Metric Spaces.

0 Notation

$A \setminus B$: difference between sets A and B, ie, $A \cap B'$
$\xi_\rho(x)$: equivalence class of x under ρ
$C[a,b]$: the set of continuous functions on the interval $[a,b]$
$A \subseteq B$: the set B includes the set A, ie, $A \cap B = A$
$A \supseteq B$: the set A contains the set B, ie, $A \cup B = A$
$f^{\leftarrow}$: inverse function of f (ie f^{-1})
$R(a)$: equivalence class of a under R
$\sup\limits_{a \leq x \leq b} f(x)$: the supremum, or least upper bound, of $f(x)$ on the interval

$N(a,r)$: the spherical neighbourhood, centre a and radius r,
defined by : $N(a,r) = \{x \mid \rho(a,x) < r\}$, ρ a metric
$K(a,r)$: the closed sphere, centre a and radius r
defined by : $K(a,r) = \{x \mid \rho(a,x) \leq r\}$, ρ a metric
$P[a,b]$ the set of polynomials on $[a,b]$

1 Set Operations, Functions & Relations

1.1 Sets satisfy the following laws:

Associativity

$(A \cup B) \cup C = A \cup (B \cup C)$ $\qquad$ $(A \cap B) \cap C = A \cap (B \cap C)$

Commutativity

$A \cup B = B \cup A$ $\qquad$ $A \cap B = B \cap A$

Distributivity

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\qquad$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan

$(A \cup B)' = A' \cap B'$ $\qquad$ $(A \cap B)' = A' \cup B'$

1.2 Functions

A function $f : A \to B$, A, B sets, is a rule which associates with each element of A one specific element of B. We write $b = f(a)$.

If $A_1 \subseteq A$, then $f(A_1) = \{ y \in B \mid y = f(x) \text{ for some } x \in A_1 \}$

If $f(A) = B$ then f maps A onto B and is called surjective
If $f(x) = f(w) \Rightarrow x = w$ then f is one - one, called injective
If f is both injective and surjective, it is called bijective

If $C \subset B$ then $f^{-1}(C) = \{ x \mid x \in A \wedge f(x) \in C \}$ (This could be empty and may not be a function).

Examples of Equivalence Relations

① In $\mathbb{Z}$, aRb iff $a-b$ is divisible by 7
(ie $a \equiv b \mod 7$)

② In $\mathbb{C}$, aRb iff $|a| = |b|$

③ In $\mathbb{N} \times \mathbb{N}$, $(a,b) R (c,d)$ iff $ad = bc$

④ In $\mathbb{C} - \{0\}$, wRz iff $|w+z| = |w| + |z|$

1.3 Equivalence Relations and Classes.

A <u>relation</u> R in a set X is a subset of $X \times X$; we write aRb iff $(a,b) \in R$

An <u>equivalence</u> <u>relation</u> on X is a relation satisfying

(i) $aRa \quad \forall a \in X$ (<u>reflexive</u>)

(ii) $aRb \Leftrightarrow bRa \quad \forall a,b \in X$ (<u>symmetric</u>)

(iii) $aRb \wedge bRc \Rightarrow aRc \quad \forall a,b,c \in X$ (<u>transitive</u>)

The <u>equivalence</u> <u>class</u> $R(a)$ is defined to be:

$$R(a) = \{ b \mid aRb \}$$

<u>THEOREM</u>: An equivalence relation partitions a set into disjoint equivalence classes

<u>PROOF</u>: Since $aRa \Rightarrow a \in R(a)$, ie $R(a)$ not empty.

Suppose $R(a)$ and $R(b)$ are not disjoint but have some common element z

Let $x \in R(a) \Leftrightarrow xRa$

and $z \in R(a) \Leftrightarrow zRa \Rightarrow aRz \Rightarrow xRz$

But $z \in R(b) \Leftrightarrow zRb \Rightarrow xRb \Rightarrow x \in R(b)$

$\Rightarrow R(a) \subseteq R(b)$ and similarly, $R(b) \subseteq R(a)$

$\Rightarrow R(a) = R(b)$

Thus if two equivalence classes are not disjoint, they are equal.

Examples of Metrics

① On $\mathbb{R}$, $\rho(x,y) = |x-y|$

② On $\mathbb{R}^2$ $\rho(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ is called the Euclidean metric, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$

③ Let $E = C[a,b]$ be the set of all $\mathbb{R}$-valued continuous functions on the interval $[a,b]$.

Let $\rho(f,g) = \sup_{[a,b]} |f(x)-g(x)|$

For eg, in $C[0,1]$ with $f(x) = x$, $g(x) = x^2$,
then $\rho(f,g) = \frac{1}{4}$ (found by maximising $|x-x^2|$)

As any cont. fn. on a closed bounded interval is bounded, proposition (i) holds.

Props (ii) – (iv) are obvious.

If $f, g, h \in C[a,b]$
then $|f(x)-h(x)| \leq |f(x)-g(x)+g(x)-h(x)|$
and $|f(x)-g(x)+g(x)-h(x)| \leq |f(x)-g(x)| + |g(x)-h(x)|$
and $|g(x)-h(x)| \leq \sup_{[a,b]} |g(x)-h(x)|$

$$\Rightarrow |f(x)-h(x)| \leq \sup_{[a,b]} |f(x)-g(x)| + \sup_{[a,b]} |g(x)-h(x)|$$

this is true $\forall x \in [a,b] \Rightarrow$ true for $\sup_{[a,b]} |f(x)-h(x)|$

$\Rightarrow$ proposition (iv) holds.

④ Any set can be made into a metric space by defining a metric

$$\rho(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

This is called the discrete metric

2 THE METRIC

2.1 <u>DEFINITION</u>: A <u>metric</u> ρ on a set E is a function $\rho : E \times E \to \mathbb{R}$ satisfying:

(i) $0 \leq \rho(x,y) < \infty \quad \forall x, y \in E$
(ii) $\rho(x,y) = \rho(y,x) \quad$ "
(iii) $\rho(x,x) = 0 \quad$ "
(iv) $\rho(x,y) = 0 \Leftrightarrow x = y \quad$ "
(v) $\rho(x,y) \leq \rho(x,z) + \rho(z,y) \quad \forall x, y, z \in E$ (TRIANGLE INEQUALITY)

The set E with the metric ρ is called a <u>metric space</u>

Ⓔ In $\mathbb{R}^2$, $\rho_0(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ is called the King's metric

$\rho_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$ is called the taxi-cab metric or lame rook's metric.

Examples

① In $\mathbb{R}$ with the usual modulus metric, the spherical nbd. is an interval $(a-r, a+r)$

② In $\mathbb{R}^2$ with the Euclidean metric, the s.n. is the interior of a disc $\left(\begin{matrix}\text{cent } a \\ \text{rad } r\end{matrix}\right)$

With the King's metric, the s.n. is the interior of a square bounded by pts $(a-r, a-r)$, $(a+r, a-r)$, $(a+r, a+r)$, $(a-r, a+r)$

With the Rooks' metric, the s.n. is a diamond centre a, diameter $2r$

③ In $C[0,1]$ with the supremum metric is an interval around the fns.

① In $\mathbb{R}$, the interval (a,b) is open, but $[a,b)$ is not as a is not an interior point of $[a,b)$

② In $\mathbb{R}$, what is $\text{Int}(\mathbb{Q})$? The empty set, as no point of $\mathbb{Q}$ is an interior point, since every s.n. around a rational contains irrationals

③ In any m.s. (E, ρ), the s.n's are open.

Let $b \in N(a,r)$ and let $x \in N(b,\delta)$ for $\delta = r - \rho(a,b)$

By $\triangle$ inequality; $\rho(x,a) \leq \rho(x,b) + \rho(b,a)$

$$\leq \delta + \rho(b,a) = r$$

$\Rightarrow x \in N(a,r) \Rightarrow N(b,\delta) \subset N(a,r)$

$\Rightarrow b \in \text{Int}(N(a,r))$

④ The intersection of any two spherical neighbourhoods is an open set. Let $b \in N(a_1, r_1) \cap N(a_2, r_2)$

Let $\delta = \min\{r_1 - \rho(a_1, b), r_2 - \rho(a_2, b)\}$

and proceed as for ③

3. Open and Closed Sets.

3.1 **Definition:** If (E, ρ) is a metric space, and $a \in E$ and $r > 0$ then the spherical neighbourhood, centre a and radius r is defined by:

$$N(a, r) = \{x \mid x \in E \text{ and } \rho(a, x) < r\}$$

3.2 **Definition:** The set $K(a, r) = \{x \mid x \in E \text{ and } \rho(a, x) \leq r\}$ is called the closed sphere, centre a and radius r

3.3 **Definition:** If A is a subset of a metric space (E, ρ), then a point $a \in A$ is said to be an interior point of A if for some $r > 0$, then $N(a, r) \subset A$

The set of all interior points of A is called the interior of A, written $\operatorname{Int}(A)$

If $A = \operatorname{Int}(A)$, ie, every point in A is an interior point, then A is said to be open.

3.4 **THEOREM:** If (E, ρ) is a metric space, then:

(i) E is open (and $\emptyset$ is open)

(ii) The union of any family of open sets is open

(iii) The intersection of any finite family of open sets is open

PROOF: (i) As any point in E is an interior point, E is open. $\emptyset$ is open by convention, as there are no elements of $\emptyset$ which are not interior points.

Examples

① 1 is an adherent point of $(0,1)$. Similarly for 0
-1 is not an " " " "

② Let $A = \mathbb{N}$. Every point is an isolated point. The adherent points are $\mathbb{N}$, the accumulation points $\emptyset$

③ The adherent points of $\mathbb{Q}$ are $\mathbb{R}$, the isolated points $\emptyset$, and the accumulation points $\mathbb{R}$.

(ii) Suppose O_i $(i \in I)$ are open

Let $a \in \bigcup_{i \in I} O_i$

Then $a \in O_{i_0}$ for some $i_0 \in I$

Since O_{i_0} is open, $\exists\, \delta \ni N(a, \delta) \subset O_{i_0}$

But $O_{i_0} \subset \bigcup_{i \in I} O_i$

$$\Rightarrow N(a, \delta) \subset \bigcup_{i \in I} O_i$$

(iii) Let O_1 and O_2 be open sets

If $O_1 \cap O_2 = \phi$ then open by (i)

If not ϕ, then let $a \in O_1 \cap O_2$

Since $a \in O_1$ which is open, $\exists\, \delta_1 \ni N(a, \delta_1) \subset O_1$

and similarly, $\exists\, \delta_2 \ni N(a, \delta_2) \subset O_2$

Let $\delta = \min(\delta_1, \delta_2)$

Then $N(a, \delta) \subset O_1 \cap O_2$

3.5 THEOREM: If A is a subset of a metric space (E, ρ), then $Int(A)$ is the largest open subset of A.

3.6 DEFINITIONS: (i) A point a is said to be an adherent point or closure point of a set A if for every $r > 0$, then $N(a, r) \cap A \neq \phi$ (note that a is not necessarily $\in A$)

(ii) A point $a \in A$ is said to be an isolated point of A if for some $r > 0$, $N(a, r) \cap A = \{a\}$

(iii) A point a is said to be an accumulation point of A if for every $r > 0$, $(N(a, r) \setminus \{a\}) \cap A \neq \phi$

Note Every point $a \in A$ is an adherent point, and is either an isolated point or an accumulation point

Examples.

① $A = (0,1)$, $\bar{A} = [0,1]$

② $A = [0,1)$, $\bar{A} = [0,1]$

③ Any interval $[a,b]$ is closed
so is $[a,\infty)$

④ $\mathbb{Q}$ is not closed, as $\overline{\mathbb{Q}} = \mathbb{R}$

⑤ $\mathbb{N}$ is closed, as $\overline{\mathbb{N}} = \mathbb{N}$.

3.7 DEFINITION : The set of all adherent points of A is called the closure of A, written $\bar{A}$. If $A = \bar{A}$, then A is said to be closed. (Note that closure is not the inverse of open, ie, closed $\not\Rightarrow$ not open, etc).

3.8) THEOREM (i) For any A, $\bar{A}$ is closed

(ii) If $A \subset B$, then $\bar{A} \subset \bar{B}$

(iii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

$\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

(iv) $\bigcup_{i \in I} \bar{A}_i \subseteq \overline{\bigcup_{i \in I} A_i}$

(v) $\bar{A}$ is the smallest closed set containing A

(vi) A finite union of closed sets is closed

An arbitrary intersection of closed sets is closed.

PROOF: (i) $A \subset \bar{A}$ for any set A, so $\bar{A} \subseteq \bar{\bar{A}}$

Let $z \in \bar{\bar{A}}$. Then z is an adherent point of $\bar{A}$

Let $N(z, \delta)$ be a s.n. around z

Since $z \in \bar{\bar{A}}$ $\Rightarrow$ $N(z, \delta) \cap \bar{A} \neq \phi$

Suppose $y \in N(z, \delta) \cap \bar{A}$

Since $N(z, \delta)$ is open, y is an interior point, and we can find $\varepsilon > 0$

such that $N(y, \varepsilon) \subset N(z, \delta)$

Since $y \in \bar{A}$, $N(y, \varepsilon) \cap A \neq \phi$

Hence $N(z, \delta) \cap A \neq \phi$

$\Rightarrow z \in \bar{A}$

(ii) If $z \in \bar{A}$, then for any $\delta > 0$, $N(z, \delta) \cap A \neq \phi$

and $N(z, \delta) \cap B \neq \phi$

$\Rightarrow z \in \bar{B}$

(iii) (a) $A \subset A \cup B \Rightarrow \overline{A} \subset \overline{A \cup B}$ by (ii)

Similarly for B, $\overline{B} \subset \overline{A \cup B}$

$\Rightarrow \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$

Let $z \in \overline{A \cup B}$, $z \notin \overline{A}$ and $z \notin \overline{B}$

Then for some $\delta > 0$, $N(z, \delta) \cap A = \phi$ and $N(z, \delta) \cap B = \phi$

But since $z \in \overline{A \cup B}$, $N(z, \delta) \cap (A \cup B) \neq \phi$

$\Rightarrow \phi \cup \phi \neq \phi$, a contradiction

$\Rightarrow z \in \overline{A}$ or $z \in \overline{B}$

$\Rightarrow \overline{A} \cup \overline{B} = \overline{A \cup B}$

(b) $A \cap B \subset A \Rightarrow \overline{A \cap B} \subset \overline{A}$

Similarly $A \cap B \subset B \Rightarrow \overline{A \cap B} \subset \overline{B}$

$\Rightarrow \overline{A \cap B} \subset \overline{A} \cap \overline{B}$

(iv) $\bigcup_{i \in I} \overline{A_i} = \overline{\bigcup_{i \in I} A_i}$

For any i, $A_i \subset \bigcup_{i \in I} A_i$

$\Rightarrow$ By (ii) $\overline{A_i} \subset \overline{\bigcup_{i \in I} A_i}$

$\Rightarrow \bigcup_{i \in I} \overline{A_i} \subset \overline{\bigcup_{i \in I} A_i}$

(v) By (i) $\overline{A}$ is closed. Suppose B is a closed set containing A

$A \subset B$

So $\overline{A} \subset \overline{B}$ by (ii)

$\Rightarrow \overline{A} \subseteq B$

$\Rightarrow \overline{A}$ is the smallest closed set containing A.

Examples

① Let $A = (0,1)$ and $B = (1,2)$

Then $\overline{A \cap B} = \overline{\phi} = \phi$

But $\overline{A} \cap \overline{B} = [0,1] \cap [1,2] = \{1\}$

Hence $\overline{A \cap B} \subsetneqq \overline{A} \cap \overline{B}$

② An infinite countable union of closed sets need not be closed:

Let $A_n = \left[\frac{1}{n+1}, \frac{1}{n}\right]$

So $A_1 = \left[\frac{1}{2}, 1\right]$, $A_2 = \left[\frac{1}{3}, \frac{1}{2}\right]$, etc

Then $\bigcup_{i=1}^{n} A_i = \left[\frac{1}{n+1}, 1\right]$ is closed

But $\bigcup_{i=1}^{\infty} A_i = (0, 1]$ is not closed.

(vi) If A and B are closed, $\overline{A \cup B} = \bar{A} \cup \bar{B}$ by (iii)

$= A \cup B$ by closure of A & B

$\Rightarrow A \cup B$ is closed.

By induction, finite unions of closed sets are closed.

Suppose $\{A_i \mid i \in I\}$ are closed

Since $\bigcap_{i \in I} A_i \subset$ any A_i

$$\Rightarrow \overline{\bigcap_{i \in I} A_i} \subset \bar{A_i} = A_i \quad \text{by (ii)}$$

$$\Rightarrow \overline{\bigcap_{i \in I} A_i} \subset \bigcap_{i \in I} A_i$$

But $\bigcap_{i \in I} A_i \subset \overline{\bigcap_{i \in I} A_i}$

$$\Rightarrow \overline{\bigcap_{i \in I} A_i} = \bigcap_{i \in I} A_i \text{ is closed.}$$

3.9) THEOREM: ① A is closed $\Leftrightarrow$ A' is open

② $\text{int}(A) = (\overline{A'})'$

③ $\bar{A} = (\text{int}\, A')'$

④ $K(a,r) = \{x \mid \rho(a,x) \leq r\}$ is closed. (the closed sphere)

PROOF: ① Suppose A is closed. Let $x \in A'$. Then $x \notin A = \bar{A}$

So we can find $r > 0$ such that $N(x,r) \cap A = \phi$

Thus $N(x,r) \subset A' \Rightarrow A'$ is open, as any point is an interior point.

Conversely, suppose A' is open. Let $x \in \bar{A}$

Suppose $x \notin A$. Then $x \in A'$. Since this is open, x is an interior point, i.e., $\exists r \ni N(x,r) \subset A'$

$\Leftrightarrow N(x,r) \cap A = \phi \Rightarrow x \notin \bar{A}$, a contradiction

$\Rightarrow x \in A \Rightarrow A$ is closed.

Examples

① $\mathbb{Q}$ is dense in $\mathbb{R}$

② $\mathbb{N}$ is nowhere dense in $\mathbb{R}$ but dense in $\mathbb{N}$.

② Let $x \in (\text{int } A)'$. Then $x \notin \text{int } A$.

$\Rightarrow$ no neighbourhood around x is contained in A

$\Rightarrow \forall r > 0, \; N(x, r) \cap A' \neq \phi$

$\Rightarrow x \in \overline{A'} \;\; \Rightarrow (\text{int } A)' \subseteq \overline{A'}$

Let $x \in \overline{A'}$. Then $\forall r > 0, \; N(x, r) \cap A' \neq \phi$

$\Rightarrow N(x, r) \not\subset A \;\; \Rightarrow x \notin \text{int } A$

$\Rightarrow x \in (\text{int } A)' \;\; \Rightarrow (\text{int } A)' = \overline{A'}$

$\Rightarrow \text{int } A = (\overline{A'})'$

③ Exchanging A and A' in ②, we have:

$\text{int } A' = (\overline{A})' \;\; \Rightarrow \;\; \overline{A} = (\text{int } A')'$

④ Let $\delta = \rho(a, x) - r$, ie the distance from x to the spher. nghbhd $N(a, r)$.

Then $K(a, r) \cap N(x, \delta) = \phi$

(if not $\exists c \in K(a, r) \cap N(x, \delta)$

then $\rho(a, c) < r$ and $\rho(x, c) < \delta$

but $\rho(a, x) = r + \delta$, contradiction of Δ inequality)

$\Rightarrow N(x, \delta) \subset K(a, r)'$

3.10) <u>DEFINITION</u>: – A subset A of a metric space E is said to be <u>dense</u> in E if $\overline{A} = E$

– A subset A of a metric space E is said to be <u>nowhere dense</u> if $\text{int } \overline{A} = \phi$

3.11) <u>DEFINITION</u>: Let (E, ρ) be a metric space. Suppose $E_0 \subset E$. Then E_0 with the metric ρ is called a <u>subspace</u> of E. The metric on E_0 is $\rho|_{E_0 \times E_0}$, read ρ <u>restricted to</u> $E_0 \times E_0$

3.12) THEOREM: $E_0 \subset E$, ρ a metric

① If $N_0(a,r)$ denotes the neighbourhood in E_0, then
$$N_0(a,r) = N(a,r) \cap E_0$$

② a set G_0 is open in E_0 iff $G_0 = G \cap E_0$ where G is open in E.

③ If $A \subset E_0$ then $\bar{A}$ (closure in E_0) $= E_0 \cap \bar{A}$ (closure in E)

PROOF: ① $N_0(a,r) = \{x \mid x \in E \wedge \rho(a,x) < r\} \cap E_0$
$= N(a,r) \cap E_0$ by defn.

② Every open set is a union of spherical neighbourhoods.
If G_0 is open in E_0,
$$G_0 = \bigcup_{i \in I} N_0(a_i, r_i)$$
$$= \bigcup_{i \in I} [N(a_i, r_i) \cap E_0]$$
$$= \left[\bigcup_{i \in I} N(a_i, r_i)\right] \cap E_0$$
$= G \cap E_0$, where G is open in E.

③

4 COMPLETENESS

4.1) DEFINITION: A sequence $\{a_n\}$ in a metric space (E,ρ) is said to ~~be~~ converge to a point $a \in E$ if $\rho(a_n, a) \to 0$
We write $a_n \to a$

NOTES:

① This is equivalent to:
$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ni n > N \Rightarrow \begin{cases} \rho(a_n, a) < \varepsilon \\ a_n \in N(a, \varepsilon) \end{cases}$$

For every open set G containing a $\exists N \in \mathbb{N} \ni n > N \Rightarrow a_n \in G$

Proof: If G is an open set with $a \in G$
then $\exists \varepsilon > 0 \ni N(a,\varepsilon) \subset G$
By assumption, $\exists N$ such that $n > N \Rightarrow a_n \in N(a, \varepsilon)$
So $n > N \Rightarrow a_n \in G$
$\Rightarrow$ every spherical neighbourhood is an open set.

② If $a_n \to a$ and $a_n \to b$, then $a = b$
Proof Suppose $a \neq b$. Then $\delta = \rho(a,b) > 0$.
Consider $N(a, \frac{\delta}{2})$ and $N(b, \frac{\delta}{2})$
$\exists N_1 \in \mathbb{N} \ni n > N_1 \Rightarrow a_n \in N(a, \frac{\delta}{2})$
$\exists N_2 \in \mathbb{N} \ni n > N_2 \Rightarrow a_n \in N(b, \frac{\delta}{2})$
If $n > \max(N_1, N_2) \Rightarrow a_n \in N(a, \frac{\delta}{2})$ and $a_n \in N(b, \frac{\delta}{2})$
But $N(a, \frac{\delta}{2})$ and $N(b, \frac{\delta}{2})$ are disjoint by $\triangle$ inequality
contradiction $\Rightarrow a = b$.

③ If $a_n = a$ for all n, then $a_n \to a$
If $a_n = a$ for all but finitely many n, then $a_n \to a$

④ If (a_n) is a sequence in a metric space and every subsequence has a sub-subsequence $\to a$, then $(a_n) \to a$

Examples

① The metric space $P[0,1]$ with $\rho(p,q) = \underset{[0,1]}{\text{lub}} |p(x) - q(x)|$

The sequence of polynomials $p_n(t) = 1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!}$

is a Cauchy sequence in $P[0,1]$. But it converges to e^t which is not a polynomial.

NB? ④ If $a_n \to a$ then $a_{n_k} \to a$, where (a_{n_k}) is a subsequence of (a_n)

⑥ If $a \in \bar{A}$, then $\exists\, a_n \in A \ni a_n \to a$.

Proof: Since $a \in \bar{A}$, $N(a, r) \cap A \neq \phi$ for all $a \in \bar{A}$

Consider $N(a_n, \frac{1}{n})$

For each n, $\exists\, a_n \in N(a, \frac{1}{n}) \cap A$

Since $\rho(a, a_n) < \frac{1}{n} \to 0 \Rightarrow a_n \to a$.

⑦ If $a_n \to a$ and $b_n \to b$ then $\rho(a_n, b_n) \to \rho(a, b)$

Proof: $\rho(a_n, b_n) - \rho(a,b) \leq \rho(a_n, a) + \rho(a,b) + \rho(b, b_n) - \rho(a,b)$

$= \rho(a_n, a) + \rho(b, b_n)$

Similarly $\rho(a,b) - \rho(a_n, b_n) \leq \rho(a_n, a) + \rho(b, b_n)$

So $|\rho(a_n, b_n) - \rho(a,b)| \leq \rho(a_n, a) + \rho(b, b_n) \to 0$.

Note: This says that ρ is continuous

⑧ If $a_n \to a$ then $\forall \varepsilon > 0\ \exists N \ni m, n > N \Rightarrow \rho(a_m, a_n) < \varepsilon$.

Proof: Since $a_n \to a$, we can find N such that

$\rho(a_n, a) < \varepsilon$ for $n > N$

So if $m > N$ and $n > N$, then

$\rho(a_m, a_n) \leq \rho(a_m, a) + \rho(a_n, a) < 2\varepsilon$

4.2 DEFINITION: A sequence of points a_n in a metric space is said to be a Cauchy sequence if $\forall \varepsilon > 0$,

$\exists N \ni \rho(a_m, a_n) < \varepsilon$ for $m, n > N$.

NOTE: ① Every convergent sequence is a Cauchy sequence

② The converse may be false (eg in the rationals, the sequence 1.4, 1.41, 1.414, etc is a Cauchy sequence which is not convergent (converges to $\sqrt{2}$, a non-rational)

4.3 DEFINITION: A metric space in which every Cauchy sequence is convergent, is said to be complete.

NOTES: ① $\mathbb{Q}$ and $P[0,1]$ are not complete

② If a Cauchy sequence $\{a_n\}$ has a subsequence which converges to a, then $a_n \to a$.

Proof: Suppose $a_{n_k} \to a$, where $\{a_n\}$ is a Cauchy sequence. Then, given $\varepsilon > 0$ we can find N such that $\rho(a_m, a_n) < \frac{\varepsilon}{2}$ for $m, n > N$. Also, we can find N' such that $\rho(a_{n_k}, a) < \frac{\varepsilon}{2}$ for $n_k > N'$.

Then if $n > \max(N, N')$, and $n_k > N'$ then:

$$\rho(a_n, a) \leq \rho(a_n, a_{n_k}) + \rho(a_{n_k}, a) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

③ If $a_n \in A$ and $a_n \to a$ then $a \in \bar{A}$

4.4 THEOREM (CANTOR): A metric space (E, ρ) is complete ~~iff for~~ every sequence of closed spherical neighbourhoods $K(a_n, r_n)$ which are contracting (ie $K(a_n, r_n) \subset K(a_{n-1}, r_{n-1})$) with $r_n > 0$, we have

$$\bigcap_{n=1}^{\infty} K(a_n, r_n) \neq \phi$$

PROOF: Suppose that (E, ρ) is complete, and $\{K(a_n, r_n)\}$ is a contracting sequence with $r_n \to 0$. We show that: (a) $\{a_n\}$ is a Cauchy sequence

(b) by completeness, $a_n \to a$

(c) $a \in K(a_n, r_n)$, $\forall n$.

Examples

① In $C[a,b]$ with $\rho(f,g) = \underset{[a,b]}{\text{lub}} |f(x) - g(x)|$

Note that $f_n \to f$ in this metric space means that f_n converges uniformly to f. We can say $f_n \to f$ iff $\rho(f, f_n) \to 0$

Also $C[a,b]$ with metric ρ is complete (cf course 2, convergence)
Suppose $\{f_n\}$ is a Cauchy sequence in $C[a,b]$. For each x, $|f_n(x) - f_m(x)| \leq \underset{[a,b]}{\text{lub}} |f_n(t) - f_m(t)|$

and it follows that $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$, which is a complete metric space, as a uniform limit of a sequence of continuous functions is continuous (ie $f \in C[a,b]$).

(a) Let $\varepsilon > 0$.

Choose $N \ni r_n < \varepsilon$ for $n > N$

If $m > n > N \Rightarrow a_m \in K(a_m, r_m) \subset K(a_n, r_n)$

so $\rho(a_m, a_n) \leq r_n < \varepsilon$, ie, sequence is a C. seq

(b) Since E is complete, we have $a_n \to a$ (every C seq over a complete space is convergent).

(c) For each m, $a_m \in K(a_n, r_n)$, if $m > n$

Let $m \to \infty$ and keep n fixed.

Since $a_m \to a$ and $K(a_n, r_n)$ is closed,

$a \in K(a_n, r_n), \forall n$

$\Rightarrow a \in \bigcap_{n=1}^{\infty} K(a_n, r_n)$ so this gives us: $\bigcap_{n=1}^{\infty} K(a_n, r_n) \neq \emptyset$.

To prove that E is complete, let $\{a_n\}$ be a Cauchy sequence in E. For each K, $\exists N_K \ni$

$m, n > N_K \Rightarrow \rho(a_m, a_n) < \frac{1}{2^{k+1}}$

Consider the closed spherical neighbourhoods $K\left(a_{N_k}, \frac{1}{2^{k+1}}\right)$

This means $K\left(a_{N_{k+1}}, \frac{1}{2^{k+2}}\right) \subset K\left(a_{N_k}, \frac{1}{2^{k+1}}\right)$

PROOF: Let $x \in LHS$. Then $\rho(x, a_{N_k}) \leq \underbrace{\rho(x, a_{N_{k+1}})}_{\leq \frac{1}{2^{k+2}}} + \underbrace{\rho(a_{N_{k+1}}, a_{N_{k+2}})}_{< \frac{1}{2^{k+2}}}$

$\Rightarrow \rho(x, a_{N_k}) \leq \frac{1}{2^{k+1}}$

By hypothesis, $\exists a \in \bigcap_{k=1}^{\infty} K\left(a_{N_k}, \frac{1}{2^{k+1}}\right)$ and clearly $a_{N_k} \to a$

So $\{a_n\}$ is a Cauchy sequence with a subsequence converging to a. So $a_n \to a$.

4.5 LEMMA: If a set A is nowhere dense (i.e. $\text{int}\,\overline{A} = \phi$) then every non-empty open set G contains a non-empty open set G' such that $G' \cap A = \phi$.

PROOF: Suppose that $\text{int}\,\overline{A} = \phi$.
Let G be an open set.
Since $\text{int}\,\overline{A} = \phi$, $\exists\, x \in G \ni x \notin \overline{A}$ [Otherwise $G \subset \overline{A}$ and $\text{int}\,\overline{A} \supset G \neq \phi$

Since $x \notin \overline{A}$, we can find $\varepsilon > 0 \ni N(x, \varepsilon) \cap A = \phi$
Now $x \in G$ (open), so we can find $\delta > 0 \ni N(x, \delta) \subset G$

Let $G' = N(x, \min\{\varepsilon, \delta\})$. Then $G' \subset G$ and $G' \cap A = \phi$ and G' is open.

4.6 THEOREM (BAIRE'S): A complete metric space cannot be expressed as a countable union of nowhere dense sets.

PROOF: Let (E, ρ) be a complete metric space.
Suppose $E = \bigcup_{n=1}^{\infty} A_n$, where each A_i is nowhere dense.

Since A_1 is nowhere dense, E (which is open) contains an open set $N(x_1, r_1')$ such that $N(x_1, r_1') \cap A_1 = \phi$.
Let $r_1 = \frac{1}{2} r_1'$
Then since A_2 is nowhere dense, E contains an open set $N(x_1, r_1)$ which contains an open set $N(x_2, r_2')$ such that $N(x_2, r_2') \cap A_2 = \phi$.
Let $r_2 = \min\{\frac{1}{2} r_2', \frac{1}{2} r_1\}$
Note that $K(x_1, r_1) \subset N(x_1, r_1')$
and in general $K(x_i, r_i) \subset N(x_i, r_i')$

We construct a sequence of neighbourhoods $K(x_k, r_k)$ such that: $K(x_k, r_k) \cap A_k = \phi$

and $K(x_{k+1}, r_{k+1}) \subset K(x_k, r_k)$

and $r_{k+1} \leq \frac{1}{2} r_k$ (so $k \to \infty \Rightarrow r_k \to 0$)

By Cantor's Intersection Theorem (4.4), since E is complete,
$\exists\, a \in \bigcap_{i=1}^{\infty} K_i(x_i, r_i)$

Now $E = \bigcup_{i=1}^{\infty} A_i$ so $a \in A_n$ for some n
but $a \in K(x_n, r_n)$
which contradicts $K(x_n, r_n) \cap A_n = \phi$

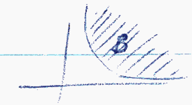
So $E \neq \bigcup_{i=1}^{\infty} A_i$, ie, E cannot be expressed as a countable union of nowhere dense sets.

<u>Note</u> Every metric space can be "completed" by embedding it into a complete metric space.

Examples

① $(0,1)$ and $(1,2)$ are seperated

② In R, $A = \{(x,y) \mid y = 0\}$
$B = \{(x,y) \mid x > 0 \text{ and } y > \frac{1}{x}\}$



are seperated sets: $\bar{A} = A$, $\bar{B} = \{(x,y) \mid x > 0 \text{ and } y \geq \frac{1}{x}\}$

③ In $\mathbb{R}$ suppose $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$. These are <u>not</u> seperated (as $\bar{A} = \mathbb{R}$)

§5 Connected Sets

5.1 **Definition:** Two sets A and B in a metric space are said to be separated if:

$$A \cap \overline{B} = \overline{A} \cap B = \phi$$

Notes: ① Separated sets are obviously disjoint (converse not nec. true)

② If $\rho(A,B) > 0$, then A and B are separated. The converse is false – in ex ②, $\rho(A,B) = 0$ but A, B are separated.

③ If A, B are separated, and $A_1 \subset A$, $B_1 \subset B$, then A_1, B_1 are separated.

④ Two closed sets are separated iff they are disjoint.

⑤ Two open disjoint sets are separated:

Proof: Suppose $A \cap B = \phi$, A, B open.
Then $A \subseteq B'$ which is closed
$\Rightarrow \overline{A} \subseteq B' \Rightarrow \overline{A} \cap B = \phi$ and similarly $A \cap \overline{B} = \phi$
$\Rightarrow A, B$ separated

⑥ If O is an open set, and $O = A \cup B$ where A, B are separated, then both A and B are open sets.

Proof: If $z \in A$, then since $A \cap \overline{B} = \phi \Rightarrow z \notin B$
So $z \notin \overline{B}$, an open set. But $z \in O$
$\Rightarrow z \in O \cap \overline{B}'$, an open set $\subset A$
$\Rightarrow z \in \operatorname{int}(A)$
Thus every $z \in A$ is an interior point $\Rightarrow A$ is open. (simil. for B)

⑦ If F is a closed set, and $F = A \cup B$ where A, B are separated, then both A and B are closed.

Proof: If $z \in \overline{B}$, then $z \in \overline{F} = F$
But $A \cap \overline{B} = \phi$. So $z \notin A \Rightarrow z \in B$.
$\Rightarrow B = \overline{B}$ is closed. Similarly, A is closed.

5.2 DEFINITION. A subset A of a metric space E is said to be *connected* if it cannot be expressed as the union of two non-empty seperated sets. A set is *disconnected* if it can be expressed in this way.

5.3 THEOREM. Suppose we have a metric space (E, ρ), then:

(i) E is connected iff the only non-empty subset of E which is both open and closed, is E itself.

(ii) A subset A of E is connected iff the only non-empty subset of A which is both open and closed in A (ie regarding A as a subspace of E) is A itself.

PROOF: (i) Suppose E is connected, and B is a subset of E, $B \neq \phi$, B both open and closed. Let $A = B'$, ie, A both closed and open. And $E = A \cup B$.

Now, $\bar{A} \cap B = A \cap B = \phi = A \cap \bar{B}$

So A, B are seperated and disjoint

Since E is connected, either A or $B = \phi$

But $B \neq \phi \Rightarrow B = E = A'$ ($A = \phi$).

(ii) If A is not connected, then $A = B \cup C$ where B, C are seperate and non-empty.

Then $A = (\bar{B} \cap A) \cup (\bar{C} \cap A)$

$(\bar{B} \cap A)$ and $(\bar{C} \cap A)$ are both closed in A, and their complements in A are open. Note however, that $(\bar{B} \cap A)' = (\bar{C} \cap A)$, hence they are both open and closed in A, and they are both non-empty (this proves the ~~negation~~ converse of (ii))

Let B be non-empty, $B \subset A$, B open and closed in A.
Then $A \cap B'$ is closed and open in A.
Now, $A = B \cup (A \cap B')$
and $\overline{B} \cap (A \cap B') = (\overline{B} \cap A) \cap B' = B \cap B' = \phi$
Similarly, $B \cap (\overline{A \cap B'}) = \phi$.

So if A is connected, either B or $(A \cap B')$ is empty.
But B is non-empty by defn, so $A \cap B' = \phi$, ie, $B = A$.

5.4 THEOREM: ① If A is connected, then so is $\overline{A}$

② If $A = B \cup C$, B, C seperated, and D is a connected subset of A, then $D \subset B$ or $D \subset C$.

③ If A is connected and $A \subset B \subset \overline{A}$, then B is connected.

④ If A, B are connected, and $A \cap B \neq \phi$, then $A \cup B$ is connected.

⑤ If A and B are connected and not seperated, then $A \cup B$ is connected.

PROOFS: ① Suppose A is connected, and $\overline{A} = B \cup C$ where B, C are non-empty and seperated.
By note 7, B and C are closed.
Now, $A = A \cap \overline{A} = A \cap (B \cup C)$
$= (A \cap B) \cup (A \cap C)$
and $(A \cap B)$ and $(A \cap C)$ are closed in A (as B, C are closed)
By THM 5.3 ②, either B or C is empty in A
The contradiction implies $\overline{A}$ is connected.

② If $D \cap C$ and $D \cap B$ are both non-empty, then
$(\overline{D \cap B}) \cap (D \cap C) \subset \overline{B} \cap C = \phi$
and $(D \cap B) \cap (\overline{D \cap C}) \subset B \cap \overline{C} = \phi$
So $D = (D \cap B) \cup (D \cap C)$ is seperated, non-empty, contradicts connectivity of D

③ Suppose A is connected, $A \subset B \subset \overline{A}$, and B is not connected. Then $B = B_1 \cup B_2$ where B_1, B_2 are non-empty and seperate.

ie, $A \subset B_1 \cup B_2 \subset \overline{A}$

By ②, $A \subset B_1$ (or $A \subset B_2$).

But then $\overline{A} \subset \overline{B_1}$

We have $\overline{B_1} \cap B_2 = \phi$; $B_1, B_2 \subset A$

But as $\overline{A} \subset \overline{B_1} \Rightarrow \overline{A} \cap B_2 = \phi$

Hence $B_2 = \phi$, ie $B_1 = B$, so B is connected.

⑤ Suppose A, B are each connected and not seperated.

If $A \cup B$ is not connected,

then $A \cup B = C \cup D$ where C, D are non-empty seperated sets

Now $A = A \cap (C \cup D) = (A \cap C) \cup (A \cap D)$

Since A is connected, and $(A \cap C), (A \cap D)$ are seperated, either $(A \cap C)$ or $(A \cap D)$ is empty.

Thus $A \subset D$ or $A \subset C$, and similarly for B.

There are 4 possibilities (2 by symmetry).

(i) $A \subset C \wedge B \subset C$

(ii) $A \subset C \wedge B \subset D$

Both of these produce contradictions, hence $A \cup B$ is connected.

④ This follows from ⑤.

We shall in fact prove that if $\{A_\alpha\}_{\alpha \in I}$ is a collection of connected sets, and $\bigcap_{\alpha \in I} A_\alpha \neq \phi$, then $\bigcup_{\alpha \in I} A_\alpha$ is connected.

Suppose $B \subset \bigcup_{\alpha \in I} A_\alpha$ is a non-empty set, which is

both open and closed in $\bigcup_{\alpha \in I} A_\alpha$. We must prove $B = \bigcup_{\alpha \in I} A_\alpha$

Since $B \neq \phi$, $B \cap A_{\alpha_0} \neq \phi$ for some α_0. Now B is open and closed in A_{α_0}, which is connected. So $B \cap A_{\alpha_0} = A_{\alpha_0}$, ie $A_{\alpha_0} \subset B$.
Let $z \in \bigcap_{\alpha \in I} A_\alpha$. Then $z \in A_{\alpha_0} \subset B \Rightarrow z \in B$.

Thus for every α, $A_\alpha \cap B \neq \phi$
The same reasoning as above gives $A_\alpha \cap B$ is non-empty, open and closed in A_α which is connected, so $A_\alpha \cap B = A_\alpha$ all α
so $A_\alpha \subset B$, all α

$\Rightarrow \bigcup_{\alpha \in I} A_\alpha \subset B$ but $B \subset \bigcup_{\alpha \in I} A_\alpha \Rightarrow B = \bigcup_{\alpha \in I} A_\alpha$

55. THEOREM: Let (E, ρ) be a metric space. Then E may be expressed in just one way as a union of closed connected sets which are mutually disjoint.

PROOF: In E, we define an equivalence relation R by:
$a R b \Leftrightarrow a$ and b are in a connected subset of E
(i) $a R a$ as $\{a\}$ is connected
(ii) $a R b \Leftrightarrow b R a$ obvious
(iii) $a R b$ and $b R c \Rightarrow a R c$
(If $a, b \in A$ and $b, c \in B$, then $a, c \in A \cup B$ which is connected, since $A \cap B \supseteq \{b\} \neq \phi$).

Let $R(a)$ be the equivalence class of a.

$R(a) = \bigcup_{\alpha \in I} A_\alpha$ where A_α is connected and $a \in A_\alpha$

so $R(a)$ is connected, as $\bigcap_{\alpha \in I} A_\alpha \neq \phi$
$\supseteq \{a\}$

These $R(a)$'s are also closed

($R(a)$ connected $\Rightarrow \overline{R(a)}$ connected, but $\overline{R(a)}$ is a connected set containing $\{a\}$, hence $\overline{R(a)} \subseteq R(a) \Rightarrow \overline{R(a)} = R(a)$).

Also, each $R(a)$ is maximal (ie, there is no bigger connected set, ie, C connected, $C \supseteq R(a) \Rightarrow C = R(a)$)

Thus $E = \bigcup_{i \in I} E_i$, E_i closed, connected and maximal (*)

To prove unique, suppose $E = \bigcup_{j \in J} F_j$, F_j's closed, connected, max.

then for each i, $\exists j$ such that $E_i \cap F_j \neq \emptyset$
but as these are non-empty, connected sets,
$E_i \cup F_j$ is closed and connected.
But as they are maximal, $E_i = F_j$

5.6 THEOREM (1) $\mathbb{R}$ is connected
(2) Every real interval is connected
(3) Every connected subset of $\mathbb{R}$ is an interval over $\mathbb{R}$.

PROOF (1) Suppose $\mathbb{R}$ is not connected,
ie, $\mathbb{R} = A \cup B$ where A, B nonempty & separated.

Let $a_1 \in A$, $b_1 \in B$. Consider $c = \frac{1}{2}(a_1 + b_1)$

$$\text{If } \begin{cases} c \in A, & \text{let } a_2 = c, \; b_2 = b_1 \\ c \in B, & \text{let } a_2 = a_1, \; b_2 = c \end{cases} \text{ then } \begin{cases} a_1 \leq a_2 \\ b_1 \geq b_2 \end{cases}$$

and $|a_2 - b_2| = \frac{1}{2}|a_1 - b_1|$ (*)

Repeat this process giving the sequence (a_n, b_n) But from (*), we have $\rho(a_n, b_n) \to 0$ as $n \to \infty$, so $a_n \to a$, $b_n \to b$
But $a_n \in A$, $b_n \in B \Rightarrow a \in \bar{A}$, $b \in \bar{A}$. But $a = b$ in the limit, hence A, B are not separate. $\Rightarrow \mathbb{R}$ is connected.

② Same argument as ① but using an interval of $\mathbb{R}$.

③ If C is connected, but not an interval, there are points $a, b \in C$ with $a < c < b$ such that $c \notin C$.

Then $\left\{\begin{matrix} C \cap (-\infty, c) \\ C \cap (c, \infty) \end{matrix}\right\}$ are non-empty, seperated & open in C.

So C cannot be connected $\Rightarrow$ hence C must be an interval

Examples

① $\{(n, n+2) : n \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$.
As each midpoint $n+1$ is covered only once by the appropriate interval, there are no subcovers, and the cover is not finite. Note $\mathbb{R}$ is not compact.

② $\{(\frac{1}{n}, 1 - \frac{1}{n}) : n \in \mathbb{N}, n \geqslant 3\}$ is an open cover of $(0, 1)$.
We can eliminate any finite number of these and be left with a subcover, as the intervals are expanding. However, there is no finite subcover. Note intervals of $\mathbb{R}$ not compact.

③ In $\mathbb{R}^2$, $\{N(\underline{0}, n) : n \in \mathbb{N}\}$ is an open cover of expanding spherical neighbourhoods about the origin $\underline{0}$. Subcovers as for ②

§6 Compact Sets.

6.1 Definition: A cover of a set A is a collection of sets $\{C_i\}_{i\in I}$ such that $\bigcup_{i\in I} C_i \supseteq A$.

An open cover of A is a cover with all C_i's open.

A finite cover of A is a cover with I finite

A subcover of a cover $\{C_i\}_{i\in I}$ of A is a collection $\{C_k\}_{k\in K}$, where $K\subset I$ and $\{C_k\}_{k\in K}$ covers A.

6.2 Definition: A subset A of a metric space (E,ρ) is said to be compact if every open cover of A has a finite subcover.

6.3 Theorem: ① A compact set is bounded and closed.

② If A is compact, and $B\subset A$, B closed, then B is compact.

Proof ① Suppose A is compact in (E,ρ). Let $x_0\in E$.
Then $\bigcup_{n=1}^{\infty} N(x_0,n)\supseteq A$ (an open cover of A).

Since A is compact, this open cover has a finite subcover $\{N(x_0,n_1),\dots,N(x_0,n_k)\}$
Let $R=\max\{n_1,\dots,n_k\}$
then $A\subset N(x_0,R)$. Hence $\delta(A)\le 2R \Rightarrow A$ is bounded.

Assume $a\notin A$. For each point $x\in A$ we can find disjoint open sets V_x, U_x with $x\in V_x$, $a\in U_x$.
Then $\{V_x \mid x\in A\}$ is an open cover of A.
Since A is compact, we can extract a finite subcover

Examples

① $\mathbb{R}$ is separable, as $\mathbb{Q}$ is a countable dense subset (although $\mathbb{R}$ is not compact).

ie, there are set $V_{x_1} \dots V_{x_n}$ such that $\bigcup_{i=1}^{n} V_{x_i} \supset A$

Consider $\bigcap_{i=1}^{n} U_{x_i}$, an open set containing a.

Then $\left(\bigcap_{i=1}^{n} U_{x_i}\right) \cap \underbrace{\left(\bigcup_{i=1}^{n} V_{x_i}\right)}_{\supset A} = \phi$.

$\Rightarrow \bigcap_{i=1}^{n} U_{x_i} \cap A = \phi$

Hence $a \notin \bar{A} \Rightarrow \bar{A} \subseteq A$, hence A is closed.

② Let $B \subset A$, B closed, A compact.

Let $\{U_i \mid i \in I\}$ be an open cover of B.

Then $\{U_i \mid i \in I\} \cup \{B'\}$ is an open cover of A

But A is compact, so there is a finite subcover say $\{U_{i_1}, U_{i_2}, \dots, U_{i_n}, B'\}$, which covers A.

Since $B \subset A$ and $B \cap B' = \phi$, $\{U_{i_1}, U_{i_2}, \dots, U_{i_n}\}$ is a finite subcover of B
$\Rightarrow B$ is compact.

6.4 THEOREM: A compact metric space is seperable (ie, has a countable dense subset)

PROOF: Suppose (E, ρ) is compact.
Consider all neighbourhoods of radius $\frac{1}{k}$, $k \in N$.
In fact, $E = \bigcup_{x \in E} N(x, \frac{1}{k})$

As E is compact, a finite subcover exists, so $E = \bigcup_{i=1}^{n_k} N(x_i^k, \frac{1}{k})$

Then the set $\{x_1^1, \ldots, x_{n_1}^1, x_1^2, \ldots, x_{n_2}^2, x_1^3, \ldots\} = A$
is countable and dense in E

for each $x \in E$ and each $\varepsilon > 0$,
$\exists\, y \in A \ni \rho(x,y) < \varepsilon$.

6.5 <u>DEFINITION</u>: A subset A of a metric space (E, ρ) is said to be <u>sequentially</u> <u>compact</u> if every sequence of points in A has a subsequence which converges to a point of A.

6.6. <u>THEOREM</u>: A set A is sequentially compact iff every infinite subset of A has an accumulation point in A.

<u>PROOF</u>: Let B be an infinite subset of A. Then B contains a sequence of distinct points $b_1, b_2, \ldots, b_n, \ldots$
Assume A is sequentially compact. Then this sequence contains a subsequence converging to a point in A

END OF 1ST QUARTER

The limit point b is a point of accumulation of A.

<u>Converse</u> Let $\{a_n\}$ be a sequence of points in A.
If $\{a_n\}$ contains a point which is repeated infinitely many times, i.e., $a_{n_k} = a$ for some subsequence, then $\{a_{n_k}\}$ is convergent to $a \in A$.
If there is no such point, then $\{a_n\}$ forms an infinite set with a point of accumulation a.
From A we can select a sequence converging to a
This gives a convergent subsequence, as required
(as if $a \in \bar{A}$, then $\exists\, a_n \in A \ni a_n \to a$)

6.7 THEOREM: A compact set in a metric space is sequentially compact.

PROOF: Let A be compact. Suppose that B is an infinite subset of A, with no point of accumulation in A.

Then for each $a \in A$, a is not an accumulation point of B, ie, $(N(a,r) \setminus \{a\}) \cap B = \phi$

So $\forall a \in A$, $\exists r_a > 0$ such that $(N(a, r_a) \setminus \{a\}) \cap B = \phi$

But $A \subseteq \bigcup_{a \in A} N(a, r_a)$

and A is compact, so this open cover has a finite subcover.

ie, $A \subset \bigcup_{i=1}^{n} N(a_i, r_{a_i})$

Each of the $N(a_i, r_{a_i})$ contains at most one point of B, (the a_i itself), so B contains at most n points. But B is infinite, so B must have accumulation point(s) in A.

6.8 COROLLARY: A (sequentially) compact metric space is complete

PROOF: Let $\{a_n\}$ be a Cauchy sequence in E, a compact metric space. Since E is sequentially compact (by 6.7), $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$.
This implies (by a previous completeness argument), that $\{a_n\}$ is convergent $\Rightarrow E$ is complete.

6.9 DEFINITION: A subset A of a metric space E is said to be totally bounded if, for every $\varepsilon > 0$, A can be covered by a finite number of spherical neighbourhoods of radius ε. (also called precompact)

6.10. THEOREM: A sequentially compact metric space is totally bounded.

PROOF: Let E be sequentially compact. Let $\varepsilon > 0$.

Let $a_1 \in E$. If $E \subset N(a_1, \varepsilon)$ then proved.

If not, then $\exists\, a_2 \in E \ni a_2 \notin N(a_1, \varepsilon)$

Consider $N(a_1, \varepsilon) \cup N(a_2, \varepsilon)$. Either this contains E (and we have proved 6.10), else, $\exists\, a_3 \in E \ni a_3 \notin N(a_1, \varepsilon) \cup N(a_2, \varepsilon)$, etc.

This process must terminate, else we would have a sequence $\{a_n\}$ of distinct points all further than ε apart. This sequence could not have a convergent subsequence, so E cannot be sequentially compact.

6.11. THEOREM: A totally bounded complete space is sequentially compact.

PROOF: Let $\{a_n\}$ be a sequence in a metric space E. We shall construct a convergent subsequence.

Since E is totally bounded, $E \subset \bigcup_{i=1}^{n} N(a_i, 1)$, some finite n.

At least one of these neighbourhoods (say N_1), will contain a subsequence $\{a_{1n}\}$ of $\{a_n\}$ (as $\{a_n\}$ is infinite and so at least one neighbourhood must contain an infinite number of a_n's.

But E (and N_1) can be covered by a finite number of spherical neighbourhoods of radius $1/2$. At least one of these neighbourhoods (say N_2) will contain a subsequence $\{a_{2n}\}$ of $\{a_{1n}\}$.

In general, we can construct a neighbourhood N_i of radius $1/i$ containing an infinite subsequence $\{a_{in}\}$.

Consider the diagonal sequence $\{a_{nn}\}$, a subsequence of a_n. We show that $\{a_{nn}\}$ converges by showing it to be Cauchy, and then using E's completeness.

In general all of the terms of $\{a_{nn}\}$ starting with a_{kk} are contained in N_k, ie, they are all within $2/k$ of each other.

Thus, for any $\varepsilon > 0$, we can choose k such that $2/k < \varepsilon$. Then the terms $\{a_{kk}, a_{k+1\,k+1}, \ldots\}$ are all in N_k and hence $\rho(a_{ii}, a_{jj}) < 2/k < \varepsilon \quad \forall i, j > k$.

So $\{a_{kk}\}$ is Cauchy and thus convergent by E's completeness.

6.12 THEOREM: Let E be a sequentially compact metric space. If $\{O_i\}_{i\in I}$ is an open cover of E, then $\exists\, \varepsilon > 0$ such that every spherical neighbourhood in E of radius ε is contained in some O_i.

PROOF: Suppose the result is false. Then $\forall \varepsilon > 0$,
$\exists N(a_\varepsilon, \varepsilon) \not\subset O_i, \ \forall i \in I$.

So for each n, $\exists N(a_n, \frac{1}{n})$ not contained in any O_i.
Since E is sequentially compact, $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$, $a_{n_k} \to a \in E$.

Since $\{O_i\}_{i\in I}$ covers E, $a \in O_{i_0}$ for some $i_0 \in I$ and since O_i is open, $\exists\, n_0$ such that $N(a, \frac{2}{n_0}) \subset O_{i_0}$

Since $a_{n_k} \to a$, all but finitely many of the sequence $\{a_{n_k}\}$ lie in $N(a, \frac{1}{n_0})$. So we can find $n_j > n_0$ so that $a_{n_j} \in N(a, \frac{1}{n_0})$

Then $N(a_{n_j}, \frac{1}{n_j}) \subseteq \underbrace{N(a_{n_j}, \frac{1}{n_0}) \subseteq N(a, \frac{2}{n_0})}$ $\subseteq O_i$

If $x \in N(a_{n_j}, \frac{1}{n_0})$, then $\rho(x, a) \leq \underbrace{\rho(x, a_{n_j})}_{< \frac{1}{n_0}} + \underbrace{\rho(a_{n_j}, a)}_{< \frac{1}{n_0}}$

which contradicts our assumption.

6.13 THEOREM: A sequentially compact metric space is compact.

PROOF: Let $\{O_i\}_{i\in I}$ be an open cover of E. By 6.12, $\exists\, \varepsilon > 0$ such that every $N(\ ,\varepsilon) \subseteq O_i$, some $i \in I$.

Since E is totally bounded, E is contained in a finite union of neighbourhoods of radius ε

ie, $E \subset \bigcup_{k=1}^{n} N(a_k, \varepsilon)$

For each k, $N(a_k,\varepsilon) \subset O_{i_k}$, hence $E \subset \bigcup_{k=1}^{n} O_{i_k}$,

a finite subcover, ie, E is compact.

NOTE: We have:

$$\text{TOTALLY BOUNDED} \; \left\{ \begin{array}{c} \text{COMPACT} \\ \Leftarrow \quad \Updownarrow \quad \Rightarrow \\ \text{SEQ. COMPACT} \end{array} \right\} \; \text{COMPLETE}$$

6.14 THEOREM: A metric space is compact $\Leftrightarrow$ for every family of closed sets with the "finite intersection property" (ie, every finite subfamily has a non-empty intersection) has a non-empty intersection.

PROOF: $\Rightarrow$:

Suppose E is compact, $\{F_i\}_{i\in I}$ a family of closed sets in E with the finite intersection property.

Assume $\bigcap_{i\in I} F_i = \phi \Rightarrow \left(\bigcap_{i\in I} F_i\right)' = E$

$\Rightarrow E = \bigcup_{i \in I} F_i'$ (an open cover of E, which is compact)

So there is a finite subcover, i.e.,

$$E = \bigcup_{k=1}^{n} F_{i_k}'$$

Taking complements, $\bigcap_{k=1}^{n} F_{i_k} = \phi$, contradicting finite $\cap$ property.

$\Longleftarrow$:

Let $\{O_i\}_{i \in I}$ be an open cover of E. We must find a finite subcover. If there is no finite subcover, then for every finite subfamily $O_{i_1}, \ldots, O_{i_n}$,

$$\bigcup_{k=1}^{n} O_{i_k} \neq E \implies \bigcap_{k=1}^{n} O_{i_k}' \neq \phi$$

Thus $\{O_i'\}_{i \in I}$ forms a family of closed sets with f.i.p.

$$\Rightarrow \bigcap_{i \in I} O_i' \neq \phi \Rightarrow \bigcup_{i \in I} O_i \neq E$$

But $\bigcup_{i \in I} O_i$ is an open cover of E, and equals E.

So there is a finite subcover, and E is compact.

Examples

① In $\mathbb{R}$ every closed bounded interval is compact.
Let $[a, b]$ be a closed bounded interval. We prove that every infinite subset A of $[a, b]$ has an accumulation point $\overset{\text{THM 6.6}}{\Longrightarrow}$ sequentially compact $\Rightarrow$ compact.

PROOF: Let $c = \frac{1}{2}(a+b)$. Then one of $[a, c]$ $[c, b]$ contains infinitely many points of A.
Whichever it is, call it $[a_1, b_1]$ and choose a $x_1 \in A \cap [a_1, b_1]$

Repeating, one of $\left[a_1, \frac{a_1+b_1}{2}\right]$, $\left[\frac{a_1+b_1}{2}, b_1\right]$ contains infinitely many points of A.
Call it $[a_2, b_2]$ and choose $x_2 \in A \cap [a_2, b_2]$
$x_2 \neq x_1$

Continuing in this way, we get a sequence $[a_n, b_n]$ of closed intervals and points $x_n \in A \cap [a_n, b_n]$ with all x_n's different, such that:

① $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$ (ie, contracting)

② $|b_{n+1} - a_{n+1}| = \frac{1}{2}|b_n - a_n|$

Then $a_1 \leq a_2 \leq \dots \leq a_n < b_n \leq b_{n-1} \leq b_{n-2} \leq \dots \leq b_1$

So $\begin{cases} \{a_n\} \text{ is an increasing sequence, bounded above, so } a_n \to a_0 \\ \{b_n\} \text{ is a decreasing sequence, bounded below, so } b_n \to b_0 \end{cases}$

But $|b_n - a_n| = \frac{1}{2^n}|b - a|$ so $a_0 = b_0$

But $a_n \leq x_n \leq b_n$ so $x_n \to a_0, b_0 \in [a, b]$
and $x_n \in A$

So a_0 is a point of accumulation of A (any neighbourhood around a_0 will contain x_n's $\neq a_0$ where $x_n \in A$)

② In R^2, every set of the form $[a,b] \times [c,d]$ is compact.

Let (a_n, b_n) be a sequence of points in $[a,b] \times [c,d]$
Then $\left.\begin{cases} a_n \in [a,b] \\ b_n \in [c,d] \end{cases}\right\}$ and each of these intervals is compact by ①

Since $[a,b]$ is compact, any sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \to x_0 \in [a,b]$

Since $[c,d]$ is compact, $\{y_{n_k}\}$ has a convergent subsequence $\{y_{n_{k_j}}\} \to y_0 \in [c,d]$.

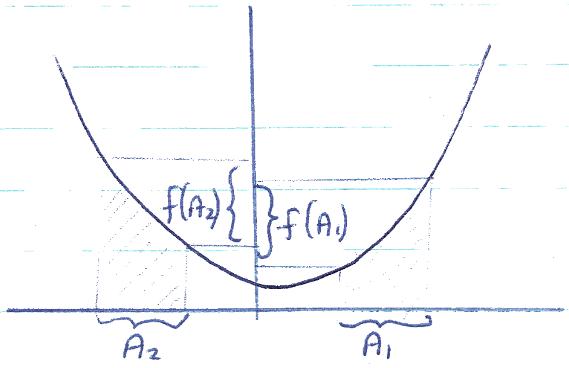
So $\{(x_{n_{k_j}}, y_{n_{k_j}})\}$ is convergent to (x_0, y_0) in $\mathbb{R}^2$

and so $[a,b] \times [c,d]$ is compact in $\mathbb{R}$.

(Note: it does not matter which metric we use)

Examples

(1)



Here $f(A_2) \cap f(A_1) \neq \phi$

$f(A_1 \cap A_2) = f(\phi) = \phi$

§7 Functions

7.1 **Definition:** Let E_1 & E_2 be sets. A *function* is a rule associating with each elt of E_1 a specific elt of E_2. If $x \in E_1$ and f is the function, $y \in E_2$, then we write $y = f(x)$.

OR A function is a subset of $E_1 \times E_2$ such that: if (x, y) and $(x, y') \in f$, then $y = y'$

Notes: If $A \subset E_1$ then $f(A) = \{ f(x) : x \in A \}$

If $B \subset E_2$, then $f^{-1}(A) = \{ x \in E_1 : f(x) \in B \}$

If $f(E_1) = E_2$ then we say f maps E_1 onto E_2 or that f is a surjection from E_1 to E_2

If $[f(x) = f(y) \Rightarrow x = y]$ then we say f is one-one or that f is injective

If f is both one-one and onto, it is called a bijection

7.2 **Properties of Functions $f : E_1 \to E_2$**

$\forall A_1, A_2 \subset E_1 \; ; \; B_1, B_2 \subset E_2$

① If $A_1 \subset A_2 \subset E_1$, then $f(A_1) \subseteq f(A_2)$

② $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

③ $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ (if f is injective then $=$)

④ $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

⑤ $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

⑥ $[f^{-1}(B)]' = f^{-1}(B')$

⑦ $f(f^{-1}(B)) \subseteq B$.

7.3 <u>DEFINITION</u>: If (E_1, ρ_1) and (E_2, ρ_2) are metric spaces, a function $f: E_1 \to E_2$ is said to be <u>continuous</u> at x_0 if for each $\varepsilon > 0$, $\exists\, \delta > 0$

such that $\rho(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$.

<u>Note</u>: this is identical to the course 1 defn for $\mathbb{R}$ using metric $\rho(x,y) = |x-y|$.

f continuous at x_0

$\overset{①}{\Leftrightarrow}$ $(\forall \varepsilon > 0)(\exists \delta > 0) \ni f(N_1(x_0, \delta)) \subset N_2(f(x_0), \varepsilon)$

$\overset{②}{\Leftrightarrow}$ For every open set $V \subset E_2$, $f(x_0) \in V$, we can find an open set $U \subset E_1$ with $x_0 \in U$ and $f(U) \subset V$.

$\overset{③}{\Leftrightarrow}$ If $\{x_n\}$ is a sequence in E_1 with $x_n \to x_0$, then $\{f(x_n)\}$ is a sequence in E_2 with $f(x_n) \to f(x_0)$.

$\overset{④}{\Leftrightarrow}$ For every open set $V \subset E_2$, $f(x_0) \in V$, $f^{-1}(V)$ is open.

PROOFS (① ⇔ ③ we do ④ later). ① ⇒ ②: If $V \subset E_2$, $f(x_0) \in V$, V open,

then $\exists\, \varepsilon > 0 : N_2(f(x_0), \varepsilon) \subset V$

Let $U = N_1(x_0, \delta)$, with δ given by ①.

② ⇒ ①: Let $\varepsilon > 0$ be given.

Then $V = N_2(f(x_0), \varepsilon)$ is an open set, $f(x_0) \in V$.

② $\Rightarrow \exists\, U$ open, $x_0 \in U$ with $f(U) \subset V$.

Now $\exists\, \delta : N_1(x_0, \delta) \subset U$

Then $f(N_1(x_0, \delta)) \subset f(U) \subset V$

① ⇒ ③: Suppose $x_n \to x_0$. Let $\varepsilon > 0$ be given.

We want a stage N such that $n > N \Rightarrow \rho_2(f(x_0), f(x_n)) < \varepsilon$

Since f is continuous at x_0,

$\exists\, \delta : \rho_1(x, x_0) < \delta \Rightarrow \rho_2(f(x), f(x_0)) < \varepsilon$.

Since $x_n \to x_0$, $\exists\, N : n > N \Rightarrow \rho_1(x_n, x_0) < \delta$

So $n > N \Rightarrow \rho_2(f(x_n), f(x_0)) < \varepsilon$.

③ ⇒ ①: By contradiction: Suppose ① is false, ③ true.

$\Rightarrow (\exists\, \varepsilon > 0) : \forall \delta > 0,\ f(N_1(x_0, \delta)) \not\subset N_2(f(x_0), \varepsilon)$

$\Rightarrow (\exists\, \varepsilon > 0)(\forall \delta > 0)(\exists\, x \in N_1(x_0, \delta) : f(x) \notin N_2(f(x_0), \varepsilon))$

Let $\delta = \frac{1}{n}$. Then $\exists\, x_n$ with $\rho_1(x_n, x_0) < \frac{1}{n}$

and $\rho_2(f(x_n), f(x_0)) \geq \varepsilon$

Then $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$, contradicting ③

① We say f is continuous on E_1 if it is continuous at each point of E_1

② $\Leftrightarrow$ For every open set V in E_2, $f^{-1}(V)$ is open in E_1
(our previous claim ④)

③ $\Leftrightarrow$ Every closed set F in E_2 has $f^{-1}(F)$ closed in E_1

④ $\Leftrightarrow$ For every non-empty $A \subseteq E_1$, $f(\bar{A}) \subset \overline{f(A)}$

PROOFS: ① $\Rightarrow$ ②: Suppose f is continuous on E_1, V open in E_2. Let $x_0 \in f^{-1}(V)$.
Since f is continuous at x_0, and $f(x_0) \in V$,
$\exists$ open set U : $x_0 \in U$ and $f(U) \subset V$.
So $x_0 \in U \subset f^{-1}(V) \Rightarrow x_0 \in \operatorname{int} f^{-1}(V)$
So $f^{-1}(V)$ is open.

② $\Rightarrow$ ③: Let F be closed in E_2. Then F' is open in E_2. By ②, $f^{-1}(F')$ is open in E_1
But $f^{-1}(F') = (f^{-1}(F))'$ is open
$\Rightarrow f^{-1}(F)$ is closed.

③ $\Rightarrow$ ④: Let $A \subset E_1$. Then $\overline{f(A)}$ is closed in E_2.
By ③, $f^{-1}(\overline{f(A)})$ is closed in E_1.
But $A \subset f^{-1}(\overline{f(A)})$
[If $a \in A$, $f(a) \in f(A) \subseteq \overline{f(A)}$
so $a \in f^{-1}(\overline{f(A)})$]
So $\bar{A} \subset f^{-1}(\overline{f(A)}) \Rightarrow f(\bar{A}) \subset \overline{f(A)}$

④ $\Rightarrow$ ①: Suppose $\forall A \subset E_1$, $f(\bar{A}) \subset \overline{f(A)}$
Suppose $x_n \to x_0$. We must show $f(x_n) \to f(x_0)$

Examples

① On $\mathbb{R}$, $\rho_1(x,y) = |x-y|$ is a metric

Set $\rho_2(x,y) = \dfrac{\rho_1(x,y)}{1+\rho_1(x,y)}$. Then ρ_2 is also a metric

These metrics are equivalent on $\mathbb{R}$.

Note first that $\rho_2(x,y) \leq \rho_1(x,y)$

So the identity map $(\mathbb{R}, \rho_1) \to (\mathbb{R}, \rho_2)$ is continuous (as any sequence convergent w.r.t. ρ_1 must by squeeze principle be convergent w.r.t. ρ_2)

Since $\rho_2(x,y) \leq \dfrac{\varepsilon}{1+\varepsilon} \Rightarrow \rho_1(x,y) \leq \varepsilon$ the identity map $(\mathbb{R}, \rho_2) \to (\mathbb{R}, \rho_1)$ is also continuous (taking $\delta = \frac{\varepsilon}{1+\varepsilon}$ in the ε-δ defn).

② The metrics on $\mathbb{R}^2$

$$\rho_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$$
$$\rho_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

are equivalent, since $\rho_2(x,y) \leq \rho_1(x,y) \leq \sqrt{2}\,\rho_2(x,y)$

If $f(x_n) \not\to f(x_0)$ then $\exists\, \varepsilon > 0$: there is a subsequence $\{x_{n_k}\}$ with $\rho_2(f(x_{n_k}), f(x_0)) \geq \varepsilon$

Let $A = \{x_{n_k}\}$ Then $x_0 \in \bar{A}$ By ④, $f(x_0) \in \overline{f(A)}$
But $N(f(x_0), \varepsilon) \cap f(A) = \phi$, contradiction.

<u>Exercise</u>: If $(E_1, \rho_1), (E_2, \rho_2), (E_3, \rho_3)$ are metric spaces and $\left.\begin{array}{l} f : E_1 \to E_2 \\ g : E_2 \to E_3 \end{array}\right\}$ are continuous functions, then

$$\begin{tikzcd} E_1 \arrow[r, bend left, "f"] \arrow[rr, bend right, "g \circ f"'] & E_2 \arrow[r, bend left, "g"] & E_3 \end{tikzcd}$$

$g \circ f$ is continuous

Find as many proofs of this as possible using various defns of continuity

---

7.4 <u>DEFINITION</u>: If $f : (E_1, \rho_1) \to (E_2, \rho_2)$ is a bijection, and f and f^{-1} are both continuous, then f is called a <u>homeomorphism</u>

<u>NOTE</u>: ① Two metrics ρ_1, ρ_2 on the same space E are <u>equivalent</u> (ie the ρ_1-$\overset{\text{closed}}{\text{open}}$ sets are ρ_2-$\overset{\text{closed}}{\text{open}}$) <u>iff</u> the identity map of (E, ρ_1) onto (E, ρ_2) is a homeomorphism.

② If $\exists\, a, b > 0$ such that $a\rho_1(x,y) \leq \rho_2(x,y) \leq b\rho_1(x,y)$ then ρ_1, ρ_2 are equivalent.

7.5 <u>THEOREM</u>: If $f : E_1 \to E_2$ is continuous and $A \subset E_1$ is connected, then $f(A)$ is connected in E_2.

Example

* (1) If $f: \mathbb{R} \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, show that $\exists c : a < c < b$ and $f(c) = 0$.

PROOF: $[a, b]$ is a connected set. By 7.5, $f([a,b])$ must be connected in $\mathbb{R}$.
Hence $f([a,b])$ is an interval in $\mathbb{R}$.
Since $f(a) < 0 < f(b) \Rightarrow 0 \in f([a,b])$
So $\exists c \in (a,b) : f(c) = 0$

PROOF: Suppose $f(A)$ is not connected. Then there is a non-empty subset $B \subset f(A)$ which is both open and closed in $f(A)$.

So $B = f(A) \cap V$ where V is open in E_2
Then $f^{-1}(B) = A \cap f^{-1}(V)$
Since f is continuous, $f^{-1}(V)$ is open in E_1
So $f^{-1}(B)$ is open in A.

We can repeat this for B closed using a set F closed in E_2, and get $f^{-1}(B)$ is closed in A.
So $f^{-1}(B)$ is non-empty, closed and open in A
so A cannot be connected, contradiction.

Exercise: Try to prove the above by showing that if $f(A) = C \cup D$ where C, D are non-empty seperated sets, then $f^{-1}(A) = f^{-1}(C) \cup f^{-1}(D)$ and these are seperated.
(Hint: use $f(\bar{A}) \subset \overline{f(A)}$ if f is continuous)

7.6 THEOREM If $f: E_1 \to E_2$ is a continuous mapping and $A \subset E_1$ is compact, then $f(A)$ is compact.

PROOF: Let $\{O_i\}_{i \in I}$ be an open cover of $f(A)$, $f(A) \subset \bigcup_{i \in I} O_i$

Then $A \subset f^{-1}\left(\bigcup_{i \in I} O_i\right) = \bigcup_{i \in I} f^{-1}(O_i)$

and each $f^{-1}(O_i)$ is open, so $\{f^{-1}(O_i)\}_{i \in I}$ is an open cover of A.

Since A is compact, there is a finite subcover

Examples.

① Suppose f is a continuous real valued function on $[a,b]$. Then since $[a,b]$ is compact, $f([a,b])$ is also compact, ie, a closed bounded subset of $\mathbb{R}$. So f is bounded and attains its sup/inf on $[a,b]$ (ie $\exists\, c,d \in [a,b]$: $f(c) = \sup\limits_{[a,b]} f(x)$ and $f(d) = \inf\limits_{[a,b]} f(x)$).

Let $K = \sup\limits_{[a,b]} f(x)$. Then $\exists\, x_n \in [a,b] : f(x_n) > K - \frac{1}{n}$ (since $K - \frac{1}{n}$ is not an upper bound)

Choose $x_{n_k} \to c \in [a,b]$ by sequential compactness of $[a,b]$. Then $f(x_{n_k}) \to f(c)$ as f is continuous. But $f(x_{n_k}) \to K$, so $f(c) = K$, ie, f attains its sup on $[a,b]$.

Similarly for inf.

② $f : (0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is continuous but <u>not</u> uniformly so on $(0,1)$.

③ $f(x) = \sin x$ is uniformly continuous as a map from $\mathbb{R} \to \mathbb{R}$. (hint: prove that $|\sin x - \sin y| \leq |x - y|$. Note max gradient of $\sin x$ is 1)

④ $f(x) = x^2$ is not unif. cont on $\mathbb{R}$.

ie, $\exists\, O_{i_1}, \ldots, O_{i_n} \ni A \subset \bigcup_{k=1}^{n} O_{i_k}$

Then $f(A) \subset f\left(\bigcup_{k=1}^{n} f^{-1}(O_{i_k})\right) = \bigcup_{k=1}^{n} f(f^{-1}(O_{i_k}))$

So $f(A) \subset \bigcup_{k=1}^{n} O_{i_k}$ is a finite subcover of $f(A)$

So $f(A)$ is compact.

[Alternative: use compact $\equiv$ sequ. compact. Pick a convergent sequence in A and prove it has a convergent subsequence in $f(A)$.]

7.7 <u>DEFN:</u> A function $f: E_1 \to E_2$ is said to be <u>uniformly continuous</u> if $(\forall \varepsilon>0)(\exists \delta>0): [\rho_1(x,y)<\delta \Rightarrow \rho_2(f(x),f(y))<\varepsilon]$ $\forall x,y \in E_1$.

Note: $\begin{cases} \text{Unif cont}: & (\forall \varepsilon>0)(\exists \delta>0)(\forall x\in E_1)(\forall y \in E_1)[\ldots\ldots] \\ \text{Normal cont}: & (\forall \varepsilon>0)(\forall x\in E_1)(\exists \delta>0)(\forall y)[\ldots\ldots] \end{cases}$

ie, uniform continuity means that for any x, ε $\exists\, \delta$ ...

normal continuity means that for any ε and <u>each</u> x $\exists\, \delta$ ....

Whether a continuous function is uniformly so or not depends ~~largely~~ entirely on whether the gradient is always finite & bounded.

7.8 <u>THEOREM:</u> If $f: E_1 \to E_2$ is continuous and E_1 is compact, then f is uniformly continuous.

<u>PROOF:</u> Let $x \in E_1$. Then f is continuous at x. Let $\varepsilon > 0$.
So $\exists\, \delta_x > 0 : \rho_1(x,y) < \delta_x \Rightarrow \rho_2(f(x), f(y)) < \frac{\varepsilon}{2}$

For each x we can find a δ_x.
Now $E_1 = \bigcup_{x\in E_1} N(x, \delta_x)$, an open cover.

But E_1 is compact $\Rightarrow E_1 = \bigcup_{i=1}^{n} N(x_i, \frac{1}{2}\delta x_i)$, a finite subcover.

Let $\delta = \frac{1}{2}\min(\delta x_1, \delta x_2, \dots, \delta x_n)$. Suppose $\rho_1(x,y) < \delta$.

Since $x \in \bigcup_{i=1}^{n} N(x_i, \frac{1}{2}\delta x_i)$, $\exists x_k : \rho_1(x, x_k) < \frac{1}{2}\delta x_k$

Then $\rho_2(f(x), f(x_k)) < \varepsilon$

and $\rho_1(y, x_k) < \rho_1(x,y) + \rho_1(x, x_k) < \delta x_k$

$$\Rightarrow \rho_2(f(x), f(y)) \leq \underbrace{\rho_2(f(x), f(x_k))}_{< \frac{\varepsilon}{2}} + \underbrace{\rho_2(f(x_k), f(y))}_{< \frac{\varepsilon}{2}} < \varepsilon.$$

7.9 <u>THEOREM</u>: If $f: E_1 \to E_2$ is a continuous bijection and E_1 is compact, then f^{-1} is also continuous

<u>PROOF</u>: To show that f^{-1} is continuous, we show that for every closed set $F \subset E_1$, the inverse image under f^{-1} is closed in E_2, ie, $(f^{-1})^{-1}(F)$ is closed in E_2.

But $(f^{-1})^{-1}(F) = f(F)$ so RTP that $f(F)$ is closed in E_2.

Since E_1 is compact and F is closed, F is compact (cf 6.3②)
Since F is continuous $f(F)$ is compact (cf 7.6)
Hence $f(F)$ is closed (cf 6.3①).

Examples

① Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, with $\sup_{x \in \mathbb{R}} |f'(x)| = k < 1$.

Then f is a contraction mapping.

We must show $|f(x) - f(y)| \leq k|x-y|$

But if $x \neq y$ then we have $\dfrac{|f(x) - f(y)|}{|x-y|} \leq k$

First MVT $\Rightarrow \dfrac{f(x) - f(y)}{x-y} = f'(\xi)$, ξ between x, y.

$$\Rightarrow \left| \frac{f(x) - f(y)}{x-y} \right| \leq k \quad \text{from defn of } f.$$

② Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined by $\begin{pmatrix} \frac{1}{2} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}$

This is a contraction mapping, as:

$$\begin{pmatrix} \frac{1}{2} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix} = \frac{5}{6} \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

and $\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ is a rotation matrix which preserves distance, and the factor $\frac{5}{6}$ provides the contraction.

③ $f(x) = x + 1$ has no fixed point in $\mathbb{R}$.
$f(x) = x^2 + 2x$ has two fixed points in $\mathbb{R}$.

Any continuous mapping of $f : [0,1] \to [0,1]$ has a fixed point (geometrically clear from graphs of $y = f(x)$ and $y = x$ – a rigorous proof uses intermediate value theorem).

8 The Contraction Mapping Theorem

8.1 Defn: A mapping $f: E \to E$ is said to be a contraction mapping if $\exists\, k$, $0<k<1$ such that $\rho(f(x), f(y)) \le k\rho(x,y)$ all $x, y \in E$.

Note: A contraction mapping is uniformly continuous (take $\delta = \varepsilon$ in defn).

8.2 Defn: A mapping $f: E \to E$ is said to have a fixed point x if $f(x) = x$.

8.3 Theorem: If (E, ρ) is a complete metric space and f is a contraction mapping on E, then f has a unique fixed point.

Proof: Let x_0 be any point of E. Let $x_1 = f(x_0)$, $x_2 = f(x_1)$ $x_{n+1} = f(x_n)$, ... We shall show that this sequence converges to a point of E, which will be the required fixed point. Since E is complete, we need only show that $\{x_n\}$ is a Cauchy sequence.

Suppose $n > m$. Then, since $\rho(x_{i+1}, x_{i+2}) \le k\rho(x_i, x_{i+1})$;

$$\begin{aligned}
\rho(x_m, x_n) &\le \rho(x_m, x_{m+1}) + \rho(x_{m+1}, x_{m+2}) + \dots + \rho(x_{n-1}, x_n) \\
&\le k\rho(x_{m-1}, x_m) + \qquad \dots + k\rho(x_{n-2}, x_{n-1}) \\
&\le k^2\rho(x_{m-2}, x_{m-1}) + \qquad + k^2\rho(x_{n-3}, x_{n-2}) \\
&\vdots \\
&\le k^m\rho(x_0, x_1) + k^{m+1}\rho(x_0, x_1) + \dots \\
&\le k^m\rho(x_0, x_1)\frac{1}{1-k}
\end{aligned}$$

But as $k^m \to 0$ as $m \to \infty$, $\rho(x_m, x_n) \to 0$ as $m \to \infty$, $n > m$.
Hence $\{x_n\}$ is a Cauchy sequence, and $\{x_n\}$ converges to a point

Examples

① If $f : [a,b] \to [a,b]$ is continuous, and $|f'(x)| \leq k < 1$ for all x in $[a,b]$, then the equation $f(x) = x$ has a unique solution in $[a,b]$

② NEWTON'S METHOD

To solve the equation $f(x) = 0$, we use the iteration $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Consider the map:

$$x \to F(x) = x - \frac{f(x)}{f'(x)}$$

. This has a fixed point when $f(x) = 0$.

Now: $$|F(x_1) - F(x_2)| = \left| (x_1 - x_2) - \left\{ \frac{f(x_1)}{f'(x_1)} - \frac{f(x_2)}{f'(x_2)} \right\} \right|$$

$$= \left| (x_1 - x_2) - (x_1 - x_2) \frac{[f'(\xi)]^2 - f(\xi) f''(\xi)}{[f'(\xi)]^2} \right|$$

(Mean value theorem: ξ between x_1 & x_2)

$$\leq |x_1 - x_2| \left| \frac{f(\xi) f''(\xi)}{[f'(\xi)]^2} \right|$$

So if $\sup_{\xi} \left| \dfrac{f(\xi) \cdot f''(\xi)}{[f'(\xi)]^2} \right| < 1$, then F is a contraction mapping on $\mathbb{R}$, and therefore has a fixed point.

x of E.
Since any contraction mapping is continuous, $f(x_n) \to f(x)$
But $f(x_n) = x_{n+1} \to x$. So $f(x) = x$ ie x is a fixed pt.

The fixed point is unique, for if x and y are two fixed points, $\rho(x,y) = \rho(f(x), f(y)) \leq k\,\rho(x,y)$, where $k < 1 \Rightarrow \rho(x,y) = 0$, ie, $x = y$.

③ The Fredholm Integral Equation

$$f(x) = g(x) + \lambda \int_a^b K(x,y) . f(y) . dy$$

$g(x)$ a given function, continuous on $[a,b]$
$K(x,y)$ a given function, continuous on $[a,b] \times [a,b]$
f unknown.

We show that for sufficiently small λ, this equation has a unique continuous solution f.

We turn the equation into a fixed point problem.
Let $(Tf)(x) = g(x) + \lambda \int_a^b K(x,y).f(y).dy$

We want to find f such that $Tf = f$, ie, find a fixed point for T. We show that for small λ, T is a contraction mapping on a complete metric space.

The space: $C[a,b]$

The metric $\rho(\Phi, \Psi) = \sup |\Phi(x) - \Psi(x)|$

Hence if $\delta = \min(\delta_1, \delta_2)$ and $|x_1 - x_2| < \delta$

then $|Tf(x_1) - Tf(x_2)| \leq |g \dots| + \lambda |\int \dots| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

So T maps $C[a,b] \to C[a,b]$.

We now prove that T is a contraction mapping for small λ.

$$\rho(Tf_1, Tf_2) = \sup_{x \in [a,b]} \left| (Tf_1)(x) - (Tf_2)(x) \right|$$

$$= \sup_{x \in [a,b]} \left| \lambda \int_a^b K(x,y) \{ f_1(y) - f_2(y) \} . dy \right|$$

$$\leq |\lambda| (b-a) \sup_{x,y \in [a,b]} |K(x,y)| . \rho(f_1, f_2)$$

this is bounded as it is continuous on a compact interval

So if λ is chosen small enough to make $k = |\lambda|(b-a) \sup_{x,y \in [a,b]} |K(x,y)| < 1$

then we have $\rho(Tf_1, Tf_2) \leq k\, \rho(f_1, f_2)$

and this is a contraction mapping.

So T has a unique fixed point f, which is the solution of the integral equation.

Problems:

p 30	1-4 6-8	
p 42	1-14	
p 59	1-6	
p 71	1-8	
p 84	1-12	not 3
p 107	1-16	not 5.

We show first that T maps $C[a,b]$ into $C[a,b]$,

where $(Tf)(x) = g(x) + \lambda \int_a^b K(x,y)\, f(y)\, dy$

Note that, as K is continuous, if $f \in C[a,b]$ then the integral is defined.

Let $f \in C[a,b]$. We must show $Tf \in C[a,b]$

Let $\varepsilon > 0$. Since g is uniformly continuous on $[a,b]$ (compact)

$\exists\, \delta_1 > 0 : |x_1 - x_2| < \delta_1 \Rightarrow |g(x_1) - g(x_2)| < \frac{\varepsilon}{2}$

We want to show $|(Tf)(x_1) - (Tf)(x_2)| < \varepsilon$ if $|x_1 - x_2| < \delta$

Now $|(Tf)(x_1) - (Tf)(x_2)| = \left| \{g(x_1) - g(x_2)\} + \lambda \int_a^b [K(x_1,y) - K(x_2,y)] f(y) dy \right|$

and: $\left| \lambda \int_a^b [K(x_1,y) - K(x_2,y)]\, f(y)\, dy \right|$

$$\leq \underbrace{|\lambda|\,(b-a) \sup_{y\in[a,b]} |K(x_1,y) - K(x_2,y)|}_{(*)} \sup_{y\in[a,b]} |f(y)|$$

Since $K(x,y)$ is uniformly continuous on compact set $[a,b]\times[a,b]$

$\exists\, \delta_2 > 0 : |K(x_1,y_1) - K(x_2,y_2)| < \frac{\varepsilon}{2|\lambda||b-a|A}$ (where $A = \sup_{[a,b]} |f(x)|$)

when $\rho((x_1,y_1),(x_2,y_2)) < \delta_2$ (ρ metric on $\mathbb{R}^2$)

In particular, if $|x_1 - x_2| < \delta_2$

then $|K(x_1,y_1) - K(x_2,y_2)| < \frac{\varepsilon}{2|\lambda||b-a|A}$

so if $|x_1 - x_2| < \delta_2$ then $(*) \leq |\lambda|.|b-a|.\frac{\varepsilon}{2|\lambda||b-a|A}.A = \frac{\varepsilon}{2}$

$\Leftarrow$

LEBESGUE MEASURE & INTEGRATION.

We want to define a "measure" on R^n which would be a generalisation of length (in R), area (in R^2) and volume (R^3). We shall denote this measure by μ. So $\mu : R^n \to R$ should intuitively satisfy:

① $\mu(A) \geq 0$
② $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \phi$
③ The domain of μ should be as wide as possible, and should include all the normal sets
④ If I is an interval in R of $[a,b]$ then $\mu I = b-a$
" " " " " R^2 of $[a,b]\times[c,d]$ then $\mu I = (b-a)(d-c)$

and in general, if $I = \prod_{i=1}^{n} [a_i, b_i]$ is an interval in R^n, then $\mu I = \prod^n (b_i - a_i)$

This measure will then be used as a basis for integration.

1. LEBESGUE MEASURE.

1.1 DEFN: A bounded interval I in R^n is a set of the form $I = \{\underline{x} : a_i \leq x_i \leq b_i,\ i = 1, 2, \dots, n\}$ (where either $\leq$ or $<$ can be used) and its measure μI is defined to be: $(b_1 - a_1)(b_2 - a_2)\dots(b_n - a_n) = \prod_{i=1}^{n}(b_i - a_i)$

NOTE: ① If $a_k = b_k$ for any k, the interval has zero measure
② The empty set is an interval with $a_i < x < a_i$ with measure 0.
③ This measure agrees with our intuitive ideas for R, R^2 and R^3.

We assume without proof:

1.2 <u>LEMMA</u> ① Given any bounded interval I, and $\varepsilon > 0$, we can find an open bounded interval J_1, such that $I \subset J_1$, and $\mu J_1 < \mu I + \varepsilon$ and a closed bounded interval J_2, such that $J_2 \subset I$ and $\mu I < \mu J_2 + \varepsilon$

② If $I, I_1, I_2, \ldots, I_n$ are bounded intervals such that $I \subset \bigcup_{k=1}^{n} I_k$, then $\mu I \leq \sum_{k=1}^{n} \mu I_k$

1.3 <u>DEFN</u>: If $E \subset R^n$, the <u>outer measure</u> of E, written $\mu^* E$, is defined by:

$$\mu^* E = \inf\left\{\sum_{r=1}^{\infty} \mu I_r : E \subset \bigcup_{r=1}^{\infty} I_r,\ I_r \text{ a bounded interval}\right\}$$

1.4 <u>THEOREM</u>: If I is a bounded interval, $\mu I = \mu^* I$.

<u>PROOF</u>: (a) Let $I_1 = I$, $I_2 = I_3 = \ldots = \phi$

Then $I \subset \bigcup_{r=1}^{\infty} I_r$ and $\mu^* I \leq \sum_{r=1}^{\infty} \mu I_r = \mu I$

(b) To show that $\mu I \leq \mu^* I$, we prove that, for any $\varepsilon > 0$, $\mu I < \mu^* I + \varepsilon$.

Suppose $I \subset \bigcup_{r=1}^{\infty} I_r$ where all I's are bounded intervals. Given $\varepsilon > 0$, we choose a closed bounded interval $J \subset I$ such that $\mu I < \mu J + \varepsilon$ by lemma. and we choose open bdd intervals $J_r \supset I$ such that $\mu J_r < \mu I_r + \frac{\varepsilon}{2^r}$ for each r.

Then $J \subset I \subset \bigcup_{r=1}^{\infty} I_r \subset \bigcup_{r=1}^{\infty} J_r$

Now, J is a closed bdd interval, and hence is compact, with an open cover $\bigcup_{r=1}^{\infty} J_r$. Thus there is a finite subcover, ie $\exists N : J \subset \bigcup_{r=1}^{N} J_r$.

But $\mu I < \mu J + \varepsilon \leq \sum_{r=1}^{N} \mu J_r + \varepsilon$ (by 1.2 ② lemma)

$$< \sum_{r=1}^{N} \left(\mu I_r + \frac{\varepsilon}{2^r}\right) + \varepsilon$$

$$< \sum_{r=1}^{N} \mu I_r + 2\varepsilon \quad \text{, since } \sum_{r=1}^{N} \frac{1}{2^r} < 1$$

So $\mu I - 2\varepsilon < \sum_{r=1}^{\infty} \mu I_r$ for any $\{I_r\}$ with $I \subset \bigcup_{r=1}^{\infty} I_r$

Hence $\mu I - 2\varepsilon \leq \mu^* I$, and since ε is arbitrary,

$$\mu I \leq \mu^* I.$$

1.5 <u>Theorem</u>: ① $\mu^* E \geq 0$ for all $E \subset \mathbb{R}^n$
② $\mu^* \phi = 0$
③ $\mu^* \mathbb{R}^n = \infty$
④ If $\{E_r\}$ is any sequence of sets, $\mu^*\left(\bigcup_{r=1}^{\infty} E_r\right) \leq \sum_{r=1}^{\infty} \mu^* E_r$

<u>Proof</u>: ①-④ are immediate from the defn. of μ^*
⑤: If $\mu^* E_r = \infty$ for any r, the result is immediate.
If $\mu^* E_r < \infty$ $\forall r$, then, given $\varepsilon > 0$ we can choose for each r a sequence $\{I_{rs}\}$ of bdd intvls such that $E_r \subset \bigcup_{s=1}^{\infty} I_{rs}$, and $\mu^* E_r \geq \sum_{s=1}^{\infty} \mu I_{rs} - \frac{\varepsilon}{2^r}$

Then $\bigcup_{r=1}^{\infty} E_r \subset \bigcup_{\substack{s=1\\r=1}}^{\infty} I_{rs}$, so:

$$\mu^*\left(\bigcup_{r=1}^{\infty} E_r\right) \leq \sum_{r,s} \mu I_{rs} \leq \sum_r \left(\mu^* E_r + \frac{\varepsilon}{2^r}\right) = \sum_{r=1}^{\infty} \mu^* E_r + \varepsilon$$

Since ε is arbitrary, the result follows.

1.6 **Defn:** A set E is said to be measurable (Lebesgue-meas) if, for any set X, $\mu^* X = \mu^*(X \cap E) + \mu^*(X \setminus E)$

Note: Since $\mu^* X \leq \mu^*(X \cap E) + \mu^*(X \setminus E)$ by 1.5 (5) a set E is measurable iff for any "test set" X, $\mu^* X \geq \mu^*(X \cap E) + \mu^*(X \setminus E)$

Since this is clearly true if $\mu^* X = \infty$, it is sufficient to consider test sets X of finite outer measure.

1.7 **Theorem** ① R^n and ϕ are measurable

② If E, F are measurable, so are: $E \cap F$, $E \cup F$, $E \setminus F$ and $F \setminus E$.
Further, if $E \cap F = \phi$, then $\mu^*(E \cup F) = \mu^* E + \mu^* F$

③ If $\{E_r\}$ is a sequence of measurable sets, then $\bigcup_{r=1}^{\infty} E_r$ is measurable, and:

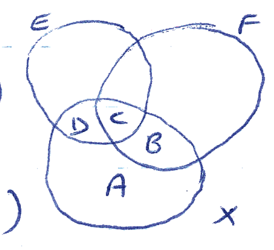
(a) If $E_r \subseteq E_{r+1}$ $\forall r$ then $\mu^*\left(\bigcup_{r=1}^{\infty} E_r\right) = \lim_{r\to\infty} \mu^* E_r$

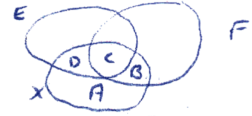
(b) If $E_r \cap E_s = \phi$ if $r \neq s$, $\forall r, s$, then
$$\mu^*\left(\bigcup_{r=1}^{\infty} E_r\right) = \sum_{r=1}^{\infty} \mu^* E_r$$

Proof ① Immediate

② We shall prove that $E \setminus F$ is measurable.
Let X be a test set. Then:

$$\mu^*(X \cap (E \setminus F)) + \mu^*(X \setminus (E \setminus F))$$
$$= \mu^* D + \mu^*(A \cup B \cup C)$$
$$\leq \mu^* D + \mu^* C + \mu^*(A \cup B)$$





Now, F is measurable, so, with test set $C \cup D$

$$\mu^*(C \cup D) = \mu^*((C \cup D) \cap F) + \mu^*((C \cup D) \setminus F)$$
$$= \mu^* C + \mu^* D$$

So $\mu^*(X \cap (E \setminus F)) + \mu^*(X \setminus (E \setminus F)) \le \mu^*(C \cup D) + \mu^*(A \cup B)$
$$= \mu^* X$$

③ Set $F_r = E_r \setminus \bigcup_{s=1}^{r-1} E_s$

$G_r = \bigcup_{s=1}^{r} E_s, \quad G_0 = \phi$

Then $F_r \cap F_s = \phi \ \forall\ r \neq s$ and $G_r \subseteq G_{r+1} \ \forall r$

Moreover, $\bigcup_{r=1}^{\infty} F_r = \bigcup_{r=1}^{\infty} G_r = \bigcup_{r=1}^{\infty} E_r$

Now, let X be any test set with $\mu^* X < \infty$

Then: $\mu^*(X \cap \bigcup_{r=1}^{\infty} E_r) = \mu^*(X \cap \bigcup_{r=1}^{\infty} F_r)$
$$= \mu^*(\bigcup_{r=1}^{\infty} (X \cap F_r))$$
$$\le \sum_{r=1}^{\infty} \mu^*(X \cap F_r) \quad \ldots Ⓐ$$

and: $\mu^*(X \setminus \bigcup_{r=1}^{\infty} E_r) = \mu^*(X \setminus \bigcup_{r=1}^{\infty} G_r) \le \mu^*(X \setminus G_r)$ any r

However: $\mu^*(X \setminus G_r) = \mu^*((X \setminus G_{r-1}) \setminus E_r)$ since $G_r = G_{r-1} \cup E_r$
$$= \mu^*(X \setminus G_{r-1}) - \mu^*((X \setminus G_{r-1}) \cap E_r) \quad \text{since } E_r \text{ is meas}$$
$$= \mu^*(X \setminus G_{r-1}) - \mu^*(X \cap F_r)$$
$$= \ldots = \mu^* X - \sum_{s=1}^{r} \mu^*(X \cap F_s) \ldots Ⓑ$$

So $\mu^*(X \setminus \bigcup_{r=1}^{\infty} E_r) \le \mu^* X - \sum_{s=1}^{r} \mu^*(X \cap F_s)$ any r

Letting $r \to \infty$: $\mu^*(X \setminus \bigcup_{r=1}^{\infty} E_r) \le \mu^* X - \sum_{r=1}^{\infty} \mu^*(X \cap F_s)$
$$\le \mu^* X - \mu^*(X \cap \bigcup_{r=1}^{\infty} E_r)$$
(from Ⓐ)

So $\bigcup_{r=1}^{\infty} E_r$ is measurable.

1.8 THEOREM: If $\{E_r\}$ is a contracting sequence of measurable sets, and $\mu^* E_1 < \infty$, then $\bigcap_1^{\infty} E_r$ is measurable, and $\mu^*(\bigcap_1^{\infty} E_r) = \lim_{r\to\infty} \mu^* E_r$

PROOF: Let $F_r = E_r \setminus E_{r+1}$. By 1.7 ② each F_r is measurable (and they are disjoint) so $\bigcup_1^{\infty} F_r$ is measurable.
Hence $E_1 \setminus \bigcup_1^{\infty} F_r = \bigcap_1^{\infty} E_r$ is measurable (1.7 ②)
Since $\mu^* E_1 < \infty$,

$$\begin{aligned}
\mu^*\left(\bigcap_1^{\infty} E_r\right) &= \mu^* E_1 - \mu^*\left(\bigcup_1^{\infty} F_r\right) \\
&= \mu^* E_1 - \lim \mu^*\left(\bigcup_1^{r} F_s\right) \quad (1.7 ③ (a)) \\
&= \lim_r \mu^*\left(E_1 \setminus \bigcup_1^{r} F_s\right) \\
&= \lim_r \mu^* E_r
\end{aligned}$$

Note: The requirement that $\mu^* E_1 < \infty$ (or at least, $\mu^* E_K < \infty$ for some K) is essential in 1.8.
For if $E_r = [r, \infty)$ in $\mathbb{R}$, $\mu^* E_r = \infty$ for all r, but $\bigcap_1^{\infty} E_r = \Phi$.

1.9 THEOREM: Bounded intervals in $\mathbb{R}^n$ are measurable.

PROOF: Bounded intervals are intersections of sets of the form $\{x ; a_i < x_i\}$, $\{x : a_i \le x_i\}$, $\{x : x_i < b_i\}$, $\{x : x_i \le b_i\}$
We show that the first of these is measurable.

Let $E = \{x : a_i < x_i\}$. Let X be a test set, and $\varepsilon > 0$.
Choose bounded intervals I_r such that $X \subseteq \bigcup I_r$, and $\sum_1^{\infty} \mu I_r \le \mu^* X + \varepsilon$. Then $\{I_r \cap E\}$ and $\{I_r \setminus E\}$ are both sequences of bounded intervals, and ~~$\mu(I_r \cap E$~~
$\mu(I_r \cap E) + \mu(I_r \setminus E) = \mu I_r$ for each r.

Since $X \cap E \subseteq \bigcup_1^\infty (I_r \cap E)$ and $X \setminus E \subseteq \bigcup_1^\infty (I_r \setminus E)$,

we have: $\mu^*(X \cap E) + \mu^*(X \setminus E) \leq \sum_1^\infty \mu(I_r \cap E) + \sum_1 \mu(I_r \setminus E)$ $= \sum_1^\infty \mu I_r < \mu^* X + \varepsilon$

Since ε is arbitrary, we have the desired inequality.

1.10 DEFN: If E is measurable we write $\mu E = \mu^* E$
– this is the Lebesgue measure of E.

NOTE: The class of measurable sets includes open intervals, and hence all open sets, closed sets, and all the sets of obtained by countable operations of union, intersection and complement from the open sets. The measure function thus defined is "countably additive" (1.7 ③) and agrees with our intuitive concepts of length, area and volume in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

2. MEASURABLE FUNCTIONS

2.1 DEFN: A function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be measurable if for every real number a $\{x : f(x) \geq a\}$ is a measurable subset of $\mathbb{R}^n$.

2.2 THEOREM: For a real-valued function f, the following are equivalent

1. f is measurable
2. For every real a $\{x : f(x) < a\}$ is measurable
3. For every real a $\{x : f(x) \leq a\}$ " "
4. For every real a $\{x : f(x) > a\}$ " "

<u>PROOF</u>: $1 \Rightarrow 2$ $\{x : f(x) < a\} = \{x : f(x) \geq a\}'$ (1.7.②)

$2 \Rightarrow 3$ $\{x : f(x) \leq a\} = \bigcap_{n=1}^{\infty} \{x : f(x) < a + \frac{1}{n}\}$ (1.7 ② & ③)

$3 \Rightarrow 4$, $4 \Rightarrow 1$ similarly.

2.3 <u>THEOREM</u>: If f and g are measurable functions on $\mathbb{R}^m$, then so are $f+g$, λf, fg, $|f|$ and $f \vee g$

<u>PROOF</u>: That λf is measurable is an easy exercise, while the measurability of $|f|$ and $f \vee g$ is very little harder. For $f+g$, note that $f(x) + g(x) > r$

$\Leftrightarrow \exists q \in \mathbb{Q} : f(x) > q$ and $g(x) > a - q$

Hence $\{x : f(x) + g(x) > r\} = \bigcup_{q \in \mathbb{Q}} \{x : f(x) > q\} \cap \{x : g(x) > a - q\}$

a countable union of measurable sets.

We next prove that f^2 is measurable:

$\{x : (f(x))^2 > a\} = \{x : f(x) > \sqrt{a}\} \cup \{x : f(x) < -\sqrt{a}\}$ $(a \geq 0)$

or $= \mathbb{R}^m$ if $(a < 0)$

Now note: $fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ which is measurable.

2.4 <u>THEOREM</u>: If $\{f_n\}$ is a sequence of measurable functions on $\mathbb{R}^m$, then $\sup_n f_n$, $\inf_n f_n$, $\lim_n \sup f_n$, $\lim_n \inf f_n$ and $\lim f_n$ are all measurable (if they exist).

<u>PROOF</u>: $\{x : \sup f_n(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > a\}$

$\{x : \lim_{n \to \infty} \sup f_n(x) > a\} = \bigcup_{n=1}^{\infty} \bigcap_{r=n}^{\infty} \{x : f_n(x) > a + \frac{1}{n}\}$

2.5 <u>DEFN</u>: A simple function on $\mathbb{R}^m$ is one which can be expressed in the form $\sum_{i=1}^{n} c_i \chi_{E_i}$, where $c_1, \ldots, c_n$ are real numbers and $E_1, \ldots, E_n$ are measurable sets of finite measure, and

$$\chi_E(x) = 1 \quad \text{if } x \in E$$
$$= 0 \quad \text{if } x \notin E$$

Note that every simple function is measurable.

2.6 <u>DEFN</u>: If $f = \sum_{i=1}^{n} c_i \chi_{E_i}$ is a simple function, we define the Lebesgue Integral, written $\int f$, to be $\sum_{i=1}^{n} c_i \mu E_i$.

<u>NOTE</u>: