History of Mathematical Ideas

Theory & Problems.

1. Babylonian Maths
   - 1.1 Sexagesimal Number System
   - 1.2 Quadratic Equations
   - 1.3 Roots to Cubic Equations
2. Egyptian Mathematics
3. Early Greek Maths
   - 3.1 Pythagoras
   - 3.2 Other
4. Later Greek Maths
   - 4.1 Archimedes
   - 4.2 Erastosthenes' Measurement of the Earth
   - 4.3 Diophantus (see also 5.8)
5. Renaissance Maths
   - 5.1 Continued Fractions
   - 5.2 Fibonacci
   - 5.4 Theorem: Existence of continued fraction for all Reals
   - 5.5 Tartaglio / Cardano Solution of $x^3 + mx = n$
   - 5.6 Fermat's Little Theorem
   - 5.7 Fermat Primes
   - 5.8 Diophantus' Equation $x^2 + y^2 = z^2$
   - 5.9 Method of Infinite Descent (Fermat / Euler)
   - 5.10 Projective Geometry
6. Eighteenth Century Maths
   - 6.1 Euler's work on Zeta Function
   - 6.2 Gauss' work on Residue Arithmetic

1 BABYLONIAN MATHEMATICS

1.1 SEXAGESIMAL NUMBER SYSTEM.

$\frac{1}{2} = 0;30$ $\qquad$ $\frac{1}{1,21} = 0;0,44,26,40$

$\frac{1}{3} = 0;20$

$\frac{1}{4} = 0;15$

1.2 QUADRATIC EQUATIONS (FROM A SELEUCID TEXT)

"Find a number such that a given number is obtained if its reciprocal is added to it"

ie, solve for x : $x + \frac{1}{x} = b$

Let b be given. Calculate $\left(\frac{b}{2}\right)^2$. Subtract 1 and take square root (ie $\sqrt{\left(\frac{b}{2}\right)^2 - 1}$).
Then add to and subtract from $\frac{b}{2}$ the required result.

(ie $\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - 1}$, which is the solution to the quadratic

$ax^2 + bx + c = 0$ where $a = 1$, and $c = 1$

ie $\frac{-b \pm \sqrt{b^2 - 4}}{2} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{2^2} - 1} = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - 1}$ if we say $b = -b$.

This negation is right, as $ax^2 + bx + c$

$= ax + b + \frac{c}{x}$

$= x + 1 + \frac{1}{x} = 0$ for $a, c = 1$

$\Rightarrow x + \frac{1}{x} = -1$, and this should be negated)

1.3 Roots of Cubic Equations. $ax^3 + bx^2 + c = 0$.

These were solved by tables for $n^3 + n^2$. The equation is first reduced to normal form, ie $x^3 + x^2 = c'$

This is done by: $ax^3 + bx^2 + c = 0$

$$\Rightarrow x^3 + \frac{b}{a}x^2 + \frac{c}{a} = 0$$

Set $x = \frac{b}{a}y$.

$$\Rightarrow \left(\frac{b}{a}\right)^3 y^3 + \left(\frac{b}{a}\right)^3 y^2 + \frac{c}{a} = 0$$

$$\Rightarrow y^3 + y^2 = -\left(\frac{b}{a}\right)^{-3} \cdot \frac{c}{a}$$

2. Egyptian Mathematics

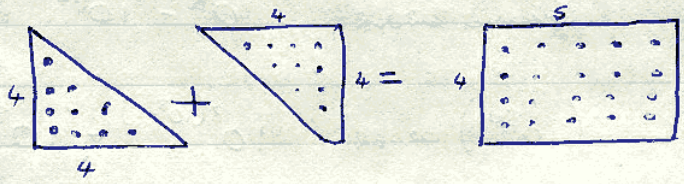
The Egyptians were restricted to using only reciprocals (and $\frac{2}{3}$?) as their fractions. They thus calculated, for example, $\frac{2}{3}x = \frac{1}{2}x + \frac{1}{6}x$

3. EARLY GREEK MATHS.

3.1 PYTHAGORAS. (b 572 BC)

Pythagoras used graphical methods to find general formulae for triangular, square and pentagonal numbers.

For example,



ie $\Delta_4 + \Delta_4 = \square_{4\times 5} \Rightarrow \Delta n = \dfrac{n(n+1)}{2}$.

The irrational square roots were also examined.
Suppose we have $\sqrt{2} = \dfrac{m}{n}$, some $m, n \in \mathbb{N}$
m, n have no common factors

$\Rightarrow 2 = m^2/n^2$, ie $2n^2 = m^2$
But $\Rightarrow 2 \mid m^2 \Rightarrow 2 \mid m$, ie m is even.
As m, n have no common factors, n must be odd.

Now suppose $m = 2m'$
then $2n^2 = 4(m')^2$ ie $2(m')^2 = n^2$

A similar argument shows n even $\Rightarrow$ contradiction.

3.2 OTHER

Euclid (330-270 BC) proofs on geometry and number theory

4. LATER GREEK MATHS.

4.1 ARCHIMEDES (287 - 212 BC)

Archimedes devised a number system for very large numbers:

FIRST PERIOD

FIRST ORDER	$1 - 10^8$
SECOND ORDER	$10^8 - 10^{16}$
$\vdots$	
(10^{8th}) ORDER	$10^{8(10^8-1)} - 10^{8 \cdot 10^8}$ (called P)

SECOND PERIOD

FIRST ORDER	$P - P \cdot 10^8$
SECOND ORDER	$P \cdot 10^8 - P \cdot 10^{16}$
$\vdots$	
10^8th ORDER	$P \cdot 10^{8(10^8-1)} - P \cdot 10^{8 \cdot 10^8}$ (say P^2)

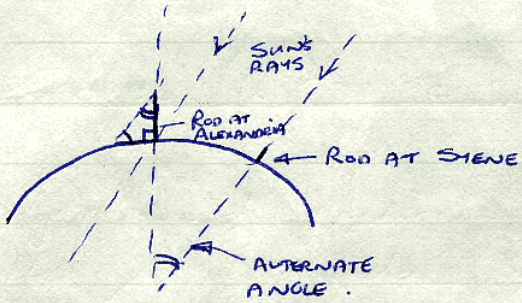
10^8th PERIOD

FIRST ORDER	$P^{10^8-1} \cdot 1 - P^{10^8-1} \cdot 10^8$
$\vdots$	$\vdots$
$(10^8$th$)$ ORDER	$P^{10^8-1} \cdot 10^{8(10^8-1)} - P^{10^8}$

ie, numbers up to 8×10^{16} zeroes.

4.2

ERASTOSTHENES' MEASUREMENT OF THE EARTH'S SIZE (c276-195BC)



Erastosthenes noticed that on Midsummer's day, a rod at Syene cabts no shadow. Thus by measuring the angle at Alexandria, and using

$$\frac{\text{Angle measured}}{\text{4 Right angles}} \times \text{Distance from Alexandria to Syene (5000 stadia)}.$$

he calculated an estimate for the circumference of the earth.

(Plus Erastosthenes' Sieve)

4.3

DIOPHANTUS (c 250 AD)

To divide a given square number into two squares: Eg. 16. The squares are x^2 and $16-x^2$.

Take a square of the form $(mx-4)^2$, m any integer, $4=\sqrt{16}$, and equate this to $16-x^2$ say $(2x-4)^2 = 16-x^2$ and solve this.

5 Renaissance Mathematics.

5.1 General - Continued Fractions.

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + 7^2 \ddots}}}$$

$$2\pi = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots\cdots$$

$$x = \sqrt{a + \frac{b}{x}} \Rightarrow x = \sqrt{a + \cfrac{b}{\sqrt{a + \cfrac{b}{\sqrt{a + \cfrac{b}{\ddots}}}}}}$$

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{\frac{1}{\sqrt{2}-1}} = 1 + \cfrac{1}{\cfrac{1}{1 + \cfrac{1}{\ddots}}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$$

5.2 Fibonacci

Let φ_n be the ratio between x_{n+1} and x_n (where x_i is the ith Fibonacci number).

Then $\dfrac{x_{n+1}}{x_n} = \varphi_n = \dfrac{x_n + x_{n-1}}{x_n} = 1 + \dfrac{1}{\varphi_{n-1}}$

$$\Rightarrow \lim_{n \to \infty} \varphi_n = \varphi = 1 + \frac{1}{\varphi}$$

ie, $\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$

5.4 <u>THEOREM</u>: Every real number can be written as a continued fraction of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}$$

with the a_i's integers.

<u>PROOF</u>: Suppose $x \in \mathbb{R}$.

Let $a_1 = \lfloor x \rfloor$

Then $x = a_1 + r_1$ where $0 \le r_1 < 1$

If $r_1 = 0$ we get a trivial expansion

Otherwise write $x_1 = a_1 + \dfrac{1}{\left(\frac{1}{r_1}\right)}$

Now put $a_2 = \lfloor \frac{1}{r_1} \rfloor$

giving $\frac{1}{r_1} = a_2 + r_2$, etc.

5.5 TARTAGLIO / CARDANO'S SOLUTION OF $x^3 + mx = n$ (16th C)

Use $(a-b)^3 + 3ab(a-b) = a^3 - b^3$

If we choose a, b : $\left\{ \begin{array}{l} 3ab = m \\ a^3 - b^3 = n \end{array} \right\}$ (*)

then $x = a - b$.

Solving (*) simultaneously gives:

$$a = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} \qquad b = \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$$

5.6 FERMAT'S LITTLE THEOREM.

If p is prime, then $\forall n \in \mathbb{N}$, $p \mid n^p - n$.

<u>LEMMA</u> (DUE TO PASCAL) If p is prime, $(a+b)^p = a^b + b^p$ + a multiple of p.

<u>PROOF:</u> By Binomial Theorem, $(a+b)^p = a^p + \binom{p}{1} a^{p-1} b + \binom{p}{2} a^{p-2} b^2 + \dots + b^p$

All the middle factors $\binom{p}{i} = \frac{p(p-1)\dots(p-i+1)}{1 \cdot 2 \cdot 3 \dots i}$ contain a factor p which cannot be cancelled, and are thus divisible by p.

<u>PROOF OF FLT</u>: From the Lemma we get:

$(1+1)^p = 1^p + 1^p$ + a multiple of p

so $2^p = 1^p + 1^p$ + " " " "

inductively $(2+1)^p = 2^p + 1^p$ + " " " "

$= 1^p + 1^p + 1^p$ + " " " "

and so on to $n^p = \underbrace{1^p + 1^p + \dots + 1^p}_{n \text{ terms}}$ + " " " "

$\Rightarrow n^p = n$ + a multiple of p

$\Rightarrow n^p - n$ is divisible by p. (QED).

5.7 FERMAT PRIMES.

Eg primes of form $p = 2^{2^n} + 1$

Fermat developed the method of infinite descent to prove that if p is prime and $p = 4n+1$, then p is a sum of two squares.

Note: – (Even no)^2 : $(2n)^2 = 4n^2$ ie divisible by 4

(Odd no)^2 : $(2n+1)^2 = 4n^2+4n+1$ ie leaves remainder of 1. when divided by 4.

So: x^2+y^2 divided by 4

- can leave 0 (x, y even)
- can leave 1 (one odd, one even)
- can leave 2 (both odd)

Can <u>never</u> leave 3.

Consequently, if p is a sum of two squares,

either $p = 2n$ (rem 0, 2)

or $p = 4n+1$ (rem 1).

This means, for example, that the sum of two odd squares cannot be a square.

5.8 DIOPHANTUS. EQUATION (USED BY FERMAT (5.a) ~~FERMAT'S GENERAL SOLUTION~~)

The most general solution of $x^2+y^2=z^2$ in integers with x, y positive and ~~at sharing~~ coprime, x even is:

$$x = 2ab$$
$$y = a^2 - b^2$$
$$z = a^2 + b^2$$

for $a, b \in \mathbb{Z}$, $a > b$, one even, one odd, a, b coprime.
Then: $(4a^2b^2) + (a^4 + b^4 - 2a^2b^2) = (a^4 + b^4 + 2a^2b^2)$
$\Rightarrow 0 = 0$.

PROOF: Let x be even, y, z odd.
so $\frac{1}{2}(z-y)$ and $\frac{1}{2}(z+y)$ are integers, which don't share common factors by assumption.

Now, $\left(\frac{x}{2}\right)^2 = \frac{z^2-y^2}{4} = \left(\frac{z+y}{2}\right)\left(\frac{z-y}{2}\right)$

$\left(\frac{x}{2}\right)^2$ is a square, so all its prime factors occur an even number of times. The two RHS factors are coprime, hence each of them must also be a square.

Let $\frac{z+y}{2} = a^2$, $\frac{z-y}{2} = b^2$ and the result follows.

5.9 METHOD OF INFINITE DESCENT.

There are no positive integral solutions of $x^4+y^4=z^2$.

PROOF: We may remove common factors and get (if a solution exists) a smallest solution. By the same reasoning used before we cant have both x, y odd

Assume x is the even one. Then we have $(x^2)^2 + (y^2)^2 = z^2$ and we can apply Diophantus' result

$$\left.\begin{aligned} x^2 &= 2ab \\ y^2 &= a^2 - b^2 \\ z &= a^2 + b^2 \end{aligned}\right\} \quad a, b > 0, \quad a, b \text{ coprime.}$$

If a is even, b odd then $y^2 = a^2 - b^2$ will be 1 less than a multiple of 4, which is impossible for a, as a (square) − (square) can also leave 0 or 1 when divided by 4. So a is odd, b even is the only possible case. Say $\boxed{b = 2c}$

$$\Rightarrow x^4 = 4ac \Rightarrow \left(\frac{x}{2}\right)^2 = ac$$

a, c are coprime so they must themselves be squares; say $a = d^2$ and $c = f^2$; by the same argument as before ($d, f > 0$, coprime)

Hence $y^2 = a^2 - b^2 = (d^2)^2 - 4c^2 = d^4 - 4f^4$

So rearranging: $(2f^2)^2 + y^2 = (d^2)^2$, and no two terms have a common factor

We once again can apply Diophantus:

$$2f^2 = 2lm$$
$$d^2 = l^2 + m^2$$

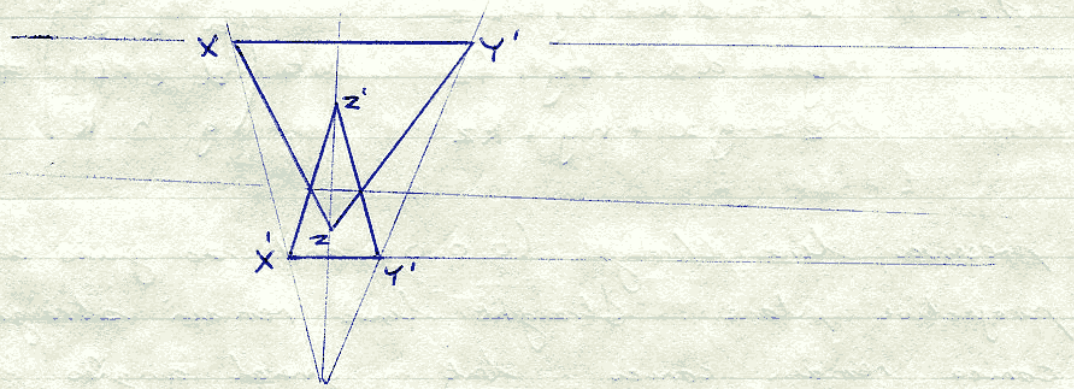
Since $f^2 = lm$, l, m coprime, l and m must themselves be squares, say $l = v^2$, $m = s^2$

Substituting into $d^2 = l^2 + m^2$ we get $d^2 = r^4 + s^4$
So $d, r, \& s$ constitute <u>another</u> solution to our original equation. But $d < d^2 = a \leq a^2 < a^2 + b^2 = z$
so we have found a smaller solution. Hence by Infinite Descent, there are no solutions in $\mathbb{Z}^+$.

5.10 PROJECTIVE GEOMETRY

Pascal's teacher Desargue formulated a theorem which is problematic in Euclidean geometry: viz

<u>THM:</u> Given two triangles XYZ and $X'Y'Z'$ such that the lines joining corresponding points are concurrent, then the intersections of corresponding sides are colinear.



In Euclidean geometry this is false, as XY and $X'Y'$ may be parallel. We introduce projective geometry where parallel lines intersect at infinity. For a sensible theory we need to introduce 1 point at infinity for each direction; thus we get the <u>projective plane</u>:
(real) $\mathbb{P}^2 = \{$ ordinary points + points at infinity $\}$

The problem is to introduce coordinates for $\mathbb{P}^2$. The basic idea (not clear who introduced it) is to study lines through the origin in $\mathbb{R}^3$, with the $\mathbb{P}^2$ plane

lying at $z=1$, so points are projected along lines through the origin in $\mathbb{R}^3$. Each line through $\underline{0}$ in $\mathbb{R}^3$ will either cut the $z=1$ plane in a unique point, or will lie in the $z=0$ plane, ie a point at infinity.

Thus $\mathbb{P}^2 := \{ \underline{l} : \underline{l} \text{ is a line through } \underline{0} \text{ in } \mathbb{R}^3 \}$

Each line is defined by the equation of the plane normal to it, which is not unique; we thus take (a,b,c) and (ka, kb, kc) as the same point. The point $(0,0,0)$ is not included in $\mathbb{P}^2$.

Given a projective plane $\mathbb{P}^2$ with homogenous coordinates (x,y,z), a linear transformation to (x',y',z') where:

$$\begin{aligned} x' &= ax + by + cz \\ y' &= dx + ey + fz \\ z' &= gx + hy + kz \end{aligned} \qquad \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \neq 0$$

preserves lines through $(0,0,0)$ and so gives another projective plane. Straight lines remain straight, conics remain conics, but in general angles, lengths and areas are not preserved. What <u>is</u> preserved are, for example, intersections of lines, etc. These preserved properties are called projective, and projective geometry is the study of these problems.

THM (BEZOUT) ($\sim$1680 first correct proof 1943)

If we have two curves in complex projective space given by polynomial equations $f(x,y,z)=0$, $\deg f = n$
$g(x,y,z)=0$, $\deg g = m$

where f and g are coprime

then the two curves intersect in mn points when the points are counted correctly (including complex points and pairs of points at tangents).

A fundamental invariant of a projective curve is its <u>genus</u>. If the curve comes from a function $f(x,y,z)$ of degree n, then:

$$\text{genus } f = \frac{(n-1)(n-2)}{2} - \#(\text{singular points})$$

(singular points are at $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$, ie no unique tangents >)

Now, the genus of a curve is unaltered by a projection transformation - in fact it is unaltered by a much more general map:

$$(x,y,z) \longmapsto \left(\frac{P_1(x,y,z)}{g(x,y,z)} \quad \frac{P_2(x,y,z)}{g(x,y,z)} \quad \frac{P_3(x,y,z)}{g(x,y,z)} \right)$$

(P_1, P_2, P_3, g polynomials).

So we find that under a rational map, a projective line gets transformed into another curve of the same genus (a so-called rational curve).

The genus of the line $ax+by+cz=0$ with no singular points is $g = \frac{1}{2}(1-1)(1-2) - 0 = 0$, and one can prove the converse holds: a curve of genus 0 can always be obtained by a rational transformation of a line. Consequently, the curves of genus 0 are precisely the

curves which can be parametrically described with a single parameter.

Consider the FERMAT CURVE $x^n + y^n - z^n = 0$

$$\text{genus} = \frac{(n-1)(n-2)}{2} - \#(\text{singular points (none)}).$$

So if $n=1$ or $n=2$ this curve is rational, else not. If $n=1$ we get a line, if $n=2$ a circle - so these are the only ones which can be parameterised by a single parameter.

Consider the circle. We usually write: $x = r\cos\theta$

$$y = r\sin\theta$$
$$z^2 = r^2$$

but this is not an algebraic parameter. We know from the theorem (*) that an algebraic parameter exists, and we can get one using:

$$t = \tan\frac{\theta}{2}$$

so $(x,y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$

This is the finite point $\left(\frac{x}{z}, \frac{y}{z}, 1\right)$ so we get a general solution in projective space of $x = 1-t^2$

$$y = 2t$$
$$z = 1+t^2$$

If we take t rational, say $t = \frac{a}{b}$, then we get $\left(\frac{b^2-a^2}{b^2}, \frac{2a}{b}, \frac{b^2+a^2}{b^2}\right)$, ie a rational point,

and since we are in $\mathbb{P}^2$ we can always multiply by a scalar, say b^2, to get an integer point

$(b^2-a^2, 2ab, b^2+a^2)$. In this way we recover the set of integer solutions of $x^2+y^2+z^2$ found by DIOPHANTUS. We can also see that only the $n=1$ and $n=2$ cases of FERMAT's problem will have infinitely many solutions (as $n>2$ means we need more than one parameter, and the variables become dependent).

MORDELL (1922) conjectured that any curve of genus >1 has only finitely many integer points on it. FALTINGS proved this (1984).

6 EIGHTEENTH CENTURY MATHS

6.1 EULER'S WORK ON SERIES & THE ZETA FUNCTION.

Euler misunderstood convergence and often worked with divergent series giving strange results. When he produced sensible results it was often because of some geometrical constraints forcing convergence. His other results have to be saved by introducing the <u>theory of summability</u>.
A <u>method of summation</u> is some technique which assigns a value to a large family of series, assigning the limit to convergent series, and some other value to divergent series

So, we get $\sum a_n \to S$ by EULER SUMMABILITY
if $\sum a_n x^n$ converges to $f(x)$ on some region R and $\lim_{x\to 1} f(x) = S$.

In general, $f(x)$ is not continuous at 1 and $\sum a_n$ need not converge. But by linking $\sum a_n$ to a function $f(x)$

in this way, we get a sensible notion throwing some light on the behaviour of $\sum a_n$.

For example, $1-1+1-1\ldots$ is summable to $\frac{1}{2}$ by Euler's method since $\lim_{x\to 1}(1-x+x^2-\ldots) = \lim_{x\to 1}\frac{1}{1+x} = \frac{1}{2}$.

Euler introduced the <u>Zeta function</u>, defined as:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s}\cdots = \sum_{n\in\mathbb{N}} \frac{1}{n^s}$$

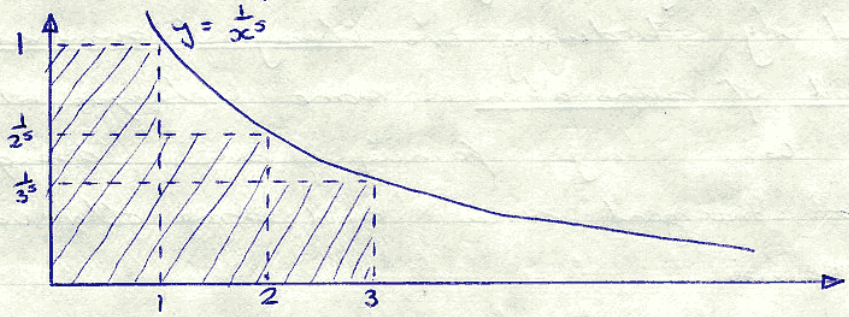
So, $\zeta(2) = \sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$

$$\zeta(4) = \sum_{n\in\mathbb{N}} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Aveny showed that $\zeta(3)$ is also irrational.

<u>PROPOSITION</u>: $\zeta(s)$ converges if $s>1$, diverges if $s=1$.

<u>PROOF</u>: We compare the series with an integral



We see that the series will be the same as the area of the step function under the curve, so:

$$1 + \frac{1}{2^s} + \frac{1}{3^s} \ldots\ldots \leq 1 + \int_1^\infty \frac{dx}{x^s}$$

$$= 1 + \left[\frac{x^{1-s}}{1-s}\right]_1^\infty = 1 + \lim_{x\to\infty}\frac{x^{1-s}}{1-s} - \frac{1}{1-s}$$

$$= 1 + \frac{1}{s-1}, \text{ which exists if } s>1.$$

EULER PRODUCT FORMULA:

$$\zeta(s) = 1 + \frac{1}{2^s} + \dots\dots = \prod_{\text{all prime } p} \left(1 - \frac{1}{p^s}\right)^{-1}$$

PROOF: We have from $(1-x)^{-1} = 1 + x + x^2 + \dots\dots$

that $\left(1 - \frac{1}{p^s}\right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{(p^s)^2} + \dots\dots$

so $\prod \left(1 - \frac{1}{p^s}\right)^{-1} = \prod \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right)$

$$= \sum_{\substack{\text{all combinations} \\ \text{of } p\text{'s}}} \frac{1}{p_1^{n_1 s}\, p_2^{n_2 s} \cdots\cdots p_r^{n_r s}}$$

$$= \sum_{\substack{\text{all combis} \\ \text{of } p\text{'s}}} \frac{1}{(p_1^{n_1}\, p_2^{n_2} \cdots p_r^{n_r})^s}$$

$$= \sum_{n \in \mathbb{N}} \frac{1}{n^s} \text{ by the unique factorisation theorem.}$$

(This was shown by Euler. The proof of the convergence of the product was found in the 19th c (given below)).

We must show that as $r \to \infty$, so $\prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right)^{-1} \to \zeta(s)$

But $\left|\zeta(s) - \prod_{i=1}^{r}\left(1 - \frac{1}{p_i^s}\right)\right| = \left|\sum_{n \in \mathbb{N}} \frac{1}{n^s} - \sum \frac{1}{(p_1^s \cdots p_r^s)}\right|$

$= \left|\sum \frac{1}{n^s}\right|$ where n is not a product of $p_1 \dots p_n$

$\leq \frac{1}{(p_r + 1)^s} + \dots\dots$

$\leq$ remainder after p_r terms in Zeta series

and since the Zeta series converges for $s > 1$ this remainder $\to 0$ as $r \to \infty$.

<u>COROLLARY</u>: There are infinitely many primes.

<u>PROOF</u>: (by DIRICHLET) If there were only a finite number of primes, then:

Since $\zeta(s) = \prod \left(1 - \frac{1}{p^s}\right)^{-1}$ is valid for all $s > 1$ we could take the limit as $s \to 1$ and since the product on the RHS would be finite, we would have $\zeta(s) \to$ finite value - contradiction of divergence of $\zeta(1)$.

RIEMANN had the idea of allowing s to be complex and he showed that $\zeta(s)$ has a unique definition as a complex function with a simple pole at $s = 1$. This allows an extension of the ζ function to the whole complex plane. The negative values are related to the positive values by the condition that:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \quad \left(\text{where } \Gamma(s) = \int_0^\infty x^{s-1} \cdot e^{-x} \cdot dx\right)$$

is unchanged by $s \to 1 - s$

$$\text{so } \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s) = \pi^{-\frac{(1-s)}{2}} \Gamma\left(1 - \frac{s}{2}\right) \zeta(1-s)$$

(the RIEMANN FUNCTION EQUATION)

Riemann also had the idea that $\zeta(s)$ should be like a polynomial, ie, determined by its roots.
Any polynomial $P(x)$ can be written as:

$$(\text{constant})\left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right) \ldots \left(1 - \frac{x}{r_n}\right)$$ where $r_1, r_2, \ldots, r_n$ are roots

So Riemann writes $\zeta(s) = f(x) \prod \left(1 - \frac{s}{p}\right)^{e^{\frac{s}{p}}}$

? where p are roots of the equation $\zeta(p) = 0$.

From this, Riemann is able to deduce all sorts of results on the distribution of primes - just knowing a bit about the roots p. From the Functional Equation we see the complex roots occur in pairs symmetrically around the line $s = \frac{1}{2}$ (the line fixed under $s \to 1-s$).

<u>THE RIEMANN HYPOTHESIS</u>: (unproven) All complex roots of $\zeta(s) = 0$ are complex numbers $s = \sigma + it$ with $\sigma = \frac{1}{2}$. All these roots lie on the line $\sigma = \frac{1}{2}$.

6.2 GAUSS' WORK ON RESIDUE ARITHMETIC.

<u>DEFN</u> $p \equiv q \pmod{n} \Leftrightarrow n \mid p-q$

We can show that the integers mod n form a commutative ring. Gauss questioned which numbers are square mod n (ie what are the quadratic residues mod n).

Eg $\mathbb{Z}_8$

$\odot$	1	2	3	4	5	6	7	0
1	1							
2		4						
3			1					
4				0				
5					1			
6						4		
7							1	
0								0

so 1 has four square roots in $\mathbb{Z}_8$ ie 1, 3, 5 and 7

This is undesirable

Gauss noted that things are better when n is prime. If n is composite we get non-zero numbers whose product is zero, and whenever $n = pq$ is composite, then $p \cdot q \equiv 0 \mod n$, so we have non-zero zero divisors.

If n is prime (say $n = p$) then each number $a \not\equiv 0 \mod p$ has an inverse, so that $a \cdot a' \equiv 1 \mod p$ and the ring $\mathbb{Z}_p$ becomes a commutative ring in which division is possible (a field) written $\mathbb{F}_p$.

<u>PROOF:</u> We must show that $\exists\, a'$ such that $a \cdot a' \equiv 1 \mod p$. Consider all the residue classes $1, 2, \ldots, p-1$. Multiply by a. Since p is prime, the result must be non-zero. Two different classes have different products since $ax \equiv ay$ means $a \cdot (x-y) \equiv 0$ which since $a \not\equiv 0$ means $x - y = 0$ so $x \equiv y$ (ie same class).

Hence the effect of multiplying our original $(p-1)$ residue classes by a is to give $p-1$ different residue classes $\not\equiv 0$. There are only $p-1$ such classes, so 1 must be among the products, ie there is some a' such that $a \cdot a' \equiv 1 \mod p$. (We have shown that for any a, if we multiply a by $1, 2, \ldots, p-1$ we will get the numbers $1, 2, \ldots, p-1$ (in a different order probably) provided p is prime. Examining $\mathbb{Z}_7$ should clarify this:

Because of this, division mod p is possible. (eg in $\mathbb{Z}_7$: $4 \div 6 \equiv 4 \cdot 6^{-1} \equiv 24 \equiv 3$)

$\mathbb{Z}_7$

$\odot$	1	2	3	4	5	6	0
1	1	2	3	4	5	6	0
2	2	4	6	1	3	5	0
3	3	6	2	5	1	4	0
4	4	1	5	2	6	3	0
5	5	3	1	6	4	2	0
6	6	5	4	3	2	1	0
0	0	0	0	0	0	0	0

Modular Arithmetic.

All numbers are divided into congruency classes mod n. ie everything is congruent to one of $0, 1, 2, 3, \ldots, n-1$. We define the sum of two congruence classes to be the class of the sums and the product of the classes to be the class of the product (ie $\hat{x} \odot \hat{y} = \widehat{x \odot y}$ etc).

If p is prime then in $\mathbb{Z}_p$ we have:

$$m \cdot m' \equiv 0 \bmod p \Rightarrow p \mid m \cdot m'$$
$$\Rightarrow p \mid m \text{ or } p \mid m'$$
$$\Rightarrow \text{either } m \equiv 0 \text{ or } m' \equiv 0 \bmod p.$$

thus we have no zero divisors except 0.

Fermat's Theorem: $x^p \equiv x \bmod p$, all x, where p is prime. (done before as $n^p = n$ + a multiple of p).

Corollary: If $x \not\equiv 0 \bmod p$, then $x^{p-1} \equiv 1 \bmod p$.

Proof: If $x \not\equiv 0 \bmod p$ then $\exists\, x'$ such that $x \cdot x' \equiv 1 \bmod p$. So we take $x^p \equiv x \bmod p$, and multiply by x' to get $x^{p-1} \equiv 1 \bmod p$.

We have two square roots of 1 in $\mathbb{F}_p$, namely ± 1. So when p is an odd prime, what is the value of $x^{\frac{p-1}{2}}$?

We must have $x^{\frac{p-1}{2}} \equiv 1$ or -1, but which?* Both cases can occur and it depends on x.

The French mathematician Legendre introduced the so-called <u>Legendre symbol</u>:

$$\left(\frac{x}{p}\right) := \begin{cases} x^{\frac{p-1}{2}} \bmod p & \text{if } x \not\equiv 0 \bmod p \\ 0 & \text{otherwise (ie } p \mid x) \end{cases}$$

It is one of Gauss' results that the Legendre symbol also tells whether x is a quadratic residue (ie square) mod p or not. – if $\left(\frac{x}{p}\right) = 1$ then x is a square mod p, but <u>not</u> if $\left(\frac{x}{p}\right) = -1$.

<u>THM:</u> (Gauss) $\left(\frac{x}{p}\right) = 1 \iff x$ is a square mod p $(x \not\equiv 0)$

<u>PROOF:</u> If $x = y^2 \bmod p$ we have:

$$\left(\frac{x}{p}\right) = x^{\frac{p-1}{2}} = (y^2)^{\frac{p-1}{2}} = y^{p-1} \equiv 1 \bmod p.$$

The converse is more difficult. ~~We need~~:

<u>LEMMA (WILSON'S THEOREM)</u> $(p-1)! \equiv -\left(\frac{a}{p}\right) \cdot a^{\frac{p-1}{2}} \bmod p$ for all $a \not\equiv 0 \bmod p$.

* This must be the case as we have shown that $x^{p-1} \equiv 1 \bmod p$ in the previous corollary, so $x^{\frac{p-1}{2}} \equiv 1^{\frac{1}{2}} \equiv \sqrt{1}$ ie ± 1.

<u>PROOF:</u> Consider $(p-1)! = (p-1)(p-2)\ldots 1$

Now, fix a and consider the equation $xx' \equiv a \bmod p$. Because division is unique in $\mathbb{F}_p$, the equation will have a unique soln for each x (and we will have $x = x'$ if a is a square).

There are two cases:

(i) a is not a square: we have $x \neq x'$, any x, and we can arrange the factors in the factorial in pairs whose product is a, so we get:

$$(p-1)! \equiv \underbrace{\underbrace{x_1 \cdot x_1'}_{a}\ \underbrace{x_2 \cdot x_2'}_{a} \ldots\ldots}_{\frac{p-1}{2}\text{ terms}} \equiv a^{\frac{p-1}{2}}$$

(ii) a is a square: in this case we get a in two different ways: $x^2 \equiv a$ and $(-x)^2 \equiv a$. so we group into pairs except for x and $-x$ which we put together. We thus get:

$$(p-1)! \equiv -x \cdot x \cdot \underbrace{a \cdot a \cdot a \ldots}_{\frac{p-3}{2}\text{ terms}}$$

$$\equiv -a \cdot a^{\frac{p-3}{2}} \equiv -a^{\frac{p-1}{2}}$$

So we get $(p-1)! \equiv -\varepsilon(a) \cdot a^{\frac{p-1}{2}} \bmod p$ where $\varepsilon(a) = \begin{cases} +1 & \text{if } a \text{ is square mod } p \\ -1 & \text{else} \end{cases}$

Now, put $a = 1$ and get $(p-1)! \equiv -1$

so $-1 \equiv -\varepsilon(a) \cdot a^{\frac{p-1}{2}}$ for all a.

and $\varepsilon(a)$ and $a^{\frac{p-1}{2}}$ are both ± 1, hence $\varepsilon(a)$ and $a^{\frac{p-1}{2}}$ must have the same sign for any particular a. This gives us Gauss' formula $\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}} = \varepsilon(a)$, and Wilson's formula

$$(p-1)! \equiv -\varepsilon(a)\, a^{\frac{p-1}{2}} \bmod p \equiv -1 \bmod p.$$

<u>COROLLARY</u>: -1 is a square mod p (an odd prime) iff $p = 4n+1$, some n (ie $p \equiv 1 \bmod 4$).

<u>PROOF</u>: By Gauss' formula, $\left(\frac{-1}{p}\right) \equiv (-1)^{\frac{p-1}{2}} = \begin{cases} +1 & \text{if } -1 \text{ is a square mod } p \\ -1 & \text{otherwise} \end{cases}$

So -1 is a square iff $\frac{p-1}{2}$ is even, say $2n$

$$\Rightarrow p-1 = 4n \quad \text{ie } p = 4n+1.$$

GAUSS' LAW OF QUADRATIC RECIPROCITY.

Gauss asks if we know that an odd prime p is a square modulo another odd prime q. So we know whether q is a square mod p? Yes, in fact $\left(\frac{q}{p}\right)\cdot\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$

GAUSSIAN ARITHMETIC.

Gauss had the idea of studying the numbers $\mathbb{Z}(i)$, where $\mathbb{Z}(i) = \{a+bi \mid a, b \in \mathbb{Z}_F\}$, the <u>Gaussian Integers</u>, by the same techniques Euclid used - to factor these into primes. Some ordinary primes remain prime (eg 3, 7, 11, 19) while others have complex integer factors:

eg $2 = -i\,(1+i)^2$
$5 = (2+i)(2-i)$
$13 = (3+2i)(3-2i)$

So in $\mathbb{Z}(i)$ there is a somewhat different list of primes to the ordinary one, but otherwise arithmetic is very similar - any Gaussian integer can be uniquely

factorised into a product of Gaussian primes and the numbers $\pm 1, \pm i$. The ordinary primes which can now be factorised are 2 and the odd primes $p = 4n+1$. This is essentially the content of the famous theorem of Fermat.

<u>FERMAT</u>: If p is a prime of the form $4n+1$ then p is the sum of two squares.

<u>PROOF</u>: From the corollary above, if $p = 4n+1$ then $\left(\frac{-1}{p}\right) = +1$ so -1 is a square mod p.
So $x^2 \equiv -1 \mod p$ for some x.
or $x^2 + 1 \equiv 0 \mod p$, or $p \mid x^2+1$ some x.

Now, if p is a prime in $\mathbb{Z}(i)$ (called a <u>Gaussian prime</u>) we would get $p \mid (x-i)(x+i)$, so it would have to divide a factor, and either $\frac{x+i}{p} = \frac{x}{p} + \frac{i}{p}$

or $\frac{x-i}{p} = \frac{x}{p} - \frac{i}{p}$ would have to be a Gaussian

integer. However, this is impossible as $\frac{1}{p} \neq 0$ and is not an integer. So p must itself have Gaussian integer factors $p = \pi_1 \pi_2 \ldots \quad \pi_i \in \mathbb{Z}(i)$

All of these factors must be complex, since if we had a real factor p would not be a real prime.

Now, if p has a factor π, then conjugation shows that $\bar{\pi}$ is a factor of $\bar{p} = p$, so the complex factors of p must occur in complex conjugate pairs.

If we group these together, we can arrange that
$p = m\bar{m}$ where $m = \pi_1 \pi_2 \ldots$ and $\bar{m} = \bar{\pi}_1 \bar{\pi}_2 \ldots$

If $m = a + ib$ then $p = (a+ib)(a-ib) = a^2 + b^2$
so p is a sum of two squares (ie FERMAT)

Gauss' pupil Eisenstein got similar results using the ring $\mathbb{Z}(\omega)$ where $\omega^3 = 1$.

HISTORY OF MATHEMATICAL IDEAS:

(NB page numbers in margins refer to previous section).

1 BABYLONIAN MATHEMATICS 1

2 EGYPTIAN MATHS 1

3 BEGINNINGS OF IONIAN MATHS & THE EASTERN GREEK SCHOOL 2

4 PYTHAGOREANS & THE WESTERN GREEK SCHOOL 3.

1 BABYLONIAN MATHEMATICS.

We know about the Babylonians from inscriptions on clay tablets. These records are uncontinuous, the majority belonging to the "Old Babylonian" (c 1800 - 1600 BC) and the Seleucid (c 300 BC)

p1 Numbers were represented in sexagesimal with place values, and the Seleucid texts use a 'zero'. (Remnants - 360°, hours, minutes, seconds). Most of the tablets are tables for multiplication, squares, etc, or of worked examples illustrating the solution of geometrical or algebraic. The geometric work was mainly area estimates, although the Pythagorean property of Δ was known. π was taken as 3. (adopted (possibly) by Old Testament).

The algebra of the Old Babylonians could solve quadratic equations (cf Theory pg 1 §1.2) by a method equivalent to the modern formula, although this rule is nowhere quoted or proved. They also handled linear equations in several unknowns and attempted
p2. to solve cubic and biquadratic equations.

2. EGYPTIAN MATHS.

By contrast to the Babylonians, the Egyptians were primitive mathematicians. Numbers up to one million were represented in a decimal system without place holders. Addition and subtraction were mechanical, with ÷ and × being repeated − and + respectively.
p2 Other than $\frac{2}{3}$, all fractions were reciprocals (ie $\frac{1}{n}$, some n)

Thus $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$. Most known Egyptian problems deal with the resolution of fractions, solution of equations by trial, mensuration of areas and volumes of simple figures, progressions, and calculations apparently relating to the slope of a pyramid. π was taken as 3.16 (actually $\frac{256}{81}$).

3 BEGINNINGS OF IONIAN MATHS & THE EASTERN GREEK SCHOOL.

The Ionians were favourably placed for the reception of foreign ideas. Eastward was Mesopotamia (which was invaded by Persia in the 6th C. BC, leading to further contact). They traded with Egypt, Phoenicia, and even reached India. The first historically known scientist was the Ionian Greek Thales of Miletus (c624-565 BC).

Thales was a successful politician and businessman who visited Egypt and Mesopotamia. Knowledge he gained enabled him to predict an eclipse at Miletus in 585 BC. The Greeks subsequently regarded Thales as the father of science (the systematic observation of nature). Thales also worked in Geometry; he discovered that the base angles of an isosceles Δ's are equal, opposite angles of intersecting straight lines are equal, that the angle subtended at the circumference of a circle by its diameter is a right angle, that the sum of the angles of a Δ is equal to two right angles, and that the sides of Δ's with equal angles are proportional. Thales was also successful in applying his knowledge, eg measuring the height of ships at sea.

His pupil, ANAXIMANDER (611-547 BC) contributed to the search for natural law in the fields of geographical and astronomical surveying, and map making. ANAXIMENES (born c 570 BC), another Miletan, extended Anaximanders ideas and philosophy. At about the same time CLEOSTRATUS of Tenedos improved the calendar, and introduced from Mesopotamia the knowledge of the signs of the Zodiac.

4 PYTHAGOREANS & THE WESTERN GREEK SCHOOL.

p3 PYTHAGORAS (born c 582 BC) was a native of Samos in Ionia, who travelled widely. About 530 he settled at Croton where he founded his sect. They held that numbers have existence outside minds, and that in the final analysis, everything was a number, including music, from which they derived their "harmony of the spheres" - observing that the pitch of musical notes depends on a simple numerical ratio in the length of the cords struck - this ratio, it was held, corresponded to the distances of the heavenly bodies from the centre of the world.

The Pythagoreans also applied numbers to geometry, naming the series 1, 1+2, 1+2+3, 1+2+3+4... ~~as~~ triangular numbers, and observing the sum of any two consecutive triangular number is a square number. They erected a system of plane geometry in which were formulated the principal theorems which concern parallels, triangles, quadrilaterals and regular polygons. They discovered many important properties of prime numbers and progressions.

5 2ND PERIOD OF GREEK MATHS - THE ELEATICS.

Paramenides vs Heraclitus ~ 6th C B.C.
Heraclitus claimed the world is unstable and changing, whereas Paramenides claimed the world was one solid "the way of truth versus the way of seeing" (155 line poem). What we see is illusion, he claimed. His scholar, Zeno of Elea (~480 BC) used paradoxes to "show" that motion is impossible, and hence change is impossible. As points and lines have no area, Zeno claimed everything has measure zero and is thus an illusion. His three paradoxes of motion were dichotomy, Achilles and the Tortoise, and the Arrow and the Stadium. Democritus agreed with Paramenides that the universe was built out of lots of small solid blocks (origin of atomic theory), but argued that vacuums also exist. He was a democrat (Marx wrote his thesis on Democritus).

6 3RD PERIOD OF GREEK MATHS - THE ACADEMY.

Plato formed the Academy in Athens in ~380 BC. Amongst his circle were Eudoxus (theory of proportions), Aristotle and Euclid ("The Elements", work on number theory and solid geometry (5 solids)). Geometry was given precedence. This was the first system of axiomatisation and proof. Plato believed maths to be perfect rationalism and authoritarianism. He was very dogmatic and wanted no paradoxes. He thought of maths as perfection and wanted life lived accordingly. He discovered that although there are infinitely many regular n-polygons, in three dimensions, there are only 5 regular (Platonic) solids (equal faces & angles).
Eudoxus was not concerned with equality of rationals $\frac{a}{b} = \frac{c}{d}$, which he wrote as $\overset{\text{is to}}{a : b} \overset{\text{as}}{=} \overset{\text{is to}}{c : d}$

5

<u>Defn of Proportionality</u>: Iff for all $m, n \in \mathbb{N}$,

$na > mb \Leftrightarrow nc > md$

or $na = mb \Leftrightarrow nc = md$

or $na < mb \Leftrightarrow nc < md$

then $a : b = c : d$.

<u>Defn</u>: A $p \in \mathbb{N}$ is prime if (i) $p \neq 1$

(ii) $p \mid ab \Rightarrow p \mid a \vee p \mid b$ $(\forall a, b \in \mathbb{N})$

<u>Consequence</u>: If $p = ab$ then $a = 1 \vee b = 1$

Euclid proved that there are infinitely many primes

<u>Proof</u>: Suppose there are only a finite number $p_1, p_2, \ldots, p_n$

Then $p_1 \cdot p_2 \cdot p_3 \ldots p_n + 1$ is not divisible by any of these, and thus either is prime or has a prime $> p_n$ which divides it.

Euclid proved by induction that all natural numbers can be factored uniquely into primes. The Greeks also noticed that there are many primes of the form $4n+1$ (eg 5, 13, 17) and $4n+3$ (eg 7, 11, 19). The infinitude of these was proved in the Renaissance.

7 <u>LATER GREEK MATHS.</u>

Archimedes (287 - 212 BC) lived in Syracuse. He was killed by a Roman soldier. 6 of his books are still in existence. He worked on military weapons such as catapults and mirrors for the defense of Sicily from Rome. He was the first person to measure the circle (Euclid showed only that circum $\propto$ radius and area $\propto$ radius^2). He approximated π and showed that the proportionality constant is the same in both cases. He worked out equations of volume and surface area of spheres and cylinders. He discovered logarithmic (Archimedean) spirals $r = a\theta$, as

p4 well as estimating the number of particles in the universe.

He developed a method of exhaustion (numerical approximation) which can be regarded as the ancestor of calculus. He also spotted an unproven unstated axiom in Euclid, the Archimedes axiom: Given any small δ we can find for any large ε a number N such that $\delta N > \varepsilon$ (ie there are no infinitesimal numbers).

Appolonius worked on conic sections; Erastosthenes developed
p5 a method for finding primes and measured the earth's size. Diophantus of Alexandria introduced subtraction (but not negative numbers), and worked on artificial sums (expressing
p5 numbers as sums of squares, etc).

LIMITATIONS OF GREEK MATHS.

Zeno & Archimedes prevented Greek thought from embracing infinity. Three problems were also pondered but never solved (since shown to be unsolvable): doubling the cube, squaring the circle, and trisecting angles. (with ruler & compass)

MATHEMATICAL ASTRONOMY.

The Hellenistic Greeks combined Greek and Egyptian superstitions and maths to produce astrology. In Babylon, artificial sums (step function approximation) was used to calculate the sums movement. Ptolemy's Almagest contained tables still used till the 18th, even though the majority of his observations were faked and were just interpolations.

8 THE DECLINE OF ANCIENT MATHEMATICS.

380 Founding of Constantinople and conversion of Rome to Christianity
476 Fall of Rome to the Goths
529 Justinian closes Plato's Academy - destruction of oral tradition
641 Fall of Alexandria to Islam & sack of Great Library
Boethius (480-584) established 4 textbooks preserving Greek philosophy - arithmetic, music, geometry and astronomy. His textbooks included Trivia - logic, grammar, rhetoric - which comprised a 'liberal education' in the Middle Ages.

The Dark Ages saw much technological achievement, especially in the field of energy (mills, horses, etc - possibly due to lack of slaves!) but little development of theoretical knowledge.

9 ORIENTAL MATHS. & MIDDLE EASTERN MATHS.

There was no real contact or relationship between eastern and Greek mathematics. The Chinese did much work on magic squares. The Hindus (Aryabatha G, Brahmagupta G, sp?...), particularly Bhaskara (born 1114) had excellent maths and astronomy. They used the decimal system, including zero and negative numbers, as well as having systematic methods (algebraic) for solving quadratics and reducing polynomials (from Bhaskara's book Lilavati - "the beautiful one"). The Arabs adopted hindu numerals (so-called Arabic numbers), arithmetic and algebra, and Greek geometry and astronomy. Tabit bin Qurra (826-901) translated Euclid; Al Khanjmi ('algorithm'); ibn Al Haitham ('Al Hazan') continued Ptolemy's work, and Umar Khayyam (13th) invented 'Pascal's Δ' and a geometrical solution of the cubic. At this time in Europe, numeracy was restricted to the elite. The arabic

number system was only adopted in 13th. The Normans, who conquered England and Sicily, were originally Norsemen who settled in France. There was a learned Greek community as well as Jews and Arabs in Sicily at the time, and many travelled to Sicily in the 12th to study 'Arab studies'. Adelard of Bath translated Euclid from Greek; Robert of Ketton also translated Greek works. The Jews in Sicily and Spain translated works of Aristotle and Geometry. All this contributed to:

10 THE RENAISSANCE OF THE 12th CENTURY

Because of all the translations, Europe had to deal with the more advanced science and religions of the arab world. The resurgence of Aristotelian philosophy posed a threat to Christianity which was met by Thomas Aquinas and his philosophy of Scholasticism.

p7 Leonardo Pisano (of Pisa), also called Fibonacci (son of Bonacci) published the Liber Abaci (Book of the Abacus) in 1202. Frederick III Hohenstaufen (the Great) held a contest between Fibonacci and two other mathematicians, and posed three problems:

- find x (rational) such that x^2+5 and x^2-5 are squares of natural numbers (he answered $\frac{41}{12}$).
- solve $x^3+2x^2+10x=20$ (Fib. showed it was <u>not</u> of the form $\sqrt{a+\sqrt{b}}$ which are numbers generated by Greek geometrical methods. He answered $\approx 1,3688081075$)

Fibonacci answered all 3 questions, no-one else answered any.

Rabbit Problem: How many rabbits can be produced in a year if each month each pair produces another pair which becomes fertile after two months (starting with one pair)? This gives the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ... $U_n = U_{n-1} + U_{n-2}$.

Some properties: U_n and U_{n+1} are mutually prime
U_n divides U_{nd}
If $d = \gcd(m,n)$ then $U_d = \gcd(U_m, U_n)$
$U_m^2 - U_{m-1} \cdot U_{m+1} = (-1)^{m-1}$

In the 14th, the School of Merton in Switzerland revived the methods of Archimedes, studied motion, and infinite sequences and series.

LATER RENAISSANCE MATHS.

In the 15th the Medici family in Italy patronised the Renaissance movement. At the time Aristotelian philosophy was being challenged by several others:

- nominalism (William of Occam)
- mysticism (Meister Eckhart)
- magic - the neoplatonic philosopher Ficino translated Platonic dialogues and antique scripts on magic. His students Pico (coined phrase 'Dignity of Man') and Bruno, who was burned at the stake in 1600 after claiming to find the essence of religions and the infinity of universes, thus contradicting the uniqueness of the resurrection and being branded an heretic.

The 15th was also the time of da Vinci, and the reception of Roman law.

The 16th was a great period for algebra. Widman introduced +, - to replace p, m. The reformation (Luther, Calvin) set the stage for an upheaval in orthodoxy. Francois Vieta discovered the equality $\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \ldots$ and the self-taught mathematician Tartaglia discovered a solution for the cubic. He kept this solution to himself but was tricked into revealing it to Cardano, who published it in

the Ars Magna ('the Great Art'). Stevinus introduced full decimal arithmetic (Belgium 1585). In 1580 Billingsley translated Euclid plus architectural works.
This was also the time of the new astronomy. Copernicus (1473-1543) was influenced by neoplatonic philosophy and Pythagoras. He wrote in Latin and was thus not widely noticed. Galileo (1564-1642) wrote in Italian on physics, maths and astronomy and provided the first widespread exposition of Copernicus' ideas.

THE LIFE OF DESCARTES

1596 - born at La Haye near Tours
1604 - sent to Jesuit school at La Flèche
1612 - leaves to study maths in Paris with Marin Mersenne.
1617 - enlists at own expense in army of Maurice of Nassau
1619 (Nov 10) - slept in a stove heated room in Germany and had three dreams (the Poêle ('stove'))
1620 - participated in army of Louis of Bavaria at the Battle of White Mountain (Kuta Hora)
1622 - returned to Paris
1629 - encountered Berulle.
wrote 'Treatise on the World'
settled in Holland, worked as optical instrument maker.
1635 - fathered illegitimate daughter Francine, who died in 1640.
1637 - Discourse on Method (Appendix - 'On Geometry', which introduced Cartesian coordinates).
1640 - teaches Princess Elisabeth of Bohemia
1641 - 'Meditations'
1644 - 'Principia Philosophia'
Vortex theory of gravity (aether)
'Passions' (geometric analysis of love and hate!)

1649 summoned to Sweden by Queen Christina to teach maths
1650 died of inflamation of lungs in Stockholm.

THE LIFE OF FERMAT.

1601 Born Beaumont de Lomagne
Studied in Toulouse, a great centre of learning in (17).
1631 Maître de Requêtes in Parlement (Supreme Court) of Toulouse
1648 Conseiller du Roi ('King's Counsellor').
1665 Dies in Castres.

Main contributions:

- analytic geometry (developed Descartes' ideas further)
- maxima / minima
- dispute with Descartes over speed of light
- number theory

p 9 ff.

PASCAL'S MATHS.

Pascal's Theorem: If you have a conic section and inscribe a (not necessarily regular) hexagon in it, and extend 'opposite' sides till they join, the three points of intersection lie on a straight line ("Mystic Hexagram")

Pascal also contributed to number theory, the calculus of probability, magic squares, and geometry (conic sections and cycloid).

ENGLISH & DUTCH MATHS IN THE 17th CENTURY.

1580 Sir Henry Billingsley translates Euclid

1560 - 1621 Thomas Harriot : notation for <, >

1616 - 1703 John Wallis, decipherer for parliament

discovers $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$

members of Royal Society

Lord Brouncker $4/\pi = 1 + \cfrac{1}{2 + \cfrac{2^2}{2 + \cfrac{3^2}{2 + \cfrac{4^2}{\ddots}}}}$

James Gregory $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

Christiaan Huygens invented the pendulum clock and spring watch, as well as showing that pendulums are most accurate when moving in a cycloid rather than a circular arc.

Nicolaas Mercator worked on maps and projections

THE LIFE OF NEWTON.

1642 - Born on Xmas day (old calendar) at Woolsthorpe

1660 - Enters Trinity College, Cambridge

Buys a book on astrology at a fair

1665/6 - 'Annus Mirabilis' (wonderful year) Plague in Cambridge for 18 months, so goes home.

Discovers binomial theorem and differential calculus.

1669 Barrow resigns Lucasian chair of mathematics in favour of Newton.

1672 Papers on optics (ridiculed in Europe, so he determines not to publish any more)

Works on chemistry, gravitation and theory of equations

1684 Halley visits Cambridge

1685/6 Writes Principia in 16 months

1687 'Philosophiæ Naturalis Principia Mathematica' published

1689 Represents the university in Parliament
1690 meets Fatio de Duillier, and begins relationship
Starts argument with Leibniz
1692 Fatio returns to Switzerland. Newton poisons himself accidentally with antimony and suffers a breakdown
1696 Warden of Mint (gets job through Halley)
1699 Master of Mint
1703 President of Royal Society
1704 publishes 'Opticks'
1705 Knighted.
1707 'Algebra Universalis', Newtons Cambridge Lecture notes published
1716 "Commercium Epistolicum" report on Leibniz' 'plagiarism'
1727 Dies after long illness, buried in Westminster Abbey.

Some achievements
- work on projective geometry & algebra
- calculus, series & Newton's method
- physics (fluxions (rates of change) and fluents (integrals)).

Other British Newtonians were Taylor, McLaurin and Stirling

THE LIFE OF GOTTFRIED WILHELM LEIBNIZ.

1646 (old calendar) born in Leipzig. Taught himself to read Greek, Latin at age 4.
1666 Completes education in law, theology and philosophy
Refused degree (too young)
Studies alchemy at Nuremburg
Works as a diplomat for the rest of his life, first for the Elector of Mainz, and then for the Duke of Brunswick at Hanover.

1672 Meets Huygens while in Paris
1673 Goes to London. FRS.
Newtons "Second letter"
1674-5 Invents calculus while in Paris. Studies
Pascal's manuscripts
1676 Returns to Hanover, works as Royal Librarian
Studies Sanskrit; writes History of the House of Brunswick
Writes 'Monadology' (Philosophy - foundation of modal logic)
Supervises mining industry
Attempts to reconcile Catholics and Protestants.
1682-92 Founds and edits journal Acta Eruditorum,
for which he writes most of his mathematical papers.
1700 Founds Berlin Academy
1707-1716 Dispute with Newton

Some achievements: Calculus $\frac{dy}{dx}$, $\int f(x)\,dx$, product rule, series
Logic (Monadology)
Determinants
Number System, convergence/divergence of series $1+1-1+1-\ldots$
Complex numbers $\sqrt{1+\sqrt{-3}} + \sqrt{1-\sqrt{-3}} = 6$
Chinese binary system

THE BERNOULLI FAMILY (Calvinist refugees from Holland, nomad into Basle aristocracy)

- JACOB 1654-1705 (most famous)
  Polar coords
  Calculus of variations
  Physics
- JOHANN 1667-1748
  L'Hospital, cycloid
  - NICOLAUS 1695-1786
    St Petersburg paradox in probability
  - DANIEL
    Hydrodynamics
  - JOHANN II
    Heat theory
    - JOHANN III
- NICOLAUS

LEONHARD EULER

1707 Born near Basle
Grows up at Reiken near Basle, where his father is a calvinist pastor.
1724 Studies at Basle University with Johan Bernoulli
Nicolaus and Daniel try to get him a job in Russia
Learns medicine
1727 Chair at St Petersburg Academy. Wins 2nd prize in Paris competition
1733 Succeeds Daniel Bernoulli in Chair of Mathematics
1736 Establishes mechanics as a (non-geometrical) science with publication of "Mechanics"
Marries. Loses sight of left eye. Has 13 children
1741 Invited to Berlin Academy by Fred the Great
1766 Return to Russia under Cathy the Great
1777 Second marriage. Goes blind in right eye.
1783 dies in Petersburg age 76.

Some achievments of Euler

$e^x = 1 + x + x^2/2! + x^3/3! + \ldots$

$e^{ix} = \cos x + i \sin x$

e the first irrational since antiquity not of form $\sqrt[n]{x}$ or $\sqrt[n]{\pi x}$

$(\cos x + i \sin x)^n = \cos nx + i \sin nx$

$e^{i\pi} = \cos \pi + i \sin \pi = -1 \Rightarrow e^{i\pi} + 1 = 0$.

Work on number theory
Newtonian mechanics, theory of solids and fluids, linear theory
Calculus of variations
Curved surfaces
Logic - Venn diagrams
Dispute over value of logs of -ve numbers
p17 Convergence (see p17)

CRISES IN 18th CENTURY MATHS.

- Berkeley (1685-1753) attack on the calculus 1734 "the ghost of departed quantities" (only laid to rest by Bolzano and Cauchy in 19th C).
- Complex numbers - existence, infinite solutions to i^i, is $i > 0$? (if $i > 0$, $i^2 > 0$ wrong; if $i < 0$, $(-i)^2 = i^2 > 0$ wrong)

Complex numbers were legitimised by Argand diagrams in $\mathbb{C}$ plane in 1787.

- The concept of function - controversy between Euler & d'Alembert (1717-1783). There was no general function $f(x)$, only specific functions. d'Alembert worked on the wave equation $\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial t^2}$

His solution was $y = \frac{1}{2} f(x+at) + \frac{1}{2} f(x-at)$. He then examined special cases. Euler pointed out that a plucked string is not differentiable, and instead wrote:

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le l \\ a-x & \text{if } l \le x \le 2l. \end{cases}$$

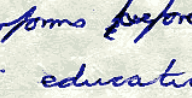
and called the old type 'closed expressions'

Fourier introduced the heat equation, and with the introduction of infinite (Fourier) series, it was seen that closed expressions are often not continuous (eg a step function can be expressed as an infinite trig series).

Bolzano distinguished between differentiability (no bumps) and continuity (no jumps), and found a nowhere differentiable function.

CARL FRIEDERICH GAUSS

1777 Born in Braunschweig (Brunswick)
Calculating prodigy
Age 14: performs before Duke of Brunswick who pays for his education

Age 19: solves problem of constructing regular polygons with ruler and compass
(eg  used complex numbers - we want to find ω, ω^2 (cube roots of 1) is solutions of
$x^3 - 1 = 0 \Rightarrow (x-1)(x^2+x+1) = 0$
(quadratic $\Rightarrow$ constructible with ruler & compass)

Keeps notebook: first entry: "Eureka num = $\Delta + \Delta + \Delta$"

1795-8 Studies at the University of Göttingen. Befriends Bolyai
1799 PhD. with Pfaff at Helmstadt. Fundamental Theorem.
1801 Writes Disquisitiones Arithmetica
1805 Marries. Works on Ceres (astronomy)
1806 Death of Duke while fleeing Napoleonic forces
1807 Called to Petersburg, then Göttingen Observatory
1809 Death of first wife
Work on curved surfaces. Theory of errors, electricity, etc.
Pupils: Dirichlet (1805-1859), Riemann (1826-66)
1855 Dies

THE AGE OF GAUSS.

Peter Lejeune Dirichlet 1805-1859 (German)
Professor at Göttingen; works on trig series, is the first to give a modern defn of function, shows (1831) ∞ many primes in an arithmetic progression

William Rowan Hamilton 1805 - 1865 (Irish)
Works on differential equations; invents non-commutative algebra; matrix algebra with Cayley.
Spoke about 15 languages when he was 16. Prolific poet, but discouraged by Wordsworth, then applied himself to physics.
Child prodigy.

PEANO AXIOMS FOR $\mathbb{N}$ (ACTUALLY BY DEDEKIND).

- one is a number
- the successor of a number is a number
- no two numbers have the same successor
- 1 is not the successor of any number.
- (Induction) If 1 has a property P, and for all $n \in \mathbb{N}$ it is the case that $Pn \Rightarrow P(n+1)$ then all $\mathbb{N}$ have property P.

From $\mathbb{N}$ we get $\mathbb{Z} = \{x \mid x = 0 \wedge x \in \mathbb{N} \wedge -x \in \mathbb{N}\}$.
$\mathbb{Q} = \{(a,b) : a, b \in \mathbb{Z},\ b \neq 0\}$.

DEDEKIND AXIOMS FOR $\mathbb{R}$.
Essentially say $\mathbb{R}$ is a commutative ring (in fact a field) which is well-ordered and <u>complete</u> (Bolzano axiom: if X is a bounded set of $\mathbb{R}$ then X has a lub.).

THREE WAYS OF CONSTRUCTING $\mathbb{R}$ FROM $\mathbb{N}$ (or $\mathbb{Q}$).

- infinite decimal expansions
- Dedekind cuts in the rationals
- Cantor's sequences - identify real numbers with Cauchy convergent sequences of rationals eg $\sqrt{2} = \{1, 1.4, 1.41, 1.414, \ldots\} \to \sqrt{2}$, and equate two sequences if their difference converges to zero.

The last method leads to new number systems, eg Hensel's Peadic number system. In general, decimal numbers have a finite integer part and infinite fractional part. In the peadic numbers we work in a prime base instead eg 5 and use a special 5-adic mod where $|5|_5 = \frac{1}{5}$ and $|\frac{1}{5}|_5 = 5$, so 5-adically 5 is small and $\frac{1}{5}$ is big. A typical 5-adic number thus has an infinite expansion to the left of the decimal point and a finite one to the right.

GEORG CANTOR 1845-1918 (teacher Dedekind)

1879 - 1895 Develops set theory, analysis of infinite sets.
Mental breakdown
1902 Russell's paradox.

Defns of cardinality, countability and enumerability.

eg Galileo $n \to n^2$

Cantor $\mathbb{Z} \to \mathbb{N}$ $\left(n \to (-1)^{n-1} \left\lceil \frac{n}{2} \right\rceil \right)$

Cantor $\mathbb{Q} \to \mathbb{N}$ (diagonal counting of lattice points in $\mathbb{Z} \times \mathbb{Z}$)

Cantor also introduced algebraic numbers (ie complex roots of polynomials with $\mathbb{Z}$ coefficients), gave a method of counting all the polynomials, and deduced that the algebraic numbers are countable.

Theorem: $\mathbb{R}$ is not countable

Proof: Use decimal expansions of $[0,1)$* and construct a number not on the list. (2nd year proof).

So we get a new infinite number, the cardinality of the continuum. Cantor asked: is the cardinality of the

* removing recurring digits

continuum the next aleph? (ie is $\# \mathbb{R} = \aleph_1$?) - the CONTINUUM HYPOTHESIS.

In 1967 P.J. Cohen showed that it depends on our notion of sets whether this is true or not.

Cantor also (after 1896) discovered paradoxes connected with the very large sets such as the 'set' of all sets.

BERTRAND RUSSELL (1902) pointed out a simple paradox. According to Cantor (as formalised by Frege), whenever one has a property P there should be a set X such that $X = \{x : Px\}$. But consider the set $\mathbb{R}$ consisting of all sets which are not members of themselves. Is $\mathbb{R}$ a member of itself?

Dirichlet's Work on the Zeta Function.

Dirichlet modified Euler's work on the Zeta function:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \prod_{\substack{\text{all primes} \\ p}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

to take into account the distribution of primes in an arithmetic progression. He introduced the so-called Dirichlet character, $\chi(p)$

$$L\chi(s) = \prod \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \sum_{\substack{\text{numbers } n \\ n = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}}} \frac{\chi(p_1)^{n_1} \cdot \chi(p_2)^{n_2} \dots \chi(p_r)^{n_r}}{n^s}$$

We will get this equal to $\sum \frac{\chi(n)}{n^s}$ if χ satisfies $\chi(a.b) = \chi(a).\chi(b)$

So Dirichlet's idea is to modify the analytic proof via zeta functions of the infinity of primes by putting in functions $\chi(n)$ as coefficients which satisfy the above multiplicative property and which take different values on different arithmetical progressions.

Example: The so-called Legendre character

$$\chi(n) = \begin{cases} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} = \begin{cases} +1 & \text{if } n \text{ is of form } 4x+1 \\ 0 & \text{if } n \text{ even (of form } 4x+2, 4x \\ -1 & \text{if } n \text{ is of form } 4x+3 \end{cases}$$

So this function is multiplicative and takes different values on the arithmetic progressions $4x+1$, $4x+3$, $2x$.

If we are interested in the AP $(ax+b)$ we want $\chi(n)$ to take the same value on all numbers in this progression ie all numbers congruent to b mod a. So we become interested in all functions $\chi : \{\text{natural nos}\} \to \mathbb{C}$

with $\chi(a.b) = \chi(a).\chi(b)$

and $\chi(a) = \chi(b)$ if $a \equiv b$

This is the same as getting a ~~number~~ function on the residue classes mod n. If we have a function $\overline{\chi} : \mathbb{Z}_n \to \mathbb{C}$ we can extend it to a function $\chi : \mathbb{N} \to \mathbb{C}$ by setting $\chi(a) = \overline{\chi}(a \bmod n)$

So the study of Dirichlet characters is really the study of all functions $\overline{\chi} : \mathbb{Z}_n \to \mathbb{C}$ which preserve multiplication

Since in $\mathbb{Z}_n$, all x satisfy $x^n \equiv x \bmod n$, we have that the $\chi(x)$ values are either 0 or $(n-1)$st roots of 1. So once again we have a special role played by Gauss sums of roots of unity. Dirichlet showed that there are only finitely many characters for each n, and that

$$\prod_{\substack{\text{all characters} \\ \text{at level } n}} L\chi(s) \to \infty \quad \text{as} \quad s \to 1$$

By then carefully examining the different coeffs for the various characters, he showed that this must imply that there are infinitely many primes in each AP $\{nx+b\}$ where b and n share a common factor.

Question: are there ~~an~~ ∞ many primes of form x^2+1?

HAMILTON'S QUATERNIONS. (first hypercomplex numbers)

$\mathbb{H} = \{a + \underline{i}b + \underline{j}c + \underline{k}d\}$

$1 ::= (1,0,0,0)$ $\qquad$ $j ::= (0,0,1,0)$

$i ::= (0,1,0,0)$ $\qquad$ $k ::= (0,0,0,1)$

1 is the unit, ie, every quaternion α has $1.\alpha = \alpha.1 = \alpha$

We define $i^2 = j^2 = k^2 = -1$, and:

$ij = -ji = k$ $\qquad$ ie mult. not commutative.

$ik = -ki = -j$

$jk = -kj = i$

So $(a,b,c,d).(a',b',c',d')$

$= a.a' + ab'i + ac'j + ad'k + ba'i - bb' + bc'k - bd'j$

$+ ca'j - cb'k - cc' + cd'i + da'k + db'j - dc'i - dd'$

$= (a.a' - bb' - cc' - dd',\ ab' + ba' + cd' - dc',$

$ac' + ca' - bd' + db',\ ad' + da' + bc' - cb')$

$(a, \underbrace{b, c, d})$

vector part.

Real, or scalar part

NON-EULIDEAN GEOMETRY

SACHERRI (1780) attempted to prove Euclid's 5th postulate: "There is only <u>one</u> line through a given point parallel to another given line". He tried contradictions:

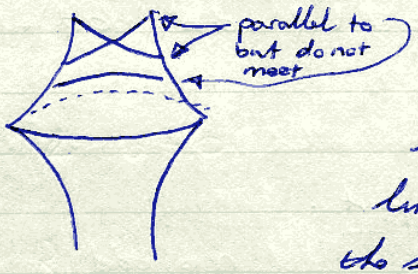
(a) no lines - he found a contradiction, although this was based on a hidden assumption (ie planar).

(b) ∞ lines.

He produced hundreds of theorems in the (b) case without finding a contradiction; nevertheless, these were published.

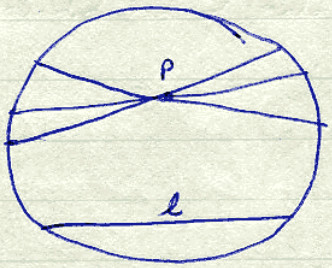
BOLYAI (1829) & LOBACHEVSKY (1826) developed this into non-Euclidean geometry (linked with hyperbolic functions.

BELTRAMI (1860) showed Lobachevsky's system L was consistent and produced a model of it - the pseudosphere.



Interpret a straight line in L to be a curved line of shortest length on the surface of the pseudosphere.

HILBERT (1890) & KLEIN (1880) gave the Gottingen / Erlangen program of axiomatisation.



Klein model of L. - interpret the 'plane' to be the interior of a fixed circle; a 'straight line' in L as a line segment in the circle. "Parallel lines" simply don't meet.

Theorems in L-geometry $\overset{\text{translate}}{\rightleftharpoons}$ Theorems in Euclidean geometry (ie contradictions / non-contradictions preserved).

Hilbert proves Euclidean geometry consistent by reducing axiomatically to properties of $\mathbb{R}$ in the Cartesian system. He also showed that Euclid omitted some axioms.

~~Th~~ Consequences of this ~~is~~ are that: axioms are not self-evident truths but useful postulates; proofs of their soundness need to be given, and logic also needs to be specified axiomatically

Hilbert's programme was to reduce all mathematics to one axiom system and show this system is consistent. (for sets of $\mathbb{N}$, which are then used to develop $\mathbb{R}$ and from this calculus, etc).

FREGE (1897) produced the first axiomatisation of logic.

Number Theory.

Peano's Axioms for $\mathbb{N}$ (actually by Dedekind)

(a) 1 is a number

(b) the successor of a number is a number

(c) no two numbers have the same successor

(d) 1 is not the successor of any number

(e) Induction: · If 1 has property P, and
· For all $n \in \mathbb{N}$, $Pn \Rightarrow P(n+1)$
then all $n \in \mathbb{N}$ have property P.

Construct $\mathbb{Z}$

$$\mathbb{Z} = \{x : x = 0 \text{ or } x \in \mathbb{N} \text{ or } -x \in \mathbb{N}\}$$

Construct $\mathbb{Q}$

$$\mathbb{Q} = \{(a,b) : a, b \in \mathbb{Z},\ b \neq 0\}$$

Dedekind Axioms for $\mathbb{R}$

Essentially, these say that $\mathbb{R}$ is a field which is well-ordered and complete (complete $\Rightarrow$ satisfies Bolzano's axiom: if X is a bounded set of reals, then X has a least upper bound)

CONSTRUCTION OF $\mathbb{R}$ FROM $\mathbb{N}$ (OR $\mathbb{Q}$)

Three ways:

- infinite decimal expansions – problematic due to ambiguity
- Dedekind cuts

  Every real r appears to split $\mathbb{Q}$ into two disjoint subsets

  (i) all $\mathbb{Q} \leq r$ (lower section)

  (ii) all $\mathbb{Q} > r$ (upper section)

- Cantor sequences

  The reals comprise the totality of all equivalence classes of Cauchy convergent series.

  eg $\sqrt{2} = \{1, 1.4, 1.41, 1.414, \ldots\} \to \sqrt{2}$

PEADIC NUMBERS – see p 19 in main notes.

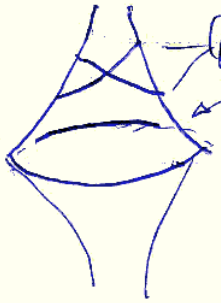
m1
4/10

Non-Euclidean Geometry.

- Sacherri (1780) attempts to prove Euclid's 5th postulate.
  'There is only one line through a given point parallel to another given line'.
  He tries contradictions (a) no lines - he found a contradiction although this was based on a hidden assumption (ie plane).
  (b) ∞ lines.
  He produced hundreds of theorems in the (b) case without finding a contradiction - even so, these were published.

Bolyai (1829) } developed this into N-E geometry.
Lobachevsky (1826) } linked with hyperbolic functions.

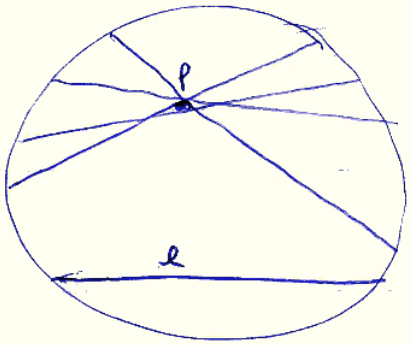
Beltrami (1860) showed Lobachevsky's system L was consistent & produced a model of it; by producing a surface called the pseudosphere.



(parallel to but do not meet)

interpret straight line in L to be a curved line of shortest length on the surface of the pseudosphere.

Hilbert (1890) & Felix Klein (1880) give Gottingen | Erlangen programme of axiomatisation



Klein model of L
interpret the 'plane' to be the interior of a fixed circle.
'straight' line in L as line segment inside circle
'2 lines ∥' means they don't meet.

Theorems in L-geometry $\overset{\text{translate}}{=}$ theorems in Euclidean geometry
(ie there will only be a contradiction in L geometry if there is already one in Euclidean geometry & vice versa)

HILBERT proves Euclidean geometry consistent by reducing axiomatically to properties of the $\mathbb{R}$ in Cartesian system. He also shows Euclid omitted some axioms.

Consequence: - Axioms not self-evident truths but useful postulates affording an economy of thought
- Proofs of soundness need to be given
- Logic also needs to be specified axiomatically

Hilbert's programme - to reduce all mathematics to one axiom system, & show this system is consistent.
(for sets of $\mathbb{N}$, which then are used to develop $\mathbb{R}$ and from this calculus, etc.)

G. FREGE (1897) produces first axiom system for logic

HISTORY OF MATHEMATICS 1985

Practical 1

1. (a) A Babylonian tablet has been discovered which gives the values of $n^3 + n^2$ for $n = 1$ to 30.
   Make such a table for $n = 1$ to 10.

   (b) Find by means of the above table a root for the cubic equation $x^3 + 2x^2 - 3136 = 0$.

   (c) The greatest modern historian of ancient mathematics OTTO NEUGEBAUER, believes that the Babylonians were quite capable of reducing the general cubic equation to the "normal form" $x^3 + x^2 = c$ although there is as yet no hard evidence that they actually did this.
   How might such a reduction be made?

2. Egyptian arithmetic

   (a) The RHIND papyrus makes the following statement "If you were asked what is $\frac{2}{3}$ of $\frac{1}{5}$ take the double and the sixfold; that is $\frac{2}{3}$ of it. One must proceed likewise for any other fraction."
   Interpret this and

   (b) prove a general result.

3. The Greeks and the Irrational

   (a) Prove $\sqrt{3}$ is irrational.

   (b) Find two irrationals whose sum is rational.

   (c) Prove $\sqrt{2} + \sqrt{3}$ is irrational.

1985

HISTORY OF MATHEMATICAL IDEAS

COMMENTARY ON THE FIRST PRACTICAL

1. (b) Hint : Put $x = 2y$ and remove a factor 8 .

   (c) To reduce $ax^3 + bx^2 + cx + d + 0$ first divide through by $a \neq 0$ to get form

   $$x^3 + \beta x^2 + \gamma x + \delta = 0 .$$

   Then remove x term by making a substitution where x is replaced by $z + \varepsilon$, where ε is a root of the quadratic function one gets as coefficient of A when one replaces x by $z + \varepsilon$.

   This reduces the equation to the form

   $$z^3 + mz^2 + n = 0 .$$

   Finally a scale change $z = mt$ can be used to get rid of the coefficient m . (provided $m \neq 0$) for one gets $t^3 + t = \frac{-n}{m^3}$.

2. If we use the notation $\dot{n}$ for the EGYPTIAN unit fraction $\frac{1}{n}$ the RHIND result is just the consequence of writing $\frac{2}{3} = \dot{2} + \dot{6}$ (not $\dot{3} + \dot{3}$) for then :

   $$\left(\frac{2}{3}\right)\dot{n} = \left(\frac{1}{2} + \frac{1}{6}\right)\frac{1}{n} = \frac{1}{2^n} + \frac{1}{6^n} = (\dot{2}n) + (\dot{3}n)$$

   A general result will be

   $$(\sum_n \dot{a}_n)b = \sum_n (\dot{a_n b}) \quad \text{or even better}$$

   $$(\sum_n \dot{a}_n)(\sum_m \dot{b}_m) = \sum_n \sum_m (\dot{a_n b_m}) .$$

3. (a) One may prove this by initiating the PYTHAGOREAN proof just replacing "even" i.e. "divisible by 2" by "divisible by 3" .

   <u>Alternatively</u> once we have proved (by induction) that prime factorization is unique, we may prove a general theorem as follows :

   <u>THEOREM</u> : If n is an integer which is not a square then $\sqrt{n}$ is irrational.

   <u>PROOF</u> Suppose $\sqrt{n} = \frac{a}{b}$, then $n = \frac{a^2}{b^2}$ or $b^2 n = a^2$. Now factor a, b, n into products of powers of primes. Since n is not a square some prime p alters in its factorization to

an odd power. Since a and b are squared in the equation $a = nb$ p alters in the factorization of the LHS an even number of times and in the RHS an odd number of times. This contradicts uniqueness of factorization. Hence there are no such a, b.

One may also prove this result by FERMAT's method of infinite descent (PRAC 3); and the unique factorization proof also extends to prove that if n is not an integral cube (or generally n th power) its cube root (respectively nth root) is irrational.

(b) Take $\sqrt{2}$ (or $\sqrt{3}$ or your favourite irrational number) and $3 - \sqrt{2}$ (or any other rational minus your favourite number). Their sum will be 3 (or the rational you chose) so one needs only to show that $3 - \sqrt{2}$ is irrational (or the difference you chose is irrational.) But this is an easy consequence of the <u>basic fact</u> :

The sum of rational numbers is rational.
(Proof : $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.)

So if $3 - \sqrt{2}$ <u>were</u> rational its sum with the rational number -3, i.e. $-\sqrt{2}$, would be rational which it clearly is not. (Similar reasoning for the general case.)

(c) If $2 + \sqrt{3}$ were rational $\left(\frac{a}{b}\right)$ the same would be true of the square $\left(\frac{a}{b}\right)$. So $2 + 2\sqrt{2}\sqrt{3} + 3. = 5 + 2\sqrt{6}$. would be rational. But $\sqrt{6}$ is irrational by point (a).

<u>REMARK</u> In points (b) and (c) we are really using the <u>good</u> properties of the rational numbers the sum (or difference) of two rational numbers is rational, the product of two rational numbers is rational. Any algebra system containing 0 and 1 and closed under an addition and a multiplication satisfying the usual laws $a + b = b + a$, $a(b + c) = ab + ac$ etc., etc., is called a <u>ring</u> . $\mathbb{Z}$ the set of all integers forms a ring and here we see that $\mathbb{Q}$ - the traditional name for the set of all rational numbers also forms a ring. The set of complex numbers $\mathbb{C}$ forms a ring but there are many rings in between $\mathbb{Q}$ and $\mathbb{C}$ gotten by allowing in various

irrational numbers e.g.

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a,b \in \mathbb{Q}\}$$

forms a ring. We will encounter further examples later in the course.

5/9/85 Thursday 7PM

HISTORY OF MATHEMATICAL IDEAS 1985

MOCK TEST

Time : 1 hour

Part A : Theory

1. Prove that $\sqrt{2}$ is irrational by the method of PYTHAGORAS.

2. Show that the limit of the ratio of two successive FIBONACCI numbers

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} \quad \text{is} \quad \varphi = \frac{1+\sqrt{5}}{2}$$

Part B : Problems

3. Show how a biquadratic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \qquad a \neq 0$$

can be reduced to one where a and b are both 1.

4. (a) Find the regular continued fraction expansion for $\sqrt{3}$.

   (b) Write down a 5×5 magic square.

Part C : History

5. Write brief notes explaining the following :

   (i) Egyptian unit fractions.

   (ii) The Quadrivium.

   (iii) Proof by Dissection.

   (iv) PASCAL's Triangle.

   (v) DESCARTES' Dream.

6. Write an essay describing the relations between Algebra, Geometry and Astronomy from the time of HAMMURABI to the 17th Century.

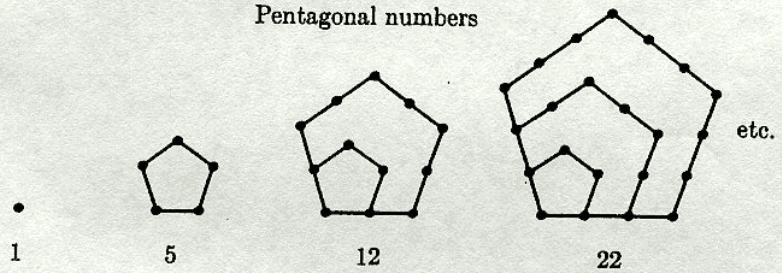
HISTORY OF MATHEMATICAL IDEAS

MATHEMATICS 2 MODULE 8

PRACTICAL 2 1985

PYTHAGOREAN MATHEMATICS

1. The pentagonal numbers 1, 5, 12, 22 ... are defined by building up dot diagrams with regular pentagons :



Prove by Pythagorean methods that the nth pentagonal number is equal to n plus three times the (n-1)st triangular number, and hence find a formula for the nth pentagonal number.

2. (a) Show that the sum of the reciprocals of all divisors of a perfect number is equal to 2 .

(b) Prove the last proposition of Book 9 of EUCLID's Elements which says :

If $2^n - 1$ is a prime then $2^{n-1}(2^n - 1)$ is a perfect number.

[Remark : Primes of the form $2^n - 1$ are known as MERSENNE primes. EUCLID's theorem shows every MERSENNE prime gives rise to an even perfect number, and EULER proved a converse : every even perfect number arises in this way. Before the advent of modern computers only 12 MERSENNE primes were known, corresponding to n = 2, 3, 5, 6, 13, 17, 19, 31, 61, 89, 107 and 127 but now quite a few further MERSENNE primes, and hence perfect numbers, have been discovered.]

EUCLID'S THEOREM ON THE INFINITY OF PRIMES

3. EUCLID's argument that there are an infinity of primes can be improved to give an estimate for $\pi(x) = \#\{\text{primes} \leqslant x\}$.

Let P_n denote the nth prime. Suppose that

$$P_n < 2^{2^n} \qquad \text{for } n = 1,2 \ldots N .$$

then EUCLID's argument shows that

$$P_{N+1} \leqslant P_1 P_2 \ldots P_N + 1 < 2^{2+4+\ldots 2^N} + 1 < 2^{2^{N+1}}$$

since our assumed inequality is true for $n = 1$ we get by induction that it holds for all n .

Suppose now $n \geqslant 4$ and

$$e^{e^{n-1}} < x \leqslant e^{e^n} .$$

then $e^{n-1} > 2^n$, $e^{e^{n-1}} > 2^{2^n}$ and so

$$\pi(x) \geqslant \pi(e^{e^{n-1}}) \geqslant \pi(2^{2^n}) \geqslant n .$$

Since $\log \log x \leqslant n$ we deduce that

$$\pi(x) \geqslant \log \log x .$$

Fill in the details of the above argument and deduce the weak estimate for $\pi(x)$:

$$\pi(x) \geqslant \log \log x \qquad (\text{for } x \geqslant 2) .$$

(This is a <u>very</u> weak estimate : if $x = 10^9$ it gives $\pi(x) \geqslant 3$ whereas the actual value is $\geqslant 5 \times 10^7$! GAUSS at the beginning of the 19th century conjectured $\pi(x) \sim \frac{x}{\log x}$. This is true, but took nearly 100 years to prove. What is significant is that prior to the modern development of the calculus there were only qualitative statements like : there exist infinitely many primes; but a specific <u>quantitative</u> conjecture about how rapidly a function like $\pi(x)$ grows is much more informative).

4. By varying EUCLID's argument slightly using the number

$$2^2 . 3 . 5 \ldots P - 1$$

prove there exist infinitely many primes of the form $4_n + 3$.

5. HINDU AND ARABIC MATHEMATICS

(a) (From Bhāskara's Līlāvati). The square root of half the number of bees in a swarm has flown out upon a jessamine bush, 8/9 of the swarm has remained behind: one female bee flies about a male that is buzzing within a lotus flower into which he was allured by the sweet odour, but is now imprisoned in it. Tell me, most enchanting lady, the number of the bees.

(b) Prove Bhāskara's identity

$$\sqrt{a \pm \sqrt{b}} = \sqrt{(a + \sqrt{a^2 - b})/2} \pm \sqrt{(a - \sqrt{(a^2 - b)})/2}$$

and use it to express $\sqrt{17 + \sqrt{240}}$ as a sum of two quadratic surds

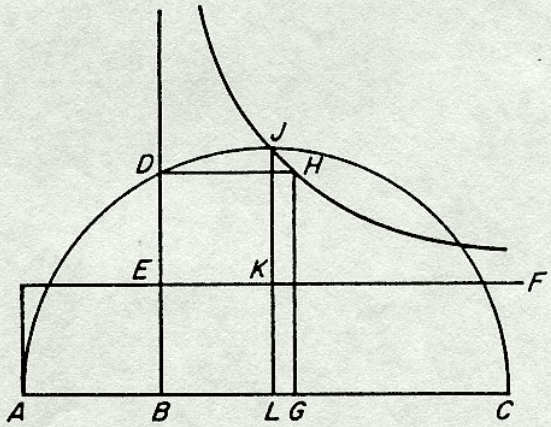
(c) Tâbit ibn Qorra invented the following rule for finding pairs of amicable numbers. IF $p = 3.2^n - 1$ $q = 3.2^{n-1} - 1$ $r = 9.2^{2n-1} - 1$ are three primes then $2^n pq$ and $2^n r$ are amicable. Verify for $n = 2$ and 4.

(d) (Optional - hard question) The diagram below gives the geometrical method of Omar Khayyam for finding the root of the cubic equation

$$x^3 + b^2 x + a^3 = cx^2$$

when it has a positive root.

Construct $AB = a^3/b^2$ (by part (a)) and $BC = c$. Draw a semicircle on AC as diameter and let the perpendicular to AC at B cut it in D. On BD mark off $BE = b$ and through E draw EF parallel to AC. Find G on BC such that $(BG)(ED) = (BE)(AB)$ and complete the rectangle $DBGH$.



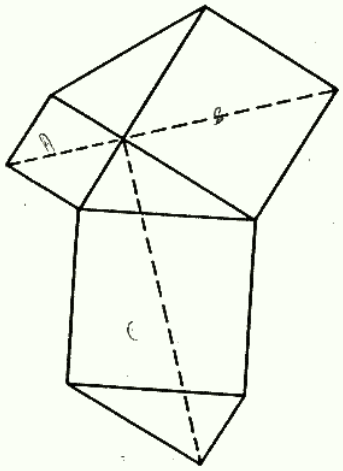
Through H draw the rectangular hyperbola having EF and ED for asymptotes, and let it cut the semicircle in J. Let the parallel to DE through J cut EF in K and BC in L. Show, successively, that: (1) $(EK)(KJ) = (BG)(ED) = (BE)(AB)$, (2) $(BL)(LJ) = (BE)(AL)$, (3) $(LJ)^2 = (AL)(LC)$, (4) $(BE)^2/(BL)^2 = (LJ)^2/(AL)^2 = LC/AL$, (5) $(BE)^2(AL) = (BL)^2(LC)$, (6) $b^2(BL + a^3/b^2) = (BL)^2(c - BL)$, (7) $(BL)^3 + b^2(BL) + a^3 = c(BL)^2$. Thus BL is a root of the given cubic equation.

RENAISSANCE MATHEMATICS

6. (a) Show that the generalized FIBONACCI sequence $a, b, a+b, a+2b, \ldots$ yields numbers which are just a linear combination of ordinary FIBONACCI numbers.

(b) Find continued fraction expansions for $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}$.

(c) Among the continuers of the PYTHAGOREAN tradition of proof-by-dissection was the painter LEONARDO DA VINCI. Can you reconstruct his proof of PYTHAGORAS Theorem from the diagram given below?



MATHEMATICS II MODULE 8

PRACTICAL 3

MORE ABOUT FIBONACCI NUMBERS

1. We can get more information about FIBONACCI numbers by using a method due to DESCARTES.

(a) Introduce the series $S = \sum u_n x^n$

$$S = 1.x + 1.x^2 + 2x^3 + 3x^4 + 5x^5 + \ldots$$

with FIBONACCI numbers as coefficients and show that

$$(x + x^2)S = S - x .$$

(b) Now expand $S = \dfrac{x}{1-x-x^2}$

in partial fractions to deduce

$$u_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$$

where $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \dfrac{1-\sqrt{5}}{2}$.

2. VIETA's Formula for $\frac{2}{\pi}$.

VIETA's own proof of his formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \ldots$$

made ingenious use of ARCHIMEDES technique of exhaustion. We can modernize this slightly using calculus.

(a) Use the $\cos(A+B)$ formula to deduce

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

and hence $\sin\frac{\theta}{2} = \sqrt{\dfrac{1-\cos\theta}{2}} = \dfrac{\sin\theta}{\sqrt{2(1+\cos\theta)}}$ for $\theta \in [0,\pi)$

(b) From the first part deduce

$$\frac{\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \frac{\theta}{\sin\theta} \cdot \frac{\sqrt{2(1+\cos\theta)}}{2}$$

hence $\dfrac{\left(\frac{\theta}{4}\right)}{\sin\frac{\theta}{4}} = \dfrac{\theta}{\sin\theta}\ \dfrac{\sqrt{2(1+\cos\theta)}}{2}\ \dfrac{\sqrt{2(1+\cos\frac{\theta}{2})}}{2}$

and by induction a formula for $\dfrac{\frac{\theta}{2^n}}{\sin\left(\frac{\theta}{2^n}\right)}$

(c) Use $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

to deduce $\dfrac{\sin\theta}{\theta} = \dfrac{\sqrt{2(1+\cos\theta)}}{2} \cdot \dfrac{\sqrt{2(1+\cos\frac{\theta}{2})}}{2} \cdot \dfrac{\sqrt{2(1+\cos\frac{\theta}{4})}}{2} \cdots$

(d) Use $2\cos\frac{\theta}{2} = \sqrt{2(1+\cos\theta)}$ to obtain nested square roots and put $\theta = \frac{\pi}{2}$ to get VIETA's formula.

MISCELLANY

3.(a) GALILEO showed that any two line segments can be put into correspondence so that one can see they have the same number of points. In particular the interval $[0,1]$ contains the same number of points as does its first half $[0,\frac{1}{2}]$. This property of being equal in size to any sufficiently large part of itself is characteristic of infinite sets, and is not confined to a continuum like $[a,b]$. Convince yourself of GALILEO's other discovery: there are as many square natural numbers as there are numbers.

(b) Solve the cubic equation

$$x^3 + 63x = 316$$

by the TARTAGLIA-CARDANO method.

(c) When NAPIER invented logarithms he worked to the base $\frac{1}{e}$, but to avoid the use of fractions also introduced a factor 10^7. Express

NAPIERIAN log of $y = 10^7 \log \frac{1}{e}\left(\frac{y}{10^7}\right)$ in terms of the $\ell n = \log_e$ function using the definition $a^{\log_a(b)} = b$ to convert from one base to another.

4. DESCARTES

(a) DESCARTES gave the following useful rule for determining the signs of the roots of a polynomial equation

$$p(x) = \sum_{0}^{n} a_m x^m = 0 \quad (a_m \in \mathbb{R})$$ known as

<u>DESCARTES rule of signs</u> :

$$\#\{\text{positive roots of } p(x) = 0\} \leq \#\{\text{changes of sign in } (a_n, a_{n-1} \ldots a_0)\}$$

$$\#\{\text{negative roots of } p(x) = 0\} \leq \#\left\{\begin{array}{l}\text{changes of sign in coefficients in } p(-x) \\ \text{i.e. in } (-1)^n a_n, \ldots a_2, -a_1, a_0)\end{array}\right\}$$

with roots counted repeatedly if they are repeated and with the inequalities possibly strict (and defective by an even number if they are.)

Use this to show that

$$x^5 + x^2 + 1 = 0$$

has 4 imaginary roots.

(b) Use DESCARTES rule of signs to show that the cyclotomic (or circle-cutting) polynomial $x^n - 1 = 0$ has exactly two real roots if n is even and only one if n is odd.

(c) Draw the graph of the FOLIUM of DESCARTES

$$x^3 + y^3 = 3\alpha xy.$$

[The line $x + y + \alpha = 0$ is an asymptote] .

(d) Find the polar equation of the FOLIUM and also a parameterization with $y = tx$. What ranges of t yield the loop, the lower arm and the upper arm of the FOLIUM?

<u>FERMAT</u>

5. We may prove that $\sqrt{2}$ is irrational also by using FERMAT's technique of infinite descent. (Some people believe that this was the first result FERMAT proved by this method.) Fill in the gaps in the following proof:

(i) Suppose $\sqrt{2} = \frac{a}{b}$ $a, b > 0$ integers then $\sqrt{2} + 1 = \frac{1}{(\sqrt{2}-1)}$

gives $\frac{a}{b} + 1 = \frac{1}{\frac{a}{b} - 1} = \frac{b}{a-b}$

and $2 = \frac{a}{b} = \frac{b}{a-b} - 1 = \frac{2b-a}{a-b} = \frac{a_1}{b_1}$ (say)

(ii) Since $1 < \sqrt{2} < 2$ we get $b < a < 2b$ hence

$$a_1 = 2b - a > 0.$$

continued on p. 4.

5.(ii) continued.

Since $b < a$, it follows $a_1 < a$, thus a_1 is a positive integer less than a. Repeating we find an infinite descending chain of positive integers, which is impossible.

6. PASCAL and PROBABILITY

The following gambling problem was put to PASCAL by the CHEVALIER DE MÉRÉ: When should one bet even money for getting a 12 in n throws of 2 dice?

(a) Show that the probability of getting a twelve on 1 throw is $\frac{1}{36}$.

(b) Hence the probability of getting no twelve is $\frac{35}{36}$. Hence the probability of being unsuccessful on n throws is $\left(\frac{35}{36}\right)^n$.

(c) Hence the probability of getting at least one twelve is $1 - \left(\frac{35}{36}\right)^n$.

(d) For even money need probability $\frac{1}{2}$ so $n = 24{,}605 \ldots$
Thus it is slightly advantageous to bet on throwing a twelve in 25 throws, slightly disadvantageous to bet on its occurrence in 24 throws.
This result was not believed by the man who raised the query. PASCAL writes to FERMAT:

"I have not time to send you the demonstration of a difficulty that astonished M. de MÉRÉ. He is very bright, but he is no geometer; and that as you know is a great defect. He told me that he had the following difficulty with numbers. If one wagers on the throwing of a six with a single die it is advantageous to wager that the 6 will come up once in 4 throws. But if one wagers on throwing 12 with two dice there is a disadvantage in wagering that it will occur in 24 tries. Nevertheless 24 is to 36 (which is the product of the number of faces of 2 dice) as 4 is to 6 (the number of faces on one die). To him it was a great scandal, and this it was which made him say loudly that arithmetical theorems are not always true and that arithmetic contradicts itself."

(e) What happens with 3 dice?

MATHEMATICS II AO MODULE 8

HISTORY OF MATHEMATICAL IDEAS

PRACTICAL NO. 4 1985

1. The Cycloid

Since so many distinguished mathematicians of the 17th Century (GALILEO, PASCAL, WALLIS, HUYGENS, NEWTON ...) worked on the cycloid it may be of interest to find equations for this curve.

(i) Suppose a circle of radius a rolls along the x-axis without slipping, and the point on the circumference whose locus defines the cycloid is taken to be the tangent point when the circle is tangent to the axis at $x = 0$ $y = 0$. Then if θ denotes the angle between the radius vector from the centre of the circle and the vertical show that

$$x = a\theta - a\sin\theta$$
$$y = a - a\cos\theta .$$

(ii) By using these formulae deduce that the cycloid is a solution to the differential equation

$$\left(y\frac{dy}{dx}\right)^2 + (a - y)^2 = a^2$$

which arises in connection with HUYGENS problem on the manufacture of accurate clocks.

2. Three and Four Squares

FERMAT stated that every number whatsoever is the sum of four squares. It is believed that he proved this with a (somewhat lengthy) proof by infinite descent. The first published proof was given by LAGRANGE only at the end of the eighteenth century. Many proofs are now known, but most are long.

However it is easy to prove that not all numbers are sums of three squares - in particular : no number of the form $8n + 7$ is a sum of three squares.

Prove this by first showing that the remainder after dividing by 8, of any square number is either 0, 1 or 4 .

3. Magic Squares

Find a 5×5 magic square by the Siamese method and another one by a different method, (e.g. putting successive integers down in the squares by a more complicated "jump" than moving along the diagonals.)

4. Projective Geometry

Every curve in $\mathbb{P}^1$ with a homogeneous equation $f(x,y,z) = 0$ can be restricted to a finite plane, say $z = 1$, by substituting

$$f(x,y,1) = 0$$

and conversely every ordinary algebraic curve

$$g(x,y) = 0$$

can be extended to the projective plane by substituting $\frac{x}{z}$ for x and $\frac{y}{z}$ for y and multiplying by an appropriate power of Z to get a homogeneous equation $z^k g(\frac{x}{z}, \frac{y}{z}) = 0$.

(a) Show the two parallel lines

$$x + y + 1 = 0$$
$$x + y - 1 = 0$$

melt "at infinity" when they are prolonged to the projective plane.

(b) NEWTON showed that just as every conic can be obtained from the circle by projection so every cubic curve can be obtained from one of the form

$$y^2 = ax^3 + bx^2 + cx + d .$$

Write down the corresponding form of the equation when this curve is extended to the projective plane.

5. Series

(a) LIEBNIZ as a young man proved

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 .$$

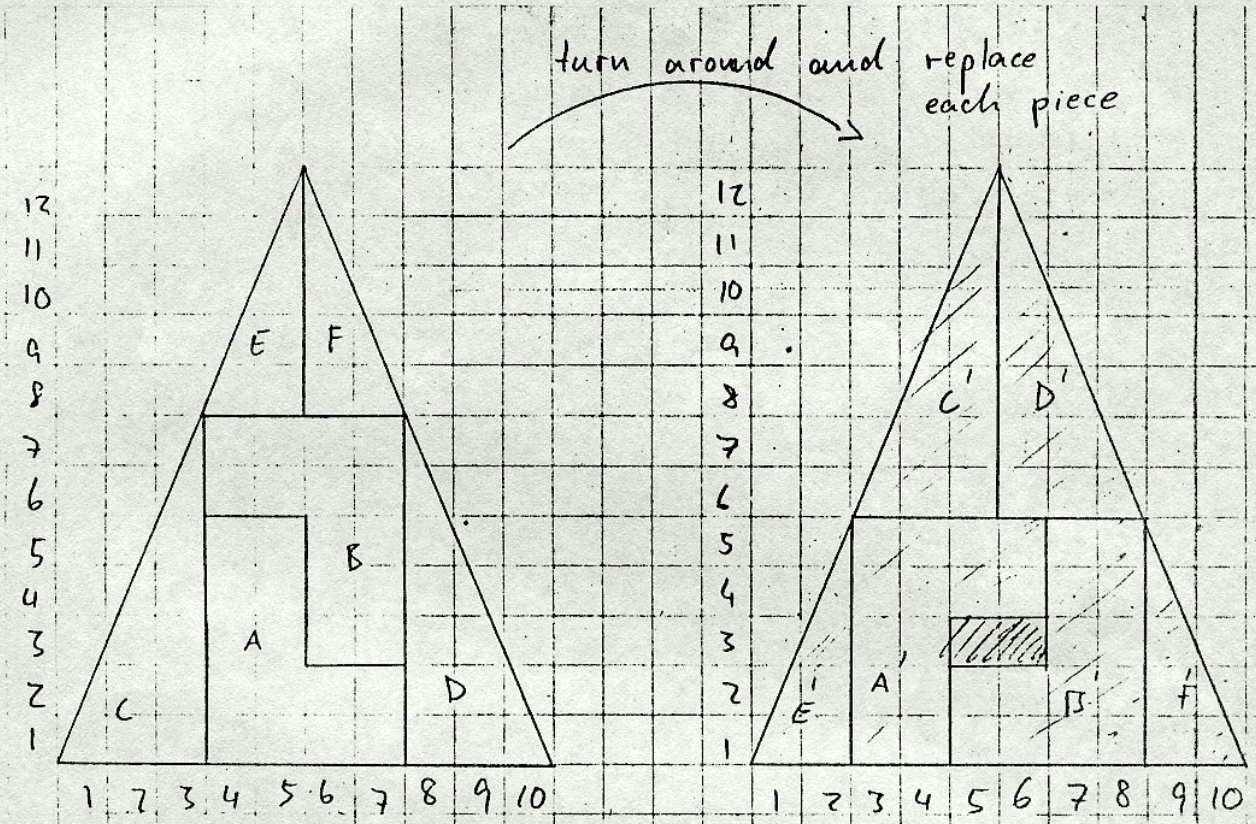
Write $\frac{1}{n(n+1)}$ as partial fractions and do the same.

(b) By integrating the series for $(1 + x)^{-1}$ prove

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)2n} = \ln 2$$

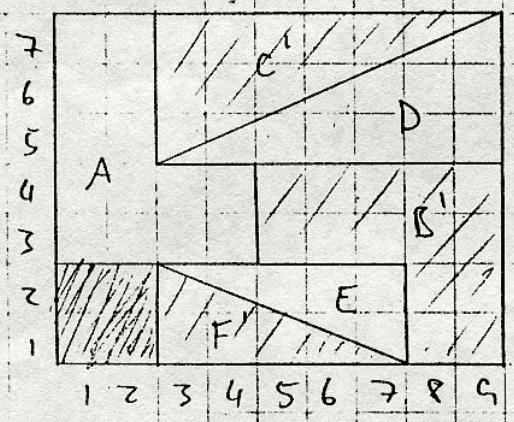
6. Fallacies

Sometimes proof-by-dissection can lead to fallacious results. Can you spot the flaw in the diagrams below which purport to prove that 60 = 58 or, if we only turn half the pieces over, more modestly that 60 = 59 ?



Area = 60 a^2

Area = 58 a^2



Area = 59 a^2

HISTORY OF MATHEMATICAL IDEAS

PRACTICAL 5

BERNOULLI NUMBERS.

1. The kth BERNOULLI number B_k is defined to be $k!$ times the kth coefficient in the MACLAURIN expansion of

$$\frac{t}{e^t - 1}$$

that is $$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

(a) Check by computation (either directly from differentiation or by using the formulae below) that the first six BERNOULLI numbers are

$B_0 = 1 \quad B_1 = -1/2 \quad B_2 = 1/6 \quad B_3 = 0 \quad B_4 = -1/30 \quad B_5 = 0$

$B_6 = +1/42$

(b) By picking out the coefficient of $t^n/n!$ in the RHS of the equation

$t = (e^t - 1)\left(\sum B_k \frac{t^k}{k!}\right) = e^t\left(\sum B_k \frac{t^k}{k!}\right) - e^t$ establish the formulae

$$B_0 + B_1 = B_1 + 1 \qquad (n = 1)$$

$$\binom{n}{0} B_0 + \binom{n}{1} B_1 + \ldots\ldots \binom{n}{n} B_n = B_n \qquad (\text{for } n > 1)$$

which can be used recursively to compute the B_k, and use it to establish that the B_k are all rational numbers.

REMARKS. The BERNOULLI numbers occur in large numbers of classical formulae. For example changing the sign (Putting $t = -x$) yields

$$\frac{e^x}{e^x - 1} = \sum (-1)^n B_n \frac{x^{n-1}}{n!} \qquad \text{and}$$

other similar manipulations can be used to get the

MACLAURIN expansion for such functions as cosec, cot, tan in terms involving powers of 2 , factorials and BERNOULLI numbers. Other major applications are to number theory. For example the geometric progression formula

$$\frac{1}{1 - e^t} = 1 + e^t + e^{2t} + e^{3t} \ldots\ldots + e^{rt} \ldots\ldots + e^{nt} + \frac{e^{(n+1)t}}{1 - e^t}$$

can be used to give the following formula for the sum of the kth powers of the first n integers

$$(k + 1)\left(\sum_{r=0}^{n} r^k\right) = \binom{k+1}{1}(n + 1) B_k + \binom{k+1}{2}(n + 1)^2 B_{k-1} + \ldots\ldots$$

$$\ldots\ldots + \binom{k+1}{r}(n + 1)^r B_{k-r} \ldots\ldots$$

from which one may deduce such familiar formulae as

$$\sum_0^n r = \frac{n(n+1)}{2} \qquad \text{(triangular numbers)}$$

$$\sum_0^n r^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_0^n r^3 = \frac{n^2(n+1)^2}{4} \qquad \text{etc.}$$

2. ZETA FUNCTIONS

The tools used in the proof of VIETA's formula (Practical 3) can be used to prove an infinite product formula due to WALLIS which as spinoff yields the connection between the zeta-function and BERNOULLI numbers.

(a) If n is an odd integer ($n = 2k + 1$ say) then $\sin nx$ can be written using DE MOIVRE's theorem and trig identities as a polynomial of degree n in $\sin x$. $\sin nx = P_n(\sin x)$

(b) $\dfrac{\sin nx}{n \sin x}$ is a polynomial in $\sin x$ vanishing when $x = \pm\dfrac{t\pi}{n}$ and with constant term 1 (by taking limits). So by the representation of a polynomial $p(z)$ as $p(0) \prod\limits_{r \text{ roots of } p(z)} (1 - \frac{z}{r})$

we get

$$\frac{\sin nx}{n \sin x} = \prod_{\substack{t=-k \\ t \neq 0}}^{t=+k} \left(1 - \frac{\sin x}{\sin t\frac{\pi}{n}}\right)$$

whence replacing

x by $\frac{\pi x}{n}$ and rearranging we get

$$\frac{\sin \pi x}{n \sin (\frac{\pi x}{n})} = \prod_{t=1}^{k} \left(1 - \frac{\sin^2 \frac{\pi x}{n}}{\sin^2 \frac{\pi t}{n}}\right) .$$

(c) Take the limit as $n = 2k+1 \to \infty$. LHS goes to $\frac{\sin \pi x}{\pi x}$ while the terms in the product go to $(1 - \frac{x^2}{t^2})$. Whence $\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2})$ from which we may deduce

$\sin \pi x = \pi x \prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2})$ and by suitable choice of x

factorizing and inverting, WALLIS's famous formula :

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots .$$

(d) By substituting a complex variable we may change to the hyperbolic sine :

$$\sinh x = \pi x \prod_{n=1}^{\infty} (1 + \frac{x^2}{n^2}) .$$

(e) Taking log of both sides get

(LHS) $\ln \sinh \pi x = \ln (1 - e^{-2\pi x}) + \pi x - \ln 2$

(RHS) $\ln \pi + \ln x + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x}{kn^{2k}}$ (log series)

$$= \ln \pi + \ln x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k}}{k} \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$$

(absolute convergence permits rearrangement)

$$= \ln \pi + \ln x + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k}}{k} \zeta(2k) .$$

(e) Now take derivatives of both sides, equate, replace x by $\frac{x}{2}$, multiply by x and rearrange to get

$$\frac{\pi x}{e^{\pi x} - 1} + \frac{\pi x}{2} = 1 + \sum^{\infty} (-1)^{k+1} \frac{\zeta(2k)}{2^{2k-1}} x^{2k}$$

whence equating coefficients of even power of x gives

$$\pi^{2k} \frac{B_{2k}}{2k!} = \frac{(-1)^{k+1}}{2^{2k-1}} \zeta(2k)$$

and we get EULER's formula

$$\zeta(2k) = (-1)^{k+1} \pi^{2k} \frac{2^{2k-1}}{2k!} B_{2k} .$$

(including as special cases $\zeta(2) = \frac{\pi^2}{6}$ $\zeta(4) = \frac{\pi^4}{90}$ etc.)

3. PROOF BY DISSECTION : PARTITIONS (EULER)

EULER gave some proofs which involved a very cunning variation on the old PYTHAGOREAN technique of proof-by-dissection. A partition of a number n is a way of writing it as a sum of positive integers. Thus

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1$$
$$= 1 + 1 + 1 + 1 + 1$$

has 7 partitions. EULER noticed that each partition could be represented graphically by a dot diagram in which the pieces are stacked by size eg.

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & & \end{array} \qquad \text{gives} \quad 10 = 3 + 3 + 3 + 1 .$$

Rearranging such a partition in a systematic way (Eg. by reflecting along an axis) one can convert one partition into another. Use this technique to prove EULER's THEOREM:

The number of partitions of n into m parts is equal to the number of partitions of n into parts the largest of which is m.

4. MODULAR ARITHMETIC (GAUSS)

(a) Find the cubes in $\mathbb{F}_7$ and hence determine the LEGENDRE symbols $\left(\frac{n}{7}\right)$, for $n \equiv 1, 2, 3, 4, 5, 6 \mod 7$.

(b) Prove $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

the multiplicative property for the LEGENDRE symbol.

5. GAUSSIAN ARITHMETIC

(a) Let $\pi = a + ib \in \mathbb{Z}[i]$ be a GAUSSIAN integer then show that if π is prime so is its conjugate $\bar{\pi} = a - ib$.

(b) GAUSSIAN factorization theory is much the same as ordinary factorization theory - the main difference is that whereas in $\mathbb{Z}$, ± 1 are the only two integers which divide 1 (and hence everything) in $\mathbb{Z}[i]$ we have four so called units (integers dividing 1) namely $\pm 1, \pm i$. Then every GAUSSIAN integer can be factorized into a product of units and GAUSSIAN primes and the factorization is unique (except possibly for permuting the factors and inserting units). Mark the GAUSSIAN primes with $|z| < 10$ on an ARGAND diagram.

(c) Ordinary primes of the form $p = 4n + 3$ remain prime we pass to $\mathbb{Z}[i]$ (whereas primes of the form $4n + 1$ factor). To see this show using part (a) that if p were not a prime it would have to be a sum of two squares, and by modular arithmetic, $4n + 3$ cannot be a sum of two squares.

6. HAMILTON's QUATERNIONS

(a) Multiply the two quaternions

$$1 + 2\underline{i} + 3\underline{j} + 4\underline{k}$$

and $5 + 4\underline{i} + 3\underline{j} + 2\underline{k}$.

together in the two possible orders.

(b) Using complex numbers to guide you find the inverse of the quaternion

$a + \underline{i}b + \underline{j}c + \underline{k}d \qquad (a,b,c,d) \neq (0,0,0,0)$.

7. MISCELLANY

(a) Show that the GAUSSIAN integers are countable.

(b) Evaluate the 5-adic number

$4\times5 + 4\times5^2 + 4\times5^3 + 4\times5^4 \ldots$

and convince yourself that every periodic p-adic number converges to an ordinary rational number.

MATHEMATICS 2A0 - MODULE 8

HISTORY OF MATHEMATICAL IDEAS

COMMENTARY ON PRACTIALS

2 - 4 1985

Corrections:

Prac 2: Page 3, 5(b): Bhaskara's identity should read :

$$\sqrt{a \pm \sqrt{b}} = \sqrt{(a + \sqrt{a^2 - b})/2} \pm \sqrt{(a - \sqrt{a^2 - b})/2}$$

5(d): line 1 of construction. Delete "by part (a)".

Prac 3: Page 3, 5(i): line 3 should read :

$$2 = \frac{a}{b} = \frac{b}{a - b} - 1 = \frac{2b - a}{a - b} = \frac{a_1}{b_1} \quad \text{(say)}$$

and in 5(ii) delete the full stop after $b < a$ on page 4.

Prac 4: Page 2, 4 Delete (I) after Projective Geometry.

The additional question on Projective Geometry has now been incorporated in my comments on the FOLIUM OF DESCARTES below.

5(b): It is the series for $(1 + x)^{-1}$ that is intended. Also LEIBNIZ spelt incorrectly.

Corrections to Lecture Notes:

From Test 1 it appears that quite a few students did not get the correction to the proof of the fact that a periodic regular continued fraction converges to a quadratic surd, namely that

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots\cdots \cfrac{}{a_n + x}}}}$$

can be written in the form $\frac{mx + n}{px + q}$. The proof of this fact can be given inductively noting that substituting an expression of this form into a fraction $a_i + \frac{1}{a_j + (-)}$ continues to give an expression of this form.

Hints to Practical 2

1. Split pentagonal diagram into 3 triangles plus bottom line with pentagonal member = 3(n-1)st triangular member + n

$$= 3\frac{(n-1)n}{2} + n = \frac{3n^2 - n}{2} .$$

2(a) If n is perfect $n = \sum_{\substack{d_i \text{ divides } n \\ d_i \neq n}} d_i$

so the sum of all divisors is given by :

$$\sum_{d_i \text{ divides } n} d_i = n + n = 2n$$

Thus $$\sum_{d_i \text{ divides } n} \frac{1}{d_i} = \frac{\sum \left(\frac{n}{d_i}\right)}{n}$$

writing all the fractions in terms of a common denominator n . But as d_i runs through all the devisors of n so does $\frac{n}{d_i}$. Thus $\sum_{d_i \text{ divides } n} \left(\frac{n}{d_i}\right) = \sum_{d_i \text{ divides } n} d_i = 2n$

and $$\sum_{d_i \text{ divides } n} \frac{1}{d_i} = \frac{2n}{n} = 2 .$$

2(b) All the divisors of $2^{k-1}(2^n - 1)$ if the second factor is prime are given by $1, 2, 2^2, \ldots, 2^{n-1}$ and $1.(2^n - 1), 2.(2^n - 1), \ldots, 2^{n-2}(2^n - 1)$ (and the number $2^{n-1}(2^n - 1)$ itself.) Now use geomertic progression to add.

Newsflash: A new Mersenne prime $2^{216091} - 1$ has been discovered in the last few weeks (Cape Times: 7 October 1985)

3. The main step which caused headaches was showing $e^{n-1} \geqslant 2^n$ for $n \geqslant 4$. But for $n = 4$ we get $e^3 > 2^4$

(by calculation) and since $e > 2$ we get $e^{n-4} \geqslant 2^{n-4}$ for $n \geqslant 4$ so multiplying in-equalities $e^{(n-4)+3} \geqslant 2^n$.

4. Suppose all primes of the form $4n + 3$ were contained in the list of primes from 2 to P . Then the number

$$2^2.3.5...P - 1$$

is of the form $4n + 3$ and cannot be the product solely of primes of the form $4n + 1$ since the product of primes of this form is always itself of the form $4n + 1$ (REMARK: In terms of modular arithmetic 3 mod 4 is not the product of numbers all 1 mod 4 .)

5(a) Disregard negative root of quadratic.

(c) Similar to 2(b).

6(a) nth generalised FIBONACCI number = $a\,u_n + b\,u_{n-1}$.

(b) Be careful to take <u>regular</u> continued fraction expansions i.e. have 1's above the line - more general continued fractions are not guaranteed to converge. Also remember to take <u>largest integer</u> less than x not just 1 , so for example $\sqrt{5}$ starts with

$$\sqrt{5} = 2 + (\sqrt{5} - 2) = 2 + \frac{1}{\left(\frac{1}{\sqrt{5} - 2}\right)}$$

$$= 2 + \frac{1}{2 + \sqrt{5}}$$

and $\sqrt{3}$ requires some care with

$$\sqrt{3} = 1 + (\sqrt{3} - 1) = 1 + \frac{2}{\sqrt{3} + 1} = 1 + \frac{1}{\left(\frac{\sqrt{3} + 1}{2}\right)}$$

$$= 1 + \frac{1}{1 + \left(\frac{\sqrt{3} - 1}{2}\right)}, \quad \text{etc.}$$

(c) One way to do this is to prove the upper hexagon containing two smaller squares congruent to lower hexagon containing big square and then "peel off" extra triangles.

Practical 3

1. Many people had difficulty with finding the partial fraction expansion for $S = \dfrac{x}{1 - x - x^2}$ but if one writes $(1 - x - x^2)$ in the form $(1 - x\varphi)(1 - x\bar{\varphi})$ (which is possible since $\varphi\,\bar{\varphi} = -1$ and $\varphi + \bar{\varphi} = 1$) one gets easily that

$$S = \frac{1}{(\varphi - \bar{\varphi})} \cdot \left[\left(\frac{1}{1 - x\varphi}\right) - \left(\frac{1}{1 - x\bar{\varphi}}\right)\right]$$

3. (a)

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \dots & n \;\dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ 1 & 4 & 9 & 16 & 25 & 36 & & n^2 \end{array}$$

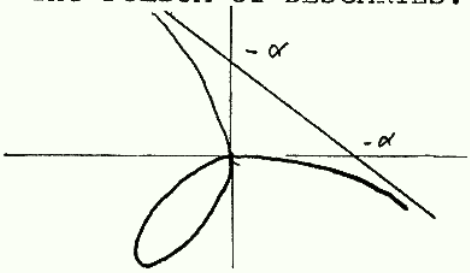
gives for each number a square and for each square a natural number.

(c) The formula for change of base of logarithms is

$$\log_a b \, \log_b c = \log_a c$$

which can be easily proved e.g. using laws of exponents.

5. (c) The FOLIUM of DESCARTES.



Crosses the axis solely at (0,0). From implicit differentiation one can find the vertical and horizontal tangents at (0,0). Symmetry in 45° axis from interchange of x,y. To get asymptote consider $|x|$ large. Then $x^3 + y^3 = 3\alpha xy$ gives $1 + \left(\frac{y}{x}\right)^3 = 3\alpha \frac{y}{x^2}$.

So as $|x| \to \infty$ $\quad 1 + \left(\frac{y}{x}\right)^2 = 0$ or $\frac{y}{x} \to -1$.

But we can do better than merely showing that asymptotically the curve goes to a straight line of slope -1: we can show this is in fact the line $x + y + \alpha = 0$. To see this consider the factorisation $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$. We get

$$(x + y)(x^2 - xy + y^2) = 3\alpha xy$$

So $\quad x + y = \dfrac{3\alpha}{\frac{x}{y} - 1 + \frac{y}{x}} \qquad$ for $(x,y) \neq (0,0)$

and as $|x| \longrightarrow \infty$ $\frac{x}{y} \longrightarrow -1$ and $\frac{y}{x} \longrightarrow -1$

so asymptotically we approach

$x + y = \frac{3\alpha}{-3} = -\alpha$ ie. the line $x + y + \alpha = 0$.

(d) The polar equation of the FOLIUM is straight-forward but the parameterization $y = tx$ is deserving of some comment. Geometrically the parameter t gives the slope of a straight line through the origin. So examining the diagram one sees that $t \in [0,\infty]$ corresponds to the loop.

$t \in [0,-1]$ corresponds to the lower arm

and $t \in (-1,-\infty)$ corresponds to the upper arm.

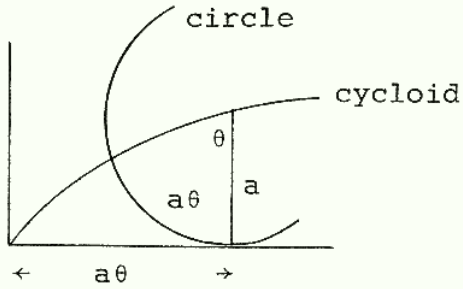
$t = -1$ does not correspond to any point on the curve <u>but</u> corresponds to a point at ∞ when we pass to the projective plane $\mathbb{P}^2$ so that effectively the direction of the asymptote (slope = -1) becomes part of the curve.

When one substitutes $y = tx$ into the equations for the FOLIUM one is able to solve for both y and x in terms of t, and thus get a complete parameterization of the curve exactly like the parameterization of the circle by $\frac{1 - t^2}{1 + t^2}$, $\frac{2t}{1 + t^2}$. Viewed from the point of view of projective geometry the reason is the same - both the circle and the folium are curves of genus 0 and so images under rational maps of a projective line. The same method use to find rational points on the circle - (or integer points on $x^2 + y^2 = z^2$) can be used to find rational points on the FOLIUM.

6. In setting up this problem it is assumed that students are aware of basic notions of probability such as the definition of probability as the ratio of favourable outcomes to all outcomes and basic combin atorial facts for counting combinations.

Practical 4

1. The main problem here is to get the drawing of the cycloid right.



2. This can be done efficiently by calculating mod 8 , but if you are not happy with modular arithmetic write $(8m + 2)^2 =$ (multiple of 8) + 4 etc.

4. (b) $y^2z = ax^3 + bx^2z + cxz^2 + dz^3$.

5. (b) $\ell n\ 2 = \int_0^1 \frac{dx}{1 + x} = \left[x - \frac{x^2}{2} + \frac{x^3}{3} \ldots\right]_0^1$.

Now rearrange.

6. The first figure is impossible - or at any rate not a triangle. For the upper triangles E,F have sloping edge with gradient $^5/_2$ the bottom triangles CD have edge with gradient $^7/_3$.

History of Mathematical Ideas: Synopsis

The Principal Schools or Traditions

(A dotted line indicates influence, a solid line indicates teacher/pupil relationship.)

Pythagoreans

Eleatics — *Parmenides* — *Zeno*

Academy — *Plato* — *Aristotle*; *Euclid*; *Eudoxus*; *Euclid* ⋮ *Archimedes*

School of Alexandria — *Appolonius, Pappus, Diophantos.*

Erastosthenes ⋮ *Ptolemy*

Arabs — *ibn Qorra* ⋮ *Omar Khayyam*

Italian Renaissance — *Tartaglia* — *Cardano*

French School — *Descartes* ⋯ *Mersenne*, *Desargues*; *Descartes* ⋯ *Pascal*; *Fermat* — *Pascal*; *Descartes* ⋮ *Huygens*; *Huygens* ⋯ *Leibniz*; *Pascal* ⋯ *Leibniz*

English School — *Harriot, Wallis, Bruncker*; *Borrow* — *Newton* — *de Moivre*; *Newton* — *Taylor*, *McLaurin*

German School — *Leibniz* — *Bernoulli family*; *Leibniz* — *Euler* ⋯ *D'Alembert*

French Academic Traditions — *D'Alembert* — *Lagrange*; *D'Alembert* ⋮ *Fourier*; *Fourier* — *Cauchy*; *Cauchy* ⋮ *Hamilton-Cayley*; *Legendre* ⋮ *Bolzano*

Götlingen Tradition — *Gauss* — *Dirichlet* — *Riemann*; *Riemann* ⋮ *Felix Klein*; *Dedekind* — *Cantor* ⋯ *Hilbert*; *Felix Klein* — *Hilbert*

CORRECTIONS

COMMENTARY on Prac. 1 3(a) "alters" should be "occurs" (twice).

COMMENTARY on Prac. 2 5(a) "negative" should be "non-integral"

6(c) The top hexagon is not strictly speaking congruent to the lower hexagon as its angles occur in the wrong order. But this can be remedied by cutting along the dotted line and flipping the top half of the hexagon over.

COMMENTARY on Prac. 4. 1. Add the following hint

$$\frac{dy}{dx} = \left(\frac{dy}{d\theta}\right) \Big/ \left(\frac{dx}{d\theta}\right)$$

Prac 5 1(b) $(e^t-1)(\sum Bkt^k/k!) = e^t \sum Bkt^k/k! - \sum Bkt^k/k!$

Correct the last term.

Development of Number System and Number Theory : Synopsis

Babylonia Sexagesimal system

Greek Mathematics

PYTHAGORAS Dot diagrams, Perfect Numbers;

EUCLID Infinity of primes; DIOPHANTOS of Alexandria.

Hindu and Arabic Mathematics

Zero, Decimal system.

IBN QORRA Amicable numbers.

Medieval mathematics

FIBONACCI series

BOM BELLI continued fractions

Renaissance

Negative and Imaginary numbers accepted by CARDANO. VIETA infinite products and roots.

STEVINUS full decimal arithmetic

17th Century ~~*19th Century*~~

FERMAT Infinite descent, $n^p \equiv n$.

All numbers sum of $3\triangle$'s or $4\square$'s .

BEZOUT Counting solutions of simultaneous polynomial equations.

LEIBNIZ Series $\sum (-1)^n$. Binary system.

18th Century

J BERNOULLI Bernoulli numbers. $\sum_{r=1}^{n} m^s$.

EULER $\zeta(s) = \Pi(1 - \frac{1}{p^s})^{-1}$. Partitions.

Proof FERMAT result on $x^4 + y^4 = z^2$.

LEGENDRE $(\frac{a}{p})$ symbol. ARGAND diagram.

LAGRANGE Proof of 4-squares theorem of FERMAT.

19th Century

GAUSS Modular arithmetic $\mathbb{Z}[i]$, $p = a^2 + b^2$ if $p = 4h + 1$. GAUSS sums.

DIRICHLET Infinitely many primes in arithmetic progression. L functions.

HAMILTON Quaternionic integers prove $4\text{-}\square$ theorem.

EINSTEIN $\mathbb{Z}[w]$ (with $w^3 = 1$) . KUMMER $\mathbb{Z}[w]$ with $w^n = 1$.

DEDEKIND general rings $\mathbb{Z}[\alpha, \beta, \gamma \ldots]$. Axioms for $\mathbb{R}$.

DEDEKIND, PEANO axioms for $\mathbb{N}$. Cut construction of $\mathbb{R}$ from $\mathbb{Q}$.

CANTOR construction of $\mathbb{R}$ from $\mathbb{Q}$. Aleph numbers.

HENSEL p-adic alternative to $\mathbb{R}$.

RUSSELL paradox regained.

Historical Development of Algebra : Synopsis

Egyptian algebra : linear equations,
Babylonian algebra : solution of quadratic equations by tabular methods.

Greek Geometric Algebra
Pythagorean dot diagrams,
Figurate numbers.

Hindu algebra
Beginnings of abstraction.

Arabic algebra
IBN QORRA, AL-KUWARISMI. Algorithm, General quadratic, Abstract methods.
OMAR KHAYYAM's triangle.

Middle Ages : FIBONACCI products of sums of 2 squares are sums of 2 squares.

Renaissance : TARTAGLIA, CARDANO Cubic, Biquadratic.
WIDMANN + and - ; HARRIOT $<$ and $>$.

17th Century

DESCARTES Power notation x^3 etc., Sine rule;
NEWTON Binomial theorem; TSCHIRNHAUSEN, LEIBNIZ General methods for reducing polynomial equations.

19th Century : GAUSS Gauss-reduction of linear systems, Modular arithmetic Gaussian integers $\mathbb{Z}[i]$. "Fundamental Theorem of Algebra"
HAMILTON Non-commutative mathematics - Quaternions.
CAYLEY Linear Algebra and Matrices.
ABEL, GALOIS impossibility of solving the quintic equation with radicals. DEDEKIND Axiomatic approach to algebra, ring theory. HILBERT : Beginnings of abstract study of rings of algebraic integers and polynomials.

Chap Ring Structures

Historical Development of Geometry : Synopsis

Primitive Measure of Areas etc.

Babylonian Angle-measure, Applications to Astronomy etc.

Greek Geometry

THALES, PYTHAGORAS,

PLATO's Solid Geometry.

EUDOXUS Theory of Proportion.

EUCLID's Axiomatic Treatment.

Arch

ERATOSTHENES measures the earth.

Later Greek Geometry

School of Alexandria

APPOLONIUS Conic sections

ARCHIMEDES, DIAPHANTOS

PTOLEMY introduces Trigonometry.

Arab Geometry

AL-HAITHAM applications to optics

OMAR KHAYYAM Geometric solution of the cubic.

Renaissance Mathematics

LEONARDO DA VINCI

17th Century : 'Century of Genius'

DESCARTES Analytic Geometry, KEPLER Star polyhedra,

PASCAL and DESARGUES beginnings of Projective Geometry,

NEWTON gives projective classification of cubics.

18 Century : EULER Curved lines $V - E + F = 2$

Age of Gauss

GAUSS Cyclotomy, Curved surfaces;

BOLYAI and LOBACHEVSKY Non-Euclidean (Hyperbolic) space;

BELTRAMI Model of LOBACHEVSKY geometry on psuedo-sphere;

RIEMANN Geometry of spherical and more general space.

KLEIN : Model of LOBACHEVSKY geometry inside a circle,

ERLANGEN programme to reduce Geometry to Algebra.

HILBERT : Axiomatisation of Geometry : reduction to $\mathbb{R}$.

Historical Development of Analysis : Synopsis

Greek Mathematics

ZENO's paradoxes about the continuum.

ARCHIMEDES method of exhaustion.

Arab Mathematics

AL-HAITHAM applies ARCHIMEDES methods to advanced problems in physics.

Medieval Mathematics

MERTON school. BRADWARDINE.

Motion with constant acceleration.

NICOLAS CUSANUS Divergence of $\sum \frac{1}{n}$

BOMBELLI Continued fractions.

Renaissance and 17th Century VIETA, WALLIS, GALILEO's work on Paradoxes. CAVALIERI develops "method of indivisibles" to find Volumes. FERMAT on maxima/minima. NEWTON secretly develops calculus. LEIBNIZ introduces $\frac{dy}{dx}$ $\int f(x)dx$. Summation of series. TAYLOR, McLAURIN, DE MOIVRE.

18th Century

BERNOULLIS Calculus of Variations; BERKELEY : Attack on Calculus;

EULER e^x, $\zeta(s)$, many new series and integrals, D.E's;

D'ALEMBERT Vibrating string problem; FOURIER Fourier Series,

Controversy over nature of functions.

19th Century

GAUSS Development of complex analysis.

BOLZANO A continuous function with both + and − values must take value 0 in between. Sorting out of paradoxes.

First correct definition of limit. Convergent/Divergent series.

CAUCHY $\epsilon - \delta$ theory of limits and continuity. Complex analysis.

DIRICHLET Rigorous approach to Fourier Series. General Functions.

RIEMANN Complex analysis of ζ -function.

WEIERSTRASS Theory of infinite products. Power series.

HILBERT Analysis must be based on Theory of real numbers $\mathbb{R}$.