REAL ANALYSIS.

§1

① PROPERTIES OF REAL NUMBER SYSTEM.

ADDITION

A_1 : $x + (y+z) = (x+y) + z$ associative

A_2 : $x + y = y + x$ commutative

A_3 : $x + \phi = x$ ϕ is additive identity

A_4 : $x + (-x) = \phi$ each real has additive inverse.

MULTIPLICATION

B_1 : $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ associative

B_2 : $x \cdot y = y \cdot x$ commutative

B_3 : $x \cdot 1 = x$ 1 is multiplicative identity

B_4 : $x \cdot \frac{1}{x} = 1$ if $x \neq 0$ each non-zero real has multiplicative inverse

C : $x(y+z) = xy + xz$ distributive

Systems that satisfy A are called groups

Systems that satisfy A, B & C are called fields.

② ORDERING

D_1 : $x \leq x$ reflexive

D_2 : $x \leq y \wedge y \leq x \Rightarrow x = y$ antisymmetric

D_3 : $x \leq y \wedge y \leq z \Rightarrow x \leq z$ transitive

D_4 : $\forall x, y \Rightarrow (x \leq y \vee y \leq x)$ total.

Systems satisfying D_1, D_2 & D_3 are called partially ordered.

(5) Systems satisfying D_1 thru D_4 are called totally ordered.

Systems satisfying A, B, C and D are called ordered fields.

THE COMPLETENESS AXIOM

This distinguishes the rationals from the reals. First we need to consider set bounds.

A set S is said to be bounded above if $\exists c \in \mathbb{R} \ni (\forall x \in S, x \leq c)$. Similarly for below.

S is bounded if it is bounded both above and below, that is, $\exists c \in \mathbb{R} \ni (|x| \leq c \ \forall x \in S)$

A set is said to have a least upper bound (l.u.b.) if $\exists \gamma \ni (\gamma$ is an upper bound $\wedge$ for any other upper bound γ', $\gamma^* \leq \gamma')$

Similarly for greatest lower bound (g.l.b.)

(6) Every non-empty set in $\mathbb{R}$ which has an upper bound, has a least upper bound in $\mathbb{R}$. This is the completeness axiom. $\mathbb{R}$ is thus a complete ordered field.

(7) CLUSTER POINTS

c is called a cluster point of a set S in $\mathbb{R}$ if every open interval containing c has infinitely many different points of S. A set S is closed if it

(8) contains all its cluster points, and open if S' (where $S' = \mathbb{R} \setminus S = \mathbb{R} - S$) is closed

A different definition for cluster points is: c is a cluster point of S iff every open interval containing c has at least one point of S different from c.

BOLZANO - WEIERSTRASS THEOREM

Let T be a bounded infinite set in $\mathbb{R}$. Then T has a cluster point.

THE ARCHIMEDEAN PROPERTY OF THE REAL NUMBERS

Consider a number ε, $\varepsilon > 0$. Then $\exists (n \in \mathbb{N}) \ni \left(\frac{1}{n} < \varepsilon\right)$

SEQUENCES.

A sequence S is an ordered set of reals, each associated with a unique cardinal positive integer, denoted $S = (a_1, a_2, \ldots) = (a_n)$

(9) SUBSEQUENCES

A subsequence S' of a sequence S is a sequence (ordered) which is contained in S. More formally, let $S = (a_n)$ and $(n_k) = (n_1, n_2, n_3, \ldots)$ etc where $n_1, n_2, \ldots$ are integers and are ordered such that $n_1 < n_2 < n_3 \ldots$ Then $S' = (a_{n_k}) = (a_{n_1}, a_{n_2}, a_{n_3}, \ldots)$ is a subsequence of $S = (a_n) = (a_1, a_2, a_3, \ldots)$

(10) A tail of a sequence $S = (a_n)$ is a sequence S' of the form $S' = (a_m, a_{m+1}, a_{m+2}, \ldots)$ where $m \geq 1$.

We call the set $\{a_n \mid n \in \mathbb{N}\}$ the set of values of (a_n)

§2

(10) CONVERGENCE OF A SEQUENCE

A sequence (a_n) of reals converges to the real number L if given any number $\varepsilon > 0$, we can find an integer $N > 0$ such that

$$(\forall n \in \mathbb{N})\ (n \geq N \Rightarrow |a_n - L| < \varepsilon)$$

Thm 1 Prof required
If $a_n \to L$ and $a_n \to L'$ then $L = L'$
If (a_n) is convergent, then (a_n) is bounded.
i.e. $\exists k \in \mathbb{R}\ \ni ((|a_n| \leq k)\ \forall (n \in \mathbb{N}))$

(11) RULES FOR LIMITS (THEOREM 2) prof required

If $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$, then

(a) $\lim_{n \to \infty} (a_n + b_n) = A + B$

(b) $\lim_{n \to \infty} (\alpha a_n) = \alpha A \qquad (\alpha \in \mathbb{R})$

(c) $\lim_{n \to \infty} (a_n b_n) = AB$

(d) $\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = A/B$ if $B \neq 0$

(e) $(\forall n \geq N)(a_n \leq b_n) \Rightarrow A \leq B$

(f) $(\exists N \in \mathbb{N})(\forall n \geq N)\wedge(a_n \leq c_n \leq b_n)\wedge(A=B) \Rightarrow \lim_{n\to\infty} c_n = A = B$

i.e. if $a_n \leq c_n \leq b_n$ for all $n \geq N$, some $N \in \mathbb{N}$, and $A = B$, then $C = A = B$ (squeeze rule).

(19) INCREASING AND DECREASING SEQUENCES

A sequence (a_n) is monotone increasing if $\forall n \in \mathbb{N}$, $a_{n+1} \geq a_n$. Similarly for monotone decreasing.

Thm 3 Proof required

If (a_n) is bounded above and monotone increasing, then it converges to its least upper bound. Similarly for decreasing.

(21) SOME USEFUL INEQUALITIES AND LIMITS

If $x \geq -1$ and $n \geq 2$ then $(1+x)^n \geq nx$

If $x \geq 0$ and $n \geq 2$ then $(1+x)^n \geq 1 + nx + \binom{n}{2}x^2$

PROPOSITION Proof required

If $|a| < 1$ then $\lim_{n\to\infty} a^n = 0$ and $\lim_{n\to\infty} na^n = 0$.

(23) If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

(24) Let a be a cluster point of the set $\{a_n \mid n \in \mathbb{N}\}$ where (a_n) is a sequence. Then there is a subsequence (a_{n_k}) of (a_n) such that $\lim_{k\to\infty} a_{n_k} = a$.

§3

(24) INFINITE SERIES OF REAL NUMBERS

Let (a_n) be a sequence. A series $\sum a_n$ is a sequence (S_N) of the form:

$$S_N = a_1 + a_2 + a_3 + \cdots + a_N = \sum_{i=1}^{N} a_i$$

Let $\sum_{n=1}^{\infty} a_n$ be a series. The sequence (S_N) is called the sequence of N^{th} partial sums of the series.

The series $\sum_{n=1}^{\infty} a_n$ is said to converge to $A \in \mathbb{R}$ if $\lim_{N\to\infty} S_N = A$ (sum of infinite series). If $\lim_{N\to\infty} S_N$ does not exist, the series is said to diverge.

(25) PROPN 1 Proof required (easy)

Note that if $\lim_{n\to\infty} a_n \neq 0$, the series diverges, but $\lim_{n\to\infty} a_n = 0$ is not a sufficient condition for convergence.

PROPN 2 Proof required (easy)

Let $A = \sum_{n=1}^{\infty} a_n$ and $B = \sum_{n=1}^{\infty} b_n$ be convergent. Let $\alpha \in \mathbb{R}$

Then (a) $S = \sum_{n=1}^{\infty} (a_n + b_n) = A + B$ is convergent.

(b) $T = \sum_{n=1}^{\infty} (\alpha a_n) = \alpha A$ is convergent.

(c) $\sum_{n=N}^{\infty} a_n$ is convergent for / any $N \in \mathbb{N}$

(27) SERIES OF POSITIVE TERMS

Suppose $a_n \geq 0$ for $\forall n \in \mathbb{N}$

Then $S_{N+1} = S_N + a_{N+1} \Rightarrow S_{N+1} \geq S_N$

ie, the series is increasing. If S is bounded above, then $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ is convergent.

COMPARISON TEST FOR SERIES OF +VE TERMS

Proof required (easy - bounded & increasing)

- if $\sum a_n$ is convergent and $b_n \leq a_n$ then $\sum b_n$ is convergent
- if $\sum a_n$ is divergent and $b_n \geq a_n$ then $\sum b_n$ is divergent

The comparison can be done from any point (see bottom of previous page)

[3.3] RATIO TEST (Proof required [3.4])

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$ exists, then

- if $r < 1 \Rightarrow \sum a_n$ converges
- if $r > 1 \Rightarrow \sum a_n$ diverges
- if $r = 1$ the test is inconclusive

[3.2] LIMIT TEST (Proof required - use comparison test)

For two series of positive terms $\sum a_n$ and $\sum t_n$

- if $\lim_{n\to\infty} \frac{a_n}{t_n}$ exists and $\sum t_n$ converges then $\sum a_n$ converges
- if $\lim_{n\to\infty} \frac{t_n}{a_n}$ exists and $\sum t_n$ diverges then $\sum a_n$ diverges

[3.5] INTEGRAL TEST (Proof required)

Let $f(x)$ be a positive decreasing function continuous on $[1,\infty)$

- if $\lim_{n\to\infty} \int_1^n f(x)\cdot dx$ exists $\sum f(n)$ converges

- if $\lim_{n\to\infty} \int_1^n f(x)\cdot dx$ does not exist $\sum f(n)$ diverges

PROCEDURE FOR TESTING SERIES OF POSITIVE TERMS

- does $a_n \to 0$?
- try comparison, ratio or limit test using $\sum r^n$ or $\sum \frac{1}{n^a}$
- try integral test

Note: $\sum \frac{1}{n^x}$ converges iff $x > 1$, else diverges
$\sum r^n$ converges for $|r| < 1$ else diverges } Prove

[3.6] GENERAL SERIES

$\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ converges.
If $\sum a_n$ converges but $\sum |a_n|$ does not, $\sum a_n$ is called *conditionally convergent*. If a series is absolutely convergent, it is convergent.

LEIBNIZ TEST

If $\sum a_n$ is a positive decreasing series then
$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges to A, where $|A - S_N| < a_{N+1}$

[3.7] PROCEDURE FOR TESTING GENERAL SERIES

- does $a_n \to 0$
- is it absolutely convergent
- failing test

POWER SERIES $\left(\sum a_n x^n \text{ or } \sum a_n (x - x_0)^n \right)$

THM 1 (Proof required)

If $\sum a_n x^n$ satisfies $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho \begin{cases} \neq 0 \\ \neq \infty \end{cases}$

then the series is absolutely convergent for $|x| < \frac{1}{\rho}$ and divergent for $|x| > \frac{1}{\rho}$

If $\rho = 0$ series is convergent for all x $\Big\}$ Proof by ratio test

If $\rho = \infty$ series is divergent for all $x \neq 0$ $\Big\}$

[3.8] For any power series, $\exists\, r \in \mathbb{R}^+ \ni$

$\begin{cases} \sum a_n x^n \text{ converges for } |x| < r \\ \sum a_n x^n \text{ diverges for } |x| > r \end{cases}$

$|x| = r$ is undecided

r is called the radius of convergence. $\{x \mid x < |r|\}$ is called the ~~radius~~ interval of convergence

---

§4 [4.1] LIMITS OF FUNCTIONS (see defn 4.1)

[4.2] $\lim_{x \to a} f(x) = A$ iff $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = A$

Suppose $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$

Then – $\lim_{x \to a} f(x)$ is unique

$-\{\exists\, \delta > 0 \wedge \exists\, C\} \ni \{|f(x)| < C \ \forall\ |x-a| < \delta\}$

$- \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = \lim_{x\to a} (f(x)+g(x)) = A+B$

$- \lim_{x\to a} \alpha f(x) = \alpha \lim_{x\to a} f(x) = \alpha A \ \forall\ \alpha \in \mathbb{R}$

$- \lim_{x\to a} f(x).g(x) = \lim_{x\to a} f(x) . \lim_{x\to a} g(x) = AB$

$- \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}$ iff $\lim_{x\to a} g(x) \neq 0$

$-$ if $f(x) \leq g(x)$ for $0 < |x-a| < \delta$ (some δ)

then $A \leq B$

$-$ if $f(x) \leq h(x) \leq g(x)$ for $0 < |x-a| < \delta$ (some δ)

and $A = B$, then $\lim_{x\to a} h(x) = A = B$

[5.1] §5 CONTINUOUS FUNCTIONS

DEFN 1

$- f(x)$ is continuous at $x = a$ iff $\lim_{x\to a} f(x) = f(a)$ exists and is true

$- f$ is continuous on $[a,b]$ if it is con. on (a,b) and $\lim_{x\to a^+} f(x) = f(a)$ & $\lim_{x\to b^-} f(x) = f(b)$

$- f(x)$ is continuous on (a,b) iff it is continuous for all $x \in (a,b)$

THM 1

If $f(x)$ and $g(x)$ are continuous at $x = a$ then so are

$f(x) + g(x)$, $\alpha f(x)\ \forall\ \alpha \in \mathbb{R}$, $f(x).g(x)$, $\frac{f(x)}{g(x)}$ if $g(x) \neq 0$

PROPN 2 (Proof reqn)
Let $f(y)$ be continuous at $y = g(a)$ and $g(x)$ be continuous at $x = a$. Then $f\circ g(x)$ is continuous at $x = a$

PROPN 3 (Proof reqn).
[5.2] If $f(x)$ is continuous at a and $f(a) > 0$ then there is a $\delta > 0$ such that $f(x) > 0$ for $x \in (a-\delta, a+\delta)$
Similarly if $f(a) < 0$ then $f(x) < 0$ for $x \in (a-\delta, a+\delta)$

THM 2
A continuous function on a closed interval is bounded.

THM 3 & 4
If $f(x)$ is continuous on $[a,b]$ then it attains its g.l.b
[5.3] and l.u.b. within that interval; as well as all intermediate values. $f[a,b]$ is also a closed bounded interval

CORR 1
If $p(x)$ is a polynomial of odd degree, then $p(x) = 0$ has a solution.

CORR 2
[5.4] For $a, n > 0$, $x^n = a$ has at least one positive solution.

UNIFORM CONTINUITY (Not required)

Given $\varepsilon > 0$ and $x, x' \in I$, I an interval
Then if there is a $\delta > 0$ such that $|f(x') - f(x)| < \varepsilon$
for all $|x - x'| < \delta$, then $f(x)$ is uniformly continuous on I

[6.1] §6 DIFFERENTIABLE FUNCTIONS

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \text{ if the limit exists.}$$

THM 1 Proof reqd.

If f is differentiable at c, f is continuous at c

THM 2 Proof reqd.

If f is differentiable on (a, b) and $\exists\ c \in (a, b)$ $\ni$ $f'(c) = 0$, then f has a local max/min at c

[6.2] THM 3 Proof

If f is differentiable on $x \in (a, b)$ and continuous on $x \in [a, b]$ and $f(a) = f(b)$, then $\exists\ c \in (a, b)$ $\ni$ $f'(c) = 0$ (ROLLE THEOREM)

THM 4 Proof

If f is differentiable on $x \in (a, b)$ and continuous on $x \in [a, b]$, then $\exists\ c \in (a, b)$ $\ni$ $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ (MEAN VALUE THEOREM)

CORR 1 Proof

If $f'(x) = 0$ on $x \in (a, b)$ then $f(x)$ is constant on (a, b)

---

§7 CAUCHY'S MEAN VALUE THEOREM AND L'HOSPITAL'S RULES

THM 1 (Prove)

Let $f(x), g(x)$ be continuous on $[a, b]$ and differentiable on (a, b), and $g(b) \neq g(a)$ and $f'(x)^2 + g'(x)^2 \neq 0$ for all $x \in (a, b)$

Then $\exists\ \xi \in (a, b)$ $\ni$ $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(\xi)}{g'(\xi)}$ (CMV TH)

If $g(a) = f(a) = 0$ or $\left|\lim_{x \to a} f(x)\right| = \infty \wedge \left|\lim_{x \to a} g(x)\right| = \infty$

then $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \lim_{x \to a} \dfrac{f'(x)}{g'(x)}$ if the limit exists

§8 TAYLOR'S THEOREM:

We wish to approximate $f(x)$ defined on an interval (a, b) by a polynomial.

Let $f(x)$ be a real-valued function such that $f^{(k)}(x)$ is defined and continuous on $[x_0 - l, x_0 + l]$ for $0 \le k \le n$. If f has $n+1$ derivatives on $I = (x_0 - l, x_0 + l)$, then for $x \in I$:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_{n+1}(x)$$

$$\text{where } R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \quad \text{for some } \xi \in (x_0, x) \text{ or } \xi \in (x, x_0)$$

Then $f(x)$ can be approximated by a polynomial

$$f(x) = a_0 + a_1(x - x_0) + \dots + a_n(x - x_0)^n + R_{n+1}(x)$$

$$\text{where } a_k = \frac{f^{(k)}(x_0)}{k!}$$

[8.3] We can take $|R_{n+1}(x)| \le \max \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| |x - x_0|^{n+1}$

MACLAURIN EXPANSION

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_{n+1}(x) \; ; \; R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$$

for some $\xi \in (0, x)$

[8.5] SOME MACLAURIN EXPANSIONS

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

[8.7] TAYLORS THEOREM IN SEVERAL VARIABLES

Thm 1 (Proof)
Let $f(x,y)$ have continuous partial derivatives of order N on a disc containing the point (x_0, y_0). Then

$$f(x_0+h, y_0+k) = \sum_{n=0}^{N-1} \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + R_N$$

$$\text{where } R_N = \frac{1}{N!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial x}\right)^N f(x_0+\tau h, y_0+\tau k)$$

for some $0 < \tau < 1$

[8.8] for 3 variables, replace (x_0, y_0) by (x_0, y_0, z_0) and $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)$ by $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)$ etc.

I If we set $h = (x - x_0)$ and $k = (y - y_0)$ we can get $f(x, y)$ in a Taylor expansion

LOCAL MAXIMA AND MINIMA OF $f(x,y)$

Let $A = \frac{\partial^2 f}{\partial x \partial x}$; $B = \frac{\partial^2 f}{\partial x \partial y}$; $C = \frac{\partial^2 f}{\partial y \partial y}$

Local maxima and minima occur when $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

If $\begin{cases} B^2 - AC < 0 \text{ then } \begin{cases} A < 0 \Rightarrow \text{max} \\ A > 0 \Rightarrow \text{min} \end{cases} \\ B^2 - AC > 0 \text{ then saddle point} \\ B^2 - AC = 0 \text{ test inconclusive} \end{cases}$

Proof Thm: Let $F(t) = f(x_0 + th, y_0 + tk)$

$$\frac{dF}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0 + th, y_0 + tk)$$

Similarly $\frac{d^n}{dt^n} F = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0 + th, y_0 + tk)$

By Taylor in 1 var

$$F(t) = \sum_{n=0}^{N-1} \frac{F^{(n)}(0)}{n!} t^n + \frac{1}{N!} F^{(N)}(\tau) t^N \quad (0 < \tau < t)$$

$$F(1) = \sum_{n=0}^{N-1} \frac{F^{(n)}(0)}{n!} + \frac{1}{N!} F^{(N)}(\tau) \quad (0 < \tau < 1)$$

ie, $f(x_0 + h, y_0 + k) = \sum_{n=0}^{N-1} \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + \frac{1}{N!} f^{(N)}(x_0 + \tau h, y_0 + \tau k)$

$(0 < \tau < 1)$

§9 THE RIEMANN INTEGRAL

DEFNS A *partition* S of $[a,b]$ is an ordered $(n+1)$-tuple of real numbers $a = x_0 < x_1 < x_2 < x_3 < \dots < x_n = b$

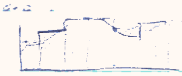
The *gauge* of S is defined as $\gamma(S) = \max_{1\le i\le n}(x_i - x_{i-1})$

Let $f: [a,b] \to \mathbb{R}$ be continuous, partitioned by S. Let

$M_i = \max\{f(x),\ x_{i-1} \le x \le x_i\}$

$m_i = \min\{f(x),\ x_{i-1} \le x \le x_i\}$

[9.2] The *upper sum* of f w.r.t. S is defined as:

$U(f,S) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ 

The *lower sum* of f w.r.t. S is defined as:

$L(f,S) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$ 

$L(f,S)$ is bounded above by $(b-a)\max_{a\le x\le b} f(x) = I(f)$

$U(f,S)$ is bounded below by $(b-a)\min_{a\le x\le b} f(x) = \underline{U}(f)$

Note $\underline{U}(f) \geq I(f)$

If the partition S' is formed from S by adding a finite number of points, then: $U(f,S') \le U(f,S)$

$L(f,S') \geq L(f,S)$

[9.3] If f is continuous, then $\underline{U}(f) = I(f)$. If $\{S_m\}$ is a sequence of partitions with $\lim_{m\to\infty} \gamma(S_m) = 0$, then

$\lim_{m\to\infty} L(f,S_m) = I(f)$; $\lim_{m\to\infty} U(f,S_m) = \underline{U}(f)$

[9.4] If f is continuous on $[a,b]$, then the definite integral of f from a to b is defined as

$$\int_a^b f(x)\,dx = \overline{I}(f) = \underline{U}(f)$$

We assume $\int_a^a f(x)\,dx = 0$

Given a partition S of $[a,b]$ and $\xi_i \in [x_{i-1}, x_i]$, we define $R(f,S) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$ to be a Riemann sum of f over S. Note that $L(f,S) \leq R(f,S) \leq U(f,S)$

If we use $\{S_m\}$, $\lim_{m\to\infty} \gamma(S_m) = 0$, then we have

$$\lim_{m\to\infty} R(f, S_m) = \int_a^b f(x)\,dx$$

[9.5] PROPERTIES OF THE DEFINITE INTEGRAL

If $a \leq c \leq b$ and $f : [a,b] \to \mathbb{R}$ is continuous, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

FIRST MEAN VALUE THEOREM: $\exists\, \xi \in [a,b] \ni \int_a^b f(x)\,dx = (b-a)f(\xi)$

THM 4 Proof required

[9.6] FUNDAMENTAL THM OF CALCULUS: Let $f(x) : [a,b] \to \mathbb{R}$ be continuous, and $F(x) = \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$ for all $x \in (a,b)$

THM 5 Proof requ.

Note: $F(b) - F(a) = \int_a^b f(t)\,dt$

[9.7] $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b f(x)+g(x)\,dx$

$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx \;,\; \alpha \in \mathbb{R}$

(Note THM 7 OMITTED, AS IT IS OBVIOUS)

[9.8] GENERALIZED MEAN VALUE THEOREM

Let $f(x)$, $g(x)$ be continuous on $[a,b]$ and suppose $g(x)\big|_{[a,b]} \geq 0$

Then $\exists\ \xi \in [a,b] \ni \int_a^b f(x)\,g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$

Proof Thm 4: $\frac{1}{h}\Big[F(x+h) - F(x)\Big] = \frac{1}{h}\Big[\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt\Big]$

$= \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(\xi)$ by the first MVT, for some $\xi \in (x, x+h)$

As $h \to 0$, $\xi \to x$ and by continuity $f(\xi) \to f(x)$

$\therefore \lim_{h\to 0} \frac{1}{h}\Big[F(x+h) - F(x)\Big] = f(x)$

Thm 5 Let $G(x) = \int_a^x F'(t)\,dt$. Then, by Thm 4

$G'(x) = F'(x) \qquad , x \in (a,b)$

Thus $(F - G)'(x) = 0$ on (a,b), so $(F - G)(x) = c$ a constant on $[a,b]$

Hence $F(x) = c + G(x)$, $F(a) = c + G(a) = c + 0 = c$

Thus $F(x) = F(a) + G(x) = F(a) + \int_a^x F'(t)\,dt$

Setting $x = b$ gives the result.

✓

§10 IMPROPER INTEGRALS

<u>DEFN</u> (a) Let $f(x)$ be continuous on $[a, c]$ for all $a \le c < \infty$.
We define the <u>improper integral</u> of f on $[a, \infty)$

written $\int_a^{\to\infty} f(x)\,dx$ to be $\lim_{T\to\infty} \int_a^T f(x)\,dx$ if it exists.

(b) Let $f(x)$ be continuous on $[a+\varepsilon, b]$ for all small $\varepsilon > 0$.
We define the <u>improper integral</u> of f on $[a, b]$

written $\int_{\to a}^{b} f(x)\,dx$ to be $\lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^{b} f(x)\,dx$ if it exists.

— If the limit exists, the integral is said to <u>converge</u>, otherwise <u>diverge</u>.

<u>Note</u> We can similarly define other forms of improper integrals, such as $\int_a^{\to b} f$ and $\int_{-\infty}^{a} f$, etc.

<u>DEFN</u>: $\int_a^{\to\infty} f(x)\,dx$ is said to be absolutely convergent if

$\int_a^{\to\infty} |f(x)|\,dx$ is convergent (similarly for others)

[10.2] <u>Note</u> Unlike series, $\lim_{x\to\infty} f(x)$ is not necessarily zero if the integral $\int_a^{\to\infty}$ exists. Also, if $\int_a^{\to\infty}$ is abs. convergent, then it is convergent. We sometimes drop the arrow when it is clear what is the case.

COMPARISON TEST FOR A.C.

If $|f(x)| \leq g(x)$ and $\int_a^\infty g(x)\,dx$ is convergent, then so is $\int_a^\infty |f(x)|\,dx$. Similarly for others

<u>Proof</u> $\int_a^T |f(x)|.dx \leq \int_a^T g(x).dx < K$

Thus $\int_a^T |f(x)|.dx$ is increasing and bounded above, and thus in the limit is convergent.

IMPORTANT COMPARISON INTEGRALS

$\int_1^\infty \frac{dx}{x^\alpha}$ is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$

$\int_0^1 x^\alpha.dx$ is convergent for $\alpha > -1$ and divergent for $\alpha \leq -1$

[10.3] LIMIT TEST FOR A.C.

(a) If $\lim_{x\to\infty} \frac{f(x)}{t(x)}$ exists and $\int t(x).dx$ is absolutely convergent

then so is $\int^\infty f(x).dx$

(b) If $\lim_{x\to\infty} \frac{t(x)}{f(x)}$ exists and $\int^\infty |t(x)|.dx$ is divergent,

then so $\int^\infty f(x).dx$ is <u>not</u> absolutely convergent

(Proof as for series)

§11 INTEGRATION OF SERIES OF FUNCTIONS

THM 1 Let $f_n(t)$ be continuous on $[a,b]$ for each n, and suppose there is a continuous function $f(t)$ such that:

$$\lim_{n\to\infty}\left(\max_{a\le t\le b}|f_n(t)-f(t)|\right)=0 \quad \text{then}$$

$$\lim_{n\to\infty}\int_a^b f_n(t)\,.dt=\int_a^b f(t)\,.dt$$

Proof:
$$\left|\int_a^b f_n(t).dt-\int_a^b f(t).dt\right|\le\int_a^b|f_n(t)-f(t)|\,.dt$$
$$\le \max_{a\le t\le b}|f_n(t)-f(t)|\int_a^b dt$$
$$\le (b-a)\max_{a\le t\le b}|f_n(t)-f(t)|\to 0$$

Note We see that it is desirable to have $\max|f_n-f|\to 0$. We define this type of convergence:

DEFN 1 (a) $(f_n(t))$ converges uniformly to $f(t)$ on $[a,b]$

if $\lim_{n\to\infty}\left[\operatorname{lub}_{a\le t\le b}|f_n(t)-f(t)|\right]=0$ (We write (f_n) is u.c)

(b) $(f_n(t))$ converges pointwise to $f(t)$ on $[a,b]$ if for each $t\in[a,b]$, $\lim_{n\to\infty}f_n(t)=f(t)$.

<u>Note</u> If $(f_n(t))$ is u.c. to $f(t)$ on $[a,b]$ then

$$\lim_{n\to\infty} \operatorname{lub}_{t\in[a,b]} |f_n(t) - f(t)| = 0$$

$$\Rightarrow \lim_{n\to\infty} |f_n(t) - f(t)| = 0 \text{ for all } t \in [a,b]$$

$$\Rightarrow \lim_{n\to\infty} f_n(t) = f(t) \text{ for all } t \in [a,b]$$

Hence, if (f_n) is u.c. it converges pointwise and its uniform limit function is the pointwise limit function.

[11.2] <u>THM 2</u> Let $f_n(t)$ be continuous on $[a,b]$ for each $n \in \mathbb{N}$ and suppose that $(f_n(t))$ converges uniformly to $f(t)$ on $[a,b]$. Then $f(t)$ is continuous.

<u>Proof</u> omitted but note that we can remove the requirement that $f(t)$ be continuous in THM 1.

Required

<u>THM 3</u> If $\sum_1^\infty f_n(t)$ is u.c. to $f(t)$ on $[a,b]$ all functions $f_n(t)$ being continuous, then

$$\int_a^b f(t)\,.\,dt = \int_a^b \left(\sum_1^\infty f_n(t)\right).\,dt = \sum_1^\infty \int_a^b f_n(t)\,.\,dt$$

<u>Proof</u> Exercise (use Thm 1). required.

[11.3] <u>Note</u> If $f_n(t)$ are continuous on $[a,b]$ and $\sum_1^\infty f_n(t)$ is u.c. on $[a,b]$ to $f(t)$, then $f(t)$ is continuous

THm 4 (Weierstrass M-test)

Suppose $|f_n(t)| \le M_n$ for all $t \in [a,b]$

Then if $\sum_1^\infty M_n$ is convergent, then $\sum_{n=1}^\infty f_n(t)$ is u.c.

on $[a,b]$.

Proof: $|f_n(t)| \le M_n \Rightarrow \sum |f_n(t)|$ is convergent by comparison test

Thus $\sum f_n(t)$ is convergent, ie, for each t $\sum_1^\infty f_n(t)$ is c.

Let the pointwise limit be $f(t)$, $t \in [a,b]$

Then $\left| f(t) - \sum_1^N f_n(t) \right| = \left| \sum_{N+1}^\infty f_n(t) \right| \le \sum_{N+1}^\infty M_n = g(N)$ say

where $\lim_{N \to \infty} g(N) = 0$ and $g(N)$ independent of t

Therefore $\text{lub}_{t \in [a,b]} \left| f(t) - \sum_1^N f_n(t) \right| \le g(N)$ and $\lim_{N \to \infty} g(N) = 0$

Hence the result.

THm 5 Let $r > 0$ be the radius of convergence of $\sum_{n=0}^\infty a_n (x - x_0)^n = f(x)$. Then

(a) for $|x - x_0| \le r'$ the power series is u.c. (any $r' < r$)

(b) $\dfrac{df}{dx} = \sum_{n=1}^\infty n a_n (x - x_0)^{n-1}$ for $|x - x_0| < r$

[11.4] (c) $\int_a^b f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}\left[(x-x_0)^{n+1}\right]_a^b$ if a and b are within the interval of convergence.

Proof omitted. Note a power series represents a continuous differentiable function in its interval of convergence. In fact, it is infinitely differentiable there.

Proof THM 2: Let $c \in [a,b]$

Given $\varepsilon > 0$, consider:

$$|f(t) - f(c)| = |f(t) - f_n(t) + f_n(t) - f_n(c) + f_n(c) - f(c)|$$
$$\leq |f(t) - f_n(t)| + |f_n(t) - f_n(c)| + |f_n(c) - f(c)| \quad (*)$$

Choose n_1 such that $\mathrm{lub}_t\, |f_{n_1}(t) - f(t)| < \frac{\varepsilon}{3}$ (exists by uniform conv.)

Choose δ such that $|t-c| < \delta \Rightarrow |f_{n_1}(t) - f_{n_1}(c)| < \frac{\varepsilon}{3}$
(δ exists by continuity of f_n)

Set $n = n_1$ in $(*)$. Then we have:

$$|t-c| < \delta \Rightarrow |f(t) - f(c)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

LINEAR ALGEBRA

§1 MATRIX ALGEBRA

DEFN 1 (a) Let S be a set. A matrix over S with m rows and n columns is an arrangement of members of S in m rows and n cols. If $S = \mathbb{R}$ it is called a real matrix, if $S = \mathbb{C}$ it is called a complex matrix (we will give a more elegant defn later)

(b) Two matrices A and B are equal if they are of the same dimensions and each $a_{ij} = b_{ij}$

NOTATION - A matrix with m rows & n cols is called $m \times n$
- The element in the i^{th} row & j^{th} col of mat A is written a_{ij} or $[A]_{ij}$
- A matrix with all elements 0 is called a zero matrix, written 0 $(m \times n)$
- The $n \times n$ matrix with 1's in the main diagonal and zeroes elsewhere is called the $n \times n$ identity matrix, written I_n. The elements of I are written δ_{ij} (the KRONECKER DELTA), so $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

[2] DEFN 2 (a) A mat A is upper (lower) triangular if $a_{ij} = 0$ for $i > j$ $(i < j)$

(b) An $n \times n$ matrix is called a square matrix.

(c) A square matrix A is diagonal if $a_{ij} = 0$ for $i \neq j$

DEFN 3 (a) Let A be a real (complex) mat $m \times n$ and let p be real (complex). Then pA is defined to be the $m \times n$ mat satisfying

$$[pA]_{ij} = p\, a_{ij} \quad \text{all } i, j \text{ in } A \quad \left(\begin{matrix}\text{SCALAR}\\ \text{MULT}\end{matrix}\right)$$

(b) Let A, B be real (complex) matrices both $m \times n$. Then $A+B$ is the $m \times n$ mat satisfying

$$[A+B]_{ij} = a_{ij} + b_{ij} \quad (\text{all } i, j \text{ in } A, B) \quad \left(\begin{matrix}\text{MAT.}\\ \text{ADDITION}\end{matrix}\right)$$

THM 1 (Properties of scalar multiplication)

If A, B are real (complex) $m \times n$ matrices and p, q are real (complex) numbers, then:

(a) $p(A+B) = pA + pB$
(b) $(p+q)A = pA + qA$
(c) $(pq)A = p(qA)$
(d) $1 \cdot A = A$

[3] Proof Prove by elements.

THM 2 (Properties of addition)

Let A, B, C be $m \times n$ real (complex) matrices. Then
(a) $A + B = B + A$ (commutative law)
(b) $(A+B) + C = A + (B+C)$ (associative)
(c) $A + 0 = 0 + A = A$ (identity)
(d) There is a mat $(-A)$ such that $A + (-A) = 0$ (inverse)

Thus matrices form an Abelian group (Thm 2) and a vector space (Thm's 1 & 2).

DEFN 4 If A is $m \times p$ and B is $p \times n$ (both real or complex) then we define AB (an $m \times n$ mat) by

$$[AB]_{ij} = \sum_{k=1}^{p} a_{ik} \cdot b_{kj} \text{ for all } i, j \text{ - } \begin{matrix} i \in [1, m] \\ j \in [1, n] \end{matrix}$$

(MATRIX MULTIPLICATION)

[4] THM 3 (Properties of matrix multiplication) (all real or all comp)

(a) Let A be $m \times p$, B be $p \times q$, C be $q \times n$.
Then $(AB)C = A(BC)$ (associative)

(b) $AI = IA = A$ (identity)

(c) Let A and D be $m \times p$ and B, C be $p \times n$.
Then $A(B+C) = AB + AC$
$(A+D)C = AC + DC$ (distributive)

Notes - Matrices need not have multiplicative inverses, thus they do not form an Abelian group under mat mult.

[5] - The cancellation law does not hold, ie, $AB = AC$ does not mean that $B = C$ or $A = 0$

- There are non-zero zero divisors, ie, matrices A, B such that $AB = 0$ but $A, B \neq 0$.

- In general $AB \neq BA$ (not commutative]

- Thm's 2 & 3 define systems called RINGS

- Thm's 1, 2 & 3 define systems called ALGEBRAS

DEFN 5 (a) Let A be a real (comp) $m \times n$ mat. Then the *transpose* of A, A^T is the $n \times m$ mat defined by
$$[A^T]_{ij} = a_{ji} .$$

(b) For a complex $m \times n$ mat A, the *conjugate* of A, $\bar{A}$, is defined by
$$[\bar{A}]_{ij} = \bar{a}_{ij} = \left(\overline{[A]_{ij}}\right)$$

(c) For a complex $m \times n$ mat A, the *adjoint* of A, A^* is the $n \times m$ mat defined by
$$[A^*]_{ij} = \bar{a}_{ji} = \left(\overline{[A]_{ji}}\right)$$

(d) If $A^T = A$, A is called *symmetric*
If $A^T = -A$, A is called *skew symmetric*
If $A^* = A$, A is called *hermitean*
If $A^* = -A$, A is called *skew hermitean*

NOTE ① If A is $m \times n$ then A^T and A^* are $n \times m$
② $A^* = (\bar{A})^T = \overline{A^T}$
③ If A has real entries only, $A^* = A^T$

[6] **THM 4**: (Properties of transpose and adjoint)
(a) $(A+B)^T = A^T + B^T$ $\quad (aA)^T = aA^T \quad (a \in \mathbb{R})$
(b) $(AB)^T = B^T A^T \quad (A\ m \times p,\ B\ p \times n)$
(c) $(A^T)^T = A$
(d) (a)-(c) hold for adjoint except in (a), $(aA)^* = \bar{a} A^*$ if $a \in \mathbb{C}$

DEFN 6 (a) B is a *left inverse* of $\overset{m\times n}{A}$ if $BA = I_n$

(b) B is a *right inverse* of $\overset{m\times n}{A}$ if $AB = I_m$

(c) B is an *inverse* of square $(n\times n)$ A if $AB = BA = I_n$

A square matrix *with* an inverse is called *non-singular*, otherwise singular

Note ① If A has both a left and right inverse, it must be square and the inverse is unique.

② A matrix need not have either a left or right inverse

③ If $m < n$ there is *no* left inverse ( )
If $m > n$ there is no right inverse ( )

[7] ④ If A is square and has a left or right inverse, it has an inverse

PROPOSITION 1: A non-singular matrix has a unique inverse (prove with associative law)

NOTATION Left inverse A_L^{-1}, right inverse A_R^{-1}, inverse A^{-1}

Note – Left & right inverses of non-square matrices will not be unique

– To show that a matrix A is non-singular, we *must find* a mat $B \ni AB = BA = I$.
If such a B exists, it is the inverse of A.

Thm 5 (a) If A is nonsingular, then so is A^{-1} and $(A^{-1})^{-1} = A$.

(b) If A & B are ", then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$

(c) If A is nonsingular, so are A^T and A^*, and
$(A^T)^{-1} = (A^{-1})^T$ and $(A^*)^{-1} = (A^{-1})^*$

PROPOSITION 2 If A has an inverse, then $A\underline{x} = \underline{b}$ has solution $A^{-1}\underline{b}$ and the solution is unique

[8] Proof Let $y = A^{-1}\underline{b}$. Then $Ay = A(A^{-1}\underline{b}) = (AA^{-1})\underline{b} = \underline{b}$ so a solution exists.

Not needed

Conversely, if $A\underline{x} = \underline{b}$, then $A^{-1}(A\underline{x}) = A^{-1}\underline{b}$, so $\underline{x} = A^{-1}\underline{b}$, implying uniqueness.

Useful Notation Let the column vectors of $(m \times n)$ A be $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$, where the $\underline{a}_i$ are m-vectors. We write $A = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n)$

Let $B = (\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_q)$ (B a $n \times q$ mat)

Then
$$AB = A(\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_q) = (A\underline{b}_1, A\underline{b}_2, \ldots, A\underline{b}_q)$$

If $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is an n-vector, then:

$$A\underline{x} = (\underline{a}_1, \ldots, \underline{a}_n)\underline{x} = x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_n\underline{a}_n$$

We write $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ position as $\underline{e}_i$

PROPOSITION 3: If $A (m \times n)$ has a right inverse $\overline{X}$, then if $\overline{X} = (\underline{x}_1 \ldots \underline{x}_m)$ we have $A\underline{x}_i = \underline{e}_i$

Conversely if $\underline{x}_i$ satisfies $A\underline{x}_i = \underline{e}_i$ for $i = 1 \ldots m$, then $\overline{X} = (\underline{x}_1, \ldots, \underline{x}_m)$ is a right inverse.

Proof If $A\bar{X} = I_m$ then $A(\underline{x}_1, \dots, \underline{x}_m) = (\underline{e}_1, \dots, \underline{e}_m)$
$\Rightarrow (A\underline{x}_1, \dots, A\underline{x}_m) = (\underline{e}_1, \dots, \underline{e}_m)$

Read Since the two matrices are equal iff their columns are equal, we just have $A\underline{x}_i = \underline{e}_i$ $(i = 1, \dots, m)$

Conversely, if the equations hold, $A\bar{X} = I_m$ follows similarly.

Notes ① Since a right inverse of a square matrix is also an inverse (proof §2 Thm 7), this gives a method of finding inverses.

② To find a left inverse $\bar{X}$, such that $\bar{X}A = I$, we may first find a right inverse A^T, i.e., we may solve $A^TB = I$ to get a matrix B. The transpose of this satisfies $(B^T)A = (A^TB)^T = I$ thus B^T is a left inverse of A.

✓

§2 LINEAR EQUATIONS

[12] DEFN An *elementary row operation* is a mapping taking $(m \times n)$ mats into $(m \times n)$ mats of one of the following forms

① $R_{pq}(c)$: Add c times q^{th} row to p^{th} row
② R_{pq} : Interchange p^{th} and q^{th} rows
③ $R_p(c)$: Multiply p^{th} row by c $(c \neq 0)$

GAUSS ELIMINATION ALGORITHM (INFORMAL DESCRIPTION)

① Find i, where i is the number of the first non-zero column of A. If no more, stop Ⓘ.
② Perform op ② if necessary to bring a non-zero element α_i into the first row of the i^{th} col. This element is called a *pivot*
③ Perform ~~elem~~ op ① to set all elements of the i^{th} col. to zero except the first.

[13] ④ Repeat steps ① to ③ until stop Ⓘ ~~[illegible]~~ leaving an upper triangular matrix
⑤ Starting with the rightmost column use ops on the whole matrix to get zeroes above the pivot (the bottom non-zero element in the col) (This is called back substitution).

GAUSS - JORDAN ELIMINATION ALGORITHM

This is similar to Gauss, except that in step 3, the elements of the col both above and below the pivot are set to zero. This eliminates the need for back substitution, but is slightly less efficient.

[14] Notes - We only used ops ① & ②. If we use type ③ as well, we can set all pivots to 1.

- The algo always work - this is part (a) of:

THM 1 (Reduction to Row-Echelon Form) No proof

(a) Every matrix A ($m \times n$) can be reduced using ops ①, ② & ③ to a matrix U satisfying:

(i) there is an integer k ($0 \leq k \leq m$) such that the first k rows of U have at least one non-zero entry.

(ii) The first non-zero entry called the leading entry in each of the first k rows is 1. The col containing the leading entry is called the i^{th} basic column. Entries above and below the leading entry are zero. (so i^{th} basic col $= \underline{e}_i$)

(iii) If $i < j$ the i^{th} basic col lies to the left of j^{th} basic col.

A matrix U satisfying (i), (ii) & (iii) is said to be in Row-echelon form.

(b) The row-echelon form of U is unique.

LINEAR EQUATIONS $A\underline{x} = \underline{b}$

DEFN 2 The integer k of THM 1 is called the rank of A, written rank A

[15] Note – For the time being we will assume without proof that k is well defined

(*) – If A is $m \times n$, rank $A \leq m$ and rank $A \leq n$.

DEFN 3 If A $(m \times n)$ and $\underline{b}$ $(m \times 1)$, then the mat with first n cols equal to those of A and $(n+1)^{st}$ col $\underline{b}$ is called the augmented matrix of A and $\underline{b}$, written $[A, \underline{b}]$.

THM 2 Let $[A_1, \underline{b}_1]$ be the result of successively applying a sequence of el. ops to $[A, \underline{b}]$. Then $A\underline{x} = \underline{b}$ and $A_1 \underline{x} = \underline{b}_1$ have exactly the same solns.

Proof by induction (informal). (not needed)

COROLLARY 1 : Consider $A\underline{x} = \underline{b}$, A $(m \times n)$ n = no of unknowns

(a) If rank $[A, b]$ > rank A, the system is inconsistent

(b) If rank $[A, b]$ = rank A, and rank $A = n$ the system has a unique solution

(c) If rank $[A, b]$ = rank A, and rank $A < n$, the system has infinitely many solutions

[16] Proof omitted : see notes (not needed)

[17] COROLLARY 2 : Consider $A\underline{x} = \underline{0}$, A $(m \times n)$

(a) $\underline{x} = \underline{0}$ is always a solution

(b) If rank $A = n$, $\underline{x} = \underline{0}$ is the unique soln

c) There is a nonzero soln iff rank $A < n$

(Proof by corr 1). (not needed)

<u>Note</u> The result that $m<n \Rightarrow A\underline{x} = \underline{0}$ has a nonzero solution follows directly from looking at the row-echelon form of A, which has more columns than rows, ~~and arguing by~~ The concept of rank is unnecessary to prove this result, and in fact it can be used to prove the existence of rank.

<u>PROPSN 1</u> Let $A\underline{x}_0 = \underline{b}$, ie, $\underline{x}_0$ is any one soln of $A\underline{x} = \underline{b}$

(a) If $\underline{x}_H$ satisfies $A\underline{x}_H = \underline{0}$, then $\underline{x}_0 + \underline{x}_H$ is a soln.

(b) Any soln has the form $\underline{x}_0 + \underline{x}_H$ for some $\underline{x}_H$ satisfying $A\underline{x}_H = \underline{0}$

[18] <u>Proof</u> (a) $A(\underline{x}_H + \underline{x}_0) = A\underline{x}_H + A\underline{x}_0 = \underline{0} + \underline{b} = \underline{b}$

(b) If $\underline{x}_1$ is any soln of $A\underline{x} = \underline{b}$, then

$A(\underline{x}_1 - \underline{x}_0) = \underline{b} - \underline{b} = \underline{0} \Rightarrow \underline{x}_H = \underline{x}_1 - \underline{x}_0$

Thus $\underline{x}_1 = \underline{x}_H + \underline{x}_0$.

<u>ELEMENTARY MATRICES.</u>

Each elementary row operation on A can be accomplished by premultiplying A by a matrix, as follows.

① $R_{pq}(c) \equiv E_{pq}(c)A$ where:

$$E_{pq}(c) = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & \cdots & 1 & c & \cdots \\ 0 & \cdots & & & 1 \end{pmatrix}$$ where the c is in the pth row, qth col.

[19] ② $R_{pq} \equiv E_{pq} A$ where : (E_{pq} is called an elementary permutation matrix)

E_{pq} = identity matrix with p^{th} and q^{th} cols transposed.

③ $R_p(c) \equiv E_p(c) A$ where

$E_p(c)$ is the identity I with $E_{pp} = c$.

Note $E_{pq}^{-1}(c) = E_{pq}(-c)$

$E^{-1}_{pq} = E_{pq}$

$E^{-1}_{p}(c) = E_p\left(\frac{1}{c}\right)$ if $c \neq 0$

DEFN 4 The matrices defined above are called elementary matrices.

PROPSN 2 (a) For each mat A, there are elementary mats $E^{(1)} \ldots E^{(\ell)}$ such that $E^{(\ell)} \ldots E^{(1)} A = U$ where U = upper triangular). Further, all the E's are of the form $E_{pq}(c)$ or E_{pq} and all $E_{pq}(c)$ are lower triangular.

Proof required

(b) If we allow $E_p(c)$ as well, then there exist E's such that the U of part a) is the row-echelon form.

[20] Note If U is the row-echelon form of A, then A has a right inverse iff U has a right inverse.

CORR 1: Suppose no row interchanges are necessary. Then $A = LU$ (L - lower triangular with 1's on diagonal, U = upper triangular).

Proof omitted, see notes. not needed

CORR 2: Let the situation be as in CORR 1. Let A be $m \times n$. Let m_{ij} = amount pivot in j^{th} col must be multiplied by before subtracting to eliminate element in i^{th} row, j^{th} col (i.e.,

$$m_{ij} = \frac{\text{elt in } (i,j)^{th} \text{ posn}}{\text{pivot}}$$

$$\text{Then } L = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & m_{32} & 1 & 0 \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ m_{m1} & m_{m2} & \cdots & m_{m\,m-1} & 1 \end{pmatrix}$$

Proof omitted not needed

[2] CORR 3 For any non singular $n \times n$ mat A, there exist matrices P: a product of elementary permutation mats

L: a lower $\triangle$ mat with 1's on diagonal

U: an upper $\triangle$ mat with nonzero elmts on diag

such that: $PA = LU$.

Proof & notes see notes (21 – 24) not needed

[25] SQUARE MATRICES & NONSINGULARITY.

PROPN 3 Let A be square $n \times n$. Then if A has row-echelon form U, A reduces to I_n iff U has no row of zeroes ① iff A has rank n. ② ③

Proof Since U is square, and is in row echelon form, it has no row of zeroes iff it has a diagonal with non-zero elements along the diagonal. (②). ① & ③ follow.

PROPN 4 A $(n \times n)$ is nonsingular ① iff it row reduces to I_n ② iff A is a product of elementary matrices ③.

Proof omitted - see notes. not required

PROPN 5 Let E be an elementary matrix $n \times n$. Then if A is $n \times n$, then $\det(EA) = (\det E)(\det A)$.
Proof omitted: see notes.

[26] THEOREM 5 Let A, B be $n \times n$. Then $\det(AB) = \det A \det B$
Proof: omitted, see notes required.

THEOREM 6: Let A be $n \times n$. Then the following are equivalent:

(a) A is nonsingular
(b) A row reduces to I_n
(c) A is a product of elementary matrices
(d) Rank $A = n$
(e) $\det A \neq 0$
(f) $A\underline{x} = \underline{0}$ has unique soln $\underline{x} = \underline{0}$

Proof (a) iff (b) iff (c) holds by PROPN 4 (not required)

(b) iff (d) holds by defn of rank.

(d) iff (f) holds by:

rank $A = n \Rightarrow A\underline{x} = 0$ has unique soln $\underline{x} = 0$ (THM 2 CORR 2 (b))

$\Rightarrow$ rank $A \geq n$

$\Rightarrow$ rank $A = n$ as we always have rank $A \leq n$.

(d) iff (e) holds by:

Since $A = E^{(1)} \ldots E^{(l)} U$

we have $\det A = \det E^{(1)} \ldots \det E^{(l)} \det U$

Since $\det E^{(i)} \neq 0$ for $E^{(i)}$ element. mat.

$\det A \neq 0$ iff $\det U \neq 0$ iff U has no row of zero iff rank $A = n$ (by PROP 3)

[27] Note By equivalence (e) iff (f), $A\underline{x} = \underline{0}$ has a non-zero soln iff $\det A = 0$ (iff A is singular).

THM 7 Let A and B be $n \times n$ and let $AB = I$.

Then A is nonsingular and $B = A^{-1}$ (so $BA = I$ also)

Proof If $AB = I$, then $\det(AB) = 1$

not reqr. By thm 5, $\det(AB) = \det A \det B \neq 0$

$\Rightarrow \det A \neq 0 \Rightarrow A$ is nonsingular.

$\Rightarrow A$ has inverse A^{-1}

$\Rightarrow A^{-1}(AB) = A^{-1}I = A^{-1} \Rightarrow IB = A^{-1} \Rightarrow B = A^{-1}$.

THM 8 Let A be nonsingular $n \times n$

Let adj A be the matrix such that $[\text{adj } A]_{ij} = A_{ji}$, the cofactor of a_{ji}. Then:

(a) $A^{-1} = \dfrac{1}{\det A} \text{adj } A$

(b) The unique soln $\underline{x}$ of $A\underline{x} = \underline{b}$ can be found from

$$x_i = \frac{\det A_i(\underline{b})}{\det A}, \text{ where } A_i(\underline{b}) \text{ is the matrix } A$$

with i^{th} col replaced by $\underline{b}$. (Cramer's rule)

proof not required

Proofs

THM 5. Suppose A is nonsingular, $A = E^{(1)} \ldots E^{(l)}$ by Prop 4 for some elem. mats. $E^{(i)}$ $\Rightarrow \det(AB) = \det(E^{(1)} \ldots E^{(l)} B)$
$= \det(E^{(1)} \ldots E^{(l)}) \det B = \det A \det B$.

If A is singular, by prop 4 A does not reduce to I_n.
Hence $A = E^{(1)} \ldots E^{(l)} U$ where the last row of U is zero.
$AB = E^{(1)} \ldots E^{(l)} (UB)$ and the last row of $UB = 0$.
$\Rightarrow \det(AB) = \det(E^{(1)} \ldots E^{(l)}) \det(UB) = 0$ since $\det(UB) = 0$
since $\det A = \det E^{(1)} \ldots \det E^{(l)} \det U$ as $\det U = 0$
by first part of proof $\det A = \det(E^{(1)} \ldots E^{(l)}) \det U = 0$
$\Rightarrow \det(AB) = \det A \det B = 0$.

PROPN 2 (a) By steps 1-4 of Gauss alg. there are elem mats $E^{(l)} \ldots E^{(1)}$ such that $E^{(l)} \ldots E^{(1)} A = U$ (U upper Δ)
Only $E_{pq}(c)$ and E_{pq} are used. All $E_{pq}(c)$ have $q < p$ (since we always eliminate elements below the pivot). Hence $E_{pq}(c)$ are lower Δ.
(b) clear.

✓

[41] §5) VECTOR SPACES

We abstract the properties of algebraic systems which have addition and scalar multiplication defined – anything with these properties is a vectorspace.

DEFN 1: A vectorspace over $\mathbb{R}$ (or $\mathbb{C}$) is a set V with equality relation '=' together with two functions $+$ and $\cdot$, $+$ being defined for pairs of elements of V, and $\cdot$ being defined for an element of V and one of $\mathbb{R}$ (or $\mathbb{C}$). satisfying addition $(+)$ and multiplication $(\cdot)$ axioms:

$$\forall u, v \in V; \ \forall a \in \mathbb{R} \ (\text{or } \mathbb{C}) :$$

A_1: $u+v \in V$ (closure) — M_1: $au \in V$ (closure)

A_2: $u+v=v+u$ (commutative) — M_2: $a(u+v)=au+av$ (distrib)

A_3: $u+(v+w)=(u+v)+w$ (associative) — M_3: $(a+b)u=au+bu$ (")

A_4: $\exists 0 \in V \ni u+0=u$ (identity) — M_4: $(ab)u=a(bu)$ (assoc)

A_5: $\exists (-u) \in V \ni u+(-u)=0$ (inverse) — M_5: $1(u)=u$ (identity)

$A_1 - A_5$ define an Abelian group.

To turn a given set V into a vectorspace over $\mathbb{R}$ (or $\mathbb{C}$) we must define $+, \cdot, =$ between elements of V satisfying these axioms

[43] PROPN 1 (a) 0 is unique (the vector 0)

(b) for any u, $-u$ is unique

(c) $0u=0$ for all $u \in V$

(d) $(-1)u=-u$ for all $u \in V$

(e) $-(au)=(-a)u$ for all $a \in \mathbb{R}$ (or $\mathbb{C}$), $u \in V$

(f) $a0=0$ for all $a \in \mathbb{R}$ (or $\mathbb{C}$) 0 is vector zero

DEFN 2 A subset S of a vectorspace V is said to be a subspace of V if S together with the same $(+, \cdot, =)$ of V forms a vectorspace. $\{\underline{0}\}$ is a subspace of every vectorspace.

PROPN 2 A subset S of V is a subspace iff

$$au + bv \in S \text{ for all } u, v \in S$$
$$a, b \in \mathbb{R} \ (\text{or } \mathbb{C})$$

[44] In $\mathbb{R}^n$, subspaces are $\{\underline{0}\}$, $\{$lines through $\underline{0}\}$, $\{$planes through $\underline{0}\}$ and higher dimensional analogues of sets of the form $x = a\underline{v} + b\underline{w} + \dots$ where $\underline{v}, \underline{w}, \dots$ are n-dimensional vectors, $a, b, \dots \in \mathbb{R}$ (or $\mathbb{C}$).

DEFN 3 Let $u_1, \dots, u_m \in V$ and $v = c_1 u_1 + \dots + c_m u_m$ where $c_1, \dots, c_m \in \mathbb{R}$. Thus v is a linear combination of $u_1 \dots u_m$. The span of $\{u_1, \dots, u_m\}$ is the set of all linear combinations of $\{u_1, \dots u_m\}$ written: $\text{span}\{u_1, \dots u_m\}$.

[45] PROPN 3: Let $u_1 \dots u_m \in V$, V a vectorspace. Then $\text{span}\{u_1 \dots u_m\}$ is a subspace of V.

Thus in $\mathbb{R}^3$, a plane is a set of the form $\text{span}\{\underline{u}_1, \underline{u}_2\}$ where $\underline{u}_1, \underline{u}_2$ are nonzero non-coincident vectors, and a line is $\text{span}\{\underline{u}_1\}$.

Proof PROPN 4: $b \in \text{span}\{\underline{u}_1 \dots \underline{u}_m\}$ iff there are $x_1, \dots x_m \in \mathbb{R}$ such that $x_1\underline{u}_1 + \dots + x_m\underline{u}_m = \underline{b}$ iff there is a soln $\underline{x} \in \mathbb{R}^m$ of $A\underline{x} = \underline{b}$

PROPN 4: Let $A = (\underline{u}_1 \dots \underline{u}_m)$, the matrix with col vects $\underline{u}_i$. Then $\underline{b} \in \text{span}\{\underline{u}_1 \dots \underline{u}_m\}$ iff $A\underline{x} = \underline{b}$ for some $\underline{x} \in \mathbb{R}^m$. Thus we have:

$$\text{span}\{\underline{u}_1 \dots \underline{u}_m\} = \{\underline{b} \mid A\underline{x} = \underline{b} \text{ for some } \underline{x} \in \mathbb{R}^m\}.$$

[46] DEFN 4: $u_1 \dots u_m \in V$ are linearly independent if there do not exist m constants $c_1 \dots c_m \in \mathbb{R}$ (or $\mathbb{C}$) such that $(c_1 u_1 + \dots + c_m u_m = 0)$
and (not all the c_i are 0)

otherwise they are linearly dependent.

To test for LI, show that if $c_1 u_1 + \dots + c_m u_m = 0$, then $c_1 = c_2 = \dots = c_m = 0 \Rightarrow$ LI

PROOFS:

PROPN 2 Suppose S is a subspace. Then A_1 & M_1 imply (a) & (b). Suppose (a) & (b) hold. Then A_1 & M_1 hold. All the rest of the axioms except A_4 & A_5 hold by virtue of the fact that they hold for all $u, v, w \in V$, $a, b \in \mathbb{R}$ (or $\mathbb{C}$) and thus also for all $u, v, w \in S$ and $a, b \in \mathbb{R}$ (or $\mathbb{C}$). For A_4: If $u \in S$, $\overset{\text{PROP 2}}{\text{implies}}$ $0u \in S$ and $0u = 0$ by PROP 1 (c). For A_5: if $u \in S$, $-u \in S$ by PROP 2.

PROPN 3: Let $v_1, v_2 \in \text{span}\{u_1 \dots u_m\}$. Then $\begin{cases} v_1 = a_1 u_1 + \dots + a_m u_m \\ v_2 = b_1 u_1 + \dots b_m u_m \end{cases}$ for some real a_i, b_i $(i = 1 \dots m)$

$v_1 + v_2 = (a_1 + b_1) u_1 + \dots + (a_m + b_m) u_m \in \text{span}\{u_1, \dots, u_m\}$

$a v_1 = a a_1 u_1 + \dots + a a_m u_m \in \text{span}\{u_1, \dots, u_m\}$ $(a \in \mathbb{R})$

$\Rightarrow \text{span}\{u_1 \dots u_m\}$ is a subspace by PROP 2

PROPN 5 $\underline{u}_1, \dots, \underline{u}_m \in \mathbb{R}^n$ are

(a) linearly dependent iff $A\underline{x} = \underline{0}$ has a solution $\underline{x} \neq 0$

(b) linearly independent iff $A\underline{x} = \underline{0}$ has unique solution $\underline{x} = \underline{0}$

where the matrix $A = (\underline{u}_1, \dots, \underline{u}_m)$.

[47] Proof As in propn 4, $A\underline{x} = 0$ iff there are $x_1 \dots x_m$ such that $x_1\underline{u}_1 + \dots + x_m\underline{u}_m = \underline{0}$. Together with defn 6, this implies (a) and (b)

LINEAR DEPENDENCE IN $C^n(a,b)$

Let $f_i(x) \in C^{n-1}(a,b)$ $(i = 1, \dots, n)$

DEFN 7 $$\tilde{W}(x) = \begin{pmatrix} f_1(x) & \dots & f_n(x) \\ f_1'(x) & \dots & f_n'(x) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{pmatrix}$$ is called the Wronskian matrix of $f_1, \dots, f_n$.

$W(x) = \det \tilde{W}(x)$ is called the Wronskian of $f_1, \dots, f_n$

PROPN 6 $\{f_i\}$ is a L.I. set on (a,b) if $W(x_0) \neq 0$ for some $x_0 \in (a,b)$.

Proof Suppose $W(x_0) \neq 0$ and the $\{f_i\}$ are L.D. Then there exist $c_1 \dots c_n$ not all 0 such that

$$c_1 f_1(x) + \dots c_n f_n(x) = 0 \quad \text{all } x \in (a,b)$$

$$\Rightarrow c_1 f_1^{(i-1)}(x) + \dots c_n f_n^{(i-1)}(x) = 0 \quad \text{all } i \in [1,n] \cap \mathbb{Z}$$

$$\Rightarrow \tilde{W}(x)\underline{c} = \underline{0} \text{ has a solution } \underline{c} \neq \underline{0}$$

$$\Rightarrow \tilde{W}(x) \text{ is singular for all } x \in (a,b)$$

But $\det(\tilde{W}(x_0)) \neq 0$, a contradiction

THM 1: Let $u_1, \dots, u_m \in \mathbb{R}^n$

(a) If $m > n$ they are L.D.

(b) If $m = n$ and they are L.I. then span $\{u_1, \dots, u_n\} = \mathbb{R}^n$

(c) If $m < n$ then span $\{u_1, \dots, u_m\} \neq \mathbb{R}^n$.

[48] CORR 1 (a) The smallest no. of L.I vectors that can span $\mathbb{R}^n$ is n.

(b) Any n L.I vectors span $\mathbb{R}^n$

(c) Any (spanning) set with $> n$ vectors is L.D.

(d) Any L.I spanning set has n vectors

Note: To prove L.I on $\mathbb{R}^n$ or $\mathbb{C}_n$ use props 4 & 5; on $C^n[a,b]$ use prop 6.

BASIS & DIMENSION

DEFN 5 (a) A L.I set $\{u_1, \dots, u_n\} \in V$ whose span is V is called a basis of V

(b) If every basis of V has exactly the same number n of vectors, then n is called the dimension of V.

We assume dimension $\{0\} = 0$.

[49] Note - dimension $(\mathbb{R}^n) = n$ is well defined by THM 1 COR 1(d)

- every vectorspace spanned by finitely many vectors has a dimension

PROPN 7 : Let $\{\underline{u}_1, \ldots \underline{u}_n\}$ be a basis of V. Let $v \in V$ and $v = c_1\underline{u}_1 + \ldots c_n\underline{u}_n$. Then $c_1, \ldots, c_n$ are unique.

Proof Suppose $v = c_1\underline{u}_1 + \ldots + c_n\underline{u}_n = c_1'\underline{u}_1 + \ldots + c_n'\underline{u}_n$
Then $(c_1 - c_1')\underline{u}_1 + \ldots + (c_n - c_n')\underline{u}_n = \underline{0}$. By L.I of $\underline{u}_1 \ldots \underline{u}_n$ we must have $c_1 - c_1' = c_2 - c_2' = \ldots = c_n - c_n' = 0$ ie. $c_1 = c_1'$, $c_2 = c_2'$. . $c_n = c_n'$.

DEFN 6 Let $B = \{\underline{u}_1, \ldots, \underline{u}_n\}$ be a basis of V and let $v = c_1\underline{u}_1 + \ldots + c_n\underline{u}_n$. Then $(c_1 \ldots c_n)$ are called the co-ordinates of v relative to B, c_i corresponding to $\underline{u}_i$ $(i = 1, \ldots, n)$.

THM 2 Let $\{u_1, \ldots, u_n\}$ be L.I in V and let $\{v_1, \ldots, v_m\}$ span V. Then $m \geq n$.

[50] CORR 1 The number of vectors in any basis of a vectorspace V is the same.

Proof Given any two bases $u_1, \ldots u_n$ and $v_1, \ldots, v_m$, both sets are spanning and L.I. Applying the theorem twice we get $m \geq n$ and $n \geq m$ hence $n = m$.

Note: If V has a basis, it then has a dimension

THM 3 If $\{u_1, \ldots u_m\}$ spans V and is L.D, there is a smaller subset that still spans V.

CORR 1 Let V be a vectorspace and let $V \neq \{0\}$
(a) If V is spanned by finitely many vectors, it has a basis
(b) Any spanning set of V contains a basis.

Proof (a) Let $V = \text{span}\{u_1, \dots u_m\}$. If $\{u_1, \dots u_m\}$ are L.I they are a basis. If not there is a smaller spanning set. Repeating the argument shows that either there
[51] is a basis or $\{\underline{0}\}$ is a spanning set. But $V \neq \{\underline{0}\}$ so V has a basis.

(b) From (a), V contains a basis.

(*) Note Any vectorspace spanned by finitely many vectors has a basis and a dimension.

PROPN 8 Let V be a vectorspace, $\dim V = n$.
Let $\{u_1, \dots, u_m\} = S \subset V$. Then

(a) If $m > n$ S is L.D.

(b) If $m = n$ and S is L.I. then S is a basis of V

(c) If $m < n$ S does not span V

PROPN 9 dimension $[\text{span}\{u_1, \dots u_m\}]$ = max no of L.I vectors in $\{u_1, \dots u_m\}$.

PROPN 10 Let V be a vectorspace, $\dim V = n$.
Let $\{u_1, \dots u_m\}$ be L.I. Then
$\exists\ V_{m+1}, \dots, V_n$ such that $\{u_1, \dots, u_m, V_{m+1}, \dots V_n\}$ form a basis of V

[52] DEFN 6 Let A be a matrix. Then

$$\text{Range } A = \{\underline{b} \mid A\underline{x} = \underline{b} \text{ for some } \underline{x}\}$$

$$\text{Nullspace } A = \{\underline{x} \mid A\underline{x} = \underline{0}\}.$$

PROPN 11 (a) Range A and nullspace A are vectorspaces.
(b) Range A = span (column vectors of A)
(c) If we row-reduce A to B then any set of L I (or L.D) columns of A remains L.I (or L.D) in B.
(d) dimension (range A) = no of leading columns of form $\underline{e}_i$ in row echelon form

Proof (a) If $\underline{b}_1, \underline{b}_2 \in$ range (A). Let $A\underline{x}_1 = \underline{b}_1$, $A\underline{x}_2 = \underline{b}_2$
Then $A(\underline{x}_1 + \underline{x}_2) = \underline{b}_1 + \underline{b}_2 \Rightarrow \underline{b}_1 + \underline{b}_2 \in$ range A (by defn of range)
Also. $A(a\underline{x}_1) = a\underline{b}_1 \Rightarrow a\underline{b}_1 \in$ range (A) for $a \in \mathbb{R}$
thus the range is a vectorspace.

If $\underline{x}_1, \underline{x}_2 \in$ nullspace A, $A(\underline{x}_1 + \underline{x}_2) = A\underline{x}_1 + A\underline{x}_2 = \underline{0}$
and $A(a\underline{x}_1) = aA\underline{x}_1 = a\underline{0} = \underline{0} \Rightarrow$ the nullspace is a vectorspace.

(b) $\underline{b} \in$ range A iff $\exists \underline{x} \ni A\underline{x} = \underline{b}$
iff $x_1\underline{a}_1 + \dots + x_n\underline{a}_n = \underline{b}$ for some $x_1 \dots x_n$
iff $\underline{b} \in$ span $\{\underline{a}_1 \dots \underline{a}_n\}$ iff $\underline{b} \in$ span {col vects of A}

Note (d) implies the number of non-zero rows in independent of the method of reduction. We thus have

CORR 1 rank A = dim (range A) = max no of L.I. cols of A.

PROPN 12 Let A be $m \times n$. Then: dim (range A) + dim (nullspace A) = n

Proof omitted by me. (see page 54).

[5] DEFN 7. Column operations for matrices are defined similarly to row operations

LEMMA 1 Let $S = \{a_1, \ldots a_m\} \subseteq V$ (V a vectorspace)

Then S is L.D. (or L.I) iff

$\{a_1, \ldots, a_{k-1}, a_k + \alpha a_i, a_{k+1}, \ldots a_m\}$ is L.D. (or L.I) for all $\alpha \in \mathbb{R}$

iff

$\{\underline{a}_1, \ldots, \underline{a}_{k-1}, \alpha a_k, \underline{a}_{k+1}, \ldots \underline{a}_m\}$ is L.D. (or L.I) for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

PROPN 13 Let $A \sim B$ under colum oper. Then rank A = rank B.

[5c] Method To find basis of $\mathbb{R}^n$ including $a_1, \ldots, a_m$ ($m < n$)

1. Col reduce $(\underline{a}_1 \ldots \underline{a}_m)$ (checking that they are L.I.)
2. Add $(n-m)$ unit vectors with 1's in those rows not of the form $\left\{\begin{smallmatrix}1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0\end{smallmatrix}\right\}$ etc
3. $\{\underline{a}_1, \ldots, \underline{a}_m\} \cup \{$new unit column vectors$\}$ is a basis

Thm 4 Rank A = Rank A^T

§6 LINEAR TRANSFORMATIONS

[6.1] **DEFN 1** Let U, V be vectorspaces both over $\mathbb{R}$ (or $\mathbb{C}$). A function f mapping U into V satisfying:

(i) $f(u_1 + u_2) = f(u_1) + f(u_2)$ $\Big\}$ all $u, u_1, u_2 \in U$

(ii) $f(au) = a\,f(u)$ $\Big\}$ $a \in \mathbb{R}$ (or $\mathbb{C}$)

is called a <u>linear transformation</u>. We write $f : U \to V$
$f(u) = v$

<u>Remark</u>. Let $f : U \to V$ be linear.

Let $\{u_1, \ldots, u_m\}$ be a basis of U
$\{v_1, \ldots, v_n\}$ a basis of V

$f(u_j) \in V$ and so can be expressed as an L.C. of $\{v_1, \ldots v_n\}$

Let $f(u_j) = \sum_{i=1}^{n} a_{ij} v_i$ where the a_{ij} are constants $\in \mathbb{R}$ or $\mathbb{C}$

Let $u \in U$. Then $u = c_1 u_1 + \ldots + c_m u_m$, ie u has coords $\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$ relative to $\{u_1, \ldots, u_m\}$

Then $f(u)$ will have co-ords $\begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix}$ relative to $\{v_1, \ldots v_n\}$ where $f(u) = c_1' v_1 + \ldots + c_n' v_n$

We can show that:

$$c_i' = \sum_{j=1}^{m} a_{ij} c_j \quad \underline{\text{or}} \quad (a_{ij}) \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix}$$

<u>THM 1</u> Let $f : U \to V$ be linear, and choose any basis $B_U = \{u_1, \ldots u_m\}$ of U and $B_V = \{v_1, \ldots v_n\}$ of V

Suppose $f(u_j) = \sum_{i=1}^{n} a_{ij} v_i$ $\quad (j = 1 \ldots m)$

[63] Then for any $u \in U$, if $\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$ are co-ords of u relative to B_u and $\begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix}$ are coords of $f(u)$ relative to B_v,

then $$(a_{ij}) \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix}.$$

(so the j^{th} col of (a_{ij}) is the vector of co-ords of $f(u_j)$ relative to B_v).

DEFN 2 The matrix (a_{ij}) of the THM is called the (matrix) representation of f relative to B_u and B_v.

PROCEDURE FOR FINDING MATRIX REPRESENTATION OF f.

$f: U \to V$, B_u is basis of U, B_v a basis of V
To find representation of A w.r.t. B_u, B_v

① Find $f(u_i) = w_i$ $\quad (u_i \in B_u)(i = 1, \ldots, m)$
② Find co-ords of w_i w.r.t. basis B_v – a_i is the col vect.
③ Mat rep A of f is $A = (\underline{a}_1, \ldots, \underline{a}_m)$.

If $V = \mathbb{R}^n$ for some n, step ② can be simplified as follows:
Write down the mat $(\underline{v}_1, \ldots, \underline{v}_n \mid \underline{w}_1, \ldots, \underline{w}_m)$ where $B_v = \{\underline{v}_1, \ldots, \underline{v}_n\}$. Row reduce it so that the LHS is I. Then the RHS is A. (this finds the coords of $\underline{w}_1 \ldots \underline{w}_m$ simultaneously)

[64] <u>PROPN 1</u> Let $B_u = \{u_1, \ldots, u_n\}$ and $B_u' = \{u_1', \ldots, u_n'\}$ be two bases for U. We can express each u_i' in terms of B_u since the latter is a basis of U.

Let the matrix P be defined by

$$u_j' = \sum_{i=1}^{n} p_{ij} u_i \qquad (j = 1, \ldots, n)$$

Let $\underline{c}$ be the co-ords of $\underline{w}$ w.r.t. B_u, and let $\underline{c}'$ be the co-ords of $\underline{w}$ w.r.t. B_u'. Then $\underline{c} = P\underline{c}'$ and P is nonsingular.

<u>DEFN 3</u> The matrix P is called the <u>change of basis</u> matrix from B_u to B_u'.

<u>THM 2</u> Let $f : U \to U$.
Let A represent f relative to B_u
A' represent f relative to B_u'

<u>Where</u>: $u_{ij} = \sum_{i=1}^{n} p_{ij} u_i \quad (j = 1, .., n)$

<u>Then</u> $A' = P^{-1} A P$

(the cols of P are the co-ords of the new basis vectors relative to the old basis).

<u>Procedure for finding P and A'</u>

Let $B_u = \{u_1, \ldots, u_n\}$, $B_u' = \{u_1', \ldots, u_n'\}$
Find the co-ords of u_i' relative to B_u, $\underline{p}_i$ say, where $\underline{p}_i$ is a col. vect. in $\mathbb{R}^n$. Then $P = (\underline{p}_1, \ldots, \underline{p}_n)$, and $A' = P^{-1}AP$.

[66] If $V = \mathbb{R}^n$, we can simplify this:
Write down the matrix $(\underline{v}_1, \dots \underline{v}_n \mid \underline{v}_1', \dots, \underline{v}_n')$ and row-reduce it until the left hand side is I. Then the RHS is P. Then row reduce until RHS is I, then LHS is P^{-1}.

FINIS

Proofs for Real Analysis §2 → §7

§2 THM 3 Since l is the lub of $\{a_n\}$, given $\varepsilon > 0$
$\exists N \in \mathbb{N}^+ \ni a_N > l - \varepsilon$. As $(a_n)\uparrow$,
$n \geq N \Rightarrow a_n > l - \varepsilon$. As l is an upper
bound of $\{a_n\}$, $(\forall n)(a_n \leq l)$

Hence $n \geq N \Rightarrow l - \varepsilon < a_n \leq l < l + \varepsilon$

ie, $n \geq N \Rightarrow |a_n - l| < \varepsilon$

Thus $\lim_{n\to\infty} a_n = l$

§3 PROP 1 $\lim_{N\to\infty} (S_N - S_{N-1}) = \left(\lim_{N\to\infty} S_N\right) - \left(\lim_{N\to\infty} S_{N-1}\right) = A - A = 0$

But $S_N - S_{N-1} = a_N \Rightarrow \lim_{N\to\infty} a_N = 0$.

PROP 2 Let $S_N = \sum_{n=1}^{N} (a_n + b_n)$ Then $S_N = \left(\sum_{n=1}^{N} a_n\right) + \left(\sum_{n=1}^{N} b_n\right)$

$\Rightarrow (S_N)$ converges to $\left(\sum_{n=1}^{\infty} a_n\right) + \left(\sum_{n=1}^{\infty} b_n\right)$

COMPARISON TEST

(a) $S_N = \sum_1^N c_n$, $T_N = \sum_1^N a_n$

Since $a_n \geq 0$, T_n is $\uparrow$. We need only show
that T_N is bounded above.

But $a_n \leq c_n \Rightarrow \sum_1^N a_n \leq \sum_1^N c_n \Rightarrow T_N \leq S_N \leq \sum_1^\infty c_n$

$\Rightarrow (T_N)$ is bounded above $\Rightarrow$ convergent

(b) Since $d_n \geq 0$, then if $S_N = \sum_1^N d_n$ (S_N) is unbounded
$d_n \leq a_n \Rightarrow S_N \leq T_N = \sum_1^N a_n \Rightarrow (T_N)$ unbounded
$\Rightarrow \sum a_n$ divergent.

RATIO TEST

Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$

① If $r < 1$ then for some n_0, $n \geq n_0 \Rightarrow \frac{a_{n+1}}{a_n} < r + \varepsilon = t < 1$ (some $\varepsilon > 0$)

We have $a_{n+1} = \frac{a_{n+1}}{a_n} \frac{a_n}{a_{n-1}} \cdots \frac{a_{n_0+1}}{a_{n_0}} a_{n_0} \leq t^{n-n_0+1} a_{n_0}$

$\leq K t^n$. $(K = a_{n_0} t^{-n_0})$

Since $t < 1$, Kt^n is the term of a C series by comparison test $\sum a_n$ is convergent

② If $r > 1$, then for some n_0, $n \geq n_0 \Rightarrow \frac{a_{n+1}}{a_n} > 1$

ie, $a_{n+1} \geq a_n$ so $a_n \not\to 0$ and series diverges

③ $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n}$ both have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. The first is conv. and the second div.

LIMIT TEST

(a) If lim exists, there is a c such that $\frac{a_n}{t_n} \leq c$ (all n)

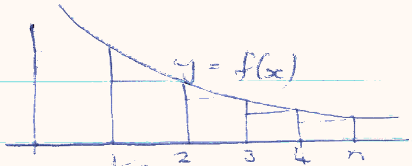
$\Rightarrow a_n \leq c\,t_n$ and $\sum (c\,t_n)$ is conv.

$\Rightarrow \sum a_n$ is conv. by comparison.

(b) If lim exists, there is a $C > 0$ such that $\frac{t_n}{a_n} \leq c$ (all n)

$\Rightarrow \frac{1}{c} t^n \leq a_n$. $\sum (\frac{1}{c} t_n)$ is divergent $\Rightarrow$ by comparison, so is $\sum a_n$.

INTEGRAL TEST



(a) Area of lower ~~rectangles~~ rectangles is $f(2)+f(3)+\dots+f(n) \leq \int_1^n f(x)\,dx$.

If $\lim_{n\to\infty} \int_1^n f(x)\,dx = A$ exists, then $\sum_{n=2}^{N} f(n) \leq A$ for all N
$\Rightarrow$ convergent.

(b) Area of upper rectangles is $f(1)+f(2)+\dots+f(n-1) \geq \int_1^n f(x)\,dx$.
If lim. RHS does not exist, RHS must be unbounded, hence so is (S_N) $\Rightarrow$ divergent

~~Exp~~ POWER SERIES

THM 1 $\left|\dfrac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\dfrac{a_{n+1}}{a_n}\right| |x| \to \rho|x|$ under the hypotheses of thm.

By the ratio test: $\rho|x| < 1 \Rightarrow$ conv.
$\rho|x| > 1 \Rightarrow$ div
If $\rho \neq 0$ or $+\infty$, the series is conv for $|x| < \frac{1}{\rho}$

and div. for $|x| > \frac{1}{\rho}$
If $\rho = 0$, then $0 = \rho|x| < 1$ always holds $\Rightarrow$ conv.
If $\rho = +\infty$, then for $x \neq 0$, $|a_{n+1}x^{n+1}| > |a_n x^n|$
for all n sufficiently large, hence terms do not $\to 0$
hence series is div.
If $x = 0$, series reduces to a_0, which is conv.

§5) PROP 2 Given $\tilde{\varepsilon} > 0$, $\exists\, \delta_1 \ni (|x-a| < \delta_1 \Rightarrow |g(x)-g(a)| < \tilde{\varepsilon})$

Given $\varepsilon > 0$, $\exists\, \delta_2 \ni (|y - g(a)| < \delta_2 \Rightarrow |f(y) - f(g(a))| < \varepsilon)$

From the first one with $\tilde{\varepsilon} = \delta_2$,

$$\exists\, \delta_1 \ni (|x-a| < \delta_1 \Rightarrow |g(x)-g(a)| < \delta_2)$$

$$\Rightarrow |x-a| < \delta_1 \Rightarrow |f(g(x)) - f(g(a))| < \varepsilon$$

PROP 3 (for $f(a) > 0$)

Since f is continuous at a, $\exists\, \delta \ni (|x-a| < \delta \Rightarrow |f(x)-f(a)| < \tfrac{1}{2} f(a))$
(taking $\varepsilon = \tfrac{1}{2} f(a)$ in the defn of cont.)

$$\Rightarrow |x-a| < \delta \Rightarrow -\tfrac{1}{2} f(a) < f(x) - f(a)$$

$$\Rightarrow x \in (a-\delta, a+\delta) \Rightarrow 0 < \tfrac{1}{2} f(a) < f(x).$$

§6) <u>Thm 1</u> Since $f'(c)$ exists, $\lim_{h \to 0} \left[\frac{f(c+h) - f(c)}{h} - f'(c) \right] = 0$

$$\Rightarrow \left[\frac{f(c+h) - f(c) - hf'(c)}{h} \right] = g(h) \text{ where } \lim_{h \to 0} g(h) = 0$$

$\Rightarrow f(c+h) - f(c) = h(f'(c) + g(h))$

$\Rightarrow \lim_{h \to 0} [f(c+h) - f(c)] = 0$ Let $x = c+h \Rightarrow h = x - c$

$\Rightarrow h \to 0 \equiv x - c \quad 0$

$x - c \to 0 \equiv x \to c$

$\Rightarrow \lim_{x \to c} [f(x) - f(c)] = 0$

$\Rightarrow \lim_{x \to c} (f(x) - f(c)) = 0$

<u>Thm 2</u> Suppose $f'(c) \neq 0$ and that f has a local max at c.

By propn 3, §5, since $f'(c) > 0$,

$\exists \, \delta_1 > 0 \ni \frac{f(c+h) - f(c)}{h} > 0$ for $|h| < \delta_1$

$\Rightarrow f(c+h) > f(c)$ if $0 < h < \delta_1$

This contradicts the fact that f has a loc. max at c.
Similarly, if $f'(c) < 0$ and f has loc. min at c.

Thm 3 (Rolle) Since f is continuous on $[a, b]$, it takes its maximum & minimum values on $[a, b]$.

2 cases (i) f takes its min or max at $c \in (a, b)$
then by Thm 2, $f'(c) = 0$

(ii) f takes max at one end and min at other or some pt.
Then $f(x) = 0$ (all x) since $f(a) = f(b) = 0$
$\Rightarrow f'(c) = 0$ all $c \in (a, b)$

Thm 4 Let $g(x) = x \dfrac{f(b) - f(a)}{b - a} - f(x)$

$$= \frac{x(f(b) - f(a)) - (b-a) f(x)}{b - a}$$

Then $g(a) = \dfrac{a f(b) - b f(a)}{b - a} = g(b)$

By Thm 3 $\exists c \in (a, b) \ni g'(c) = 0$

ie, $f'(c) - \left[\dfrac{f(b) - f(a)}{b - a}\right] = 0$.

§7 Thm 1 Let $H(x) = g(x)[f(b) - f(a)] - f(x)[g(b) - g(a)]$

By (i) and (ii), $H(x)$ satisfies the continuity and differentiability conditions of Rolle's theorem

$$H(a) = g(a)[f(b) - f(a)] - f(a)[g(b) - g(a)]$$
$$= g(a) f(b) - f(a) g(b)$$

$$H(b) = g(b)[f(b) - f(a)] - f(b)[g(b) - g(a)]$$
$$= f(b) g(a) - f(a) g(b)$$
$$= H(a).$$

$\Rightarrow$ By Rolle, $\exists\ \xi \in (a,b) \ni H'(\xi) = 0$

ie, $0 = H'(\xi) = g'(\xi)[f(b) - f(a)] - f'(\xi)[g(b) - g(a)]$

Since we cannot have both $g'(\xi)$ and $f'(\xi) = 0$, and since $g(b) - g(a) \neq 0$, it follows that $g'(\xi) \neq 0$

$$\Rightarrow \frac{(f(b) - f(a))}{(g(b) - g(a))} = \frac{f'(\xi)}{g'(\xi)}$$

Thm 2 ~~By $g(a) = f(a) = 0$ and $g(x)$ and $g'(x)$~~

Let $f(x)$, $g(x)$ be continuous on $[a, b]$ and differentiable on (a, b), with $g(0) = f(0) = 0$

* and $g(x)$ and $g'(x)$ do not vanish for $0 < x < c$

* and $f'(x)$ and $g'(x)$ are continuous on $0 < x < c$.

Then $\lim_{x \to 0^+} \frac{f(x)}{g(x)} = L$ if $\lim_{x \to 0^+} \frac{f'(x)}{g'(x)} = L$.

Proof by (I) we can apply thm 1 with $a = 0$ and $b = x$

So $$\frac{f(x)}{g(x)} = \frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f'(\xi(x))}{g'(\xi(x))} \qquad 0 < \xi(x) < x$$

As $x \to 0$, $\xi(x) \to 0$

Then $$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = L \text{ if } \lim_{x \to 0^+} \frac{f'(\xi(x))}{g'(\xi(x))} = L$$

$$\text{if } \lim_{x \to 0^+} \frac{f'(x)}{g'(x)} = L.$$