COMBINATORICS

0 NOTATION

$|A|, \#A$ – the number of elts in the set A.
$\mathbb{N} = \{0, 1, 2, \dots\}$
$\mathbb{N}^+ = \{1, 2, 3, \dots\}$
$[n] = \{1, 2, 3, \dots, n\}$ for $n \in \mathbb{N}^+$
$Y^X = \{f \mid f : X \to Y\}$ for sets X, Y, functions f.
$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \wedge a_2 \in A_2\}$ for sets A_1, A_2.

$\prod_{i=1}^{m} A_i = A_1 \times A_2 \times \dots \times A_m = \{(a_1, a_2, \dots, a_m) \mid a_i \in A_i\}$ for sets $A_1, A_2 \dots A_m$
(this is called the <u>product</u> of the sets). Each $(a_1, a_2, \dots a_m)$ is called an <u>m-list</u> or <u>m-tuple</u>

$\prod_{i=1}^{n} a_i = a_1 . a_2 . \dots . a_n$ for $a_i \in \mathbb{R}$.

$\mathcal{U}_k^n = \{k\text{-arrangements from } [n]\}$

$I_k^n = \{f \mid f \in [n]^{[k]} \wedge f \text{ is injective}\}$

$\psi : I_k^n \to \mathcal{U}_k^n$ (i.e. $f \to (f(1), f(2), \dots, f(k))$)

$A \simeq B \iff |A| = |B|$ Two sets are equivalent if there is a bijection between them.

1. ELEMENTARY COUNTING TECHNIQUES.

Elementary counting problems involve ~~solving~~ finding the size of a set whose elements are either lists or sets.
Typical problems:

① Find the number of ways we can select k elts from a set of cardinality n. with no repetition
② Problem ① but with repetitions allowed
③ As ① and ② but with the order of the selection being important.

② 1.1 THEOREM: $\left| \prod_{i=1}^{m} A_i \right| = \prod_{i=1}^{m} |A_i|$

PROOF: $\left| \prod_{i=1}^{m} A_i \right| = \# \{ (a_1, a_2, \ldots a_m) \mid a_i \in A_i \}$

as we have $|A_i|$ choices for a_i

$= |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m| = \prod_{i=1}^{m} |A_i|$

This is known as the multiplication principle. We can use it to solve problem ②:

$\# \{ (a_1, a_2, \ldots, a_k) \mid a_i \in A \} = |A \times A \times \ldots \times A| = |A|^k = n^k$

③ 1.2 COROLLARY: $|Y^X| = |Y|^{|X|}$

PROOF: $|Y^X| = \# \{ f \mid f \text{ is a function} \wedge f : X \to Y \}$

Let $|X| = m$, $|Y| = n$.

We can let $X = [m]$, $Y = [n]$, as this can be mapped back to the original sets easily.

Now each $f : [m] \to [n]$ is completely specified by an m-list of elements of $[n]$, since we can write:

$f = (f(1), f(2), \ldots, f(m))$ where $f(i) \in [n]$

$$\Rightarrow |Y|^X = \#\{f(1), f(2), \ldots, f(m)) \mid f(i) = [n]\}$$

$$= |[n] \times [n] \times \ldots \times [n] = n^m = |Y|^{|X|}.$$

④ If we have m - sets $A_1, A_2, \ldots, A_m$ with $|A_i| = n_i$, then the number of ways of choosing an elt from A_1 and an elt from A_2, and ... and an elt from A_m is $\prod_{i=1}^{m} n_i$

In other words $\left|\prod_{i=1}^{m} A_i\right| = \prod_{i=1}^{m} |A_i| = \prod_{i=1}^{m} n_i$ MULTIPLICATION PRINCIPLE

The number of ways of choosing an elt from A_1 or an elt from A_2 or ... or an elt from A_m is $\sum_{i=1}^{m} n_i$ (the or being exclusive). In other words, if the A_i are mutually disjoint, then $\left|\bigcup_{i=1}^{m} A_i\right| = \sum_{i=1}^{m} |A_i| = \sum_{i=1}^{m} n_i$

THE ADDITION PRINCIPLE

⑤ ARRANGEMENTS

We define a k-arrangement from $[n]$ to be a k-list $(a_1, a_2, \ldots, a_k)$ where:

(i) $i \neq j \Rightarrow a_i \neq a_j$ (ie no repetitions)

(ii) $a_i \in [n]$ for all i.

Let $\mathcal{U}_k^n = \{k\text{-arrangements from } [n]\}$

We see that if $(a_1, a_2, \ldots, a_k) \in \mathcal{U}_k^n$ and $f : \begin{array}{l}[k] \to [n] \\ i \mapsto a_i\end{array}$

then $f \in [n]^{[k]}$ and f is injective by (i)

So each $(a_1, \ldots a_k) \in \mathcal{U}_k^n$ can be identified with an injective $f \in [n]^{[k]}$. Conversely given an injective $f \in [n]^{[k]}$, then $(f(1), \ldots f(k)) \in \mathcal{U}_k^n$. (see diagram opposite)

Examples

① There are p_i objects of type i for $i \in [k]$. Find the number of ways of selecting a collection from the $\sum_{i=1}^{k} p_i$ objects.

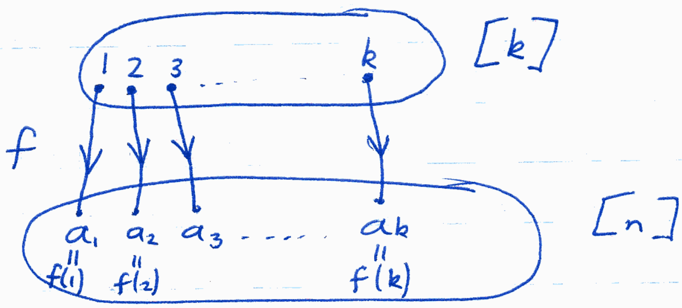
Code a selection as a k-list $(a_1, a_2, \dots, a_k)$
where $a_i = \#\{\text{elts of type } i\}$
So $(0,0,\dots,0)$ represents the empty collection
$(1,1,\dots,1)$ represents the selection with 1 of each type, etc.

We seek $\#\{(a_1,\dots,a_k) \mid a_i \in \{0,1,\dots,p_i\}\} = \prod_{i=1}^{k}(p_i+1)$

If there are m objects of type i for $i \in [k]$ (ie $p_i = m$ all i)
then $\#\{(a_1,\dots,a_k) \mid a_i \in \{0,1,\dots,m\}\} = (m+1)^k$

② There are 3 movies and 2 plays on. If we can see a movie <u>or</u> a play, the no. of choices is $3+2=5$. If we can see both a movie <u>and</u> a play, we have $3\cdot 2 = 6$ choices.

③ A man can either wear one of 3 jackets and go to one of 15 bars or wear one of his 2 costumes and go to one of 4 beaches. He has $3\cdot 15 + 2\cdot 4 = 53$ choices.



In other words, k-arrangements from $[n]$ and injective functions $f \in [n]^{[k]}$ are really the same thing, differing in name only. More precisely:

Let $I_k^n = \{ f \in [n]^{[k]} \mid f \text{ is injective} \}$

let $\psi : I_k^n \to \mathcal{U}_k^n$

$f \to (f(1), f(2), \ldots, f(k)\}$, then we claim that:

<u>ASSERTION</u>: ψ is bijective

<u>PROOF</u>: (a) let $f, g \in [n]^{[k]}$ and $f \neq g$

then $\exists\, i \in [k] \ni f(i) \neq g(i)$

$\Rightarrow (f(1) f(2), \ldots, f(k)) \neq (g(1), g(2), \ldots, g(n)\}$

$\Rightarrow \psi(f) \neq \psi(g)$

$\Rightarrow \psi$ is injective (one-one)

(b) If $(a_1, \ldots, a_n) \in \mathcal{U}_k^n$

let $f : [k] \to [n]$, $i \to a_i$, then $f \in [n]^{[k]}$

and f is injective

6 Since $i \neq j \Rightarrow a_i \neq a_j$

ie $i \neq j \Rightarrow f(i) \neq f(j)$

So $\psi(f) = (a_1, \ldots, a_n)$ and hence ψ is surjective.

If two sets A and B have a bijection $\psi : A \to B$ we write $A \simeq B$ and call A and B equivalent. Now $A \simeq B \Leftrightarrow |A| = |B|$. So if we seek $|A|$ and we establish that $A \simeq B$ where the elts of B are, for some reason, easier to count, we have simplified the task. In other words, we replace the problem by an equivalent one which is easier to handle.

1.2 THEOREM: $\boxed{|\mathcal{U}^n_k| = \frac{n!}{(n-k)!}}$

PROOF: $|\mathcal{U}^n_k| = \#\{(a_1, a_2, \ldots, a_k) \mid a_i \in [n]$ with no repetitions $\}$

There are n choices for a_1, $(n-1)$ for a_2, and so on.

$\Rightarrow |\mathcal{U}^n_k| = n.(n-1).(n-2).\ \ldots (n-k+1) = \frac{n!}{(n-k)!}$

This allows us to solve some type ③ problems: the number of selections of k elts from a set A, $\#A = n$, in order with no repetitions is $|\mathcal{U}^n_k|$.

An n-arrangement from $[n]$, ie, an elt $(a_1, a_2, a_3 \ldots a_n) \in \mathcal{U}^n_n$ is called a <u>permutation</u> of $[n]$

For example $(1, 4, 3, 2)$ is a permutation of $[4]$, and corresponds to an injective $f \in [4]^{[4]}$. We could write this as: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ (ie $\begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$)

The permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$ is called the <u>identity permutation</u> for $[n]$.

THEOREM: The number of distinct permutations on a set of finite cardinality n is $n!$

PROOF: This is $|\mathcal{U}^n_n| = \frac{n!}{1!} = n!$

The <u>product</u> $P \circ Q$ of two permutations P and Q on the same set is defined to be the single permutation on the set equivalent to the successive performance of P first followed by Q.

Examples

① What is the probability of selecting the horses which occupy places 1, 2, 3 and 4 out of a field of 15 horses?

$|\mathcal{A}^{15}_{4}| = 15.14.13.12$, so we have $\frac{1}{15.14.13.12} \simeq 0.00003$.

So a permutation of $[n]$ is just a rearrangement of the n distinct elts of $[n]$ amongst themselves. From Thm 1.2 we obtain:

1.3 <u>COROLLARY</u>: $|\mathcal{U}^n_n| = n!$ (Same as previous theorem).

Up till now we have dealt with distinct objects. Consider the following problem:

- Find the number of ways of rearranging 3 black balls and 4 white balls in a row.

Number the balls from 1–7 and let S be the set of permutations of the balls, ie $|S| = 7!$

However, the ~~[illegible]~~ balls are not truly distinct. Each ordering of white balls has $4!$ indistinct "equivalent" orderings, and the black balls $3!$. Hence the number of ways to arrange the balls so that the white/black ordering determines distinguishability (phew!), is: $\dfrac{7!}{3!\,4!}$, where $3!\,4!$ are the number of ~~elts~~ possible elts in each distinguishable arrangement. In other words, the $7!$ elts of S are partitioned into subsets, each with $3!\,4!$ elts, each subset corresponding to one arrangement of the non-distinct balls.

A similar argument yields:

1.4 If there are m_i objects of type i where $i \in [k]$ then the number of arrangements of the objects is:

$$\frac{\left(\sum_{i=1}^{k} m_i\right)!}{\prod_{i=1}^{k} (m_i!)}$$

Examples

The number of words obtainable from the letters MISSISSIPPI is $\dfrac{11!}{4!4!2!}$

The number of 10 digit sequence obtainable from 2 zero's, 3 one's and 5 two's is $\dfrac{10!}{2!3!5!}$

[BF 38-40]

r - ARRANGEMENTS (Duplicate - same as k-arrangement)

DEFN. An r-arrangement of n distinct objects is an ordered selection of r of the objects. We write $P(n,r)$ for the number of r-arrangement of n-objects.

BF.1 THEOREM Let $P(n,r)$ be the no. of r-arrangements of n distinct objects. Then:

$$P(n,r) = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

BF.2 THEOREM: The number of arrangements of n distinct objects taken r at a time with repetitions is n^r

BF.3 THEOREM: The number of cyclic ordered arrangements of n-objects is $(n-1)!$

NOTE: A cyclic arrangement has relative positions only as important, so, for eg. $ABCD \equiv BCDA \equiv CDAB \equiv DABC$.

BF.4 THEOREM: The number of n-arrangements of n objects of which exactly p are alike of one kind and q are alike of another kind, with the remainder all different, is $\dfrac{n!}{p!\,q!}$

BF.5 THEOREM: The number of ways of making a non-empty selection by choosing some or all of $p_1+p_2+\dots+p_k$ objects, where p_1 are alike of one kind, p_2 alike of a second kind, etc, is:

$$(p_1+1)(p_2+1)\dots(p_k+1) - 1$$

Example.

① The six 3-arrangements of $\{A, B, C\}$ are:

ABC, ACB, BAC, BCA, CAB, CBA

② $A = \{a, a, a, b, b\}$. The number of distinct 5-arrangements is $\dfrac{5!}{3!\,2!} = 10$.

PROOFS:

BF.1 – There are n-choices for the first object, $n-1$ for the second, ..., $n-r+1$ for the r^{th}. Apply the multiplication principle to yield result.

BF.2 – There are n-choices for each of r-objects, so the total is $\underbrace{n \times n \times \ldots \times n}_{r\text{-terms}} = n^r$.

BF.3. This is simply $P(n, n)$, where the number of objects is $n+1$. Our first choice is arbitrary – we then have n choices for the second, $n-1$ for the third, etc ... $= (n-1)(n-2)\ldots(1) = (n-1)!$

BF-4 – Let x be the required number of arrangements. Let the n-objects be labeled $x_1, \ldots, x_n$, with the p alike of one kind labeled $x_1, x_2, \ldots, x_p$ and the q alike of the other kind labeled $x_{p+1}, \ldots, x_{p+q}$. Then there will be $n!$ n-arrangements of these (now distinct) labeled objects, and so: $n! = x\,p!\,q!$
as the first p-alike can be exchanged in $p!$ ways, etc.

BF-5 – The p_i like objects yield p_i+1 selections for each i since we can choose either $0, 1, 2, \ldots, p_i$ objects from that subset. Our specification that the selection must be non-empty eliminates one choice, so we must subtract 1 from the product of the (p_i+1)'s.

Combinations.

A <u>k-combination</u> from $[n]$ without repititions is a set $K \subset [n]$ with $\#(K) = k$. This is also known as an <u>unordered selection</u> (without repititions) from $[n]$
(Note <u>ordered</u> selection is a <u>k-list</u>; <u>unordered selection</u> is a <u>k-subset</u>)

Let $\binom{n}{k} = \#\{K \mid K \subseteq [n], |K| = k\}$ (ie, the number of k-subsets of $[n]$).

[Note: BF uses $C(n,k)$]

1.5 **THEOREM** $\boxed{\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}}$

Proof: Let $P_k([n]) = \{K \mid K \subseteq [n], |K| = k\}$

Let $\psi : \mathcal{A}^n_k \to P_k([n])$
$(a_1, \dots a_k) \to \{a_1, \dots, a_k\}$

Then ψ is onto, and for each $\{a_1, \dots, a_k\}$ there are $k!$ ordered lists
(ie $|\psi^{-1}(\{a_1, \dots, a_k\})| =$ the number of k-arrangements of $\{a_1, \dots, a_k\}$
$= k!$

So $\mathcal{A}^n_k$ is partitioned into $\dfrac{|\mathcal{A}^n_k|}{k!} = \dfrac{n!}{(n-k!)k!}$ subsets,

one for each elt of $P_k([n])$

So $|P_k([n])| = \binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$

THE BINOMIAL THEOREM

BF.1 THEOREM: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

$$\boxed{(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}}$$

DERIVATION: Consider the product $(x+a)(x+a)\ \ldots\ (x+a)$ of n factors $(x+a)$. In the expansion of this product the term $x^i a^{n-i}$ whenever x is chosen from i of the n factors. Thus there are $\binom{n}{i}$ terms of the form $x^i a^{n-i}$, and the theorem follows for n a positive integer.

If the terms in the expansion of $(x+a)^n$ for n a positive integer are written down, then for given x and a there will be one or more terms with the greatest value, predicted by the following theorem:

BF.2 THEOREM: For fixed n, if $ax > 0$, the maximum term in the expansion of $(a+x)^n$ is the term

$$\binom{n}{r} x^r a^{n-r} \text{ for which } n-r = \lfloor \alpha \rfloor, \text{ and}$$

$$\alpha = \frac{n+1}{(x/a)+1}$$

Example

① $(2+1)^3 = \binom{3}{0}2^3 + \binom{3}{1}2^2 \cdot 1 + \binom{3}{2}2 \cdot 1^2 + \binom{3}{3}1^3$

$= 8 + 12 + 6 + 1$

If $x = 1$, $a = 2$, $n = 3$ we have:

$$\alpha = \frac{4}{\frac{1}{2}+1} = \frac{8}{3} \Rightarrow \lfloor \alpha \rfloor = 2 \Rightarrow r = n - \lfloor \alpha \rfloor = 1$$

The max term is the $\binom{3}{1} x \cdot a^2 = 12$, as above.

The Binomial Coefficients

Pascal's Formula: $\boxed{\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}}$

Derivation: $\binom{n}{r}$ is the number of r-subsets of a set of n distinct objects. For a particular object P, there are $\binom{n-1}{r-1}$ subsets with P and $\binom{n-1}{r}$ subsets excluding P. This establishes the formula. Alt we can prove the formula from $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Pascal's Theorem. The binomial coefficients in the expansions of $(x+y)^n$ for $n \in \mathbb{N}$ may be written in an array known as Pascal's triangle, where the $(r+1)$th entry in the $(n+1)$th row is $\binom{n}{r}$. This array can be constructed and extended by means of Pascal's formula.

Theorem. $\boxed{\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}}$

Proof: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

Differentiating: $n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k x^{k-1} = \sum_{k=0}^{n-1} \binom{n}{k+1}(k+1)x^k$

Compare the coefficients of x^{k-1}: $\binom{n}{k-1}(k-1) = \binom{n}{k} k$

We can also prove this by using $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$= \frac{n\,(n-1)!}{k\,(k-1)!\,((n-1)-(k-1))!} = \frac{n}{k}\binom{n-1}{k-1}$$

Or even by setting $x = 1$

The point is that we can generate identities involving the Binomial coefficients. The tricks are: differentiation, comparing coefficients, assigning values to x, etc.

Examples

① The set $\{1,2,3,4\}$ has six 2-subsets:
$\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$.

The first three contain $\{1\}$, the rest do not. The first three are formed by fixing 1 and choosing one elt from $\{2,3,4\}$, the last three are formed by choosing two elts from $\{2,3,4\}$. Thus $\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$.

② Setting $x=1$, we get $\underline{2^n = \sum_{k=0}^{n} \binom{n}{k}}$

Combinations with Repetitions.

Let $|N| = n$, $P_k(N) = \{K \mid K \subseteq N, |K| = k\}$ and $P(N) = \{K \mid K \subseteq N\}$

1.6 **Theorem**. $\{0,1\}^N \simeq P(N)$.

We are claiming that the set of functions $\rho : N \to \{0,1\}$ is equivalent to the set of subsets of N. We seek: $\psi \; P(N) \to \{0,1\}^N$ with ψ a bijection.

For $K \in P(N)$, we define the <u>characteristic</u> <u>function</u> of K, denoted χ_K, as follows

$$\chi_K : N \to \{0,1\}$$
$$i \to \left.\begin{array}{ll} 0 & \text{if } i \notin K \\ 1 & \text{if } i \in K \end{array}\right\}$$

For example, if $N = [5]$ and $K = \{1, 4, 5\}$ then:
$\chi_K(1) = 1$, $\chi_K(2) = 0$, $\chi_K(3) = 0$, $\chi_K(4) = 1$, $\chi_K(5) = 1$
so we can write χ_K as a 5-list of 0's and 1's.
ie, $\chi_K = (1,0,0,1,1) \in \{0,1\}^{[5]} = \{0,1\}^N$

In general $\chi_K \in \{0,1\}^N$ (an n-list of 0's and 1's)

Thus we let $\psi : P(N) \to \{0,1\}^N$ which is bijective (exercise)
$$K \to \chi_K$$

From this we can also show

1.7 **Corollary**: $2^n = \sum_{k=0}^{n} \binom{n}{k}$

PROOF: $P(N) = \bigcup_{k=0}^{n} P_k(N)$

and $k \neq l \Rightarrow P_k(N) \cap P_l(N) = \emptyset$ (ie the sets of subsets are disjoint)

$$\Rightarrow |P(N)| = \sum_{k=0}^{n} |P_k(N)| = \sum_{k=0}^{n} \binom{n}{k} = |\{0,1\}^N| = |\{0,1\}|^{|N|} = 2^n$$

(12) We have seen that a k-subset $K \subseteq N$ can be identified with $p \in \{0,1\}^N$ and we note that p has the property $\sum_{i \in N} p(i) = k$, as we have k - 1's in K.

So $P_k(N) \simeq \{p \mid p \in \{0,1\}^N \text{ and } \sum_{i \in N} p(i) = k\}$

We also called a k-subset of N an unordered selection of k elts (without repetitions) from N.

If we allow repetitions in our unordered selection, we call this a <u>k-multiset</u> from N.

Let $Q_k(N) = \{k\text{-multisets from } N\}$. Then:

<u>THEOREM</u>: $\boxed{Q_k(N) \simeq \{p \in \mathbb{N}^N \mid \sum_{i \in N} p(i) = k\}}$

<u>PROOF</u>: Let $\psi : Q_k(N) \to \{p \mid p \in \mathbb{N}^N \wedge \sum_{i \in N} p(i) = k\}$

$K \mapsto p_K$

where $p_K(i) = \#\{\text{repetitions of } i \text{ in } K\}$

then $\sum_{i \in N} p_K(i) = k$ and it is easy to show

that ψ is bijective.

Examples:

① How many ways can we form an unordered selection of 5 balls from a large collection of red, white and blue balls?

A typical solution has form $\{r, r, w, b, b\}$, a 5-multiset from $\{r, w, b\}$. We can also write it as:

$$p = (2, 1, 2) \in \mathbb{N}^{[3]} \text{ with } \sum_{i \in [3]} p(i) = 5$$

See page 14 for method of solution.

⑬

In general if a collection of objects have properties $\{1, 2, \dots, n\}$ then the number of ways in which we can form an unordered selection of k objects with repetitions allowed is $|Q_k([n])|$. We abbreviate this to Q_k^n

18 THEOREM: $\boxed{|Q_k^n| = \binom{k+n-1}{n-1}}$

PROOF: Each k-multiset from $[n]$ has form:

$$p = (p(1), p(2), \dots, p(n)) \in \mathbb{N}^{[n]} \text{ with } \sum p(i) = k.$$

We can recode each $p(i)$ as a list of 1's, and separate these lists with 0's. This gives us a list with k 1's and $n-1$ 0's, ie, a total length of $n+k-1$

Conversely, every $(k+n-1)$-list with k 1's and $(n-1)$ 0's represents an elt of Q_k^n.

$$\text{Thus } Q_k^n \simeq \{p \in \{0,1\}^{[k+n-1]} \mid \sum p(i) = k\}$$

$$\simeq \{K \mid K \subset [k+n-1] \wedge |K| = k\} = P_k([k+n-1])$$

$$\text{So } |Q_k^n| = \binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

⑮ Note (see ex ②, ③) The number of ways of distributing k indistinct balls into n distinct boxes such that (number in box i) $\geq q_i$, $\forall i \in [n]$, is:

$$\binom{k - \sum q_i + n - 1}{n-1} = \binom{k+n-1 - \sum q_i}{n-1}$$

Proof: Distribute the $\sum q_i$ balls first, then distribute the remaining $k - \sum q_i$ balls.

Examples

① Find the number of ways of colouring 5 balls with 3 colours: If we have colours $\{1, 2, 3\}$ then a colouring is a 5-multiset from $[3]$

$$|Q_5^3| = \binom{7}{2}$$

② In general, we can colour k balls with n colours in $\binom{k+n-1}{n-1}$ ways. An equivalent problem is that of distributing k balls into n boxes. This can be a useful way of looking at such problems (see ex ③)

③ Find the number of ways of colouring 5 indistinct balls with 3 distinct colours if each colour must be used once. Let the colours be $\{1, 2, 3\}$. A typical colouring is a 5-multiset.

(a) Since $\{1, 2, 3\}$ must belong to any such multiset, we have $K = \{1, 2, 3\} \cup L$, where L is a 2-multiset from $[3]$. So we have $|Q_2^3| = \binom{2+3-1}{2} = \binom{4}{2}$

(b) Else we can code a colouring as a distribution of 5 indistinct balls into 3 distinct boxes such that each box has at least 1 ball in it. We put one ball into each box and are left with two to distribute into 3 boxes, ie $|Q_2^3| = \binom{4}{2}$

<u>Example</u>: Find the number of ways of arranging n people in a row if there are p of these people whose mutual dislike is so strong that there must be at least l places between them.

Number the places $1, 2, \ldots, n$. We first find places for the people who dislike each other.
We seek sets $P = \{i_1, i_2, \ldots i_p\} \subseteq [n]$ such that:

$$\underbrace{1, 2, 3, \ldots\ldots,}_{\geqslant 0} i_1, \underbrace{\ldots\ldots,}_{\geqslant l} i_2 \ldots \ldots \ldots\ldots i_p \underbrace{\ldots}_{\geqslant 0} n$$

Each set P can be coded as a distribution of $n-p$ balls (ie empty seats) into $p+1$ distinct boxes, with the restrictions
$$\begin{cases} \# \text{ box } 1 \geqslant 0 \\ \# \text{ box } p+1 \geqslant 0 \\ \# \text{ box } i \geqslant l \quad \text{for } i \in \{2, \ldots, p\} \end{cases}$$
(or each set P can be identified with a function $\rho : [p+1] \to \mathbb{N}$ where $\rho(i) \geq l$ for $i = 2, 3, \ldots, p$)

The number of such distributions is $\binom{n-p-(p-1)l+p+1-1}{p+1-1} = \binom{n-(p-1)l}{p}$

For each set $\{i_1, \ldots, i_p\}$ of places there are $p!$ ways of arranging the p people in those places and $(n-p)!$ arrangments of the rest in their places. So the number of arrangments is (by the multiplication principle)

$$\binom{n-(p-1)l}{p} p!\,(n-p)!$$

(17)
[BF73]

Solutions of Linear Equations

Recall: $|Q_k^n| = \binom{k+n-1}{n-1} = \#\{k\text{-multisets from } [n]\}$

combinations with repetitions.

We can interpret $|Q_k^n|$ as the number of distributions of k indistinct balls into n distinct boxes, ~~or~~ as the number of colourings of k indistinct balls with n distinct colours. There is another interpretation which is useful:

If we have an equation $x_1 + x_2 + \ldots + x_n = k$, a solution in +ve integers is just an n-list where $x_i \in \mathbb{N}\ \forall i$, and so we can say:

$$|Q_k^n| = \#\{(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in \mathbb{N}^n \wedge \sum_{i=1}^{n} x_i = k\}$$
$$= \#\{\text{solutions in positive integers of } \sum_{i=1}^{n} x_i = k\}$$

(18) If we impose restrictions, we simply deal with these first before seeing how many choices we have remaining. If we have the restrictions $x_i \geqslant q_i$, then we are removing $\sum_{i=1}^{n} q_i$ objects from our selectable objects, ie:

$$\#\{(x_1, \ldots x_n) \in \mathbb{N}^n \mid \sum_{i=1}^{n} x_i = k,\ x_i \geqslant q_i\} = \binom{k - \sum_{i=1}^{n} q_i + n - 1}{n-1}$$

(N.B. - the number of solns of $x_1 + \ldots + x_n = k$ in general (not restricted to $\mathbb{N}$) is: $\binom{k-1}{n-1}$ Proof BF 74)

Examples

① A barrel contains marbles of 8 different colours. If we take a handful of 7 marbles, what is the probability of them all being white?
Code a selection as an 8-list $(x_1, \dots, x_8) \in \mathbb{N}^8$, where $x_i = \#\{\text{marbles of colour } i\}$.
The number of selections is $|Q^8_7| = \binom{14}{7}$.
Only one selection has all white marbles, thus the probability is $1/\binom{14}{7}$.

② The number of ways of selecting 5 people from a group from 4 different countries is the no of solns in positive integers of $x_1 + x_2 + x_3 + x_4 = 5$, ie $|Q^4_5| = \binom{8}{3}$

③ Find the number of 10-man teams that can be selected from a large group of Yanks, Brits & Russkies if there are to be at least two Russkies and one Yank and one Briton: $\binom{10-4+3-1}{3-1} = \binom{8}{2}$

THE TYPE OF A FUNCTION.

Let $\rho : [k] \to [n]$

We define the *type of ρ*, written $Typ(\rho)$ or $T\rho$, by:

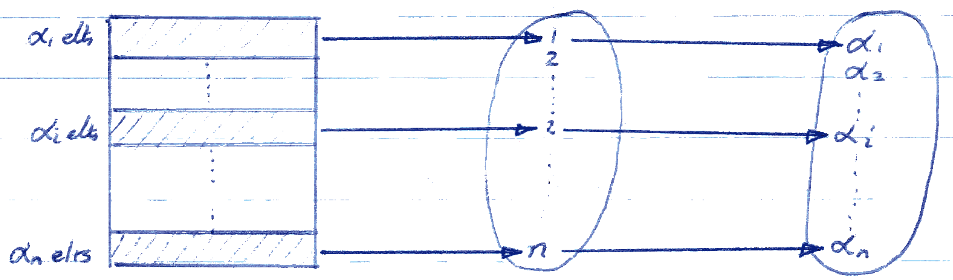
$$T\rho = \left(|\rho^{-1}(\{i\})| \text{ such that } i \in [n] \right)$$

$$= (\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ where } \alpha_i = |\rho^{-1}(\{i\})|$$

$$= \text{no of pre-images of } i.$$

Graphically

$$[k] \xrightarrow{\rho} [n] \xrightarrow{T\rho} \mathbb{N}$$



Note this is a list, not a set.

Note that $\sum_{i=1}^{n} \alpha_i = k$, and thus $T\rho \in Q_k^n$

As a special case consider $\rho : [n] \to \{0,1\}$, ie, splitting $[n]$ into two subsets indexed by 0 and 1, and let k be the number of elts in the second subset (so there are $n-k$ elts in the first). Then $T\rho = (n-k, k)$, and $\rho \in \{0,1\}^{[n]}$. The number of such partitions is

$$\#\{\rho \in \{0,1\}^{[n]} \mid T\rho = (n-k, k)\} = \binom{n}{k},$$

since each $\rho \in \{0,1\}^{[n]}$ with $T\rho = (n-k, k)$ identifies a k-subset of $[n]$.

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ has $\sum_{i=1}^{n} \alpha_i = k$, then we define the <u>multinomial coefficient</u> of α, denoted $\langle \alpha \rangle$, by:

$$\langle \alpha \rangle = \binom{k}{\alpha_1, \dots, \alpha_n} := \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_n!} = \frac{(\sum \alpha_i)!}{\prod (\alpha_i!)}$$

20 <u>THEOREM</u>: $\boxed{\# \{ p \in [n]^{[k]} \mid Tp = (\alpha_1, \alpha_2, \dots \alpha_n) \} = \langle \alpha \rangle}$

<u>DERIVATION</u>: We can chose α_1 preimages for 1 in $\binom{k}{\alpha_1}$ ways, and can choose α_2 preimages for 2 in $\binom{k-\alpha_1}{\alpha_2}$ ways, etc.

Thus the number of ways of choosing the preimages for all $[k]$ is: $\binom{k}{\alpha_1}\binom{k-\alpha_1}{\alpha_2}\binom{k-(\alpha_1+\alpha_2)}{\alpha_3} \cdots \binom{\alpha_n}{\alpha_n} = \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_n!} = \langle \alpha \rangle$

Example:

① If $\alpha = (1,2,3,4)$, then $\langle\alpha\rangle = \binom{4+3+2+1}{4,3,2,1} = \dfrac{10!}{3!2!4!}$

② Find the number of k-subsets of $[n]$.
This is $\#\{p \in \{0,1\}^{[n]} \mid Tp = (n-k,k)\} = \dfrac{n!}{(n-k)!k!} = \binom{n}{k}$

③ Find the number of arrangements of 3 red balls and 4 black balls. Arrangements are of form $p \in \{R,B\}^{[7]}$ with $Tp = (3,4)$.
$$\#\{p \in \{R,B\}^{[7]} \mid Tp = (3,4)\} = \frac{7!}{3!4!}$$

④ If there are α_i objects of type i, $i \in [n]$, $\sum_{i=1}^{n} \alpha_i = k$, then the number of arrangements of the k-objects is $\dfrac{k!}{\alpha_1!\alpha_2!\dots\alpha_n!} = \binom{k}{\alpha_1 \dots \alpha_n}$ since each arrangement is a k-list

$$p = (p(1),\dots,p(k)) \in [n]^{[k]} \text{ with } Tp = (\alpha_1,\dots,\alpha_n)$$

THE MULTINOMIAL THEOREM

We intend to generalise the binomial theorem, but first we need:

LEMMA:
$$\boxed{\prod_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{p \in [m]^{[n]}} \prod_{j=1}^{n} x_{p(j)\,j}}$$

PROOF:
$$\prod_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = (x_{11}+x_{21}+\dots+x_{m1})(x_{12}+\dots+x_{m2})\dots(x_{1n}+\dots+x_{mn})$$

Choose from

When we multiply this out, we get a sum of terms of the form $x_{p(1),1}\, x_{p(2),2}\, x_{p(3),3} \dots\dots x_{p(n),n}$, where $p : [n] \to [m]$

Each p corresponds to one such term, and we sum these over all $p \in [m]^{[n]}$.

Recall the Binomial Theorem: $(x_1+x_2)^n = \sum_{k=0}^{n} \binom{n}{k} x_1^k x_2^{n-k}$

We can write this as: $(x_1+x_2)^n = \sum_{\alpha \in Q_n^2} \binom{n}{\alpha_1\ \alpha_2} x_1^{\alpha_1} x_2^{\alpha_2}$

since each $\alpha = (\alpha_1, \alpha_2) \in Q_n^2$ has $\alpha_1 + \alpha_2 = n$.
Letting $\alpha_1 = k$, $\alpha = (k, n-k)$ is completely specified by k.

Generalising, we get:

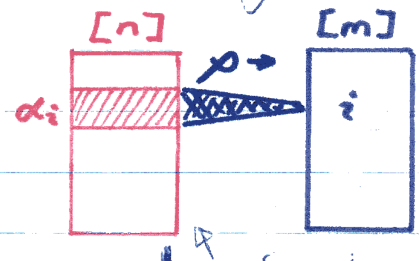
THE MULTINOMIAL THEOREM.

$$\boxed{(x_1+x_2+\dots+x_m)^n = \sum_{\alpha \in Q_n^m} \binom{n}{\alpha_1 \dots \alpha_m} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m}}$$

PROOF: $(x_1 + x_2 + \dots + x_m)^n = \prod_{j=1}^{n} \sum_{i=1}^{m} x_{ij}$ where $x_{ij} := x_i, \ \forall j \in [n]$
the j identifies the factor
from which x_i is taken

$$= \sum_{\rho \in [m]^{[n]}} \prod_{j=1}^{n} x_{\rho(j)j} \qquad (*)$$

We are summing over all $\rho \in [m]^{[n]}$, which is equivalent to summing over all ρ such that $T\rho = (\alpha_1 \dots \alpha_m)$ where there are $\binom{n}{\alpha_1 \dots \alpha_m}$ such ρ, and then summing over all $\alpha \in Q_n^m$.



If $T\rho = \alpha = (\alpha_1 \dots \alpha_m)$

then $\prod_{j=1}^{n} x_{\rho(j)j} = x_{\rho(1)1}\, x_{\rho(2)2} \cdots x_{\rho(n)n}$ | $\alpha_i = \#\{j \in [n] \mid \rho(j) = i\}$

$$= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$$

So $(*)$ becomes $\sum_{\alpha \in Q_n^m} \binom{n}{\alpha_1 \dots \alpha_m} x_1^{\alpha_1} \cdots x_m^{\alpha_m}$

COROLLARY:

$$\boxed{m^n = \sum_{\alpha \in Q_n^m} \binom{n}{\alpha_1 \dots \alpha_m}}$$

PROOF: Set all x's to 1.

Note: $m^n = |[m]^{[n]}| = \#\{\rho \mid \rho : [n] \to [m]\}$

$= \#\{n\text{-lists of } m \text{ distinct objects}\}$

$= \#\{\text{ordered selections of } n \text{ elts from } m\}$

$= \sum \#\{\alpha_i \text{ for } i \in [m]\}$

The latter comes from the addition principle: for each $\alpha \in Q_n^m$ represents a different selection.

Example:

① $(x_1+x_2+x_3)^4 = \binom{4}{1,0,0} x_1^4 + \binom{4}{0,1,0} x_2^4 + \ldots + \binom{4}{3,1,0} x_1^3 x_2^1 + \ldots$

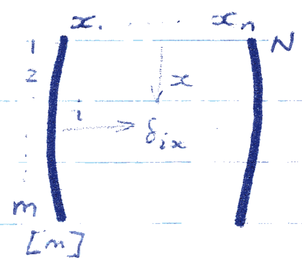
$+ \binom{4}{2,1,1} x_1^2 x_2^1 x_3^1 + \ldots + \binom{4}{1,2,1} x_1^1 x_2^2 x_3^1 + \ldots$ etc.

(23)

PARTITIONS OF A SET

If N is a set and $A_i \subset N$, $\forall i \in [m]$, then we call $(A_1, A_2, \dots, A_m)$ or $(A_i \mid i \in [m])$ a *family* of subsets of N, indexed by $[m]$, often shortened to (A_i). Often, the $i \in [m]$ are properties that some elts of N have, and $A_i = \{x \in N \mid x \text{ has property } i\}$.

If $|N| = n$, we can represent this by a matrix $(m \times n)$ with elts $0, 1$ with $\delta_{ix} = \begin{cases} 1 & \text{if } x \in A_i \\ 0 & \text{if } x \notin A_i \end{cases}$



For $x \in N$ let $A^x := \{i \in [m] \mid x \in A_i\}$, then $(A^x \mid x \in N)$ is called the dual of the family $(A_i \mid i \in [m])$. The family represents the rows, the dual the columns. Both (A_i) and (A^x) are determined by the incidence matrix $[\delta_{ix}]_{m \times n}$

Of course, if it is convenient, we can use any suitable index set I that we please instead of $[m]$. In this case, we write $(A_i : i \in I)$.

(24) We call a family $(A_i : i \in I)$ an ordered partition of N, or a division of N, iff $\begin{cases} i \neq j \Rightarrow A_i \cap A_j = \emptyset \\ \bigcup_{i \in I} A_i = N \end{cases}$

If $|A_i| = \alpha_i$, then we call $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ the type of the division

Examples

① $N = [10]$, $A_1 = \{x \in N \mid 2 | x\}$ ($y | x$ iff y divides x)
$A_2 = \{x \in N \mid 3 | x\}$

then

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

so $A_2 = \{3, 6, 9\}$, $A^4 = \{1\}$, $A^6 = \{1, 2\}$, $A^5 = \phi$, etc.

② $(\{1, 3\}, \{2, 5, 4\}, \{6, 7\})$ is a division of $[7]$, with type $(2, 3, 2)$

THEOREM. The number of divisions of N of type $\alpha = (\alpha_1, \dots, \alpha_m)$ is
$$\binom{n}{\alpha_1 \alpha_2 \dots \alpha_m}$$

PROOF: $\# \{ (A_1, A_2, \dots A_m) : |A_i| = \alpha_i, \ i \in [m] \}$

$$= \binom{n}{\alpha_1}\binom{n-\alpha_1}{\alpha_2}\binom{n-(\alpha_1+\alpha_2)}{\alpha_3} \dots \binom{n-(\alpha_1+\dots+\alpha_{m-1})}{\alpha_m}$$

$$= \frac{n!}{\alpha_1! \alpha_2! \dots \alpha_m!}$$

OR each division $(A_1, \dots, A_m)$ of type $(\alpha_1, \dots, \alpha_m)$ specifies a unique function $p : N \to [m]$ with $Tp = (\alpha_1, \dots, \alpha_m)$ and conversely every $p \in [m]^N$ with Tp specifies a unique division $(A_1, \dots, A_m)$ where $A_i = p^{-1}(\{i\})$ and we know that
$$\# \{ p \in [m]^N : Tp = (\alpha_1, \dots, \alpha_m) \} = \binom{n}{\alpha_1 \dots \alpha_m}$$

(25) The type $\alpha = (\alpha_1, \dots, \alpha_m)$ of a division of N is an element of Q_n^m and so there are $|Q_n^m| = \binom{n+m-1}{m-1}$ types of divisions of N.

If $N = [n]$ is a set of n people to be split into m distinct groups, then

(i) if there must be α_i in each group i for $i \in [m]$, then there are $\binom{n}{\alpha_1, \dots, \alpha_m}$ groups of this type

(ii) there are $\binom{n+m-1}{m-1}$ ways of forming ~~teams~~ groups (some can be empty), ie there are $\binom{n+m-1}{m-1}$ types of groups.

(iii) there are m^n ways of dividing n distinct objects into m distinct groups (incl. empty groups).

~~These~~ last case can be seen directly by counting n-lists $(a_1, a_2, \dots a_n)$ where a_i is the number of the group with object i in, or recalling

$$m^n = \sum_{\alpha \in Q^m_n} \binom{n}{\alpha_1 \dots \alpha_m}$$

Similarly, there are m^n ways of:

- distributing n distinct balls into m distinct boxes
- colouring " " " with " " colours

and there are $\binom{n+m-1}{m-1}$ ways of specifying $(\alpha_1, \dots, \alpha_m) \in Q^m_n$

where $\alpha_i = \begin{cases} \text{no of balls in box } i \\ \text{no of times colour } i \text{ is used} \end{cases}$

and there are $\binom{n}{\alpha_1 \dots \alpha_m}$ ways of:

$\begin{cases} \text{distributing } n \text{ distinct balls into } m \text{ dist. boxes with } \alpha_i \text{ balls in box } i \\ \text{colouring " " " with " " colours with colour } i \text{ used } \alpha_i \text{ times} \end{cases}$

Note that $\binom{n+m-1}{m-1}$ is the number of ways of distributing n indistinct balls into m distinct boxes, since each distribution can be coded as an m-list $(\alpha_1, \dots \alpha_m)$ where α_i = no of balls in box i.

(26)

The Principle of Inclusion and Exclusion.

If $A_1, A_2 \subset N$, then $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Similarly, $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3|$
$- |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

If $(A_i)_{i \in [m]}$ is a family of subset of N, for $K \subset [m]$,

let: $$A_K := \bigcap_{i \in K} A_i$$

$$S_k := \sum_{|K| = k} |A_K| \quad \text{for } k > 0; \quad S_0 := |N|$$

In terms of this, $|A_1 \cup A_2 \cup A_3| = S_1 - S_2 + S_3 = \sum_{k=1}^{3} (-1)^{k-1} S_k$

<u>Theorem</u> If $(A_i)_{i \in [m]}$ is a family of subsets of N,

then $$\left| \bigcup_{i \in [m]} A_i \right| = \sum_{k=1}^{m} (-1)^{k-1} S_k$$

<u>Proof:</u> By induction on m. $m = 2$ is trivial to show
Assume true for $(A_i)_{i \in [k]}$ with $k < n$.

(27) $$\left| \bigcup_{i \in [m]} A_i \right| = \left| A_1 \cup \bigcup_{i=2}^{m} A_i \right| = |A_1| + \left| \bigcup_{i=2}^{m} A_i \right| - \left| A_1 \cap \bigcup_{i=2}^{m} A_i \right|$$

$$= |A_1| + \left| \bigcup_{i=2}^{m} A_i \right| - \left| \bigcup_{i=2}^{m} A_1 \cap A_i \right|$$

(Proof ommitted - vague and probably won't be examined).

Examples

① Out of 100 people interviewed, 47 smoked, 29 chewed gum, and 18 did both. How many did neither? How many did one only? At least one?

Let $A = \{\text{smokers}\}$, $B = \{\text{gum chewers}\}$.

$|A \cap B| = 18$, $47 = |A|$, $|B| = 29$.

Let C = ~~set~~ of people who do neither.

Then $100 = A \cup B \cup C = 47 + 29 + |C| - 18$
So $|C| = 42$

The number of people who do at least one is $|A \cup B|$
ie $|A \cup B| = |A| + |B| - |A \cap B| = 47 + 29 - 18 = 58$.

The number who do one only is
$(|A| - |A \cap B|) + (|B| - |A \cap B|) = 47 + 29 - 18 - 18 = 40$.

Applications of the Principle

We will be given a set N whose elts have some, all, or none of a set of m properties. The standard procedure is to let $A_i = \{x \in N \mid x \text{ has property } i\}$.

Then $S_0 = |N|$

$S_k = \sum_{|K|=k} |A_K|$ = the sum of the orders of the sets whose elts <u>have at least</u> k properties.

$e_k = \#\{x \in N \mid |A^x| = k\}$ = the number of elts with <u>exactly</u> k of the properties.

In particular, $e_0 = \#\{x \in N : |A^x| = 0\}$ = the number of elts with <u>none</u> of the properties.

$$\Rightarrow e_0 = |N \setminus \bigcup_{i=1}^m A_i| = |N| - |\bigcup_{i=1}^m A_i| = S_0 - (S_1 - S_2 + S_3 - \dots + (-1)^{m-1} S_m)$$

$$\Rightarrow e_0 = S_0 - S_1 + S_2 - \dots + (-1)^m S_m = \sum_{k=0}^m (-1)^k S_k$$

(29) <u>Theorem</u>: If $n = \prod_{i=1}^m p_i^{\alpha_i}$ is the prime decomposition of $n \in \mathbb{N}$, then $\varphi(n) = n \prod_{i=1}^m \left(1 - \frac{1}{p_i}\right)$

<u>Proof</u>: Let $N = [n]$ and $A_i = \{x \in N : p_i \mid x\}$

Then $\varphi(n) = \#\{x \in N : hcf(x,n) = 1\} = \#\{x \in N : p_i \nmid x \ \forall i \in [m]\}$
$= |N \setminus \bigcup_{i=1}^m A_i|$

So $\varphi(n) = e_0 = \sum_{k=0}^m (-1)^k S_k$

For $K \subset [m]$, $A_K = \{x \in N : p_i \mid x \ \forall i \in K\} = \{x \in N : \prod_{i \in K} p_i \mid x\}$

Let $k = \prod_{i \in K} p_i$, then $A_K = \{k, 2k, \dots, n = \frac{n}{k} k\}$

$$\Rightarrow |A_K| = \frac{n}{k} = \frac{n}{\prod_{i \in K} p_i}$$

Example - THE EULER ϕ FUNCTION

Recall any integer can be decomposed uniquely into a product of powers of primes, eg $180 = 2^2 3^2 5$

In general if $n \in \mathbb{N}$ and $p_1, p_2, \dots, p_m$ are the primes which divide n, then $n = \prod_{i=1}^{m} p_i^{\alpha_i}$.

We say that k, l are mutually prime iff $hcf(k, l) = 1$.

Let $\phi(n) := \# \{ k \in \mathbb{N} : 1 \le k < n \quad hcf(k, n) = 1 \}$ - the Euler ϕ function

So $\phi(1) = 1 \quad \phi(2) = 1 \quad \phi(3) = \# \{1, 2\} = 2 \quad \phi(4) = \# \{1, 3\} = 2$

$\phi(5) = \# \{1, 2, 3, 4\} = 4 \quad \phi(6) = \# \{1, 5\} = 2 \quad \phi(7) = 6 \quad \phi(8) = \# \{1, 3, 5, 7\} = 4$

Let us try to find $\phi(180)$ without listing the elts of

We know $180 = 2^2 3^2 5$

Let $N = [180]$

$A_i = \{ x \in N : i \mid x \}$

We seek e_0

$$\left.\begin{array}{ll} \text{Now } A_2 = \{2, 4, 6, \dots, 180\} & \Rightarrow |A_2| = 90 \\ A_3 = \{3, 6, 9, \dots, 180\} & \Rightarrow |A_3| = 60 \\ A_5 = \{5, 10, 15, \dots, 180\} & \Rightarrow |A_5| = 36 \end{array}\right\} S_1 = 186$$

$$S_2 = |A_2 \cap A_3| + |A_2 \cap A_5| + |A_3 \cap A_5|$$
$$= \# \{6, 12, \dots, 180\} + \# \{10, 20, \dots, 180\} + \# \{15, 30, \dots, 180\}$$
$$= \frac{180}{2 \times 3} + \frac{180}{2 \times 5} + \frac{180}{3 \times 5} = 30 + 18 + 12 = 60$$

$$S_3 = |A_2 \cap A_3 \cap A_5| = \# \{30, 60, \dots, 180\} = \frac{180}{30} = 6$$

$$\Rightarrow e_0 = S_0 - S_1 + S_2 - S_3 = 180 - 186 + 60 - 6 = 48.$$

So $S_k = \sum_{|K|=k} |A_K| = \sum_{|K|=k} \frac{n}{\prod_{i\in K} p_i}$

$$\Rightarrow e_0 = \sum_{k=0}^{m} (-1)^k S_k = \sum_{k=0}^{m} (-1)^k \sum_{|K|=k} \frac{n}{\prod_{i\in K} p_i}$$

$$= n \sum_{k=0}^{m} \sum_{|K|=k} \frac{(-1)^k}{\prod_{i\in K} p_i}.$$

$$= n \sum_{k=0}^{m} \sum_{|K|=k} \prod_{i\in K} \left(\frac{-1}{p_i}\right) = n \sum_{K\subset[m]} \prod_{i\in K} \left(-\frac{1}{p_i}\right)$$

$$= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_m}\right)$$

Solution of Equations Bounded Above

From pg 16, we see $\#\{(x_1, x_2, \dots, x_n) \in N^n : \sum_{i=1}^{n} x_i = k,\ x_i \geq q_i,\ \forall i\}$

$$= \binom{k - \sum_{i=1}^{n} q_i + n - 1}{n-1}$$

We now seek $\#\{(x_1, \dots, x_n) \in N^n : \sum_{i=1}^{n} x_i = k,\ q_i \leq x_i \leq p_i,\ \forall i\}$

$$= \#\{(x_1, \dots, x_n) \in N^n : \sum_{i=1}^{n} x_i = k - \sum_{i=1}^{n} q_i,\ x_i \leq p_i,\ \forall i\}$$

So we first examine $\{(x_1, \dots, x_n) \in N^n : \sum_{i=1}^{n} x_i = k,\ x_i \leq p_i\ \forall i\}$ (see example)

We seek $\#\{(x_1, x_2, x_3) \in N^3 : \sum_{i=1}^{3} x_i = 4,\ x_1 \leq 2,\ x_2 \leq 3,\ x_3 \leq 4\}$

Let $N = \{\underline{x} \in N^3 : \sum_{i=1}^{3} x_i = 4\} = Q_4^3 \Rightarrow S_0 = |N| = \binom{4+2}{2} = 15$

$A_1 = \{\underline{x} \in N^3 : x_1 > 2\}$ $\qquad A_3 = \{\underline{x} \in N^3 : x_3 > 4\}$

$A_2 = \{\underline{x} \in N^3 : x_2 > 3\}$

Then $S_1 = \binom{4-3+2}{2} + \binom{4-4+2}{2} + \binom{4-5+2}{2} = 3 + 1 + 0 = 4$

$S_3 = S_2 = 0$, so $e_0 = 11 = 15 - 4 = S_0 - S_1 + S_2 - S_3$

Examples

① Find $\#\{(x_1,x_2,x_3)\in\mathbb{N}^3 : \sum_{i=1}^{3} x_i = 7,\ x_1\le 3,\ x_2\le 4,\ x_3\le 5\}$

Let $N = \{\underline{x}\in\mathbb{N}^3 : \sum_{i=1}^{3} x_i = 7\}$

$A_1 = \{\underline{x}\in\mathbb{N}^3 : x_1 > 3\}$

$A_2 = \{\underline{x}\in\mathbb{N}^3 : x_2 > 4\}$

$A_3 = \{\underline{x}\in\mathbb{N}^3 : x_3 > 5\}$

We seek $e_0 = \left|N\setminus \bigcup_{i=1}^{3} A_i\right| = \#\{\underline{x}\in N : x_1\le 3, x_2\le 4, x_3\le 5\}$

30 $S_1 = |A_1| + |A_2| + |A_3| = \binom{7-4+3-1}{3-1} + \binom{7-5+3-1}{3-1} + \binom{7-6+3-1}{3-1} = \binom{5}{2}+\binom{4}{2}+\binom{3}{2} = 19$

$S_2 = |A_1\cap A_2| + |A_1\cap A_3| + |A_2\cap A_3| = \binom{7-9+2}{2} + \binom{7-10+2}{2} + \binom{7-11+2}{2} = 0+0+0=0.$

$S_3 = |A_1\cap A_2\cap A_3| = \binom{7-15+2}{2} = 0.$

So $|N| = s_0 = \binom{7+2}{2} = \binom{9}{2} = 36$

$\Rightarrow e_0 = s_0 - s_1 + s_2 - s_3 = 36 - 19 + 0 - 0 = 17.$

② Find $\#\{(x_1,x_2,x_3)\in\mathbb{N}^3 : \sum_{i=1}^{3} x_i = 7,\ 1\le x_1\le 3,\ 1\le x_2\le 4,\ 1\le x_3\le 5\}$

Let $N = \{\underline{x}\in\mathbb{N}^3 : \sum_{i=1}^{3} x_i = 7,\ x_i \geq 1,\ \forall i\}$

$A_1 = \{\underline{x}\in\mathbb{N}^3 : x_1 > 3\}$ $\quad A_2 = \{\underline{x}\in\mathbb{N}^3 : x_2 > 4\}$

$A_3 = \{\underline{x}\in\mathbb{N}^3 : x_3 > 5\}$

Then $S_1 = |A_1| + |A_2| + |A_3| = \binom{7-6+2}{2} + \binom{7-7+2}{2} + \binom{7-8+2}{2} = 3+1+0 = 4$

$S_2 = |A_1\cap A_2| + |A_2\cap A_3| + |A_1\cap A_3| = \binom{7-10+2}{2} + \binom{7-11+2}{2} + \binom{7-12+2}{2} = 0.$

$S_3 = |A_1\cap A_2\cap A_3| = \binom{7-15+2}{2} = 0$

$S_0 = |N| = \binom{7-3+2}{2} = 15$

$\Rightarrow e_0 = 15 - 4 + 0 - 0 = 11$

OR $\longrightarrow$

In general, the number of ways of colouring k indistinct balls with n distinct colours, where

$q_i \le$ (no of balls with colour i) $\le p_i$, is

$$\#\{(x_1, \dots x_n) \in \mathbb{N}^n : \sum_{i=1}^{n} x_i = k, \ q_i \le x_i \le p_i\}$$

$$= \#\{(x_1, \dots x_n) \in \mathbb{N}^n : \sum_{i=1}^{n} x_i = k - \sum_{i=1}^{n} q_i, \quad x_i \le p_i - q_i\}$$

We obtain this by letting

$$N = \{\underline{x} \in \mathbb{N}^n : \sum_{i=1}^{n} x_i = k - \sum_{i=1}^{n} q_i\}$$

with $A_i = \{\underline{x} \in \mathbb{N}^n : x_i > p_i\}$

and finding $S_0, S_1, \dots, S_n$, and hence e_0

PERMUTATIONS.

Recall that a permutation of $[n]$ is an n-arrangement of $[n]$
so a permutation of n has form $(\sigma(1), \sigma(2), \dots, \sigma(n)) = \sigma$
ie $\sigma : [n] \to [n]$, a bijection.

Let $S_n := \{\sigma \in [n]^{[n]} : \sigma \text{ is bijective (ie a permutation)}\}$
so $|S_n| = |a_n^n| = n!$

A <u>derangement</u> of $[n]$ is a $\sigma \in S_n$ such that $\sigma(i) \neq i, \ \forall i \in [n]$

<u>Example</u>: Find the number of permutations of $[n]$ which have no two adjacent integers consecutive.

If $\sigma \in S_n$ has two adjacent integers consecutive, then σ has form:

$\sigma = (\sigma(1), \sigma(2), \dots, \underbrace{\sigma(i), \sigma(i+1)}_{\sigma(i+1) - \sigma(i) = 1}, \dots, \sigma(n))$ for some $i \in [n-1]$

so let $A_i = \{\sigma \in S_n : \exists i \in [n-1] \text{ st } \sigma(i+1) - \sigma(i) = 1\}$

Examples

① $(3,4,2,1)$ is a permutation of $[4]$

$S_2 = \{(1,2), (2,1)\}$

$S_3 = \{(1,2,3), (1,3,2), (2,1,3)\ (2,3,1)\ (3,1,2)\ (3,2,1)\}$

② If we have n students who place their student cards in a box, then each draw a card, what is the probability that each person will get someone else's card?

A draw sequence is a σ. We seek $\#\{\sigma \in S_n : \sigma \text{ is a derangement}\}$

Let $D_n = \{\sigma \in S_n : \sigma \text{ is a derangement}\}$

$A_i = \{\sigma \in S_n : \sigma(i) = i\}$, $\forall i \in [n]$

so A_i is a family in S_n.

Observe $\tau \in D_n$ iff $\sigma \in S_n \setminus \bigcup_{i \in [n]} A_i$

so $|D_n| = |S_n \setminus \bigcup_{i\in[n]} A_i| = e_0$

We first find $S_k = \sum_{|K|=k} |A_K|$, so let $K \subset [n]$ with $|K| = k$ be fixed

Then $|A_K| = |\bigcap_{i \in K} A_i| = \#\{\sigma \in S_n : \sigma(i) = i, \forall i \in K\}$

$= \#\{\text{permutations of } [n] \setminus K\} = (n-k)!$

Now $\#\{K \subset [n] : |K| = k\} = \binom{n}{k}$

So $S_k = \sum_{|K|=k} |A_K| = \binom{n}{k}(n-k)! = \frac{n!}{k!}$

$\Rightarrow e_0 = \sum_{k=0}^{n} (-1)^k S_k = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} = n!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \dots + \frac{(-1)^n}{n!}\right) \approx \frac{n!}{e}$

$\Rightarrow$ The probability a draw sequence will be a derangement is $\frac{|D_n|}{|S_n|} \approx \frac{n!}{e\,n!} = \frac{1}{e}$

(Note $e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots\dots$)

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We seek $\#\{\sigma \in S_n : \sigma \notin A_i \ \ \forall i \in [n-1]\} = |S_n \setminus \bigcup_{i=1}^{n-1} A_i| = e_0$

For $K \subset [n-1]$, $|A_K| = \#\{\sigma \in S_n : \sigma(i+1) - \sigma(i) = 1, \ \forall i \in K\}$

thus for each $i \in K$, $\sigma(i+1)$ is determined by $\sigma(i)$

$$\sigma = (\sigma(1), \sigma(2), \ldots, \underset{\substack{\uparrow \\ K}}{\sigma(i_1)}, \sigma(i_1+1), \ldots, \underset{\substack{\uparrow \\ K}}{\sigma(i_2)}, \sigma(i_2+1) \ldots \underset{\substack{\uparrow \\ K}}{\sigma(i_k)} \ldots \sigma(n))$$

k fixed values for each choice of $\{i_1, i_2 \ldots i_k\} = K \subset [n-1]$

As k values are fixed, $|A_K| = (n-k)!$

$$\Rightarrow S_k = \sum_{|K|=k} |A_K| = \binom{n-1}{k}(n-k)!$$

$$\Rightarrow e_0 = \sum_{k=0}^{n} (-1)^k S_k = \sum_{k=0}^{n} (-1)^k \binom{n-1}{k}(n-k)!$$

SURJECTIONS.

Let $Sur(n,m) := \{\rho \in [m]^{[n]} : \rho \text{ is a surjection}\}$
We want to find $|Sur(n,m)|$

We define a family $(A_i)_{i \in [m]}$ in $[m]^{[n]}$ by
$$A_i = \{\rho \in [m]^{[n]} : \rho^{-1}(\{i\}) = \Phi\}$$
ie, so $\rho \in A_i$ is not a surjection as it does not have i in its range.

So $\rho \in Sur(n,m)$ iff $\rho \in [m]^{[n]} \setminus \bigcup_{i=1}^{m} A_i$. Let $e_0 = |Sur(n,m)|$
Let $K \subset [m]$, $|K| = k$, then $|A_K| = |([m] \setminus K)^{[n]}| = (m-k)^n$

$$\Rightarrow S_k = \sum_{|K|=k} |A_K| = \binom{m}{k}(m-k)^n$$

and $$e_0 = \sum_{k=0}^{m} (-1)^k S_k = \boxed{|Sur(n,m)| = \sum_{k=0}^{m} (-1)^k \binom{m}{k}(m-k)^n}$$

Partitions

A partition of a set N is a class $(A_i : i \in [m])$ of sets $A_i \subset N$ where
$$\begin{cases} A_i \neq \emptyset & \forall i \in [m] \\ i \neq j \Rightarrow A_i \cap A_j = \emptyset \\ \bigcup_{i=1}^{m} A_i = N \end{cases}$$

Let $P_m(N) = \{$partitions of N into m nonempty disjoint sets$\}$

ie $P_m(N) = \{ (A_i : i \in [m]) : A_i \neq \emptyset,\ i \neq j \Rightarrow A_i \cap A_j = \emptyset,\ \bigcup_{i=1}^{m} A_i = N \}$

The numbers $S(n,m) := |P_m([n])|$ are called the <u>Stirling numbers</u> of m-partitions of $[n]$.

Now for each partition of $[n]$, there are $m!$ surjections $p \in [m]^{[n]}$

so $$S(n,m) = \frac{|Sur(n,m)|}{m!} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

From this we see that the number of ways of distributing n distinct objects into m distinct boxes such that no box is empty is $|Sur(n,m)| = m!\, S(n,m)$.

THE INVERSION FORMULAE

Let $(A_i : i \in [m])$ be a family of subsets of N. We have defined $S_0 = |N|$, $S_k = \sum_{|K|=k} |A_K|$ where $A_K = \bigcap_{i \in K} A_i$, $|K| = k$.

$$e_k = \#\{x \in N : |A^x| = k\}$$

We have seen that $e_0 = \sum_{k=0}^{m} (-1)^k S_k$, so we can obtain e_0 in terms of the S_k's. We now obtain expressions for each e_k in terms of the S_k's and vice-versa. We first need some technical lemmae:

LEMMA 1: $\boxed{\binom{m}{r}\binom{r}{k} = \binom{m}{k}\binom{m-k}{r-k}}$

PROOF: $\binom{m}{r}\binom{r}{k} = \#\{(A_1, A_2) : A_2 \subset A_1 \subset [m], |A_2| = k, |A_1| = r\}$ obtained first by choosing $A_1 \subset [m]$ in $\binom{m}{r}$ ways, then choosing $A_2 \subset A_1$ in $\binom{r}{k}$ ways.

Alternatively, choose $A_2 \subset [m]$ in $\binom{m}{k}$ ways, then choose $A_1 \setminus A_2$ from $[m] \setminus A_2$ in $\binom{m-k}{r-k}$ ways.

LEMMA 2: $\boxed{\sum_r (-1)^{r-k}\binom{m}{r}\binom{r}{k} = \delta_k^m := \begin{cases} 1 & \text{if } m = k \\ 0 & \text{if } m \neq k \end{cases}}$

Note that $\sum_r (-1)^{r-k}\binom{m}{r}\binom{r}{k} := \sum_{r=k}^{m} (-1)^{r-k}\binom{m}{r}\binom{r}{k}$ by our convention.

PROOF: $\sum_r (-1)^{r-k}\binom{m}{r}\binom{r}{k} = \binom{m}{k}\sum_r (-1)^{r-k}\binom{m-k}{r-k} = \binom{m}{k}\sum_h (-1)^h \binom{m-k}{h}$

$$= \binom{m}{k}\sum_h (-1)^h \binom{m-k}{h} 1^{m-k-h} = \binom{m}{k}(1+(-1))^{m-k} \text{ by the Binom Thm.}$$

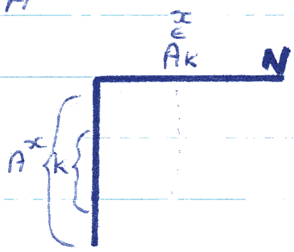
$$= \begin{cases} 0 & \text{if } m \neq k \\ 1 & \text{if } m = k. \end{cases}$$

THEOREM.

$$\boxed{\begin{array}{ll}(a) & S_k = \sum_r \binom{r}{k} e_r \\ (b) & e_k = \sum_r (-1)^{r-k} \binom{r}{k} S_r\end{array}}$$

PROOF: (a) Let $K \subset [m]$, $|K| = k$.

then $x \in A_K$ iff $x \in A_i,\ \forall i \in K$ iff $K \subset A^x$



$$\Rightarrow |A_K| = \#\{x : x \in A_K\} = \#\{x \mid K \subset A^x\}$$

Let $\delta_K(x) = \begin{cases} 1 & \text{if } K \subset A^x \\ 0 & \text{if } K \not\subset A^x \end{cases}$

Then $|A_K| = \sum_{x \in N} \delta_K(x)$ and so $S_k = \sum_{|K|=k} |A_K| = \sum_{|K|=k} \sum_{x \in N} \delta_K(x)$

$$\Rightarrow S_k = \sum_{x \in N} \sum_{|K|=k} \delta_K(x) = \sum_{x \in N} \binom{|A^x|}{k} \quad \ldots \quad (*)$$

In $(*)$, we are summing over all $x \in N$, and we can decompose this by first summing over all x which have $|A^x| = r$ (write this as $\sum_{|A^x|=r}$), then summing over all r.

$(*)$ becomes: $\sum_r \underbrace{\sum_{|A^x|=r}}_{e_r \text{ terms}} \binom{r}{k} = \sum_r \binom{r}{k} e_r$

(b) $e_k = \sum_h \delta_k^h e_h = \sum_h \sum_r (-1)^{r-k} \binom{h}{r}\binom{r}{k} e_h$

$$= \sum_r \left((-1)^{r-k} \binom{r}{k} \left(\sum_h \binom{h}{r} e_h \right) \right)$$

$$= \sum_r (-1)^{r-k} \binom{r}{k} S_r \quad \text{by (a)}.$$

END OF 1ST QUARTER

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Example – The Game of Rencontres

Balls $i \in [n]$ are drawn from a box in a sequence, and if ball i appears on the i^{th} draw of the sequence, we have a recontre (meeting). Find the probability of exactly r recontres occurring in a draw sequence.

Each sequence is an elt $\sigma = (\sigma(1), \ldots, \sigma(n)) \in S_n$

Let $A_i = \{\sigma \in S_n : \sigma(i) = i\} = \{\sigma \in S_n : \sigma \text{ has a recontre on the } i^{th} \text{ draw}\}$

Then $A^{\sigma} = \{i \in [n] ; \sigma(i) = i\}$

$$\Rightarrow e_r = \#\{\sigma \in S_n : |A^{\sigma}| = r\}$$

Now $S_k = \sum_{|K|=k} |A_K| = \binom{n}{k}(n-k)! = \frac{n!}{k!}$

$$\Rightarrow e_r = \sum_k (-1)^{k-r}\binom{k}{r} S_k = \sum_k (-1)^{k-r}\binom{k}{r}\frac{n!}{k!} = \frac{n!}{r!}\sum_k \frac{(-1)^{k-r}}{(k-r)!}$$

$$\Rightarrow e_r = \frac{n!}{r!}\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} \cdots\right) \approx \frac{n!}{r!\,e}$$

$\Rightarrow$ The probability of r recontres is $\approx \frac{e_r}{|S_n|} = \frac{1}{r!\,e}$

Generating Functions.

Let $N = \{x_1, x_2, \dots, x_n\}$ be a set of n commuting variables. REIN+

Let :

$$\boxed{\sigma_k(x_1, x_2, \dots, x_n) = \sigma_k(N) := \sum_{|K|=k} \prod_{i \in K} x_i}$$

We call $\sigma_k(N)$ the <u>k^{th} symmetric function</u> on N, and

define $\sigma_0(N) := 1$

Note that $\sigma_k(N)$ actually lists the k-subsets of N if we identify 1 with ϕ, $\cdot$ with "and" and $+$ with "or"

Note also that $\sigma_k(1, 1, \dots, 1) = \sum_{|K|=k} 1 = \binom{n}{k}$ (counts 1 for each k-subset)

so setting each x_i to 1 counts the k-subsets of N.

Observe that :

(i) $(1 + x_1 t)(1 + x_2 t) = 1 + (x_1 + x_2) t + x_1 x_2 t^2 =$
$= \sigma_0(x_1, x_2) + \sigma_1(x_1, x_2) t^1 + \sigma_2(x_1, x_2) t^2$

(ii) $(1 + x_1 t)(1 + x_2 t)(1 + x_3 t) = 1 + (x_1 + x_2 + x_3) t + (x_1 x_2 + x_1 x_3 + x_2 x_3) t^2 + (x_1 x_2 x_3) t^3$
$= \sigma_0(x_1 x_2 x_3) + \sigma_1(x_1 x_2 x_3) t + \sigma_2(x_1 x_2 x_3) t^2 + \sigma_3(x_1 x_2 x_3) t^3$

In general $\boxed{\prod_{i=1}^{n} (1 + x_i t) = \sum_{k=0}^{n} \sigma_k(x_1, x_2, \dots, x_n) t^k}$ .... (1)

Example.

① $\sigma_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$

$\sigma_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$

$\sigma_3(x_1, x_2, x_3) = x_1x_2x_3$

② $N = \{x_1, x_2, x_3\}$

then $\mathcal{P}_2(N) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$

$\sigma_2(N) = x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3$

Proof: $\displaystyle\prod_{i=1}^{n}(1+x_i t) = \sum_{K\subset[n]} \prod_{i\in K}(x_i t)$

$$= \sum_{k=0}^{n} \sum_{|K|=k} \prod_{i\in K}(x_i t)$$

$$= \sum_{k=0}^{n} \Big(\sum_{|K|=k} \Big(\prod_{i\in K} x_i \Big)\Big) t^k$$

$$= \sum_{k=0}^{n} \sigma_k(x_1, \dots, x_n)\, t^k$$

If $g(t)$ is a polynomial in t let:

$$\mathcal{C}^k g(t) := \text{the coefficient of } t^k \text{ in } g(t)$$

We have just seen that $\mathcal{C}^k \prod_{i=1}^{n}(1+x_i t) = \sigma_k(x_1, \dots, x_n)$ ... ②

Set $x_i = 1$, $\forall i \in [n]$, then: $\mathcal{C}^k (1+t)^n = \sigma_k(1, \dots, 1) = \binom{n}{k}$ ... ③

If I is some index set and $(a_i : i \in I)$ is a sequence indexed by I, then we call $g(t)$ a <u>generating function</u> for $(a_i : i \in I)$ <u>iff</u> $\mathcal{C}^k g(t) = a_k$, $\forall k \in I$.

Returning to ①: $\displaystyle\prod_{i=1}^{n}(1+x_i t) = \sum_{k=0}^{n} \sigma_k(x_1, \dots, x_n)\, t^k$

We can take the left hand side as:

$$\underset{\left(\text{NOT CHOOSE } x_1 \text{ OR CHOOSE } x_1\right) \text{ AND}}{(1 + x_1 t)} \cdot \underset{\left(\text{NOT CHOOSE } x_2 \text{ OR CHOOSE } x_2\right) \text{ AND}}{(1 + x_2 t)} \cdot \dots \cdot \underset{\dots \text{ AND } \left(\text{NOT CHOOSE } x_n \text{ OR CHOOSE } x_n\right)}{(1 + x_n t)}$$

and this just spells out how subsets of $\{x_1, \dots, x_n\}$ can be chosen. The right hand side lists the subsets in the form of symmetric functions $\sigma_k(x_1, \dots, x_n)$ $k = 0, 1, \dots, n$.

Examples.

① $\prod_{i=1}^{n}(1+x_i t)$ is a generating function for $(\sigma_k(x_1, \dots, x_n) \; k\in[n])$

② $(1+t)^n$ is a gen fn. for $(\binom{n}{k} : k\in[n])$

③ $(1-t)^{-n} = 1+t+t^2+\dots$ for $|t|<1$

$\Rightarrow (1-t)^{-n}$ is a generating function for $(1,1,1,\dots)$

If $N = \{x_1, x_2, \ldots, x_n\}$ and we are to choose a k-multiset from N then:

$$(1 + x_1 t + x_1 t^2 + \ldots + x_1 t^k) \cdot (1 + x_2 t + \ldots + x_2 t^k) \ldots (1 + \ldots + x_n t^k)$$

(NOT CHOOSE x_1 OR CHOOSE x_1 ONCE OR CHOOSE x_1 TWICE OR … OR CHOOSE x_1 k TIMES)

prescribes how a k-multiset is chosen from N. Setting all x_i's to 1 will count the number of ways in which a k-multiset can be chosen.

$$\text{ie } \mathcal{C}^k (1 + t + t^2 + \ldots + t^k)^n = |Q^n_k| = \binom{k+n-1}{n-1}$$

(See example 1 for validation)

If $N = \{x_1, x_2, \ldots, x_n\}$ and we are to choose a k-multiset from N where there are restrictions on the allowable repetitions of each x_i, say, $R_i = \{j : x_i \text{ can be repeated } j \text{ times}\}$

As before $\begin{cases} \mathcal{C}^k \prod_{i=1}^{n} \sum_{j \in R_i} x_i t^j & \underline{\text{lists}} \text{ the } k\text{-multisets} \\ \mathcal{C}^k \prod_{i=1}^{n} \sum_{j \in R_i} t^j & \underline{\text{counts}} \text{ the } k\text{-multisets} \end{cases}$

So: $\boxed{\#\{k\text{-multisets from } N \text{ with } R_i = \{\text{reps of } x_i\}\} = \mathcal{C}^k \prod_{i=1}^{n} \sum_{j \in R_i} t^j}$ … (4)

(4) Five balls must be chosen from a large collection of black, white & red balls. What is the probability of selecting an even no of red & odd no of white?

We have $R_{1B} = \{0, 1, \ldots\}$, $R_{2R} = \{0, 2, 4, \ldots\}$, $R_{3W} = \{1, 3, 5 \ldots\}$

We seek $\mathcal{C}^5 (1 + t + t^2 + \ldots)(1 + t^2 + t^4 + \ldots)(t + t^3 + t^5 + \ldots)$

$= \mathcal{C}^5 (1 + t + t^2 + \ldots)(1 + t^2 + t^4 + \ldots)^2 t = \mathcal{C}^4 (1 + t + \ldots)(1 + t^2 + t^4 \ldots)^2$

$= \mathcal{C}^4 (1 + t + t^2 + \ldots)(1 + 2t^2 + 3t^4 + \ldots) = 3 + 2 = 5$

The total number of selections is $|Q^3_5| = \binom{7}{2} = 21 \Rightarrow$ probability $= \frac{5}{21}$.

Examples

① We know $\frac{1}{1-t} = 1 + t + t^2 + \dots$ for $|t| < 1$

$\Rightarrow (1 + t + t^2 + \dots)^n = \left(\frac{1}{1-t}\right)^n = (1-t)^{-n} = g(t)$ say.

Now, $g'(t) = n(1-t)^{-(n+1)}$

$$g''(t) = (n+1)(n)(1-t)^{-(n+2)}$$

$$g^{(k)}(t) = (n+k-1)\dots n(1-t)^{-(n+k)} = \frac{(k+n-1)!}{(n-1)!}(1-t)^{-(n+k)}$$

$\Rightarrow$ By Taylor's theorem: $g(t) = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} t^k$ or $\mathcal{C}^k g(t) = \binom{k+n-1}{n-1}$

i.e., $g(t)$ is a generating function for $\left(\binom{k+n-1}{n-1} : k \in \{0, 1, 2, \dots\}\right)$

② If $R_1 = \{1, 2, 3\}$ then x_1 can be chosen 1, 2 or 3 times

If $R_2 = \{0\}$ then x_2 cannot be chosen

If $R_3 = \{1, 3, 5, \dots\}$ then x_3 can only be chosen an odd number of times

③ (See pg 26, eg①) Find the number of ways of colouring 7 indistinguishable balls with 3 distinct colours where $1 \le r_1 \le 3$, $1 \le r_2 \le 4$, $1 \le r_3 \le 5$

Here $R_1 = \{1, 2, 3\}$, $R_2 = \{1, 2, 3, 4\}$, $R_3 = \{1, 2, 3, 4, 5\}$

We seek: $\mathcal{C}^7 (t + t^2 + t^3)(t + \dots + t^4)(t + \dots + t^5) = \mathcal{C}^7 t^3 (1 + t + t^2)(1 + \dots t^3)(1 + \dots t^4)$

$= \mathcal{C}^4 (1 + 2t + 3t^2 + 3t^3 + 2t^4 + t^5)(1 + t + t^2 + t^3 + t^4) = 1 + 2 + 3 + 3 + 2 = 11.$

Example Find the number of ways of colouring r indistinguishable balls with $p+q$ colours, where:

$$\left.\begin{array}{l} \text{colours } 1,2,\dots p \text{ can be used } \leq 2 \text{ times} \\ \text{colours } p+1,\dots,p+q \text{ can be used } \leq 1 \text{ times} \end{array}\right\} \quad \dots (i)$$

Let $N=[p+q]$, then a colouring is an r-multiset from N with restrictions (i)

ie, a colouring is a $p+q$ list $\varphi = (\varphi(1), \varphi(2) \dots \varphi(p), \dots, \varphi(p+q))$

$$\text{where} \begin{cases} \varphi(i) \leq 2 & \text{for } i \leq p \\ \varphi(i) \leq 1 & \text{for } p+1 \leq i \leq p+q \end{cases}$$

$$\text{and } \sum_{i=1}^{p+q} \varphi(i) = r$$

We seek $\mathcal{C}^r (1+t+t^2)^p (1+t)^q$ and this is $\sum_k \binom{p}{k}\binom{p+q-k}{r-2k}$

because:

We have $\underbrace{(1+t+t^2)(1+t+t^2)\dots(1+t+t^2)}_{p \text{ terms}}\underbrace{(1+t)(1+t)\dots(1+t)}_{q \text{ terms}}$

We can choose k t^2's from the p-terms in $\binom{p}{k}$ ways.
then choose $r-2k$ t's from the remaining $(p+q-k)$ factors in $\binom{p+q-k}{r-2k}$ ways. We sum these over all k, $0 \leq k \leq \text{int}\left(\frac{r}{2}\right)$.

Generating functions can be used to enumerate k-lists from N with restrictions on the repetitions, but we shall not go into this here. See BF 6.3 pg 116 for application to establishing combinatorial identities.

① GRAPH THEORY

If we have a finite set V with a relation $\rho \subset V \times V$, we can represent ρ on a diagram by drawing a directed line from x to y for each $x, y \in \rho$. The diagram we obtain is called the <u>directed graph of ρ</u> and it is characterised by:

(i) A set V called the <u>vertices</u> of the graph

(ii) A set $E \subset V \times V$ called the <u>edges</u> of the graph.

We define a <u>directed graph G</u> to be a pair $G = (V(G), E(G))$ where:

① $V(G)$ is a finite set called the vertices of G

② $E(G) \subset V(G) \times V(G)$ is a finite set of edges of G.

The geometric object obtained by representing each vertex as a point in the plane and each edge (x, y) as a directed line from x to y is called a <u>diagram</u> of G, but we often also call it a directed graph even though there are many diagrams corresponding to any directed graph. If we remove the arrows from the edges of a directed graph, the resulting object is called a (diagram of a) graph.

More precisely, a <u>graph</u> is a pair $G = (V(G), E(G))$ where:

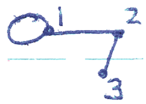
① $V(G)$ is a finite set called the vertices of G

② $E(G) \subset Q_2(V(G))$ is a finite set of edges of G.

② An edge of form $\{x, x\}$ is called a <u>loop</u>.

We often simplify notation by assigning symbols e to the edges

Examples

① 

$$G = (\{1,2,3\}, \{\underbrace{\{1,1\}}_{\text{loop}}, \{1,2\}, \{2,3\}\})$$

If $e_1 = \{1,1\}$, $e_2 = \{1,2\}$, $e_3 = \{2,3\}$, then $G = (\{1,2,3\}, \{e_1, e_2, e_3\})$

If $e = \{x, y\} \in E(G)$ we say that e joins x and y.
If $\{x, y\} \subset V(G)$ we say that x and y are adjacent
iff $\{x, y\} \in E(G)$
If $\{e, f\} \subset E(G)$ we say that e and f are adjacent
iff $\exists \{x, y, z\} \subset V(G) : e = \{x, y\}$ and $f = \{y, z\}$.
If $e = \{x, y\} \in E(G)$ we say that x and y are incident
to e (and vice-versa)
A graph which has multiple edges is called a multigraph.

More precisely, a multigraph is a triple $G = (V(G), E(G), \psi_G)$
where $\psi_G : E(G) \to Q_2(V(G))$

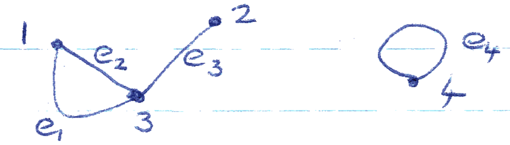
A graph with no loops is called a simple graph. An edge which is not a loop is called a link. We call $|V(G)|$ the order of G and $|E(G)|$ the size of G. If $|E(G)| = 0$ we call G empty.

(3) If $G = (V(G), E(G), \psi_G)$ and $H = (V(H), E(H), \psi_H)$ are multigraphs and there are bijections:
$$\theta : V(G) \to V(H)$$
$$\varphi : E(G) \to E(H)$$
such that $\psi_G(e) = \{x, y\}$ iff $\psi_H(\varphi(e)) = \{\theta(x), \theta(y)\}$ for all $e \in E(G)$, we say that G and H are isomorphic, denoted $G \cong H$. The pair (θ, φ) is called the isomorphism between G and H.

If $G = (V, E, \psi_G)$ is a multigraph, let:
$$\delta : V \times E \to \{0, 1, 2\}$$
$(v, e) \to$ the number of times that v is incident to e.

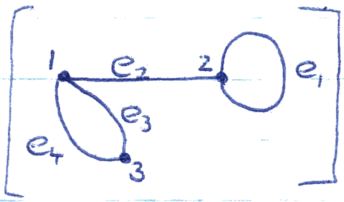
We call the matrix $M(G) := [\delta(v, e)]_{|V| \times |E|}$ the incidence matrix of G.

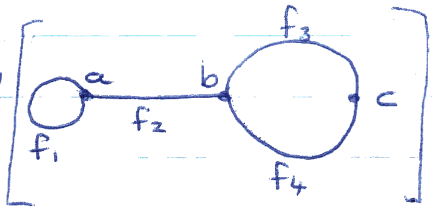
Examples

① 

$V(G) = \{1, 2, 3, 4\}$ $\quad E(G) = \{e_1, e_2, e_3, e_4\}$

$$\psi_G : E \to Q_2([4])$$
$$e_1 \to \{1, 3\}$$
$$e_2 \to \{1, 3\}$$
$$e_3 \to \{2, 3\}$$
$$e_4 \to \{4, 4\}$$

② 

and 

are isomorphic

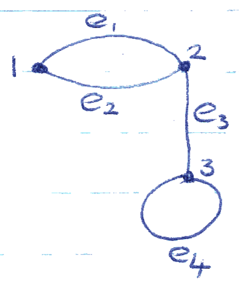
$\theta : V(G) \to V(H)$ $\quad \varphi : E(G) \to E(H)$

$1 \to b$ $\quad e_1 \to f_1$

$2 \to a$ $\quad e_2 \to f_2$

$3 \to c$ $\quad e_3 \to f_3$

$e_4 \to f_4$

③ 

$$m(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(4)

Let $\alpha : V \times V \to \mathbb{N}$
$(u,v) \to$ the number of edges joining u and v.

We call the matrix $A(G) := [\alpha(u,v)]_{|V| \times |V|}$ the

<u>adjacency matrix</u> of G.

If $G = (V(G), E(G), \psi_G)$ is a multigraph we say that $H = (V(H), E(H), \psi_H)$ is a subgraph of G, denoted $H \subset G$ iff

$$\begin{cases} V(H) \subset V(G) \\ E(H) \subset E(G) \\ \psi_H = \psi_G|_{E(H)} \end{cases}$$

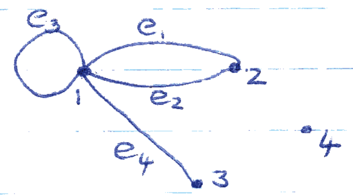
If $\phi \neq V_1 \subset V$, the subgraph $G[V_1]$ is defined to be that subgraph of G which has V_1 as vertex set and whose edge set is the set of edges $\{x,y\} = e$ of G which have $\{x,y\} \subset V_1$. We call $G[V_1]$ the subgraph of G <u>induced</u> by V_1.

We write $G - V_1$ for $G[V \setminus V_1]$ and $G - v$ for $G - \{v\}$

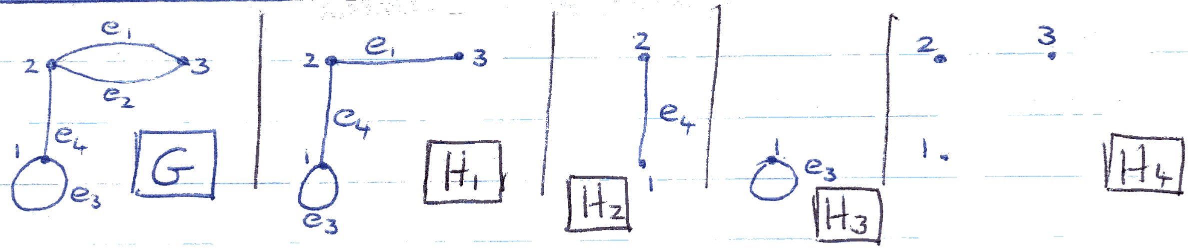
(5) If $\phi \neq E_1 \subset E$, the subgraph $G[E_1]$ is defined to be that subgraph of G which has E_1 as edge set and whose vertex set is $\bigcup_{\{x,y\} \in E_1} \{x,y\}$. We call $G[E_1]$ the subgraph of G induced by E_1. We write $G - E_1$ for $G[E \setminus E_1]$ and $G - e$ for $G - \{e\}$.

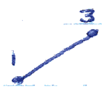
We call H a spanning subgraph of G iff $H \subset G$ and $V(H) = V(G)$

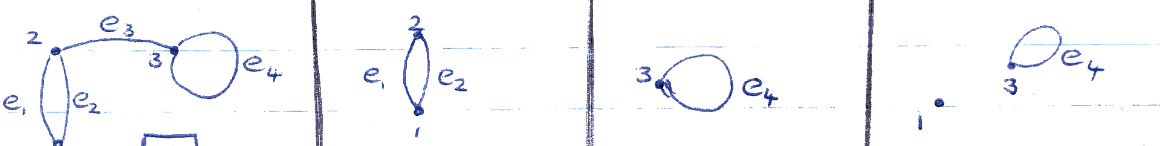
Examples

① 

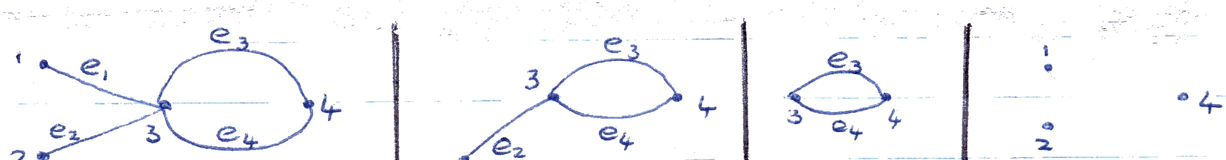
$$A(G) = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

② 

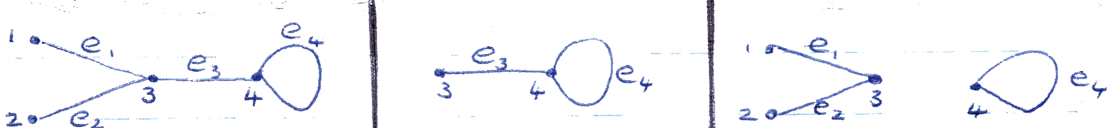
H_1, H_2, H_3, H_4 are subgraphs of G, but  is not.

③ 

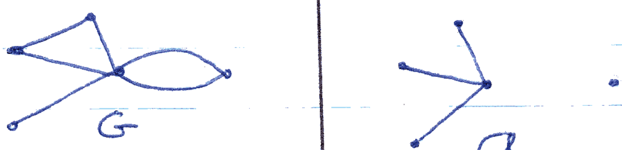
G $\quad$ $G[\{1,2\}]$ $\quad$ $G[\{3\}]$ $\quad$ $G[\{1,3\}]$

④ 

G $\quad$ $G-1$ $\quad$ $G-\{1,2\}$ $\quad$ $G-3$

⑤ 

G $\quad$ $G-\{e_1,e_2\}$ $\quad$ $G-e_3$

⑥ 

G $\quad$ A spanning subgraph of G.

For $v \in V(G)$, we call $d_G(v)$ the <u>degree</u> of the vertex v in G, defined as:

$$\boxed{d_G(v) := \text{the no. of edges incident to } v.}$$

If no confusion arises, we simply write $d(v)$.

<u>THEOREM</u>: $\boxed{\sum_{v \in V(G)} d(v) = 2|E(G)| \text{ for any multigraph } G.}$

<u>PROOF</u>: Consider $M(G) = [\delta(v,e)]_{|V| \times |E|}$

Let $|E(G)| = m$

Consider $v \in V(G)$: $v\,[\delta(v,e_1) \ldots \delta(v,e_m)]$

Then $\sum_{i=1}^{m} \delta(v,e_i) = d(v)$

$$\Rightarrow \sum_{v \in V} d(v) = \sum_{v \in V} \sum_{e \in E} \delta(v,e) = \sum_{e \in E} \sum_{v \in V} \delta(v,e) = \sum_{e \in E} 2 = 2|E|$$

as the number of vertices incident to any edge is 2.

(6) <u>COROLLARY</u>: $\boxed{\text{The number of vertices of odd degree is even.}}$

<u>PROOF</u>: Let $V_1 = \{v \in V : d(v) \text{ is odd}\}$.

Then $\sum_{v \in V} d(v) = \sum_{v \in V_1} d(v) + \sum_{v \in V \setminus V_1} d(v) = 2|E|$, even

Obviously $\sum_{v \in V \setminus V_1} d(v)$ is even, so $\sum_{v \in V_1} d(v)$ must be even.

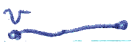
But $d(v)$ is odd for each $v \in V_1$, so $|V_1|$ is even.

A graph is <u>k-regular</u> iff $d(v) = k$ for each $v \in V$.
A graph is <u>regular</u> iff it is k-regular for some k.

Examples

① 

$d(v) = 3$



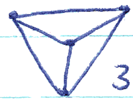
$d(v) = 1$

$d(v) = 0$

$d(v) = 4$

② 
0-regular


2-regular


3-regular.

A graph is <u>complete</u> iff it is simple and each pair of distinct vertices is joined by an edge.

$K_n :=$ a complete graph on n vertices

A graph G is <u>bipartite</u> iff $V(G) = X \cup Y$ where each edge has one vertex in X and the other in Y. Call (X, Y) a <u>bipartition</u> of G.

A <u>complete bipartite graph</u> is a simple graph with bipartition (X, Y) such that each vertex in X is joined to each vertex in Y. If $|X| = m$, $|Y| = n$, the graph is denoted $K_{m,n}$

<u>Connected Graphs</u>

If G is a multigraph and $u, v \in V(G)$ then:

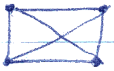
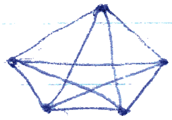
- a <u>u-v walk</u> in G is a sequence $u = v_0 e_1 v_1 e_2 \dots e_n v_n = v$ where $v_i \in V(G)$, $e_i \in E(G)$ and $\psi_G(e_i) = \{v_{i-1}, v_i\}$, $\forall i$. The number n of edges is the <u>length</u> of the walk.
- a <u>u-v trail</u> in G is a u-v walk such that the edges e_i are distinct
- a <u>u-v path</u> in G is a u-v walk such that the vertices v_i are distinct

Note: a path is a trail is a walk.

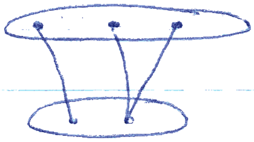
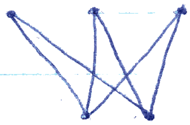
Two vertices $u, v \in V(G)$ are <u>connected</u> iff there is a u-v path or $u = v$. ~~He~~

If we write $u \sim v$ to denote that u and v are connected then we can easily see that $\sim$ is an equivalence relation on $V(G)$

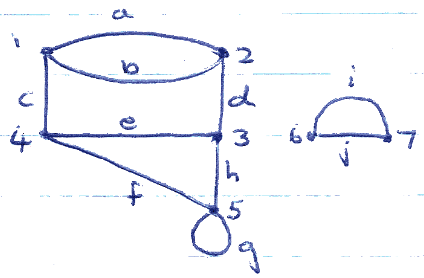
Examples

①   

K_3 K_4 K_5

②  

Bipartition (X, Y) $K_{3,2}$.

③ 

4c1a2b1c4f5g5 is a 4-5 walk, length 6

1a2b1c4e3h5 is a 1-5 trail, length 5

1b2d3h5 is a 1-5 path of length 3

1, 2 are connected, 1, 6 are not.

(see next page) $[1] = [2] = \ldots = [5]$ and $[6] = [7]$

G is not connected, as it has two connected components $G[[2]]$ and $G[[6]]$

4e3h5f4 is a cycle

4f5g5h3e4 is a circuit

5g5 is a cycle

3d2b1a2d3 is a closed walk, but not a circuit

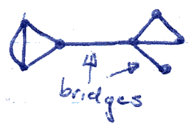
This means that $V(G)$ is partitioned into equivalence classes. The induced subgraphs on these equivalence classes are called the <u>connected components</u> of G. We say G is connected if it has only one connected component. (ie every pair of vertices is connected)

If $[u] := \{v \in G : v \sim u\}$ then G is <u>connected</u> iff $V(G) = [v]$, $\forall v \in V(G)$.

(see eg ③ previous page)
If G is not connected, it is <u>disconnected</u>

If a u-v walk has $u = v$ we call it <u>closed</u>
A closed trail of positive length is called a <u>circuit</u>
If a circuit has all its vertices (except initial/final) distinct, it is a <u>cycle</u>

If G is a connected graph, then the edge e is a bridge iff $G - e$ is disconnected.

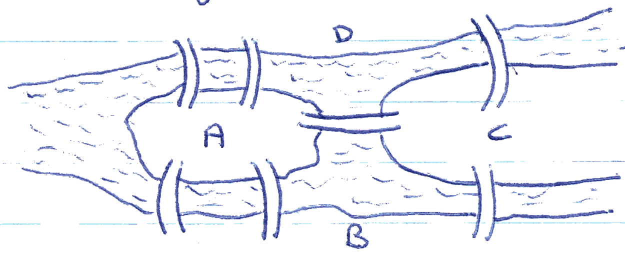


<u>THEOREM</u>: If G is a connected multigraph and $e \in E(G)$ then e is a bridge iff e does not lie on any cycle of G.

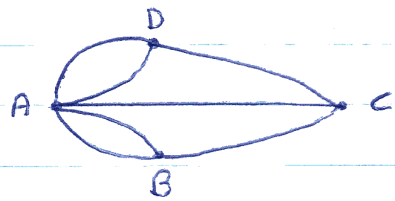
<u>PROOF</u>: $\Rightarrow$: Let e be a bridge and suppose that e does ~~not~~ lie on a cycle K of G.
Let $K = v_0 e_1 v_1 e_2 \dots e_n v_n$ and $e = e_k$.
Now, $G - e$ is disconnected, with v_k belonging to one^{G_1}, and v_{k-1} the other^{G_2}, of two connected components.
But $v_k \sim v_{k+1} \sim \dots \sim (v_n = v_0) \sim v_1 \sim \dots \sim v_{k-1}$
ie $v_k \sim v_{k-1}$
So if $u \in V(G_1)$ and $v \in V(G_2)$ then $u \sim v_k$ and $v \sim v_{k-1}$
$\Rightarrow u \sim v$ (transitivity) which contradicts the disconnectivity of $G - e$

THE KÖNIGSBERG BRIDGE PROBLEM

Seven bridges cross the river Pregel in Königsberg:



Euler proved that a continuous walk is impossible.
Represent the situation with G:



<u>Assertion</u> There is no trail in G containing all the edges of G.
<u>PROOF:</u> Note $d(A) = 4$, $d(B) = d(D) = d(C) = 3$.

Suppose that there is a trail containing all the edges of G. At least one of the vertices B, D, C is neither the initial nor the final vertex in the trail. Call this vertex v. Since $d(v) = 3$ the trail must pass through v once and upon returning must end at v. Thus v is the final vertex – a contradiction

$\Leftarrow$: Let e lie on no cycle of G and suppose e is not a bridge. Let $\psi(e) = \{u, v\}$

Then $G - e$ is connected

$\Rightarrow$ there is a $u-v$ path $u = v_0 e_1 v_1 \dots e_n v_n = v$

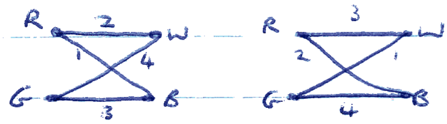
$\Rightarrow$ $u e_1 v_1 \dots e_n v_n e u$ is a cycle containing e.

contradiction.

---

each colour appears once on one ~~each~~ face and once on the opposite face.

In this example:



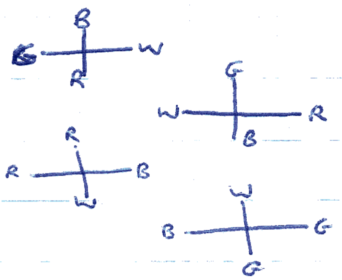
The stacking goes as follows:

first : cube 1

second : cube 3

third : cube 2

fourth : cube 4



← forced (if we stacked cube 2 now we would have a choice:

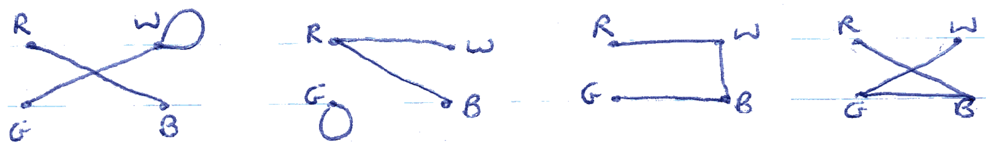
Instant Insanity

The puzzle consists of four cubes. Each of the six faces is coloured with one of four colours. The problem is to stack the cubes so that the vertical edges of the rectangular prism constructed contains each of the four colours.

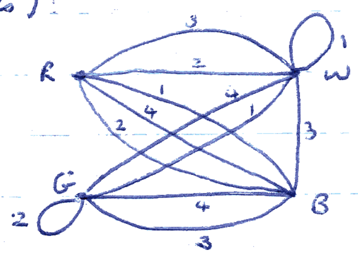
For example, if the cubes are coloured:

Represent the colouring of a cube as a graph on the vertices $\{R, B, G, W\}$ where $\#\{\text{edges connecting } u \text{ and } v\}$ = the number of times u and v are opposite in the cube.

We obtain:



Superimposing these (labelling edges to keep track of the blocks):



We seek two subgraphs F and L of G representing the front-back colouring and the left-right colouring of the prism.

Each subgraph must have 4 edges labelled with distinct numbers; each vertex must have degree 2 since

(46) Recurrence Relations.

Examples • Tower of Hanoi: number of moves required for n discs is $S_n = 2S_{n-1} + 1$, $S_1 = 1$

• Fibonacci numbers $a_n = a_{n-1} + a_{n-2}$, $a_0 = a_1 = 1$

These equns are called recurrence relations or difference equns together with their boundary conditions.

If $F(a_n, a_{n-1}, \dots, a_{n-r}) = 0$ is a difference equation and $a_1 = \gamma_1, \dots, a_r = \gamma_r$ is a set of boundary conditions, we call a function $f: \mathbb{N}^+ \to \mathbb{R}$ a solution if $f(n) = a_n \ \forall n$.

Finding Solutions by Inspection.

Eg. Towers of Hanoi.

n	1	2	3	4
S_n	1	3	7	15

Guess: $S_n = 2^n - 1$

Prove by induction $S_n = 2(2^{n-1}) - 1 = 2(2^{n-1} - 1) + 1$

$= 2S_{n-1} + 1 \longrightarrow$

Alternative: list $a_n, \frac{a_n}{n}, \frac{a_n - 1}{n}, \frac{a_n + 1}{n}, \frac{a_n + k}{b_n}$, etc and look for some relationship.

Linear Recurrence Relns. with Constant Coeffs.

We examine recurrence relns of the form:

$$c_0 a_n + c_1 a_{n-1} + \dots + c_r a_{n-r} = 0, \quad c_i \text{ constants}$$

with r given boundary conditions $a_1 = \gamma_1, \dots, a_r = \gamma_r$.

Eg Fibonacci numbers $a_n - a_{n-1} - a_{n-2} = 0$

$a_1 = a_2 = 1$

We try to find $c, \alpha \in \mathbb{R}$ such that $f(n) = c\alpha^n$ is a soln.

so $c\alpha^n - c\alpha^{n-1} - c\alpha^{n-2} = 0$

$\Rightarrow c\alpha^{n-2}(\alpha^2 - \alpha - 1) = 0$ satisfied by $\underset{\text{fails boundary conditions}}{\alpha = 0}$ or $\alpha^2 - \alpha - 1 = 0$

so let $\alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha = \frac{1 \pm \sqrt{5}}{2}$ (α_1 & α_2)

Thus $f_1(n) = c_1\alpha_1^n$ and $f_2(n) = c_2\alpha_2^n$ are two solns.

So $f_1(n) + f_2(n) = f(n) = c_1\alpha_1^n + c_2\alpha_2^n$ is also a soln.

To show this is a general soln, we must show that it satisfies any set $\{\gamma_1, \gamma_2\}$ of boundary conditions for suitable c_1, c_2 (omitted).

In our case: $f(1) = c_1\alpha_1 + c_2\alpha_2 = 1$

$f(2) = c_1\alpha_1^2 + c_2\alpha_2^2 = 1$

so $\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_1^2 & \alpha_2^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow c_1 = \frac{\begin{vmatrix} 1 & \alpha_2 \\ 1 & \alpha_2^2 \end{vmatrix}}{\alpha_1\alpha_2(\alpha_2 - \alpha_1)} = \frac{\alpha_1}{\sqrt{5}}, \; c_2 = \frac{\begin{vmatrix} \alpha_1 & 1 \\ \alpha_1^2 & 1 \end{vmatrix}}{\alpha_1\alpha_2(\alpha_2 - \alpha_1)} = -\frac{\alpha_2}{\sqrt{5}}$

We now investigate linear recurrence relations of the form

$c_0 a_n + c_1 a_{n-1} + \dots + c_r a_{n-r} = g(n)$ (eg Tower of hanoi $S_n - 2S_{n-1} = 1$)

We ① find a general soln $h(n) = a_n$ to $c_0 a_n + \dots + c_r a_{n-r} = 0$

② " " particular " $p(n) = a_n$ to " " " $= g(n)$

We then claim that $h(n) + p(n)$ is the general soln.

Finally, we show that we can find c's to satisfy the boundary conditions γ.