract quantitative understanding of all natural phenomena. Although every subject within the realm of mathematics has some application and is connected with one another, algebra is targeted as the most common problematic area for many students. algebra is often referred to as a “gatekeeper” course because completing it is strongly correlated with taking advanced mathematics in subsequent years and admittance to 4-year colleges (Atanda, 1999). This is because algebra is the basic foundation on which competency in advanced math, science, and engineering can be developed (Harel, Selden & Selden, 2006). Without a thorough understanding of this core subject, students cannot successfully take more advanced classes which require this key prerequisite. Furthermore, algebra is important in people's personal lives because it helps them develop their ability to think about and solve real problems with a more abstract approach, allowing them to find solutions that are not concrete (Taylor, 1998).

Problem Statement

Solving algebraic problems often involves reorganizing an equation in many successive steps in order to isolate an unknown variable. A solid understanding of an equation's structure is a vital component in this process as expressed in the Common Core Standards section A-SSE (NGA 2010). The problem is that there is no commonly accepted method to clearly show learners how expressions are nested in one another to make up the equations structure. The act of parsing equations is always done subconsciously and mastered through practice with many problems in order to condition the student. There is no absolute training in this core skill needed to solve algebraic problems.

The skill of breaking down an equation is unwittingly gained by students through practice, but could easily be taught before the confused student is forced to struggle through problems without explicit training. Just as the order of operations has provided an invaluable guide for simplifying numerical expressions, a thorough understanding of the general structure of equations would be extremely useful for students solving algebraic equations.

Optimally, algebra students would look at any equation and immediately attempt to decipher its structure rather than struggle through a cryptic mess of glyphs. The intimidation of algebra can be greatly alleviated once the equation is understood as a construct of well understood expressions. Explicit instruction in the matter has the potential of making equations much more user friendly and as a result, decrease the fear of math while increasing algebraic competency in the general public.

Problem background. Up until the Common Core Standards were officially adopted in California in 2010, the California State Standards did not include an understanding of the structure of equations as a core requirement in Algebra 1 classes (SCOE, 2010). At first glance, it's a bit surprising that such an imperative part of the subject would be left out of a curriculum. Upon closer inspection of the state standards, it seems that the schema for the composition of equation – although never explicitly taught – is subtly embedded into the curriculum and expected to be understood and internalized through practice with solving problems. Research in the area of math education (e.g. Bush, 2011; Lewis, 1981; Matz, 1982; Sleeman, 1984) has shown however, that errors commonly occur due to a misunderstanding of the equation's underlying structure.

The Common Core Standards section A-SSE (NGA 2010) entitled “Seeing Structure in Expressions” is comprised of two parts: “Interpret the structure of expressions” and “Write expressions in equivalent forms to solve problems.” The latter is the most common skill at the heart of algebra and is used throughout the class as well as every subsequent math class. The inclusion of the interpretation of the structure of expressions however is surprising because it is not commonly listed as a standard to be taught in detail. The fact that students are struggling with this aspect of algebra is becoming more clear to those creating the standards by which instructional units are developed. It has always been known that an appropriate interpretation of the structure of any given expression is vital to its use of course, but it is becoming more evident that more focus must be placed on explicitly teaching this skill to students. With a firm method of teaching the parsing of expressions, students will be able to effectively interpret them.

Possible Causes. Currently, each operation is taught in a lesson separate from the others. The only time these operations are explicitly related to one another is when teaching the order of operations (more commonly referred to as the popular mnemonic PEMDAS for Parentheses, Exponents, Multiplication, Division, Addition and Subtraction). As a result, the common method of approaching an algebraic problem for students is to go though the steps with the order of operations as a guide. Although this works very well for simplifying expressions, the simple list by itself is not enough to solve equations that require heavy manipulation of expressions (Glidden, 2008). In order to efficiently solve problems, students must understand the structure of the equation, using the order of operations merely as a tool to determine precedence rather than as an overall guide.

Although the structure of any equation can be deciphered by a student with enough exposure to them, it is not readily clear to the untrained student. Aside from the parentheses which specify sub-expressions, there are no visual cues that show the nesting of expressions in one another. This is because the visual display of mathematics was never historically created to be intuitive. It was simply meant to hold the information immediately necessary. The syntax has changed slightly over the course of time, but has help up very well. It can possibly be supplemented by visual marks that provide information regarding the structure of the equation for students who are just learning the subject. In this way, the display of equations can have a middle-ground between the very abstract yet sleek classical method of writing equations and a model specific diagram which can hold all the information necessary but couldn't be universal.

Research Questions

1. What are alternative methods of visualizing equations that explicitly convey their hierarchical structure?
2. What is the effect of learning each visualization method on a student's ability to learn the basic competencies of elementary algebra?

Chapter 2

Review of the Literature

Best Practices

Unfortunately, the same features of algebraic equations that allow for the powerful abstraction of real problems into symbolic models also makes it very difficult for students to understand. Often, this subject is just taken as a set of rules for rearranging glyphs, a skill that has no practical application in the real world. The use of these rules without knowledge of the underlying ideas can lead to a grave misunderstanding of the subject and careless mistakes.

Particularly when learning a subject that builds on itself such as this, it is especially important to steer clear of misleading information as it can easily contaminate the learner's knowledge of the subject (McNeil & Alibali, 2005). Even for seasoned students who have taken more advanced courses and are expected to properly use their knowledge of basic algebraic rules, misconceptions can manifest themselves in very inappropriate settings. For example, Pappanastos, Hall, & Honan, 2002 tests the performance of business students on very basic questions based on the order of operations, discovering that these students who may soon make important business decisions have a poor grasp of these basic algebraic rules. In a similar study, Glidden, 2008 tests prospective elementary teachers on the same topic. Although these future teachers were expected to be significantly more comfortable with the subject matter, this study reached similar conclusions. This misunderstanding of basic concepts in prerequisite subjects has given the mathematics taught in the US the stigma of being "a mile wide and an inch deep" (Schmidt, McKnight, & Raizen, 1997, p. 122).

The goal of any classroom is to provide students with the means to not only learn, but also retain the core ideas and skills in a particular subject they will need throughout their lives. Willis 2006 discussed the neurological theory behind the processing of information into long term and short term memory storage, noting that long term storage of memory is associated with its deeply rooted connections to other memories. She goes on to explain that building a network of nerve pathways to important information can be done very effectively through the use of graphic organizers to visually create meaningful connections. The use of pictures has been used as a very effective tool for conveying information and facilitating learning, particularly in computer aided learning (Rieber, 1994). Graphical representations provide students with another perspective of the system in question, which is particularly helpful for visual learners.

Multimedia. Presently, there are many different forms of media which could serve a rich medium for the transfer of information such as mathematical models (Mayer, 2003). Over the years, each form of media that has been created has been incorporated into classrooms in some way or another in an attempt to utilize every unique method of information storage, transfer and presentation. Each of these forms of media has its own strengths for different subjects, but it is important to have a robust method of translation between these models in order to effectively create multiple connections of understanding to the abstract mathematical constructs (Lesh, Landau & Hamilton, 1983). The utilization of varied media in their respective, appropriate situations can be used to create very effective models for learning.

Professional Literature Summary

Cognitive Information Process Theory has been used in educational psychology to explore the idea that memory is processed as different codes in two interconnected systems: verbal and visual (Clark & Paivio, 1991). Although both of these systems are important in the learning of any subject, evidence shows that algebra is intrinsically more visually oriented. One such example is that people who suffer from severe global aphasia or semantic dementia retain the ability to perform algebraic calculations (Zamarian, Karner, Benke, Donnemiller & Delazer, 2006). This means that even a complete reduction in written and spoken language has no direct effect on one's ability to not only learn, but effectively use algebra. Neurological researchers studying the brain activity during engagement in mathematics concluded that “mathematical syntax, although arising historically from language competence, becomes 'compiled' into visuo-spatial areas in well-trained mathematics students” (Maruyama, Pallier, Jobert, Sigman & Dehaene, 2012, p. 1444). This evidence supports the historic belief by many mathematicians and physicists that mathematics is naturally a predominately visuo-spatial skill. It is said that experts in advanced math understand and utilize the power of visual representations to explore a system in question in depth because it allows for a great versatility in mapping a real problem onto a simplified model, while the less experienced tend to skip visualization and miss valuable solution planning (Stylianou & Silver, 2004).

Syntax of Algebra. The use of visuals has been used to model mathematical phenomena before any formal syntax was ever created. The use of formal syntax, however, is not unlike the use of informal pictures as spacial metaphors for abstract concepts (Lakoff, & Nunez, 2000). It can be argued that there is no real difference between using the internationally accepted conventions of displaying math and the creation of a symbolic diagram besides the communication issues. Algebraic reasoning already requires developing an interpretation of abstract symbolic representations. This relational thinking depends on a graphic mapping of relationships as its form of storage and communication (Greenes & Findell, 1999). The human ability to assess configurations of symbols through perceptual-motor mechanisms allows for a diagrammatic view of the complex relationships depicted by equations (Landy & Goldstone, 2007a, 2007b & 2010). This ability to diagram mathematical models allows us to understand a complex network of related variables as a compact equation. More interestingly, it allows us to analyze the model in a concrete medium which uses formal rules to prevent errors during complex transformations. This modern use of equations in algebra has become so convenient that students often memorize visual transformations rather than understand them (Kirshner & Awtry, 2004). Yet research in alternative methods of displaying and interacting with equations "...suggest that mal-rules need not be a natural occurrence when students operate in an environment that supports explicit attention to expressions' structures, and where structure also imposes constraints on students' actions." (Thompson & Thompson, 1987, p. 5).

Alternative Graphics of Equations. The prospect of a more concise method of illustrating the natural hierarchical structure within equations has lead many researchers (e.g. Sleeman, 1984; Ernest, 1987; Thompson & Thompson, 1987; Kirshner & Awtry, 2004) to utilize a visualization technique involving tree diagrams. Sleeman noticed that students often learn incorrect algebraic transformations, known as mal-rules, which lead to the bad habit of chronically using these transformations when solving problems. One of the errors he categorized was the parsing error which could be explained through the use of a tree diagram such as the following diagram from Sleeman 1984:

The tree diagram of equations allowed students to gain a deeper understanding of the structure of an equation when properly drawn, but each equation that was to be understood in this manner would have to be physically drawn out individually. The use of a computer program to generate the graphical tree as well as allow for interactivity for algebraic transformations was described in Thompson & Thompson 1987.

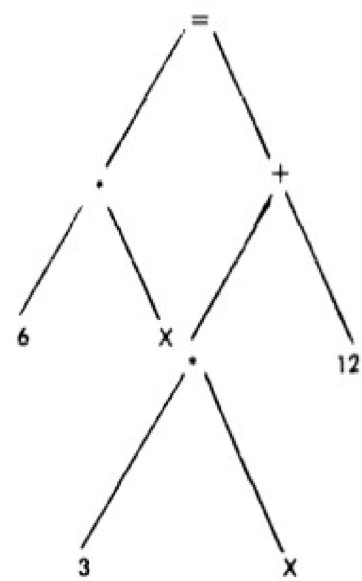


Figure 7. (a). The correct parse tree for the equation 6 * X=3 * X+12

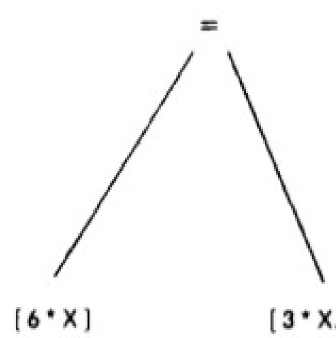


Figure 7. (b). A "two-level" representation for the same equation where [ ], following Bundy (personal communication, 1982) represents a "plus bag," where all the entities in the bag are operated on by the addition operator.

Conclusion

All the literature mentioned earlier shows a gradual improvement in the resources available to algebra education. Particularly with the use of technology, it is now possible to create very useful tools that may aid learners of this historically difficult subject in their classes. It is clear from the research on alternative graphics that there is potential for the improvement of a deeper understanding of the structure of equations through the use of graphics that show their hierarchy (Sleeman, 1984; Ernest, 1987; Thompson & Thompson, 1987; Kirshner & Awtry, 2004). This heightened level of intuitive understanding of the equations composition could help students interpret an equation as a mathematical translation of a natural model or phenomena. It could also make algebraic transformations more clear as a group of logically sound changes rather that arbitrary rules to be followed without question.

By adding an explicit visual component to the structure of equations, algebra could be given a concrete feature which makes the classically abstract subject a bit more understandable, tangible and human friendly. The method by which this graphical addition can be accomplished has been experimented with by the research mentioned earlier with a tree diagram like visualization method. Further development of this method and possibly other such methods can potentially have a positive impact on the field of algebra education.

Chapter 3

Research Methodology

Overview of the Research Methodology

A mixed method analysis involving two experiments is used to first generate and gather possible visualization methods and to subsequently test their effectiveness. The first experiment will be done with a purely qualitative design including an open-ended questionnaire followed by a more focused interview with teachers and selected students based on their questionnaire answers. This test gathers narrative data, the students' own equation designs and their visual associations with each operation. This data is coded and categorized to develop distinct visualization methods to be used as variables in the lesson plan in the second experiment.

The second test quantitatively examines the teaching power of each visualization method by testing the related algebra competencies of students who are exposed to each variable. Students go through a set lesson plan which is the same in all ways except the visual diagram of the equation. This visual is either the classic standard form of the equation (control) or one of the newly designed visualization methods (variable). An assessment of the competencies of interest is used to measure the student's progress.

Research Questions

This study ultimately aims to examine the most effective methods to teach students the basic structural units that make up equations and how to parse through them. It should be noted that only the visual component of parsing equations is tested because it has been shown to be much more important to these skills than the verbal component (Maruyama, Pallier, Jobert, Sigman & Dehaene, 2012). Although a verbal explanation is also required during the learning process, this portion of the lesson plan will be kept constant for this study. The following questions will be the center of focus throughout the research.

1. What are alternative methods of visualizing equations that explicitly convey their hierarchical structure?
2. What is the effect of learning each visualization method on a student's ability to learn the basic competencies of elementary algebra?

Hypotheses

The first of the research questions will be examined through a qualitative analysis involving both students and teachers to gain as many different perspectives as possible. Answering the second question will involve putting these visualization methods through a quantitative test with the following hypotheses:

Null Hypothesis. There will be no significant difference between groups who were each exposed to a different visualization technique in their performance on algebraic problems requiring the parsing of an equation.

Alternative Hypothesis. There will be a significant difference between groups who were each exposed to a different visualization technique in their performance on algebraic problems requiring the parsing of an equation.

Definition of Terms

Parse. The act of analyzing the structure of an equation or expression into a hierarchy of its constituent sub-expressions. This can be done consciously but is often done naturally by anyone who has experience solving equations algebraically.

Tree Diagram. One of the visualization methods for diagramming the structure of an equation under investigation in this study. The basic design consists of a branching tree which displays the children expressions at each successive node. This schema has been used in a number of research studies (eg. Sleeman, 1984; Ernest, 1987; Thompson & Thompson, 1987; Kirshner & Awtry, 2004) and has shown some potential for helping students understand common structural misunderstandings.

Stacked Diagram. This visualization technique was developed through the qualitative portion of this study. The design consists of a stack of the same equation differentiating in the highlighted portion which shows the nested expression at each part of the stack. It is similar to the tree diagram in that the inner expressions are separated vertically for clear display. The main difference with the tree diagram is that the entire equation is displayed at each level, making the diagram less abstracted from the classical look of the equation of interest.

Overlaid Diagram. This method of drawing an equation with explicit structural cues is the closest of all the techniques in this study to the classic method. The only alteration to the classical design of equations are the overlaid rectangular shapes that group the expressions in the background. The addition of this design element is similar to the unnecessary use of parentheses and other grouping symbols to explicitly show every nested expression.

Research Design

The questionnaire is given only to students who have completed an Algebra 1 course as they have been exposed to equations long enough to understand the prompt. These students are also young enough to use their creative talents and draw upon their recent experience with learning algebra. The survey is given out during class but is not meant to be performed during class time out of respect for the classroom. A brief explanation of the study and the prompt is given during class. The students are encouraged to create a visualization method that they themselves would have found useful and could potentially benefit future algebra students. The responses are collected the day after being distributed. Each response is initially scanned before the interviews.

Interviews of the teachers are taken during the breaks and conference hours while the interviews of students are taken during class time. Only a select number of students are interviewed from each class based on their survey responses and teacher suggestions. The interviews typically last about ten to fifteen minutes each for students and the entire break or conference hour for teachers. Narrative data is gathered in notes for further analysis while subjects are encouraged to draw their ideas out.

The quantitative portion of the study is designed to test the efficiency of each visual in helping students learn how equations are structured. The students are randomly assigned into one of four test groups in this true experimental design and are not told of the groupings. The only difference between the groups is that they are given different passwords for the supplemental web site and a different version of the final assessment which use one of the variable visual methods or the control using the classic view of equations. The web based lessons are entirely similar for each group except for the equation visual area in the practice test page, while the classroom lessons are done together to prevent extraneous variability. These lessons include information about the structure of equations, how to parse them and basic algebra transformations. Students become familiar with the structure of equations through the lessons which are aided by the visual method they are assigned to. The web site can be found at http://algebra.scienceontheweb.net where all the supplementary information resides. Clicking the “practice test” link will prompt the user for their “test password” which will either be “control”, “tree”, “overlay” or “stack” depending on their assigned test group. The unit assessment at the end tests their proficiency in the subject and allows for a comparison between the assigned groups. There are four versions of the test which all have the same questions, but vary in the equation display.

Participants

The entire group of participants consists only of middle school students from two separate Geometry classes and their teachers. The qualitative portion of the study involves one class and both teachers while the quantitative portion involves the students of the other class. Each of these classes contain roughly thirty students who will all be asked to participate in the study. The volunteering teachers will hand out permission slips that will specify that the data to be collected from students will in no way affect their grade. The qualitative data can be freely inquired by anyone but the quantitative data will be kept confidential.

These post-algebra classes were chosen because students who have had recently taken Algebra would be likely to provide useful information regarding possible visualization methods and their effectiveness. Since the majority of students do not take this class in middle school, these students are often either more motivated, more mathematically oriented or overall more studious than the rest of the student body. This is a very appropriate trait for this experiment because more enthusiastic students would be more likely to provide more useful responses.

Demographic. The students in this class are often much more diverse in cultural background than other classes, suggesting that the class would contain a more generally diverse pool of students than most other classes on campus. The local schools are very heavily dominated by one race which greatly influences the schools' dynamic. These issues are not brought up in the geometry classroom however because there is much more diversity within this population. Heritage is not a major topic of concern in this classroom in general.

Characteristics. The students here seem to recognize the importance of the classroom as a learning environment and act with more professionalism than they do during the breaks. The students in this class also demonstrate more respect and courtesy toward one another, seeing each other as fellow gifted students who band together in the exclusion of the more popular groups among the school. They seem to realize the fact that they have more similarities than differences as compared to the rest of the school. It almost seems as though the shared culture in this class is one that is indifferent to nationality and more concerned with a common yearn for intellectual stimulation. This environment helps students grow and learn from one another without the need for commonality is a heritage centric culture.

Permissions. The students and their parents were informed of the study being conducted through an Informed Consent Form (Appendix A) handed out to the students during class. The handout described the project by stating its purpose, the curriculum and the expectations of the student. The benefits and risks involved in participating in the study were also described including the benefit of reviewing important concepts of algebra and the risk of internet use by the young participants. It was assured that the confidentiality of all results would be maintained by keeping the individual results private and aggregating all the reported data. It was also made clear that no data collected in this study will affect the students grade, class or school standing in any way.

The Informed Consent Form included a section that made very clear that the participation in the study is not expected for the course, instead it is a voluntary act that may be withdrawn from at any time. They were informed that regular classroom activities are still mandatory, but the surveys, interviews and quizzes are not required and will not effect the students classroom standing if withdrawn from. With this in clear print, all the students and parents still volunteered to participate without any problems.

Evaluation Methods and Tools

The first instrument used in this study is a single question survey asking students to draw an equation with a new design that explicitly displays its structure. A copy of the prompt is found in Appendix B of this document. A specific equation was chosen which includes a moderately complex nesting of the basic operations. This allows the students to think about and create a visual schema with all these basic operations in mind. The prompt specifically mentions that this visual would be helpful to students new to Algebra and should be designed to be intuitive enough for someone to pick up naturally. It also gives a few questions for the student to think about to help spark ideas and help guide the design. These questions remind the student to keep it simple enough to prevent overwhelming complexity, encourage freedom in creating a schema for the operations, and encourage the design to be relevant to the classic equation.

The other instrument in the first test is an interview used to further delve into the student's ideas. The interview is of course open ended, but a list of sample questions can be found in Appendix C. The interviewees are the math teachers who have experience in the subject and a selection of students who show the most creativity and enthusiasm in the survey. The researcher helps guide the interview by asking about the student's responses as well as what they think about other designs.

The instrument used in the second experiment is an assessment which is designed to measure the ability of students to solve problems requiring the skills taught in the instructional unit. The final visualization methods from the first experiment are the variables used in this instrument. The participants are randomly assigned to one visualization method which are embedded into the lesson plan and final assessment to provide the element to be tested. The lesson plan itself teaches the structural units and algebraic transformations to all students consistently except for the visualization method. The four versions of the assessment can be found in Appendix D. These questions were chosen to cover all the basic algebra competencies that require an understanding of the structure of an equation. The topics will be introduced during the classroom lesson without variability, but the web based lessons which contain the visuals will be done individually. Students are also given the appropriate version of the unit test according to their randomly assigned test group. All four versions have the same questions but the associated equations are displayed with the related visualization method. This assessment can therefore be used to compare the difference in related competencies between students who were assigned to each visual.

Reliability and Validity. Several measures were taken to ensure the reliability and validity of the experiments in this study. The survey was written to elicit design elements that appropriately diagram expression types. It was therefore important for the equation in question to have multiple versions of the common expression types nested in every variation in order to test the inter-item correlation and sampling validity of each operation. Designs of a type of expression that are consistent through different nesting variations are considered more internally reliable and more valid than those that break down under different circumstances. The wording of the survey prompt and interview questions were clear to state the true purpose of the study in order to establish construct validity. This combination of survey coupled with an interview had the synergistic increase in the criterion-related validity of student responses.

In the quantitative experiment, questions were chosen broadly to increase the assessment's sampling validity by testing the students ability to solve many different problems that require the ability to effectively parse expressions. Although this competency is required for most algebra problems, only questions that were solvable with the competencies contained in the unit of instruction were candidates for the test for obvious reasons. This experiment's construct validity is met by ensuring that the only variability between test groups is the visualization methods on the parallel web sites. The test groups had the instructional unit in the same class and were never told there was any difference between them.

Data integrity. All raw data was collected in an appropriate manner to ensure its integrity. Student responses in all experiments within this study remain confidential through the anonymity of the participants. Although students included their names on their survey response and test, all data was aggregated before being reported. Student names were only used when choosing the participants of the interviews from the selected survey responses.

Method Adoption

The survey responses with narrative data are collected, sorted and categorized. Distinct categories are created by grouping together responses based on their structural similarity. In this step, schematic preferences such as color and shape of units are not considered; only the structural design is used to categorize the responses. These categories are used as the basis for the different equation visualization methods and are refined further after the interviews. The distribution of responses over categories is not important, only that all the unique structural designs are categorized. Operational identifiers such as color and shape are analyzed separately by frequency counts to find any possible trends. The most representative response for each category is used as a sample during the interview stage.

The interviews are used to discuss in detail and brainstorm the design of each structural visual. The interviewees are encouraged to draw their ideas out in order to collect their ideas in a concrete form to be analyzed similarly to the survey responses. All opinions on each category are taken into consideration and noted as strengths and weaknesses. These discussions are used to further refine the categories and aggregate each into a single representative visualization method.

The visualization methods are tested against one another and the control by an assessment which collects data of student performance. The score for every question is tabulated based on the test group and competency tested. The scores are analyzed by ANOVA to determine if there is any significant difference between the visualization methods. The scores of each competency are compared between the test groups to see if there is any significant difference within any particular competency. This same analysis will be done with an aggregation of the test results to see if there is an overall difference in results in addition to each competency tested.

Differences. One of the visualization designs that seemed promising has begun in development but was not implemented in the lesson plan or assessment due to logistical constraints. The design involved an interactive view of an equation which would allow students to focus in of a single expression at a time. This would give students the ability to physically parse the equation in order to learn the mental skill more easily. Although the development of this program has begun, time consuming issues have prevented its implementation in this study. Further studies will include this method once fully operational. A demo has been uploaded on the development web site at http://johngralyan.appspot.com.

The original methodology for the quantitative experiment was to create a self guiding web program teach students the entire unit of instruction in order to allow student to learn at their own pace, be more exposed to the visuals being tested and to be less invasive in their normal classroom activities. The creation of this program proved to be more difficult than initially expected. The basic instruction was eventually done in the conventional method in a classroom while the equation visualizations remain on the web site for individual use. Students were still encouraged to pay attention to the diagrammatic visuals on the web sites by labeling them as "practice tests". The revised method did however increase the experiment's construct validity by forcing the students to learn at the same pace in the classroom.

Another deviation was the class that fit as the subject for the quantitative experiment. Originally, an algebra class was targeted since the instruction is algebra based. Algebra classes were rather hectic however and unable to change the busy schedule. A geometry class however was very cooperative since the class was ahead of schedule and would benefit from an algebra refresher. It also seemed more appropriate for students to have had more experience with the basics of algebra.

Summary

This study aims to further add to the body of algebra education research pertaining to equation structure. The first part utilized a qualitative approach to survey and interview students in order to create new designs that may help students effectively visualize the structure of a given equation. These visualization methods are then tested in a quantitative experiment to determine if there is any significant difference in performance on an algebra assessment between students who use each of the different methods. These results are compared overall and by specific competencies to pinpoint any particular differences.

Chapter 4

Findings

Results Overview

The qualitative tests including the survey and interview process resulted in the design of several methods of visualizing the structure of equations. These methods have been categorized into four distinct designs, three of which are shown in Figure 1. The last design was unfortunately not implemented in the study. The non-interactive designs however, were implemented in the rest of the study as the different versions of the quantitative test.

The results of the quantitative assessments were initially collected as the raw tabulated score of every question for each student. These scores were then aggregated by question and analyzed by ANOVA to find if there is a difference between the visualization methods. The scores were also compiled by competency in order to compare the results of each test group based on the competency involved in the question.

Data Driven Findings Summary

The survey resulted in student suggestions for the visualization method designs in the for of hand drawings (n=28). These ideas were sorted and categorized in order to draw out unique features. One of the main features common to many of the drawings was the use of grouping symbols and color differentiation to distinguish a nesting of expressions. The highlighting of separate parts of the equation allowed for an easy mental parsing of the equation in it's original context. This was the main idea behind the overlay diagram (Figure 1). Another common trait was the use of space to show the equation's structure much like a graph. These spacial arrangements came in two forms: the first was a hierarchical tree like the branching diagrams in the literature, and the other is the use of multiple forms of the equation as if splaying the overlay diagram vertically. These ideas became the core of the tree and stack diagrams respectively.

A thorough discussion of different design elements took place during the interviews where students gave their input on the advantaged and disadvantages of the visuals. The use of grouping symbols to encase expressions was discouraged as it cluttered the equation and made it more difficult to understand than before. The addition of symbols was therefore minimized to only simple rectangles for highlighting. The use of color for distinguishing parts of the equation was much more pleasing to students. The overlay diagram suffered from the cluttering problem with symbols in a tight space. This problem was somewhat alleviated with the use of colored rectangles with no stroke. The tree and stack diagrams were not tight because they graph the groupings vertically, but students complained that this lead to other problems. The tree diagram mainly faults by abstracting too far from the original equation, making it an extra step of conversion. The stack diagram has many copies of redundant data in order to keep the original context. All this meta data adds bulk to the simple classical equation.

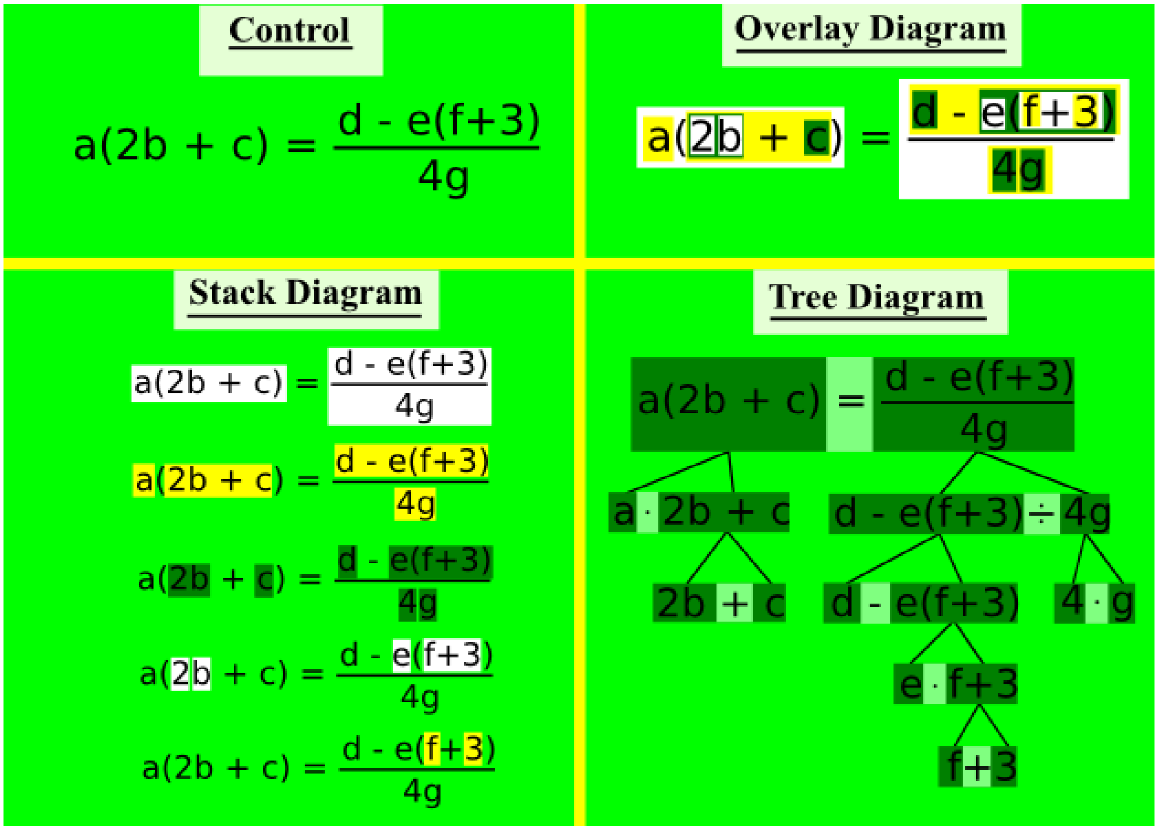
The results of the assessment were meant to test whether the alterations to the equation lead to any differences in performance on simple algebraic problems. In a single experiment with a classroom of geometry students (n=31), the visualization methods would be tested to compare their effects on an algebraic assessment. The true experimental design included randomly assigning each student to one of four test groups which consisted of a control and three visualization methods. An ANOVA analysis of the test results concluded an F value of 0.6834. The full tabulation of this analysis can be found in Table 1. The assessment results of each group is also tabulated by competency in Table 2 which shows that most students scored perfectly on most competencies. Although it is nice to know the students' algebra skills are up to par and that the visuals did not adversely effect their performance, a significant difference was not found in the test scores.

Data Analysis

Data from the qualitative experiment included information such as the types of cues that would help students visualize the nested groupings of expressions. It was clear from the interviews that students understood that parentheses were the standard symbol for grouping expressions, but the overuse of parenthesis didn't necessarily help clear things up as the equations still looked linear. Even with the use of varying grouping symbols such as brackets and curly braces, the addition of extra symbols added unnecessary clutter to the page. The inclusion of many visual cues with different shapes had a similar effect: adding excess bulk to the equation.

Students responded best to differentiation of nested groups through color coding and spatial transposition. There were no trends in colors, only favorites of each student. It is only important that they be different to effectively show the deferences. A rectangular shape is used to highlight groups because it was most simple and least likely to interfere with the shapes of the symbols used in the equation. The other common method of displaying the structure was to splay the inner groups vertically similar to the tree diagram discussed in the literature. The finalized designs have been used to create example visuals using the survey equation as shown in Figure 1. Additional examples of their implementation can be found in the assessments in Appendix D.

Figure 1



The quantitative assessment raw data was in the form of numerical scores. Each data point value was either a one for correct or zero for an incorrect answer to the question. A single data point was taken for each of the eight questions on every one of the students test (n=31). The collection of these student scores by question number in each test group resulted in four lists of eight datum. Since there was an uneven number of students in each group, it was important to normalize these results to ratios with respect to the total possible scores in order to compare them. Once these four lists of scores were calculated, they were analyzed by the statistical method of comparing lists known as ANOVA to determine if the difference between the list means were by chance or if they were significantly different. The results of this analysis are shown in Table 1 below.

Table 1

ANOVA Results of Quantitative Assessment Scores

Source	SS	DF	MS	F
Between	152.6	3	50.85	0.6834
Error	2083	28	74.41	
Total	2236	31		

Note: The lists analyzed were the four lists of compiled scores by question of each test group. The probability of these results assuming the null hypothesis is 0.57

Besides the overall test results, it was also essential to analyze the scores per competency to see if there was any significant difference with any visualization method. The original raw data was compiled by competency for each test group. The competencies tested for on the assessment included simplification by combining like terms, cancellation, factoring and distribution as well as solving by numerical evaluation and relational evaluation. All these skills involve the correct parsing of the equation. A total overall score was also calculated for a more complete understanding of the grand scheme of the test. The data was also normalized by converting them to percentages for a more clear comparison. Table 2 below contains the tabulated results of this analysis. There does not seem to be any clear significant difference between the use of any visualization method on any of the specific competencies being tested.

Table 2

Quantitative Assessment Scores by Competency

Competencies	Control		Tree		Stack		Overlay	
	Raw	Percent	Raw	Percent	Raw	Percent	Raw	Percent
Combining Like Terms	8/8	100%	7/8	87.5%	8/8	100%	7/7	100%
Cancellation	7/8	87.5%	8/8	100%	8/8	100%	7/7	100%
Factoring	8/8	100%	8/8	100%	8/8	100%	6/7	85.7%
Distribution	8/8	100%	7/8	87.5%	6/8	75%	7/7	100%
Solving	29/32	90.6%	28/32	87.5%	31/32	96.9%	27/28	96.4%
Overall	60/64	93.8%	58/64	90.6%	61/64	95.3%	54/56	96.4%

Note: The scores presented here are the points given over points possible aggregated by the test group and competency.

Answers to the Research Questions

The first research question pertains to the different visualization methods that can by used to more clearly show the nesting of expressions within an equation. Although this question can never be fully answered, the qualitative portion of this study has produced a few such designs. An example of each of the three visualization methods found in this study can be found in Figure 1 above and in the different versions of the quantitative assessments in Appendix D.

The second research question asking what the effects of learning the different visualization methods are on a students performance was the focus of the quantitative experiment in this study. The experiment compared the results of the four test groups with the null hypothesis that there will be no significant difference in the results of the basic algebra assessment. The analysis of these results concludes that there is a 0.57 probability that the results would have appeared if the null hypothesis was true. There in no sufficient evidence to reject the null hypothesis. What differences manifest in students learning these different methods is still unknown, but these test results show no advantage for students of learning the new equation structure designs.

Findings Summary

This study included a qualitative portion that examined potential designs that show the structure of equations more clearly by surveying students on their ideas and subsequently interviewing these students to refine the designs to unique visualization methods. The final designs were named the tree, overlay, and stack diagrams which are shown by example in Figure 1 and in the separate assessment versions in Appendix D. The potential for an interactive version of the diagram seems promising but was not implemented in this study for logistic reasons.

The quantitative experiment of this study included a comparison of the test scores both overall by ANOVA and by individual competency as summarized in tables 1 and 2 respectively. The calculated F value of 0.6834 and tabulated test scores do not show a clear difference in using any visualization method. None of the visualization methods seem to have any advantage according to the test results.

Chapter 5

Discussion and Conclusion

Overview

This study, using a mixed methods research design, explored the area of equation structure visualization in algebra education. The qualitative portion first elicited ideas using qualitative research methods to survey and interview students to develop new designs. These experiments successfully brought to light a number of design element ideas that were incorporated into different visualization methods. The quantitative portion then tested the effectiveness of the final visualization methods against one another and a control through an algebra assessment.

Problem Solutions

The drawings suggested in the survey responses provided different design elements which can be used to further develop and refine such visualization methods. The refined designs created after input from the interview process as seen in Figure 1 depict the most clear and student friendly methods of displaying the structure of equations found in this study. These solutions showed the most potential for helping students learn to effectively parse equations based on the interview responses.

In addition to these designs, another solution is currently being developed which was not tested in the quantitative portion of this study. This more technologically advanced visualization method used a web program to allow students the ability to shift focus through the hierarchical nesting of expressions within an equation in order to understand its structure and eventually learn how to parse equations mentally. See the further investigation section of this chapter for more details.

The quantitative portion of the study concluded with test results summarized in Table 1 and Table 2 found in the analysis section of chapter 4. These results suggest that the null hypothesis be accepted by default as there is no conclusive evidence to reject it. According to the analysis of these results, there is no reason to believe using the alternative visualization methods tested would be beneficial to students learning algebra.

Strengths

This project benefited from a mixed method approach to furthering research in this area of intuitive approaches to algebra education. The separate tests utilized the strengths of each experimental method to both elicit and test novel methods of displaying equations that may help students build an understanding of their structure. The qualitative portion alone would only lead to theory without a concrete test of the findings. The quantitative test of only previously discovered methods would be very limiting. This mixed method approach therefore allowed for an expansion of the theoretical designs as well as tested their effectiveness.

Another strength is the strategically decided topic of the study which lies at the heart of an academic subject that is notorious for being very troublesome for students, important to other subjects and a prerequisite for many other classes. Further developments in this area of research can potentially lead to resources and tools that students can use to more easily grasp key concepts of algebra which are vital to many other areas of their math and science education.

Weaknesses

One major weakness of this study is the small sample sizes involved in both studies. Although the cooperation of the students and teachers involved was greatly appreciated, a larger group of participants and more time with each would have strengthened this research. More in depth interviews or focus group sessions through an iterative design and feedback loop would be very beneficial to the further development of the visualization methods and creation of new designs. The quantitative assessments would have also benefited of course because the sample size is a key part of any statistical analysis.

The sample population for the assessments was a particularly poor choice. Since all the participants had already passed an algebra class, they were already very familiar with basic algebra concepts at the focus of this study. The instructional unit further provided a refresher for these students that made the assessments too easy for them. The test results were so good that there was naturally very little difference between the scores. The only difference the alternative visuals could have made would be to confuse and lower the test scores. The target audience should have been students who were unfamiliar with the competencies to be tested such as pre-algebra or algebra students who could potentially learn the most through the lesson plans.

Problems Encountered

Many problems arose during the development of the interactive computer program that was meant to be one of the variable visualization methods. Anyone familiar with this laborious task knows that issues manifest themselves sporadically throughout the process. Aside from expected set backs commonly associated with the development of any web based program, there were major logistical issues involving the experiment itself. The inclusion of this method meant that a computer was required for the students in that test group. The addition of any new technology would supply an unfair novelty effect. It also required an explanation of how to use the program which would inadvertently give those students extra instruction in the structure of equations which would skew the results. These issues prevented a seamless addition of this variable into the test. This was particularly unfortunate because it was the the idea that intrigued students the most and seemed most promising since it fixed problems that plagued the other designs.

The creation of a fully self guided web-based assessment could potentially have resolved the problems encountered when incorporating the interactive diagram into the experiment. This was attempted with what is now the practice test section of the supplementary web site where users are directed to a separate page depending on their test group. This branching allows students access to the main pages of the site where common information is available to all users, but information specific to a test group could only be reached by members of the appropriate test group. Although this worked well with the practice test, the actual test would require saving information from the user and unideal test conditions. For the purposes of this study, it was only reasonable to revert to the classical method of administering the test.

Influential Factors

The single most influential factor in this study was that which clustered the results of the quantitative portion of this study beyond expectations. It was the fact that the student participants have all mastered the basic competencies of algebra so well that there was almost no deviation from the perfect scores achieved on the assessment. Other influential factors included biases for the designs of the visualization methods including the prompt on the survey.

Although the survey was written with the intention of evoking the most uninfluenced variety of responses possible, it is expected that the results would be swayed based on the exact details of the prompt. In this case, the base equation would have a great effect on the students' imagination because it provides a concrete example from which to grow ideas. In order to encourage universal design elements that would be appropriate to any possible equation, the question provides an equation that contains many combinations of expression type nests. It is still limited of course, by the single equation provided and the extraneous details of the prompt.

Implications

From the variety of survey responses with common design elements, one could conclude that there are many similar ways to refine any given visualization method. It may sometimes simply be a matter of preference, but these alternate details may convey important meaning for different people. As an example, students all used different coloring schemes with no clear preference for any single operation or entity type throughout the results. This however, does not necessarily suggest that the color scheme is not important, it simply means that there is no single standard. It has been known that the same colors can arouse different responses from people. It could be possible then, that the customization of such details may allow students to create their own meaning in their designs. Although this study focused on the idea that there could be a visualization method that is more efficient in helping most students, it could be the case that each student has their own preference. This could imply that individually created designs by each student could be the most appropriate solution if plausible.

Limitations

It goes without saying that no single study could ever discover and test all the possible ways to portray equations because it is a limitless search bound only by our imaginations. This study is constrained by the number of people who provided their input and the time it takes to develop the visualization methods. There is an unlimited number of ways to display equations which could all be tested in just as many ways, making this study merely a glimpse into the possibilities which can only be thoroughly examined through an iterative research and development process.

An implementation of these learning devices into classrooms is also an important part of testing the long term effects and retention. The test for effectiveness in this study is limited to one lesson plan. It is possible to optimize the potential effectiveness of any visualization method with an appropriate lesson plan that takes advantage of it's particular strengths.

It should be noted that this project tests only the visual presentation of equation structures as the visuo-spatial portion of the brain has been shown to have the most neurological activity when parsing equations because “...mathematical expressions are quickly parsed at an early visual level in bilateral ventral occipito-temporal cortices.” (Maruyama, Pallier, Jobert, Sigman & Dehaene, 2012). Other factors besides the visual cognition of an equation's structure are also important for the skills required to appropriately utilize mathematical tools and pose valid research subjects.

Recommendations

Improvement. The most critical area of improvement for this study in particular would have been to select a more appropriate sample population for the quantitative experiment. The participants of this test were not suitable as a good sample of the true target population of this study. It is clear from their outstanding test scores that they have already achieved a level of understanding in the subject that does not require any further intervention. The clumping of test results together on one end of the spectrum made it impossible for the experiment to fairly examine the differences between test groups. It is very clear that this sample does not fall within the target population of students who would have performed at more varying levels. The optimal sample of participants would have been composed of students who have not yet become so familiar with the concepts tested such as early algebra or pre-algebra students.

Another area of improvement in this study would be to integrate the visualization methods into the unit of instruction more. Although it would weaken the randomness of the variable group samples and convert the quantitative test to a quasi-experimental design, having students in each test group learn the concepts of algebra through their assigned visualization method may provide more important information of their differences. This experiment would allow one to more thoroughly examine the differenced in performance of the current assessment as well as other differences that may occur during the lessons such as anxiety and engagement in the subject.

Further Investigation. The most auspicious idea presented in this study is the possibility of using an interactive program to demonstrate the nesting of expressions to students struggling with the concept. Using spacial input by mouse clicks or touch, users would be able to focus in on a particular expression within the structural hierarchy in order to simulate the parsing process. With practice it may help students learn to parse equations mentally and use equations more effectively. This method has clear advantages over the static diagrams tested. The cluttering problem experienced in the other diagrams is obsolete because the visual cues are conditional to the single expression of focus. There is also less need to abstract away from the universally accepted method of writing equations and therefore less interference with the body of existing mathematics. A demo of this program can be found at the development test web site at http://johngralyan.appspot.com which will be used in further studies.

Another area of investigation is the possibility that each student may benefit the most from their own customized version of a visualization method. Throughout the interview process, the designs were refined to accommodate the suggestions from the majority of participants. It is impossible however for any designers to agree on the perfect scheme because there is no ideal that suits everyones preferences. Although this study reduced the designs down to only a few different visualization techniques, there is an infinite number of such designs. Each possible detail may convey meaning that is different to each student and should therefore be considered beneficial when used by the appropriate student. Certain core components of the methods must be retained such as how the space is to be used, but details such as color, stroke thickness, texture and other such visual cues are easily interchangeable and potentially meaningful when used correctly. It may be worth creating a program or adding functionality to the interactive diagram where students can customize such details to their own desired attributes.

Master's Degree Experience

During my masters degree experience, I have further developed insight into many areas of education such as instructional design, educational research and the appropriate use of technology in education. With a more thorough understanding of effective instructional design that encourages learning with intended goals, I am more confident in my ability to create an efficient unit of instruction. The skills I have gained throughout the courses of this domain will be very useful when designing any curriculum in the future. These skills are very well complimented by deeper familiarity with the topic of educational research which always pushed the frontiers of the body of knowledge available on the subject. The experience with educational technology production has also allowed me to gain skills that will be very useful when putting all the theory to actual practice. With all these new competencies gained, the field of learning and technology has become a vital part of my area of expertise.

Implementation

The skills I have acquired through this program of study will be of tremendous value in my career path. The advanced theories in future research studies coupled with experience with actual technology can potentially lead to many practical applications of such knowledge. I now feel more comfortable with all the areas of education that may arise in my career. Whether it be the development of a new instructional unit of the creation of a new technological resource, I am now confident in my ability to take any challenge that is set before me.

Project Summary and Conclusion

Throughout the course of this study, a mixed methods approach was used to further research the area of intuitive equation design. In the first portion, two qualitative experiments were used to develop multiple methods of visualizing equations in order to explicitly convey the hierarchical structure of expressions. This nesting of expressions is intrinsic to any equation or expression and understood by the trained individual, but not obvious to those unfamiliar with algebra. The second portion of the study involved a quantitative experiment whereby the new designs would be tested to see if there is any difference when using any of the variable visuals or original equation. After an assessment of the test groups, test scores were collected and analyzed to see if any method made a significant difference overall or on any competency tested.

The analysis of the test results concluded with no significant difference between any of the visualization methods including the classical depiction of the equations. The student scores were all very close together and showed no clear sign of advantage with any method. The post-algebra student sample population was clearly so comfortable with the skills being tested that the majority had a perfect score. The pool of participants was very clearly poorly chosen as they do not represent the target population which should have been unfamiliar with the subject. Further investigation is needed with a more appropriate sample to provide more appropriate data.

The study was not at a complete loss however. The qualitative portion of the study has lead to the creation of a few interesting visualization methods including a novel design for an interactive diagram of equations. This idea seems to be the most promising as it overcame the major problems of the static designs and provides very useful functionality. See the further investigation section of this chapter for more details.

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Appendix A

INFORMED CONSENT FORM

Western Governors University

Master in Education in Learning and Technology
Illuminating the Hidden Structure of Equations
John Gralyan

Introduction

You are invited to participate in a research project being conducted by a graduate study from Western Governors University. John Gralyan is conducting research to determine the most effective ways to teach the structure of algebraic equations.

Description of the project:

- *The purpose of this project is to aid in the development of alternative methods of visualizing the structure of algebraic equations which could help students parse through, understand and solve such equations more effectively.*
- *In the course of a few lessons, students will learn the components of equations in terms of their relation to one another. Students will be randomly be assigned to one of multiple visualization methods to assess the effectiveness of each in helping students learn basic algebra.*
- *The research will be conducted in the students math class and online between May 20-31 2013.*
- *All students are expected to participate fully in all routine classroom activities.*
- *The participant will be required to do a survey and possibly an interview or two short quizzes in addition to regular participation in the curricular activities.*

Benefits and Risks of this study:

The benefits of participating in this study include supplementary instruction in the most commonly problematic areas of elementary Algebra. Regardless of the group they will be assigned to, all students will gain some extra exposure to this very core subject. Possible risks include access to the internet which should be closely monitored to avoid any deviation from the intended educational materials.

Confidentiality:

All participant's confidentiality will be maintained. Records will only be seen by the researcher and all data that is reported will be aggregated. No data collected in this study will affect the students grade, class or school standing in any way.

Voluntary participation and withdrawal:

Students are expected to participate in any regular classroom instruction but may choose to voluntarily participate or withdraw from surveys, interviews or quizzes.

Participants may withdraw at any time from non-regular classroom instruction and will not be penalized for non-participation.

Participants may request that their individual results be excluded from the final report.

Questions, Rights and Complaints:

The researcher can be contacted at any time by the participants or their legal guardians. Please direct all questions to:

John Gralyan
(818) 624-1932
jgralya@my.wgu.edu

Participants and legal guardians have a right to the results of the study.

Consent statement:

In signing this informed consent, participants and legal guardians agree to participate in the research.

______________________	______________________
Signature of Participant	Signature of Legal Guardian
______________________	______________________
Typed/printed Name	Typed/printed Name
______________________	______________________
Date	Date

Appendix B

Questionnaire

As you already know, equations are made up of different parts in a hierarchical structure. For example, consider the following equation:

$$a(2\mathrm{b}+c)=\frac{d-e(f+3)}{4\mathrm{g}}$$

The left side $a(2\mathrm{b}+c)$ is a product of a and $2\mathrm{b}+c$ which is a sum of $2\mathrm{b}$ and c

The right is a fraction $\frac{d-e(f+3)}{4\mathrm{g}}$ of a sum $d-e(f+3)$ of d and a product $e(f+3)$

… and so forth.

This structure of elements inside elements can be very confusing to students who are first learning about equations. Can you draw this equation in a way that makes sense to a new Algebra student? Use the sheets provided to draw the equation in a way that would be understandable by anyone.

Here are some things to think about:

-How would you show the hierarchical nesting clearly without cluttering the page?

-What color, shape or other notation would show an operation like a sum, product or exponent?

-How can it be similar enough to the regular looking equation to see the similarities?

Appendix C

Interview question examples

-How did you show the structure of the equation?

-How did you show the different operations (eg. Sum, product, exponent)?

-How would you show algebraic transformations such as:

Combining like terms

Distribution

Factoring

Cancellation

Solving

-Does this visual make algebraic transformations more clear?

-Is this visual similar enough to the classic that the similar elements are easily distinguished?

-Would this have made more sense to you when you were first learning Algebra? Why?

-Do you think other students would learn better using your visual than the classic? Why?

-What do you think about these other visuals your classmates have made?

-Which would you say is most useful to beginning algebra students? Why?

-Is there anything else that you think might help a beginning algebra student?

Appendix D

Qualitative Study Assessments

Unit Test

Version: Control

Simplify each expression

1) $5a-6b+3b$

2) $\dfrac{5a}{a(5-a)}$

Factor

3) $3a+3b$

Distribute

4) $5(a+3)$

Solve each equation for x

5) $4-x=24$

6) $3x-6x=18$

Solve each equation for x in terms of y

7) $12y-3x=9$

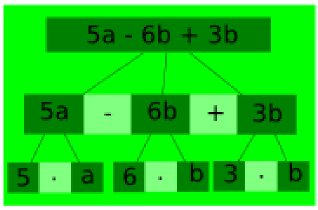
8) $8y+4x=-12$

Unit Test

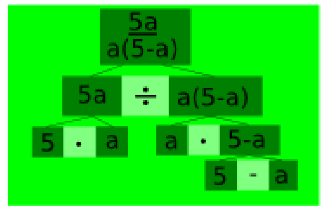
Version: Tree

Simplify each expression

1)

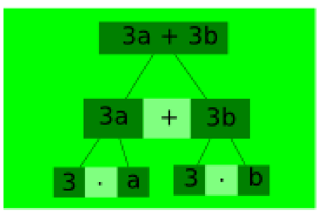


2)



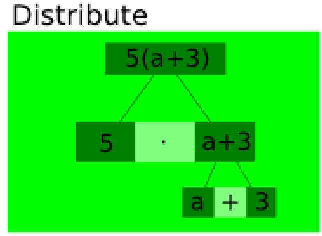
Factor

3)



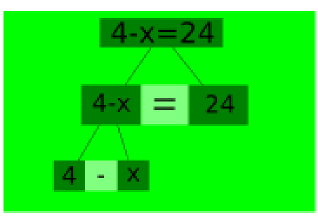
Distribute

4)

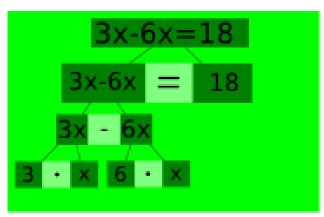


Solve each equation for x

5)

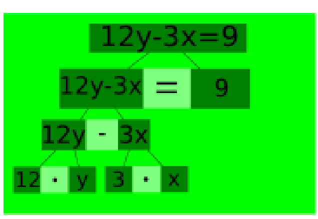


6)

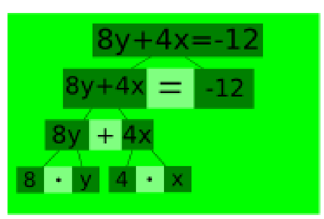


Solve each equation for x in terms of y

7)



8)



Unit Test

Version: Ovarlay

Simplify each expression

1) $5a - 6b + 3b$

2) $\frac{5a}{a(5-a)}$

Factor

3) $3a + 3b$

Distribute

4) $5(a+3)$

Solve each equation for x

5) $4-x=24$

6) $3x - 6x=18$

Solve each equation for x in terms of y

7) $12y-3x=9$

8) $8y+4x=-12$

Unit Test

Version: Stack

Simplify each expression

1) $5a - 6b + 3b$

$5a - 6b + 3b$

2) $\frac{5a}{a(5-a)}$

$\frac{5a}{a(5-a)}$

Factor

3) $3a + 3b$

$3a + 3b$

Distribute

4) $5(a+3)$

$5(a+3)$

Solve each equation for x

5) $4-x=24$

$4-x=24$

6) $3x-6x=18$

$3x-6x=18$

$3x-6x=18$

Solve each equation for x in terms of y

7) $12y-3x=9$

$12y-3x=9$

$12y-3x=9$

8) $8y+4x=-12$

$8y+4x=-12$

$8y+4x=-12$

Appendix E

Curriculum Unit

General Description

Many sources including standardized test score data show a great need for more substantial instruction of Algebra, the pre-requisite to most math and science classes (CDE, 2011). Prior research in the area of Algebra education has shown a great deal of misunderstanding of the order of precedence in Algebraic expressions (Glidden, 2008; Pappanastos & Hall, 2002; Ameis, 2011). Students don't seem to be grasping the basic concepts of equations due to confusion in it's structure. The structure of equations is never explicitly taught however, it is expected to be understood through experience and practice. This is the main focus of this instructional unit because it seems to have the most potential of positively effecting the students understanding of Algebra.

Instructional Goal

This instructional unit has been created to give learners a thorough understanding of the structure of equations. Topics of interest will cover a review of basic components that are constructed into expressions, relationships between these entities and the structure behind the order of precedence. Students will then use this knowledge to better understand simplification of expressions and finally to solve simple equations. It should be noted that this unit is not intended to replace any Algebra course, but rather as a compliment to one. Any and all parts of this unit may freely be incorporated into other units.

Intended Audience

Success in the instructional unit requires knowledge of prerequisite math concepts to build on and use correctly in the appropriate situations. These basic concepts include simplification of elementary number operations, like terms, exponents, and a basic understanding of the distribution property and simple factoring. Elementary operations are often taught in elementary schools and should be mastered by students very early in their math careers. It is not a prerequisite of much concern because students usually have a good working understanding of these requirements by the time they reach middle school. Simplification of like terms can be a bit more difficult for students and is often taught in pre-algebra. Students often have a good grasp of this ability by the time they enter Algebra 1. The exponents and factoring are the most advanced of these subjects. These concepts are often introduced in Algebra 1 to advance the students ability to simplify complex expressions. Even though a simplified form of the instructional unit can be made excluding these components, the full instructional unit would include all these components to thoroughly explain the full structure of almost any algebraic equation. It should be noted that differential equations, equations with non-real components and infinite sums will not be addressed in this instructional unit as these topics are much more advanced than the scope of this instruction and will not be expected as prerequisite knowledge.

Length

This unit of instruction in intended to be given is a classroom setting over the course of seven separate lessons lasting approximately one hour each. The unit may be extended if the students are lacking the prerequisites which would be noticeable by the end of the first lesson if not clear from the pre-test. Although much of the earlier material is review, it is important to go over the equation components so the student can get a sense of their similarities and differences in terms of how they impact the equations structure.

Delivery Approach

Depending on the technology available in the classroom, the delivery of instruction may be done in one of two ways. If a classroom set of computers is not available, the traditional method of delivering content through a lecture format and allowing students to practice problems from a book will be necessary. This can of course be done in a number of ways such as individually, in small groups or all together depending on the classroom dynamic. The use of technology in this case would be limited to work outside class where the students may have access to computers such as at home or the library.

Classes with a classroom set of computers will be able to use the technology based tools available during the lessons and can therefore take a slightly more independent approach to learning. Students may choose to further research the topic of discussion when the fundamental concepts are not quite clear to them, allowing them to supplement the content delivered though lectures. Students may also choose to practice problems from a multitude of different places, allowing them to stay in a difficulty range which is still difficult but not overbearing to them.

Instructional Sequence

The following is a hierarchical list of the topics to be addressed in this unit of instruction:

1 Structure
  1.1 Elementary Components

- 1.1.1 Constants
- 1.1.2 Variables
- 1.1.3 Operators
- 1.2 Clusters
  - 1.2.1 Expressions
    - 1.2.1.1 Sums
    - 1.2.1.2 Products
      - 1.2.1.2.1 Term
      - 1.2.1.2.2 Parentheses
    - 1.2.1.3 Fractions
    - 1.2.1.4 Exponents
      - 1.2.1.4.1 Roots
    - 1.2.1.5 Elementary Functions
- 2 Simplification
  - 2.1 Combining Like Terms
  - 2.2 Distribution
  - 2.3 Factoring
  - 2.4 Cancellation
- 3 Equations
  - 3.1 Direct Variation
  - 3.2 Inverse Variation
  - 3.3 Joint Variation
  - 3.4 Solving
    - 3.4.1 Numerical Evaluation
    - 3.4.2 Relational Evaluation

Material list for the unit

Depending on the supplies available in the classroom, either of the following groups of materials will suffice. If a class set of computers is not available, students will be required to use the technology based tools at home or at the library.

	Computer Based	**Paper Based**
Teacher	Computer with projector	Whiteboard/Chalkboard
	Any basic digital paint program	Colored markers/chalk
		Eraser

Students	Individual computers	White unlined paper
	Any basic digital paint program	Colored pencils
	Any Algebra book or program with examples	Any Algebra book

Lesson Overviews

This instruction unit aims to help learners gain the ability to perform mostly intellectual skills - those which involve the ability to use previous knowledge to create solutions to new problems not previously encountered (Gagné 1985). Although some verbal skills will be reviewed, it is assumed that most of this knowledge has already been mastered in pre-requisite math courses. Specifically, students will learn math concepts that they will then use as tools to eventually solve equations.

Since the lessons are given in smaller increments at a time, they have been broken into seven lessons equal in length and difficulty. The following is an overview of each lesson:

Lesson 1 – Components of Equations

1. Structure
    - 1.1 Elementary Components
        - 1.1.1 Constants
        - 1.1.2 Variables
        - 1.1.3 Operators
    - 1.2 Clusters
        - 1.2.1 Expressions
            - 1.2.1.1 Sums
            - 1.2.1.2 Products
                - 1.2.1.2.1 Term
                - 1.2.1.2.2 Parentheses

Objectives:

Given an algebraic expression or equation, students will be able to locate and distinguish all the constants contained 100% of the time.

Given an algebraic expression or equation, students will be able to locate and distinguish all the variables contained 100% of the time.

Given a linear algebraic expression or equation, students will be able to locate and distinguish all the basic operations contained 100% of the time.

Given an algebraic expression or equation, students will be able to locate and distinguish all the sums and their respective constituent terms 100% of the time.

Lesson 2 – Expression Patters

1.2.1.3 Fractions
1.2.1.4 Exponents
  1.2.1.4.1 Roots
1.2.1.5 Elementary Functions

Objectives:

Given an algebraic expression or equation, students will be able to locate and distinguish all the fractions and their respective numerator and denominator 100% of the time.

Given an algebraic expression or equation, students will be able to locate and distinguish all the exponentiation operands 100% of the time.

Given an algebraic expression or equation, students will be able to locate and distinguish all the radical operands 100% of the time.

Given an algebraic expression or equation, students will be able to locate and distinguish all the elementary functions including trigonometric, logarithmic and factorial functions contained 100% of the time.

Lesson 3 – Introduction to Simplification

2. Simplification
   2.1 Combining Like Terms

Objective:

Given an algebraic expression or equation containing like terms, students will be able to simplify by combine all like terms whenever possible 90% of the time.

Lesson 4 – Distribution and Factoring

2.2 Distribution
2.3 Factoring
Objectives:

Given an algebraic expression or equation containing a product of a sum and a term, students will be able to use the distributive property whenever possible 90% of the time.

Given an algebraic expression or equation containing a factorable polynomial, students will be able to factor out like terms whenever possible 90% of the time.

Lesson 5 - Cancellation

2.4 Cancellation

Objective:

Given an algebraic expression or equation containing a fraction with cancel able terms, students will be able to cancel the appropriate terms whenever possible 90% of the time.

Lesson 6 – Equation Overview

3. Equations
   - 3.1 Direct Variation
   - 3.2 Inverse Variation
   - 3.3 Joint Variation

Objectives:

Given a simple algebraic function, students will be able to categorize each variable as varying with either a direct, inverse or joint relationship 100% of the time.

Given an algebraic expression or equation, students will be able to distinguish it as either an expression or equation 100% of the time.

Given an algebraic expression or equation, students will be able to analyze and distinguish its constituent components 80% of the time.

Lesson 7 – Solving Equations

- 3.4 Solving
  - 3.4.1 Numerical Evaluation
  - 3.4.2 Relational Evaluation

Objectives:

Given a linear algebraic equation with a single variable, students will be able to manipulate the equation's structure to isolate and numerically solve for the variable 80% of the time.

Given a linear algebraic equation with multiple variables, students will be able to manipulate the equation's structure to isolate and symbolically solve for a given variable 80% of the time.

Plan of Instruction

Title: LESSON 1: Elementary Components of Equations

Lesson Overview: This very brief lesson will familiarize students with the most basic components of any mathematical equation's. The concepts of constants, variables and operations will be explained in detail in order to give students a basis for the relationships between these entities discussed in subsequent lessons.

Resources Needed: Whiteboard/Chalkboard, markers/chalk and an eraser. Any other method of display would also suffice such as a computer with projector or overhead projector. Paper and pencils for students. Any Algebra 1 book or resource with practice questions. Students can use the accompanying website http://algebra.scienceontheweb.net during or after class.

Lesson Objectives :

- Given an algebraic expression or equation, students will be able to locate and distinguish all the constants contained 100% of the time.
- Given an algebraic expression or equation, students will be able to locate and distinguish all the variables contained 100% of the time.
- Given a linear algebraic expression or equation, students will be able to locate and distinguish all the basic operations contained 100% of the time.

Time : Approximately 0.5 hour

Step 1: Pre-instructional activities:

- Give an example of a problem involving a variable
- Example: Jody has 5 gold stars in class. Suzy has 3 more gold stars than Judy. How many gold stars does Suzy have?

• Explain that since we are told how many gold stars Judy has, this number is considered constant because it will not change for the duration of the problem. The other constant present is the number of additional gold stars that Suzy has than Judy. This also shouldn't change in the time period considered. The number of gold stars that Suzy has however is unknown to us. Even though this number isn't changing during the problem, it is not considered a constant because it could be anything until we solve the problem. This number is therefore considered a variable. These constants and variables are related to each other in such a way that describes the model presented. The relationships can be specified by operations which connect all the pieces together.

Step 2: Content presentation:

• We will be discussing some of the different types of basic constants, variables and operators. Show expressions containing these components and distinguish them as you go. If on a board, you could circle, box, or underline components. If on a computer, you could outling components in different colors on a basic painting program.

• Constants are already very basic, they are essentially what we commonly refer to as numbers. We are all familiar with simple integers since elementary school, but it is important to note that there are other ways to write numbers out there. Review the ideas of negatives, decimals, and fractions with your students.

• There are also constants that are too big to write down. For example, imagine trying to write a number that described how many particles there are in the universe. This number would be so big that you couldn't write it on a single page. For this reason, some constants you encounter may look different. They are usually in the form of scientific notation which looks like this: 5x10^100. This is 5 googols which is equivalent to writing 5 with s hundred zero's behind it. We wont be discussing scientific notation in detail, just remember that these are just numbers and can be treated like any other number.

- Another basic component of math is variables which are very similar to constants except they are unknown. They represent some number but since we don't know what number it is, we write a letter to take its place. The most common way of representing a variable is with the letter x. If an unknown number is ever encountered in a problem, you could always write x to take its place until you find out what it is.
- The component that relates these components together is the operator. In previous classes you encountered operations such as addition, subtraction, multiplication and division. These are the most basic operations which link two constants or variables together. It is important to remember that operators don't represent any value the way constants and variables do, they only relate these values together.

Step 3: Learner Participation:

- Students will take the problem from the beginning of the lesson and write it using mathematical symbols instead of words.
- Find an array of different equations and expressions throughout the math book or web program. Allow students to break up into small groups to discuss the symbols making up the equation. Students should be able to distinguish the constants, variables and basic operators in each equation or expression.

Step 4: Assessment:

- With the list of equations and expressions still at hand, assign them the task of distinguishing all the components in a different manner.

- Use a specific convention such as: circle all the constants, put a box around all the variables and underline all the operations. This will hopefully allow them to see the constants and variables as valued entities while the operations as relational symbols.
- You could also choose a coloring convention if the students have colored pencils at home.

Step 5: Follow-through activities:

- Use this convention throughout the unit of instruction to remind students what they are working with more visually. These visual cues may help students see the hierarchical structure of equations beyond the linear string of symbols. This convention can be used as a helpful crutch when students are first exploring the structures of expressions and their proper manipulations.
- Since students already get used to performing algebra by rules without understanding the reasoning, these visualizations may help them procedurally as well as making the connection to the reasoning behind the manipulations.

Title: LESSON 2: Expression Patterns

Lesson Overview: The topic of structure within equations is first seen in this lesson. We discuss the basic types of expressions in order to develop an understanding of these patters. The elementary components in the previous lesson will be related by sums, terms, fractions, exponentials and radicals.

Resources Needed: Whiteboard/Chalkboard, markers/chalk and an eraser. Any other method of display would also suffice such as a computer with projector or overhead projector as long as there is freedom to freely draw. Paper and pencils or computers with a basic paint program for each student. Any Algebra 1 book or resource with practice questions. Students can use the accompanying website http://algebra.scienceontheweb.net during or after class.

Lesson Objectives :

- Given an algebraic expression or equation, students will be able to locate and distinguish all the sums and their respective constituent terms 100% of the time.
- Given an algebraic expression or equation, students will be able to locate and distinguish all the fractions and their respective numerator and denominator 100% of the time.
- Given an algebraic expression or equation, students will be able to locate and distinguish all the exponentiation operands 100% of the time.
- Given an algebraic expression or equation, students will be able to locate and distinguish all the radical operands 100% of the time.
- Given an algebraic expression or equation, students will be able to locate and distinguish all the elementary functions including trigonometric and logarithmic functions contained 100% of the time.

Time : Approximately 1.5 hours

Step 1: Pre-instructional activities:

- Look at the assessment from the last lesson. Many of the symbols have been distinguished since they are the basic building blocks of algebraic equations. Some of the symbols however will be new to students. This lesson will introduce the other common symbols and the patterns of expressions they make up.

Step 2: Content presentation:

- The first structure we will consider is the sum. A sum is the addition or subtraction of groups. Every time you see an addition or subtraction sign, both groups on either side of the operation are considered parts of the sum. A sum can consist of many operands such as 1 + 2 + 3 + 4 which is a sum of all the consecutive numbers 1-4 inclusive. The groups between operands are considered terms and can be a bunched up mess of products and fractions such as 2a + 3ab + 6a/5b. Each of the groupings between addition signs are terms. In a sum, the addition or subtraction operator should be considered an impenetrable barrier between terms, which is why the spacing around them is often wider.
- Another structure is the fraction students should already be familiar with. The parts of this structure are the numerator and denominator. It should be noted that a fraction and a division are actually the same thing. Cutting a pie into a fraction for example is the exact same thing as dividing it by the same amount.
- A new operation called the exponential is also a structure in which the base is multiplied by itself the same number of times as the exponent. The exponent is distinguished as a smaller component at the upper right corner of the base.

- The opposite of the exponent is the radical which has a very unique distinguishing symbol covering the base (draw it). This is ofter a square root, using a radical of 2, which is often omitted but can be anything.
- The last structure we will consider are basic functions. These are the only unary operators we will consider, meaning that they only operate on one component at a time (instead of two as is the case with the rest of these operators). Basic functions are those that are common enough to deserve their own symbols. They are usually a set of letters with parentheses Around the operand. Examples of this would be sin(x), cos(x), tan(x), log(x), ln(x).

Step 3: Learner Participation:

- Ask the students to look at the expressions and equations from the previous lesson and use the new knowledge from this lesson to distinguish the other components of the expressions and equations. They should now be able to distinguish the entire sum and terms as opposed to just the operators.
- Allow them to discuss in groups what the symbols might mean and how the equations and expressions are structured.
- Be sure to clarify any misconceptions and questions they may have.

Step 4: Assessment:

- By this point, the students should be able to distinguish the different components of most expressions and equations. They should be able to distinguish the structure and relationships behind all the expressions as well. This assessment should test their ability to analyses these parts of expressions. Assemble a list of expressions that contain the components discussed.
- Ask the students to perform a similar analysis as the previous lesson with the new

knowledge attained. Other conventions can be made to take into account he new content in this lesson.

- Use the performance objectives to assess the level of understanding of the subject.

Step 5: Follow-through activities:

Much like the last lesson, the information presented in this lesson will be used throughout the rest of this unit and future math classes the student might take. Encourage the student to use their understanding of the structure behind the equation in their future use of equations. This knowledge should help them make sense of the next few lessons in particular.

- Always refer back to the structure when introducing new manipulations of equations or when giving feedback on the material.

Title: LESSON 3: Combining Like Terms

Lesson Overview: The sum expressions will be discussed in more detail in this lesson as we first encounter the topic of manipulation of expressions with variables. Students will be introduced to the concept of like terms and how those in the same sum can be combined.

Resources Needed: Whiteboard/Chalkboard, markers/chalk and an eraser. Any other method of display would also suffice such as a computer with projector or overhead projector as long as there is freedom to freely draw. Paper and pencils or computers with a basic paint program for each student. Any Algebra 1 book or resource with practice questions. Students can use the accompanying website http://algebra.scienceontheweb.net during or after class.

Lesson Objectives :

- Given an algebraic expression or equation containing like terms, students will be able to simplify by combine all like terms whenever possible 90% of the time.

Time : Approximately 1 hour

Step 1: Pre-instructional activities:

- This will be the first lesson in which we will be discussing the manipulation of expressions. Remember to refer back to the structure when thinking about these manipulations to stay grounded in the structure. Explain to the students the idea of simplifying an expression. Tell them to look at all the equations they had to use in the previous lessons and think about how nice it would be if we could simplify these equations to smaller, simpler, more manageable ones. This is only possible with the power of simplification. The most common type of which will be combining like terms

Step 2: Content presentation:

- The definition of a term should be re-instated here. Also show some different terms to look at such as 4a, 2xy, 4x/3a. Also explain that a term must be any product or fraction at the most top level of the structure.
- Next, explain the idea of like terms. Like terms are and two terms that have some similar component. Examples are 4a and 3a where the a is shared between the two terms. Another more complex example could be axy/3 and 2xy/3 in which the xy/3 is shared between the terms. You could choose to skip down to the learner participation to give students a chance to work play with like terms first hand. Before presenting the idea of combining like terms
- Refer back to the idea of simplifying like terms. Give them an example of two terms with real models to think about.
- Example: A box has 5 marbles, a bag has 7 marbles. How many marbles are there.
- This is a very simple example that clearly demonstrates the combination. In this case, using marbles as the variable "m" we could write 5m + 7m which could be simplified to 12m. Remind students that "m" could be anything including units, numbers, or even other structures that are part of the term. Show a few more examples of combining like terms such as:
- 3xy + 2xy = 5xy
- 5(6+x/3) + 9(6+x/3) = 14(6+x/3)
- x5a + 2ax = 5ax + 2ax = 7ax

Step 3: Learner Participation:

- Tell students to look through the equations from previous lessons and look for like terms. Students could work in groups on this exercise with a challenge between groups. One example of a challenge with an incentive is to tell students that the first group who find all the pairs of like terms in the entire list gets to be known as group number one.

- Find the chapter in the book related to combining like terms and decide on a list of problems that are challenging enough for your students. Let students work individually to solve all these problems. This work shouldn't be done in groups because some students find it much easier than others.

Step 4: Assessment:

- Give the students a similar assignment to the individual one assigned during the learner participation. This task should involve the simplification of many different types of terms to get students comfortable with a wide range of terms.
- It should be noted that this assessment, much like the group work, should be done individually so the level of each student can be accurately determined.

Step 5: Follow-through activities:

- Tell students to reflect on the lesson and how it fits into their toolbox of knowledge about math. Explain to them that they can now not only perform operations with simple numbers, but entire clusters as well. They can simplify large equations into simple, compact ones that they could more easily analyze.
- Also note the idea that even though this new tool allows them to re-organize the structure of an equation, it does not change it as a model, it only represents it in a more compact version.

Title: LESSON 4: Distribution and Factoring

Lesson Overview: This slightly more advanced topic will be introduced to students in order to further their understanding of the manipulation of equations. Only factoring and distributing like terms will be discussed because doing so with quadratics and binomials respectively is beyond the scope of this lesson. In this lesson in particular, the product with a sum will be discussed to show the distributive property. The reverse of this property, factorization, will also be discussed, giving the students the ability to interchange the structure as appropriate for the problem.

Resources Needed: Whiteboard/Chalkboard, markers/chalk and an eraser. Any other method of display would also suffice such as a computer with projector or overhead projector as long as there is freedom to freely draw. Paper and pencils or computers with a basic paint program for each student. Any Algebra 1 book or resource with practice questions. Any Algebra 1 book or resource with practice questions. Students can use the accompanying website http://algebra.scienceontheweb.net during or after class.

Lesson Objectives :

- Given an algebraic expression or equation containing a product of a sum and a term, students will be able to use the distributive property whenever possible 90% of the time.
- Given an algebraic expression or equation containing a factorable polynomial, students will be able to factor out like terms whenever possible 90% of the time.

Time : Approximately 2 hours

Step 1: Pre-instructional activities:

- In the last lesson, we learned about combining like terms. These problems usually had a term that involved two terms with a similar group of variables and different number such as

- $4x + 2x = 6x$
- Now we will be expanding this idea to include all terms, with any like group and and different group. Take for example, the following simplification:
- $4a + 4b$
- In this case, the variables are not the same, they are different. The coefficients however are the same. It's almost as if the variables and coefficients have switched roles. Take a minute to think about how this problem simplifies.

Step 2: Content presentation:

- In the problem given earlier, we will use combining like terms as an analogy to solve the new version of the problem. Let's look at the sample problem in a little more detail:
- $4x + 2x$
- When we notice that the variables are similar, we know we will be combining these variables into one. The coefficients will be added together in the simplification. We could write an intermediate step before the complete simplification as such
- $(4+2)x$
- Which simplifies to $6x$
- Now lets see if we can perform the same analysis with the new expression:
- $4a + 4b$
- In a similar but opposite manner, we will consolidate the similar component (which in this case would be the coefficient) and we will add the different components together (which in this case would be the variables). If you did the sample problem in the pre-instructional activity correctly, you should have found this simplifies to:

- $4(a+b)$
- This simplification is much like combining like terms, but is actually part of a different type of simplification called factoring. Factoring a similar component out of any polynomial is possible, for example:
- $ya + yb + yab= y(a + b + ab)$
- $2ax + 4ay + yx = 2a(x + 2y) + yx$
- Distribution is the exact opposite of factoring. It has its name from the idea that you are distributing a component over a sum of terms. You can imagine the idea of factoring is analogous to collecting books from students since it is the only thing students have the exact same of. Distribution in this case would be to distribute books to all of the students again. You can distribute any product of something and a sum. It's also possible to distribute two sums with one another but we will not be discussing that here. In order to distribute a term into a sum, simply multiply that term with each term in the sum. For example:
- $y(a + b + ab) = ya + yb + yab$
- $2a(x + 2y) + yx = 2ax + 4ay + yx$
- Notice that this example is the exact opposite of the factoring examples

Step 3: Learner Participation:

- Find the chapter in the book involving an introduction to simplification by factoring and distribution. Look only for the lessons on factoring and distributing like terms. Factoring quadratics and distributing binomials is a slightly more advanced topic and thus, beyond the scope of this lesson.
- Allow the students to work in groups to discuss and simplify these expressions together.

They should be assigned only a few problems from each part, factoring and distributing. While going around to each group, remind them that the two skills being learned are the opposites of one another. Encourage them to look at the reverse after they complete each problem.

Step 4: Assessment:

- Find a list of practice problems much like the ones in the learner participation section to assign to the students. The students should be able to factor out components from any like terms and distribute components into any sum of terms.
- Use the lesson objectives to assess their ability to do so. If they have acquired skills and able to perform both factoring and distributing of like terms at a 90% proficiency level, go on to the next lesson. If not, review the common mistakes made in this lesson.

Step 5: Follow-through activities:

- Review the skills of combining like terms, factoring and distributing to give students a sense of how they relate to one another. These skills are very well intertwined and will be used throughout the problem solving process to manipulate the structure in strategic ways so they can be solved.
- Remind the students that these simplification techniques are not a means to their own end, but simply tools they will use to solve complex problems. These skills will be expanded to more advanced manipulations and mastered through practice as they will be seen throughout the rest of their math and science classes.

Title: LESSON 5: Cancellation

Lesson Overview: One of the most powerful simplification techniques will be the topic of this lesson. Cancellation within a fraction will be an important skill when simplifying. This powerful tool will make it very clear why factorization is an important skill.

Resources Needed: Whiteboard/Chalkboard, markers/chalk and an eraser. Any other method of display would also suffice such as a computer with projector or overhead projector as long as there is freedom to freely draw. Paper and pencils or computers with a basic paint program for each student. Any Algebra 1 book or resource with practice questions. Students can use the accompanying website http://algebra.scienceontheweb.net during or after class.

Lesson Objectives :

- Given an algebraic expression or equation containing a fraction with cancel able terms, students will be able to cancel the appropriate terms whenever possible 90% of the time.

Time : Approximately 1 hour

Step 1: Pre-instructional activities:

- The past few lessons have introduces different methods of manipulating algebraic expressions. In this lesson we will discuss the property of expressions that most truly allows for simplification, namely cancellation. In order to understand this concept, lets think about an analogy. Imagine a situation in which there has been a mathematical over complication without justification.
- An example of this would be something like, if two friends owe each other money. In this case, there is no point in making two large transactions when the friend who owes more could just pay the difference.

• Another example would be counting something individually when it never comes individually. This could be a shoe store counting by the individual shoe instead of by the pair. A much more silly example would be counting rice by the grain rather than by a more reasonable unit of measure.

• On a more humerus note, one anecdote describes a situation where a soda pop company decided to add twice the ingredient into their product in order to increase the concentrated flavor of the soda pop. Upon concerns from health conscious individuals, the company decided to make a diet version whereby the product was diluted to half the concentration. This is an example of cancellation of the effort made to change the concentration of their product.

• Of course we don't actually do these things, we simplify it so the math is more manageable. This sounds obvious in real examples but is not so obvious when encountering a new mathematical model. Let's see how cancellation can simplify expressions.

Step 2: Content presentation:

• The most simple form of cancellation that you are already familiar with is the nullification by subtraction. In the example stated earlier, if two friends owe each other $5, then they should agree that both debts cancel out. Mathematically, you would see this as $5 - 5 = 0$. In a multiplicative example, multiplying anything by 0 will result in 0, even if it's a complex expression with other variables. Similarly, we know that multiplying anything by 1 is useless as it will always result in the other operand.

• The last examples are basics you already know, but the one form of cancellation that will come in handy the most is probably a new idea to you: cancellation in a fraction. Let's say for example that a shoe company starts out counting shoes by the single rather than by the pair. In this case, they would be double counting every pair of shoes. If instead they moved to a system whereby they would count the shoes by the pair, they would cut their counts in half and therefore simplify all their math.

- Lets look at a realistic example of this. Lets say a company decided that 12 out of their 16 products should be red. This may get confusing when ordering the colored dyes so they could simplify the fraction by dividing the numerator and denominator by the common denominator, thus 12/16 = 3/2. Now they can more easily keep track of this fraction in the ordering department.
- Simplifying fractions of numbers is the same as with terms. Here are some examples:
- $2a/2b = a/b$
- $xy/5zx = y/5z$
- $x(2 + a)/(2 + a) = x$
- In the last example, we see the use of having the numerator in factored form. The distributed form $(2x + ax)/(2 + a)$ wouldn't be as easy to simplify without factoring it first.

Step 3: Learner Participation:

- In this session we will be doing a slightly different activity than just drill practices. Although practice will also be important in the mastering of the subject, it is also important to see the significance of simplifying mathematics in daily life by cancellation.
- Ask the students to get into groups and discuss amongst themselves any place they've encountered a need for a complete or even partial cancellation. A partial cancellation would be something like a store instant rebate where the price is purposefully obfuscated to confuse the customers.

- Ask each group to share the most interesting stories with the class. This may prove to be a difficult assignment for students who can't think of any examples, but it should be a lifelong lesson that understanding the math you're using is an important skill.

Step 4: Assessment:

- The learner participation section didn't involve any explicit drill practice problems because the assignment stated was a more important life lesson and because the material should be simple enough to understand and perform without too much practice. This assessment however, will not be verbal as the learner participation exercise because it's still important to be able to perform cancellations.
- Find the section in the Algebra book involving cancellation. The students should have had enough experience with manipulating simple expressions by now to be able to accurately cancel all the expressions appropriately.
- Be sure to check each problem yourself to see if any require knowledge such as manipulations not yet covered.

Step 5: Follow-through activities:

- The learner participation exercise should have allowed the students to explore and share real life situations where simplification of the underlying math would make the situation more easily manageable. To keep this same idea throughout their real life experience with math, encourage the students to simplify everything and anything they can. The smallest adjustments in a mathematical model could make a complicated mess into a more easily understandable model. Of course, this idea doesn't just stop at math, but it's the most easy to prove a simplification in mathematical terms. This type of thinking could make the most impact of all the information in this instructional unit.

Title: LESSON 6: Equations Overview

Lesson Overview: This lesson finally gives equality to expressions, allowing students to display relationships between variables. This is the most top level piece of the structure of equations which finally allows one to calculate answers to problems. The concept of functions and its common relationships will also be discussed to complete the idea of equation's.

Resources Needed: Whiteboard/Chalkboard, markers/chalk and an eraser. Any other method of display would also suffice such as a computer with projector or overhead projector as long as there is freedom to freely draw. Paper and pencils or computers with a basic paint program for each student. Any Algebra 1 book or resource with practice questions. Students can use the accompanying website http://algebra.scienceontheweb.net during or after class.

Lesson Objectives :

- Given a set of variables with relationships, students will be able to develop an equation describing all the relationships 70% of the time.
- Given a simple algebraic function, students will be able to categorize each variable as varying with either a direct, inverse or joint relationship 100% of the time.

Time : Approximately 1.5 hour

Step 1: Pre-instructional activities:

- Teacher: "In this unit we will be learning about equations. Some problems like the ones you've been working on can be solved without the use of equations, but in more complex situations, equations will prove to be a very helpful."
- Give an example of a complex word problem requiring an equation to solve.
- Example: You car gets 28 miles per gallon. Your driving average is about 150 miles

per week. A new car came out recently that boasts the fuel economy of 63 miles per gallon. Your car dealer says that you could trade in you car and an extra $5,000 for the new car and it would "pay for itself". If gas prices are expected to stay the same at $3.56 per gallon, how long would it take for the savings to cover the expense.

- Explain to the students that even though this problem can be solved in multiple ways, setting up an equation will be an immense help. Real life problems like this will come up, it would be wise of you to solve them to make the right decisions such as whether or not getting the new car is really going to bring you any savings.

Step 2: Content presentation:

- Give a word problem to students demonstrating the importance of variable relationships is a function.
- Example: Keeping turtles is hard work. Every turtle eats five pellets every day. Pellet bags come in 100 pellets per bag. If you had 2 turtles in your tank, how many pellet bags would you need this month? Year? Lifetime of turtle (5 years)?
- Discuss the different ways to solve this problem. The easiest way is to use simple arithmetic and logic. Solve the problem for each of the three time intervals.
- 30 days * 2 turtles * 5 pellets/turtle*day * 1 bag/100 pellets = 3 bags
- 365 days * 2 turtles * 5 pellets/turtle*day * 1 bag/100 pellets = 36.5 bags
- 1825 days * 2 turtles * 5 pellets/turtle*day * 1 bag/100 pellets = 182.5 bags
- Discuss the fact that this method, although effective, is very slow and redundant. We did the dame steps in all three iterations with only the time interval changing. Instead it would be much more efficient to create a function that would allow us to do this for any

time interval. In this case, we could write the number of bags needed (B) as a function of the number of days (D) giving us:

B = D*2*5/100

or in its simplified form

- B = D/10
- Also discuss the basic relations in a function which includes:
- Direct relationship: f(x) = kx where the function directly relates to the variable, that is, they vary the same proportionally to one another.
- Inverse relationship: f(x) = k/x where the function inversely relates to the variable, that is, they vary in opposing directions - as one goes up, the other goes down - proportionally.
- Joint relationships: f(x, y) = kxy where the variables are inversely related to one another.
- These examples include an arbitrary constant k.

Step 3: Learner Participation:

• The students will now try to solve the problem presented in the pre-instructional activity. Explain that this is a common problem in real life. “Your decisions are your own but should be based on concrete analysis. Try to solve this problem to see is the car dealers justification of 'it'll pay for itself' is really justifiable.”

• Allow the students to work in groups to solve the problem and discuss the methods they used to solve the problem.

• One slow method would be to find the savings per week and then see how many weeks adds up to a $5,000 savings such as:

• 3.56 $/g * 1 g/28 mi * 150mi/wk = 19.0714 $/wk

- 3.56 $/g * 1 g/63 mi * 150mi/wk = 8.4761 $/wk
- Savings per week: 19.0714 $/wk – 8.4761 $/wk = 10.595 $/wk
- 5,000 $ * 1wk/10.595 $ = **471.9 weeks**
- The other method is to develop an equation. Allow the students ample time to develop their answers and discuss their findings.

Step 4: Assessment:

- Find the lesson in the book dealing with functions. Assign all the word problems to the students asking them to create an equation that describes the model presented. The students should be able to develop a single equation that represents the system in question. This is the first performance objective which is expected to be done with at least 70% accuracy. The correctness of the response will be whether or not the equation created is synonymous with the actual equation describing the model.

Step 5: Follow-through activities:

- In order to gain a first hand, real life perspective of the use of functions, students will be asked to create a function describing something important in their own life they could use. They should be able to create a mathematical model of some simple common phenomenon in order to better understand it. They will also need to classify the relationships of the variables as direct, inverse or jointly varied. The examples seen in this lesson should be of great help. Give some examples around the classroom to help them see this.
- Example 1: The time until the school lunch hour as a function of the current time. Note that this only works on a 24 hour clock. If lunch was at 12:00, the function would be:
- Time until lunch(hr) = current time(hr) – 12(hr)

- Example 2: The amount of water you should bring to school as a function of the outside temperature. This might look something like:
- Amount of water to bring (mL) = 200 (mL/C) * Temperature + 20C

Title: LESSON 7: Solving Equations

Lesson Overview: With a thorough introduction to the structure of equations, and their manipulations, students will now be able to develop their skill of solving problems. This lesson will introduce students to isolating variables to numerically evaluate equations with a single variable and solve it as a function in equations with multiple variables.

Resources Needed: Whiteboard/Chalkboard, markers/chalk and an eraser. Any other method of display would also suffice such as a computer with projector or overhead projector as long as there is freedom to freely draw. Paper and pencils or computers with a basic paint program for each student. Any Algebra 1 book or resource with practice questions. Students can use the accompanying website http://algebra.scienceontheweb.net during or after class.

Lesson Objectives :

- Given a linear algebraic equation with a single variable, students will be able to manipulate the equation's structure to isolate and numerically solve for the variable 80% of the time.
- Given a linear algebraic equation with multiple variables, students will be able to manipulate the equation's structure to isolate and symbolically solve for a given variable 80% of the time.

Time : Approximately 2 hours

Step 1: Pre-instructional activities:

- Now that we've learned to set up an equation and simplify its constituent expressions, we can start to solve these equations algebraically.
- Give an example of an important, relevant equation to solve. The most relevant and

motivating would be financially related.

- Example: For your birthday you received $100 which you want to spend on a new bicycle. The bicycle you like is $90 plus tax. What is the maximum the tax could be that would still allow you to buy the bicycle?
- You could solve this problem by creating an equation to model the problem such as:
- cost + cost*tax = total
- Which in this case would translate to:
- $90 + 90*x = 100$
- $90(1+x) = 100$ factor out the 90 from the left side binomial
- $1 + x = 100/90$ divide both sides by 90
- $1 + x = 10/9$ simplify the fraction on the right side
- $x = 10/9 - 1$ subtract 1 from both sides
- $x = 10/9 – 9/9$ Find common denominator
- $x = 1/9 = .1111 = 11.11\%$ simplify
- Problems like this and more can be solved by first creating an algebraic equation and solving it for the unknown variable.

Step 2: Content presentation:

- Most of the problem in the pre-instructional activity involved things we covered in previous lessons. Tis includes setting up an equation, factoring, and other types of simplification. The only really new step was the one in which we subtracted one from both sides. This is a very common step in solving of equations we will discuss now.

- Solving an algebraic equation for a particular variable is usually a combination of many simplification steps along with methods of isolating the specific variable in question to eventually end up with an equation involving the unknown variable equaling something. In order to isolate this variable, it is necessary to cancel out the other components around the variable to leave the unknown alone on a single side which could arbitrarily be the left or right side. The lesson on cancellation should help you get an idea on the techniques that may be involved in isolating a variable.
- Recall that two components will cancel out if they are in the same sum with the same magnitude (absolute value) and opposing signs of one another (such as 5 – 5). We can use this same idea to cancel out any term in a sum from one side of an equation simply by adding its opposite to both sides of the equation. It is important that anything foreign operation such as this one be done on both sides of the equation to keep it balanced.
- Here is an example of canceling out a positive term
- x + 2*3 = 1 Note that isolating x involves canceling the 2*3 term
- x + 2*3 - 2*3 = 1 – 2*3 Subtract the local term from both sides
- x = 1 – 2*3 The local terms cancel out
- x = - 4 Simplify to evaluate
- This case involved canceling a positive term so it was necessary to subtract the same term from both sides. Canceling a negative term would be very similar but the term would be added to both sides instead.
- Canceling in a product or fraction is a similar process, here's an example:
- 2x + x = 6 Note that the left side has two instances of the variable

- 3x = 6 Simplify by combining like terms
- (3x)/3 = 6/3 Divide both sides by 3
- x = 6/3 The 3's on the left side cancel
- x = 2 Simplify to numerically evaluate
- Along with numerically evaluating an equation with only one variable, you can also solve an equation for a variable by finding it's relation to another or solving it as a function of another. In this last example, we'll see an example of canceling a denominator in order to isolate x to evaluate it as a function of y.
- 3x/y = 2 Note that x 3 and y should be removed
- (y/3)(3x/y) = (y/3)(2) Multiply both sides by y/3
- x = 2y/3 y/3 cancels with 3/y, isolating x
- Note that the example above canceled out two components at the same time by multiplying the fraction with its reciprocal. This problem could be solved by doing one of those operations at a time as well. Also note that even though we didn't get a single numerical answer for x, we found it as a function of y which we can evaluate for any given y.

Step 3: Learner Participation:

- Since this lesson is much more heavily based on a procedural task and creative application of the intellectual skills learned, we will not be doing a learner participation session in groups. Rather, each student should actively participate in solving a list of equations individually.
- Gather a list of equations that may be appropriate at the level the students are currently able to solve based on your experience with them. Allow them to solve the equations

individually in silence so as not to disrupt their thought process. In order to give them some incentive and challenge, you may organize a contest whereby the students who get the most correct in the allotted amount of time will be given special privileges. This could be something as simple as the ability to leave early for lunch. Treats such as candy however should be avoided whenever possible. After this quiet session, let students come up to the board to explain their process in solving the problem.

Step 4: Assessment:

- This assessment should be the considered the most important one of the entire unit of activity because this lesson is the culmination of all the other skills presented in previous lessons throughout the unit of instruction. It should be graded with care, making sure to put emphasis on the steps rather than the answer. Answers without the proper steps or explanation should not be considered correct. While there is not single set of correct steps, the steps taken should reflect proper algebraic manipulations. Partial credit should be given based on the intermediate steps in the equation. Follow the lesson objectives in grading.
- Find the chapter in the book involving solving simple equations or any collection of equations you students can handle. Allow them to use any notes or materials to solve the equations except programs in which problems are automatically solved for you (such as wolfram alpha).

Step 5: Follow-through activities:

- This lesson completes the skills necessary to solve any simple algebraic equation. Your students should now be able to analyze and and solve a plethora of simple algebraic equations. Encourage the students to use these skills to now solve any equation they come across. Students should feel confident facing new math problems because they will be able to model the situation in an equation and use the appropriate methods to solve for any variable. Encourage your students to use the new skills to analyze common problems around such as the cost effectiveness.