Monte-Carlo. Part 1

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Motivation

We often need to estimate integrals because they are often not easy to calculate analytically.

One obvious example is of course the expectation of a random variable.

Thus, we need an approach to estimate integrals, and the most popular one is Monte-Carlo.

Signed measure and Radon-Nikodym derivative

Definition (Signed measure)

Let $(\mathcal{X}, \Sigma)$ be a measurable space. Then $\nu : \Sigma \to [-\infty, +\infty]$ is called a signed measure if $\nu(\varnothing) = 0$ and ν is σ-additive.

Definition (Radon-Nikodym derivative)

Let $(\mathcal{X}, \Sigma)$ be a measurable space, ν a signed measure on it, and μ a measure on it. A measurable function f is called the Radon-Nikodym derivative of ν with respect to μ if for $\forall A \in \Sigma,\ \ \nu(A) = \int_A f d\mu$.
The notaion for it is $\frac{d\nu}{d\mu}$.

Absolute continuity

Definition

Let $(\mathcal{X}, \Sigma)$ be a measurable space, ν a signed measure on it, and μ a measure on it. ν is called absolutely continuous w.r.t. μ if $\mu(A) = 0$ implies $\nu(A) = 0$.
The notaion for it is $\nu \prec \mu$.

Theorem

Let $(\mathcal{X}, \Sigma)$ be a measurable space, ν a signed measure on it, and μ a measure on it.
The Radon-Nikodym derivative of ν with respect to μ exists iff ν is absolutely continuous w.r.t. μ. Moreover, the derivative is unique μ-a.s.

Monte-Carlo for evaluation of integrals. General case

Let $(\mathcal{X}, \Sigma)$ be a measurable space, ν a finite signed measure on it. Let's say we want to evaluate $I = \nu(\mathcal{X})$

Consider a probability space $(\Omega, \mathcal{F})$ and a random variable $\xi : (\Omega, \mathcal{F}) \to (\mathcal{X}, \Sigma)$. Denote the distribution of ξ as $\mathcal{P}_\xi$.

Let ν have the Radon-Nikodym derivative w.r.t. $\mathcal{P}_\xi$ and denote it as f. Then

$$\mathbb{E}(f(\xi)) = \int_{\mathcal{X}} f d\mathcal{P}_\xi = \nu(\mathcal{X})$$

Then, if we have a sequence of i.i.d. ξ_i, by the WLLN:

$$\hat{I}_n = \frac{f(\xi_1) + ... + f(\xi_n)}{n} \xrightarrow{\mathbb{P}} \nu(\mathcal{X})$$

Confidence Intervals

Assume,

$$\mathbb{E}f(\xi)^2 < \infty, \ \ \sigma^2 = Var(f(\xi)) < \infty$$

Then, by the CLT,

$$\frac{\sqrt{n}(\hat{I}_n - I)}{\sigma} \xrightarrow{d} N(0,1)$$

It implies,

$$\mathbb{P}\left(|\hat{I}_n - I| < \frac{\sigma x}{\sqrt{n}}\right) \to \Phi(x) - \Phi(-x)$$

$$\mathbb{P}\left(\hat{I}_n - \frac{\sigma x_\alpha}{\sqrt{n}} < I < \hat{I}_n + \frac{\sigma x_\alpha}{\sqrt{n}}\right) \to 1 - \alpha$$

, where x_α is the solution of $\Phi(x) - \Phi(-x) = 1 - \alpha$

σ is then replaced by a consistent estimator

Complexity

To obtain $\mathbb{P}\left(|\hat{I}_n - I| < \epsilon\right) \approx 1 - \alpha$ we need $n = \frac{\sigma^2 x_\alpha^2}{\epsilon^2}$.

Thus, to estimate I we need to simulate ξ_i and calculate $f(\xi_i)$ $\frac{\sigma^2 x_\alpha^2}{\epsilon^2}$ times.

If t_ξ is the average time needed to simulate ξ_i and t_f is the average time needed to calculate $f(\xi_i)$, then the total time needed is:

$$n(t_\xi + t_f) = \frac{\sigma^2 (t_\xi + t_f) x_\alpha^2}{\epsilon^2}$$

Simultaneous estimation of multiple integrals

Let $(\mathcal{X}, \Sigma)$ be a measurable space, ν_1, ..., ν_m finite signed measures on it. Let's say we want to evaluate $I^{(1)} = \nu_1(\mathcal{X})$, ..., $I^{(m)} = \nu_m(\mathcal{X})$

Suppose we can simulate a random variable ξ such that every signed measure ν_i is absolutely continuous w.r.t. the distribution of ξ.

Thus, we can use the same r.v. to estimate $(I^{(i)})_i^m$ and obtain $(\hat{I}_n^{(i)})_i^m$

Denote $\mathcal{I} = (I^{(i)})_i^m$, vector of integrals and $\hat{\mathcal{I}}_n = (\hat{I}_n^{(i)})_i^m$ vector of estimators.

By the CLT:

$$\sqrt{n}(\hat{\mathcal{I}}_n - \mathcal{I}) \xrightarrow{d} N(0, \mathbb{V})$$

Interestingly, big absolute values of $corr(m_k(\xi_i), m_s(\xi_i))$ make estimation a bit easier, but we are not going to talk about it now. (Part 2)

References

- V. Nekrutkin. Basics of Monte-Carlo. 2018.