Linear Regression with Centrality Measures

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- Want to know how network position of an agent affects economic outcome
- Network is high dimensional $\Rightarrow$ summarize using centrality measures
- Use OLS to study relationship between outcome and centrality

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- When is linear regression with centrality measures statistically valid?

Examples

- Better-networked venture capital firms are more profitable (Hochberg, Ljungqvist, and Lu, 2007)
- Greater take-up when seeding microfinance to central villagers (Banerjee, Chandrasekhar, Duflo, and Jackson, 2013)
- Central families overrepresented in political offices in the Philippines (Cruz, Labonne, and Querubin, 2017)

Linear Regression with Centrality Measures

- Researchers study:

$$Y_i = \underbrace{C_i}_{\text{centrality}} \beta + \underbrace{X_i'}_{\text{controls}} \gamma + \varepsilon_i \quad , \quad E[X_i \varepsilon_i] = 0, E[C_i \varepsilon_i] = 0$$

- β is parameter of interest

- Different centrality measures capture different ways of being important

- Estimate β by OLS; conduct inference using t-test

Statistical Challenges

- Networks may be sparse
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$\Rightarrow$ Weaker signals harder to separate from noise

When is linear regression on centrality measures statistically valid?

- Degree, diffusion and eigenvector centralities
- Cross-sectional: one large network
- Novel asymptotic framework with sparse, proxy networks
- More similar to data $\Rightarrow$ asymp. approx. more accurate in finite sample

Contributions

1. Show that OLS can become inconsistent with sparse, proxy networks
    - Characterize threshold at which inconsistency occurs
    - Show that eigenvector is less robust than degree and diffusion
    - Comparing significance involves both economic *and* statistical properties
    - Rule-of-Thumb for sparsity regime

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2. Distributional theory with sparse, proxy networks
    - Even when consistent, OLS estimators are asymptotically biased
    - Asymptotic bias can be large relative to variance
    - Slower rate of convergence than reflected by robust standard errors
    - Usual confidence intervals and tests may not be valid

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3. Novel bias correction and inference methods

Related Literature

- Linear Regression with Centrality Measures
  - **Eigenvectors:** Le and Li (2020); Cai, Yang, Zhu, Shen, and Zhao (2021)
- Estimation of Centrality Statistics with Proxy Networks
  - **Simulations:** Costenbader and Valente (2003); Borgatti, Carley, and Krackhardt (2006)
  - **Theory:** Dasaratha (2020); Avella-Medina, Parise, Schaub, and Segarra (2020)
- Econometrics with Sparse Networks
  - **Network Formation:** Graham (2017); De Paula, Richards-Shubik, and Tamer (2018); Jochmans (2018); Graham (2020b); Menzel (2022)
  - **Network Recovery:** Manresa (2016); Rose (2016); Wang (2018); De Paula et al. (2020)
  - **Network Moments:** Bickel, Chen, and Levina (2011); Bhattacharyya and Bickel (2015); Leung and Moon (2019); Menzel (2021); Matsushita and Otsu (2021); Green and Shalizi (2022)

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Centrality Measures

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- Network data is symmetric adjacency matrix A
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Centrality Measures

- Network data is symmetric adjacency matrix A
- A_{ij} records intensity of relationship between i and j
- Centrality measures summarize network positions into "importance"
- When A is known, centrality measures exactly computable
- Many different ways to measure importance $\Rightarrow$ many centrality measures
- Focus on degree, diffusion and eigenvector

Degree and Diffusion Centralities

- Degree is the total intensity of direct connections:

$$C^{(1)}(A) = A\iota$$

Degree and Diffusion Centralities

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$$C^{(1)}(A) = A\iota$$

- Diffusion reflects ability to broadcast messages in a network:

$$C^{(T)}(A) = \left(\sum_{t=1}^{T} \delta^t A^t \right) \iota$$

- T, δ are chosen by researchers

Eigenvector Centrality

- Eigenvector is the leading eigenvector of A, scaled

$$C^{(\infty)}(A) = a_n v_1(A)$$

- Friends of important agents are themselves more important

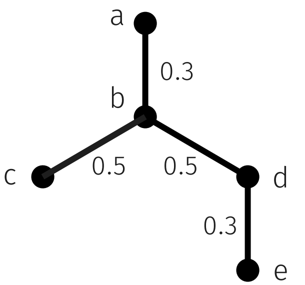
Eigenvector Centrality

- Various choices of a_n in literature
- a_n turns out to matter for statistical properties
- Today: $a_n = \sqrt{\lambda_1(A)}$

	Applied Work	Econometrics
$a_n = 1$	Banerjee et al. (2013) Cruz et al. (2017)	Dasaratha (2020)
$a_n = \sqrt{n}$	Chandrasekhar et al. (2018) Banerjee et al. (2019)	Avella-Medina et al. (2020) Cai et al. (2021)

Table 1: Examples of a_n in econometric theory and empirical work

Centrality Statistics



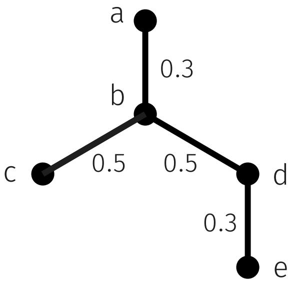
$$\begin{pmatrix} 0 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0.3 & 0 \end{pmatrix}$$

Centrality Statistics

$$\begin{pmatrix} 0.3 \\ 1.3 \\ 0.5 \\ 0.8 \\ 0.3 \end{pmatrix}$$

Degree

$C^{(1)}$



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$$\begin{pmatrix} 0.3 \\ 1.3 \\ 0.5 \\ 0.8 \\ 0.3 \end{pmatrix} \qquad \begin{pmatrix} 0.28 \\ 0.94 \\ 0.46 \\ 0.64 \\ 0.24 \end{pmatrix}$$

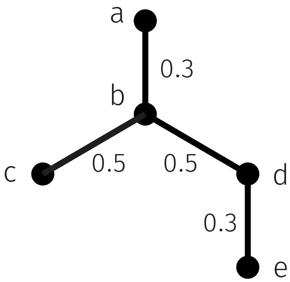
Degree

$C^{(1)}$

Diffusion

$C^{(T)}$

$(T = 3, \delta = 0.5)$



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$$\begin{pmatrix} 0.3 \\ 1.3 \\ 0.5 \\ 0.8 \\ 0.3 \end{pmatrix} \qquad \begin{pmatrix} 0.28 \\ 0.94 \\ 0.46 \\ 0.64 \\ 0.24 \end{pmatrix} \qquad 0.89 \cdot \begin{pmatrix} 0.25 \\ 0.68 \\ 0.42 \\ 0.50 \\ 0.18 \end{pmatrix}$$

Degree

$C^{(1)}$

Diffusion

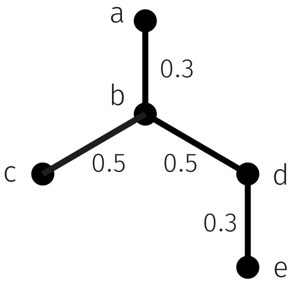
$C^{(T)}$

$(T = 3, \delta = 0.5)$

Eigenvector

$C^{(\infty)}$

$$\left(a_n = \sqrt{\lambda_1(A)}\right)$$



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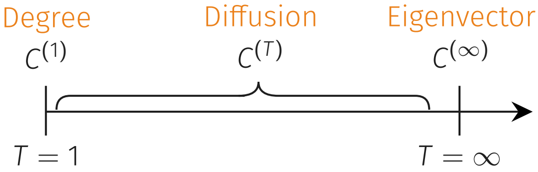
- When $T = 1$, $C^{(1)}(A) \propto C^{(T)}(A)$
- Banerjee et al. (2019): If δ is larger than the inverse of leading eigenvalue of A,

$$\lim_{T \to \infty} C^{(T)}(A) \propto C^{(\infty)}(A)$$

- The statistics we study can thus be represented on a line:



Model and Assumptions

Model and Assumptions

- For simplicity, consider regression without other covariates

- For $d \in \{1, T, \infty\}$:

$$Y_i = \beta^{(d)} C_i^{(d)} + \varepsilon_i^{(d)}$$

- Will make enough assumptions so $\beta^{(d)}$ is slope of CEF

- DGP yields i.i.d. draws of $\{(\varepsilon_i, U_i)\}$

- $U_i \sim U[0, 1]$ is unobserved latent type used to construct network

True Network

- Let A be the $n \times n$ symmetric adjacency matrix
- For $f : [0,1]^2 \to [0,1]$, $p_n \in (0,1]$ and $j > i$, let

$$A_{ij} := p_n f(U_i, U_j)$$

  Symmetry: $A_{ij} = A_{ji}$; normalisation: $A_{ii} = 0$

- When A_{ij} is large, agents i and j have a strong relationship
- $p_n \to 0$ reflects sparsity

Examples

Informal Insurance and Consumption Smoothing

- Y_i: variance in consumption expenditure
- A_{ij}: probability that i lends j money or vice versa
- U_i: social class

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Graphons (f)

- "Full" Insurance: $f\left(U_i, U_j\right) = 1$
- Assortative: $f\left(U_i, U_j\right) = 1 - \left(U_i - U_j\right)^2$

True Network

- When A is observed, can exactly compute centrality measures
- Form the estimator

$$\tilde{\beta}^{(d)} = \frac{Y' C^{(d)}}{\left(C^{(d)}\right)' C^{(d)}}$$

Proxy Network

- When A is not observed, use $\hat{A}$: for $j > i$,

$$\hat{A}_{ij} \,\middle|\, \mathbf{U} \overset{\text{iid}}{\sim} \text{Bernoulli}\left(A_{ij}\right)$$

  Below the diagonal, $\hat{A}_{ij} = \hat{A}_{ji}$

- Proxy error is “white noise”

Proxy Network

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  Below the diagonal, $\hat{A}_{ij} = \hat{A}_{ji}$

- Proxy error is “white noise”
- Use $\hat{A}$ as plug-in for A to compute $\hat{C}^{(d)}$
- Estimate

$$\hat{\beta}^{(d)} = \frac{Y'\hat{C}^{(d)}}{\left(\hat{C}^{(d)}\right)'\hat{C}^{(d)}}$$

Examples

Informal Insurance and Consumption Smoothing

- Y_i: variance in consumption expenditure
- A_{ij}: probability that i lends j money or vice versa
- U_i: social class

Proxy Networks

- Social: $\hat{A}_{ij} = 1$ if either i or j reports the other as a friend
- Financial: $\hat{A}_{ij} = 1$ if i borrows from or lends to j

Sparsity

- Want to understand the properties of $\tilde{\beta}^{(d)}$ and $\hat{\beta}^{(d)}$ when networks are sparse

- Theoretical device: $p_n \to 0$ as $n \to \infty$:

$$A_{ij} := p_n f(U_i, U_j) \quad , \quad \hat{A}_{ij} = \text{Bernoulli}\left(p_n f(U_i, U_j)\right)$$

- Weak interaction in true network: $A_{ij} \to 0$

- Sparse proxy networks: many $\hat{A}_{ij}$'s are 0

- Rate at which $p_n \to 0$ reflects different amounts of sparsity

Model and Assumptions

- Proxy Networks: $\hat{A}$ is observed but A economically meaningful
  - **Node-Level Regressions**: Auerbach (2022), Le and Li (2020), Cai et al. (2021)
  - **Dyadic Models**: see e.g. De Paula (2017) Section 3, Graham (2020a) Section 6

- Bickel-Chen Model of Sparsity (Bickel and Chen 2009)
  - Graham (2020a): "The Bickel-Chen model is the default one in the nonparametric statistics and machine learning literatures on random graphs."
  - **Dyadic Models**: Jochmans (2018), Graham (2020b)
  - **Network Moments**: Bickel, Chen, and Levina (2011); Bhattacharyya and Bickel (2015); Matsushita and Otsu (2021); Green and Shalizi (2022)

- Standard models of proxy networks and sparsity to study linear regression

Preview of Results

- Main question: how do $\tilde{\beta}^{(d)}$ and $\hat{\beta}^{(d)}$ behave at as we vary p_n?

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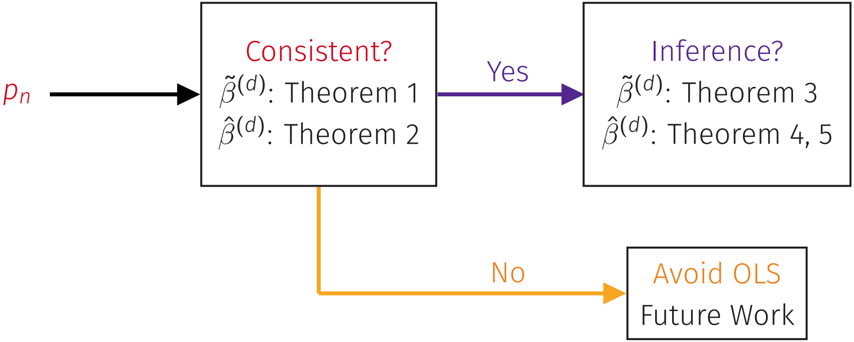


Theoretical Results

Consistency with True Network A

Theorem 1

$\tilde{\beta}^{(1)}$ and $\tilde{\beta}^{(T)}$ are consistent if and only if

$$p_n \gg n^{-\frac{3}{2}} \ .$$

$\tilde{\beta}^{(\infty)}$ is consistent if $a_n = \sqrt{\lambda_1(A)}$.

- $n \cdot n^{-3/2} \to 0$
- Inconsistent only under extreme sparsity
- Consistency requires $\|C^{(d)}\|_2 \to \infty$ w.p.a. 1
- a_n inflates eigenvector, counters sparsity

Consistency with Proxy Network $\hat{A}$

Theorem 2

$\hat{\beta}^{(1)}$ and $\hat{\beta}^{(T)}$ are consistent if and only if

$$p_n \gg n^{-1} \ .$$

- $\hat{\beta}^{(1)}$, $\hat{\beta}^{(T)}$ less robust to sparsity than $\tilde{\beta}^{(1)}$, $\tilde{\beta}^{(T)}$
- Consistency with proxy networks in dense regime

More Phases

Consistency with Proxy Network $\hat{A}$

Theorem 2

$\hat{\beta}^{(\infty)}$ *consistent if*

$$p_n \gg n^{-1}\sqrt{\frac{\log n}{\log\log n}}.$$

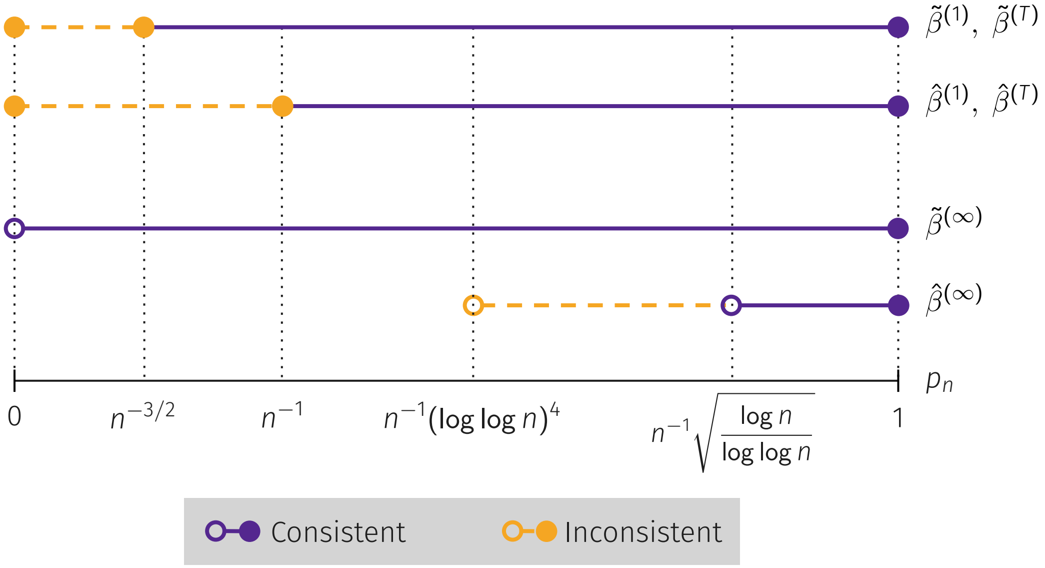
It is inconsistent if

$$n^{-1}(\log\log n)^4 \ll p_n \ll n^{-1}\sqrt{\frac{\log n}{\log\log n}}.$$

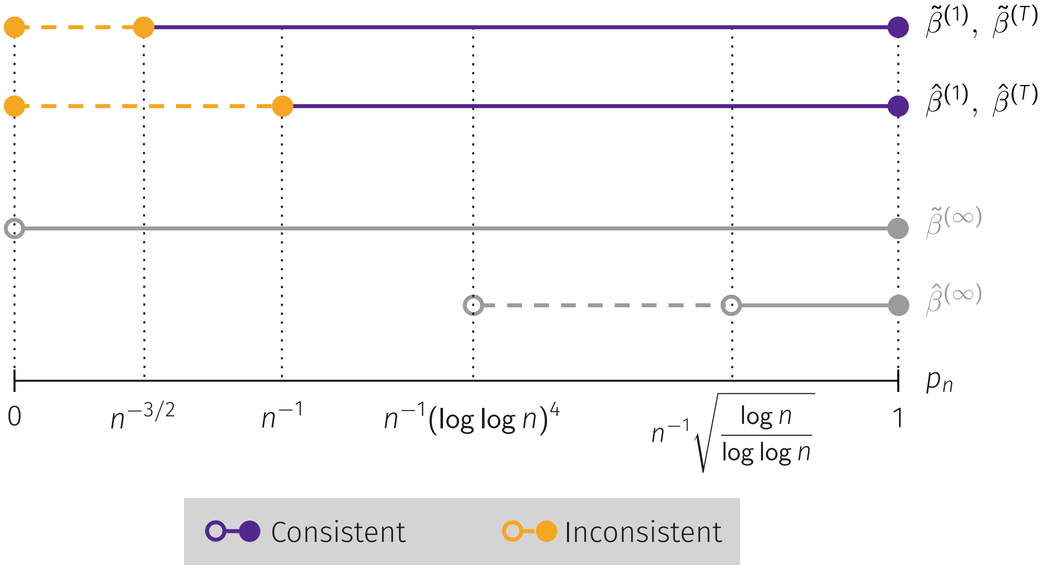
Consistency with Proxy Networks

- With proxy networks, p_n now matters
- Within the inconsistency thresholds, eigenvalues of $\hat{A}$ corresponding to informative eigenvectors are small (Alt et al. 2021a,b)
- Leading eigenvector not informative about eigenvectors of A
- Not clear if eigenvectors of $\hat{A}$ have structure below lower threshold

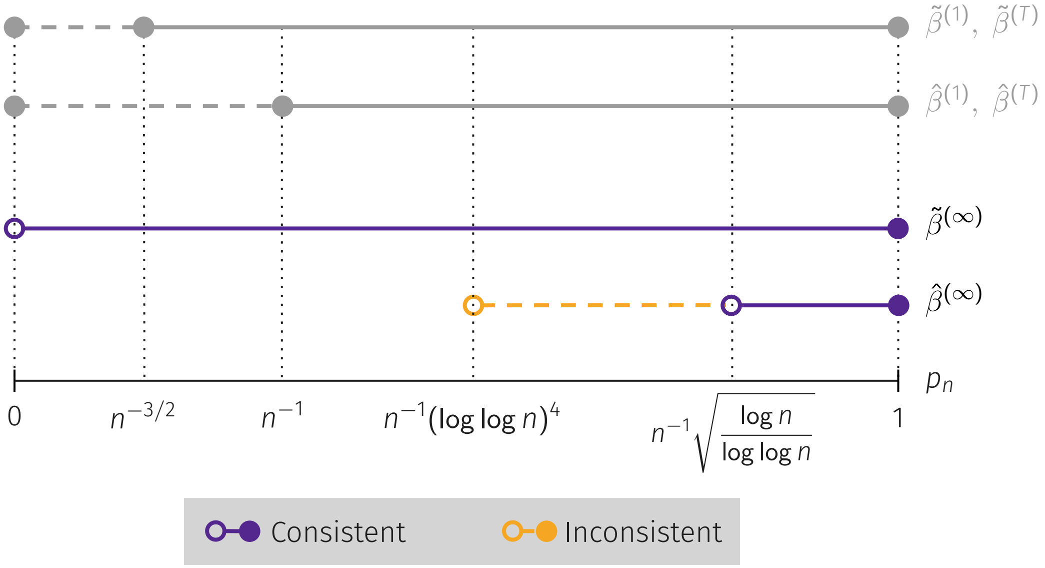
Consistency Thresholds



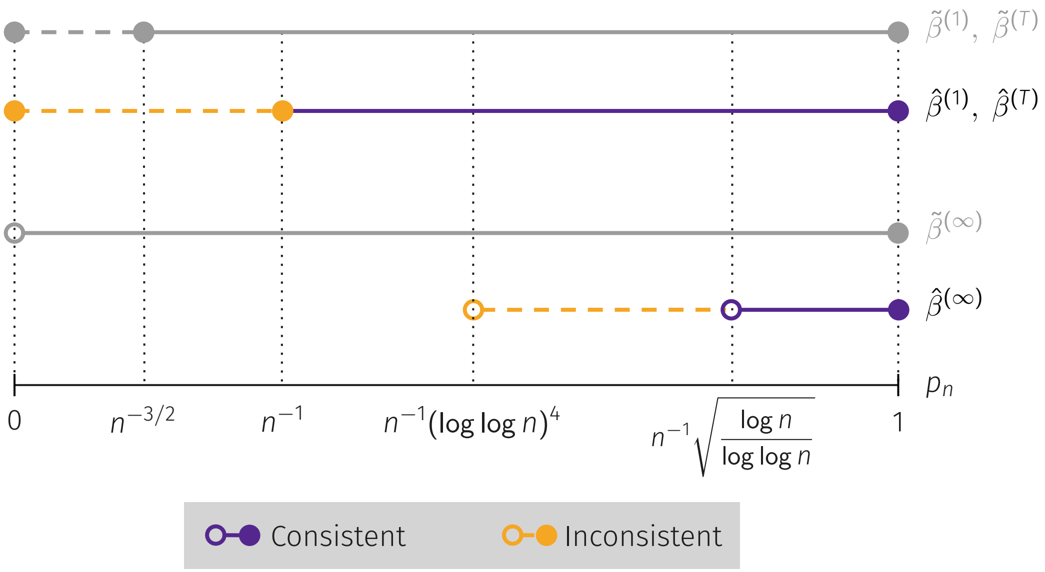
Consistency Thresholds - Degree and Diffusion



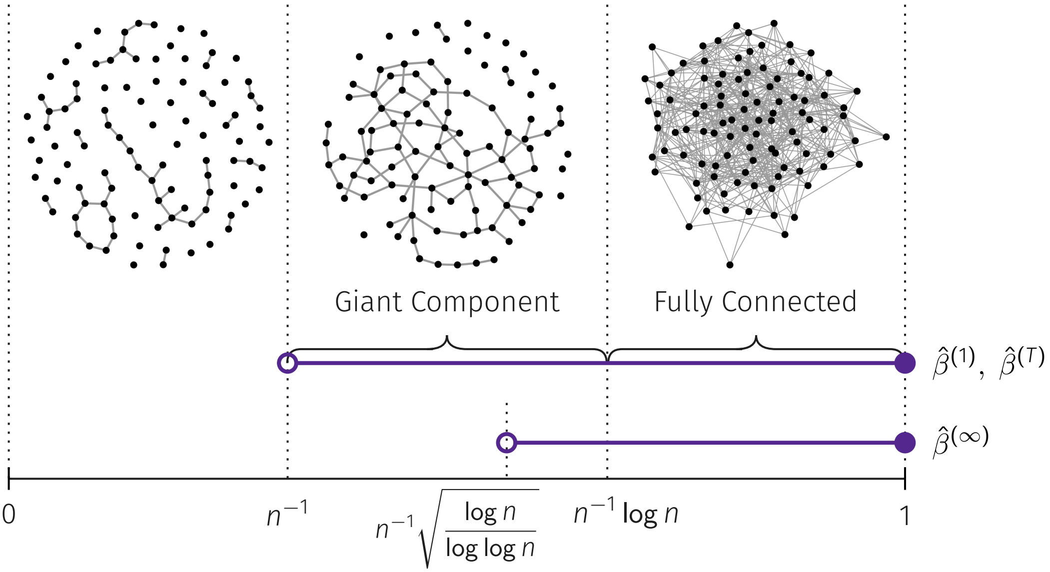
Consistency Thresholds - Eigenvector



Consistency Thresholds - Proxy Networks



Rule of Thumb for Consistency



▸ Formal Statement

Theoretical Results

Distributional Theory

Theorem 3

For $d \in \{1, T, \infty\}$, suppose $\tilde{\beta}^{(d)}$ is consistent. Then,

$$\frac{\tilde{\beta}^{(d)} - \beta^{(d)}}{\sqrt{V_0^{(d)}}} \xrightarrow{d} N(0,1) \ ,$$

where $V_0^{(d)} = E\left[\left(C_i^{(d)}\right)^2\right]^{-2} E\left[\left(C_i^{(d)}\varepsilon_i\right)^2\right].$

Distributional Theory with Proxy Network $\hat{A}$

Theorem 4

Suppose for $d \in \{1, T, \infty\}$ that $\hat{\beta}^{(d)}$ is consistent. Then if $\beta^{(d)} = 0$,

$$\frac{\hat{\beta}^{(d)}}{\sqrt{V_0^{(d)}}} \xrightarrow{d} N(0,1)$$

where $V_0^{(d)} = E\left[\left(C_i^{(d)}\right)^2\right]^{-2} E\left[\left(C_i^{(d)}\varepsilon_i\right)^2\right]$.

- Plug-in estimation works for $V_0^{(d)}$ in the cases above
- Robust/hc t-statistic appropriate for $\tilde{\beta}^{(d)}$ for any null
- Appropriate for $\hat{\beta}^{(d)}$ if null is $\beta^{(d)} = 0$.

Distributional Theory with Proxy Network $\hat{A}$

Theorem 4

For $d \in \{1, T\}$, suppose $\hat{\beta}^{(d)}$ is consistent. If $\beta^{(d)} \neq 0$,

$$\frac{\hat{\beta}^{(d)} - \beta^{(d)} \left(1 - B^{(d)}\right)}{\sqrt{V^{(d)}}} \xrightarrow{d} N(0, 1)$$

Distributional Theory with Proxy Network $\hat{A}$

$$\underbrace{\sqrt{V_0^{(d)}}}_{O_p\left(n^{-3/2}p_n^{-1}\right)} \ll \underbrace{\sqrt{V^{(d)}}}_{O_p\left(n^{-1}p_n^{-1/2}\right)} \overset{p_n \to 0}{\ll} \underbrace{B^{(d)}}_{O_p\left(n^{-1}p_n^{-1}\right)}$$

- Using proxy network slows down rate of convergence
- Bias can be much larger than variance if $p_n \to 0$
- Bias correction necessary for obtaining non-degenerate limit distribution
- hc/robust t-statistic not appropriate when $\beta^{(d)} \neq 0$
- t-statistic based confidence intervals are invalid

Distributional Theory with Proxy Network $\hat{A}$

- $\hat{B}^{(d)}$ and $\hat{V}^{(d)}$ in paper

- Bias-corrected estimator:

$$\check{\beta}^{(d)} = \frac{\hat{\beta}^{(d)}}{1 - \hat{B}^{(d)}}$$

- Adjusted tests and confidence intervals in paper

- No additional data requirement; p_n need not be specified

- Bias estimation is challenging since $B^{(d)}$ much larger than $\sqrt{V^{(d)}}$

▸ Bias Corr. Details

Distributional Theory with Proxy Network $\hat{A}$

Theorem 5

Suppose f has rank $R < \infty$. Suppose also for $\eta > 0$ that

$$p_n \gg n^{-1}\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{2}+\eta} \tag{1}$$

and that $a_n = \sqrt{\lambda_1(A)}$. Then,

$$\frac{\hat{\beta}^{(\infty)} - \beta^{(\infty)}}{\sqrt{V_0^{(\infty)}}} \xrightarrow{d} N(0,1) \ , \tag{2}$$

where

$$V_0^{(\infty)} = E\left[\left(C_i^{(\infty)}\right)^2\right]^{-2} E\left[\left(C_i^{(\infty)}\varepsilon_i\right)^2\right]$$

Rank Assump.

Eigenvector Centrality

- Chose a_n so that distribution is easy to characterize
- Usual hc/robust t-statistic valid for all null hypotheses
- Trade-off: choosing a model with slower, known rate of convergence for one with faster, unknown rate
- Using $\sqrt{\lambda_1(\hat{A})}$ does not change result

Inference under Sparsity

	Proxy Network		True Network
	$\beta^{(1)} / \beta^{(T)}$	$\beta^{(\infty)}$	$\beta^{(1)} / \beta^{(T)} / \beta^{(\infty)}$
$H_0 : \beta^{(d)} = 0$		Dense: t-test*	
$H_0 : \beta^{(d)} = b$			
Confidence Intervals			

Table 2: *: $p_n \gg n^{-1/2}$ (Le and Li, 2020). For $\beta^{(\infty)}$, $a_n = \sqrt{\lambda_1(A)}$.

	Proxy Network		True Network
	$\beta^{(1)} / \beta^{(T)}$	$\beta^{(\infty)}$	$\beta^{(1)} / \beta^{(T)} / \beta^{(\infty)}$
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Confidence Intervals			t-stat based

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Key: Theorem 3,

	Proxy Network		True Network
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$H_0 : \beta^{(d)} = 0$	t-test	Dense: t-test*	t-test
$H_0 : \beta^{(d)} = b$	New Method		
Confidence Intervals	New Method		t-stat based

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	$\beta^{(1)}$ / $\beta^{(T)}$	$\beta^{(\infty)}$	$\beta^{(1)}$ / $\beta^{(T)}$ / $\beta^{(\infty)}$
$H_0 : \beta^{(d)} = 0$	t-test	Dense: t-test* Sparse: t-test	t-test
$H_0 : \beta^{(d)} = b$	New Method	t-test	
Confidence Intervals	New Method	t-stat based	t-stat based

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Key: Theorem 3, Theorem 4, Theorem 5

Simulations

Simulations

- Suppose $f = 1$ so that

$$A_{ij} = \begin{cases} p_n & \text{if } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

- Our regression model is:

$$Y_i = \beta C_i^{(d)} + \varepsilon_i^{(d)}$$

$$\varepsilon_i^{(d)} \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, 1) \text{ and } \varepsilon_i^{(d)} \perp\!\!\!\perp \hat{A}_{jk} \text{ for all } i, j, k \in [n].$$

Inconsistency with Proxy Networks

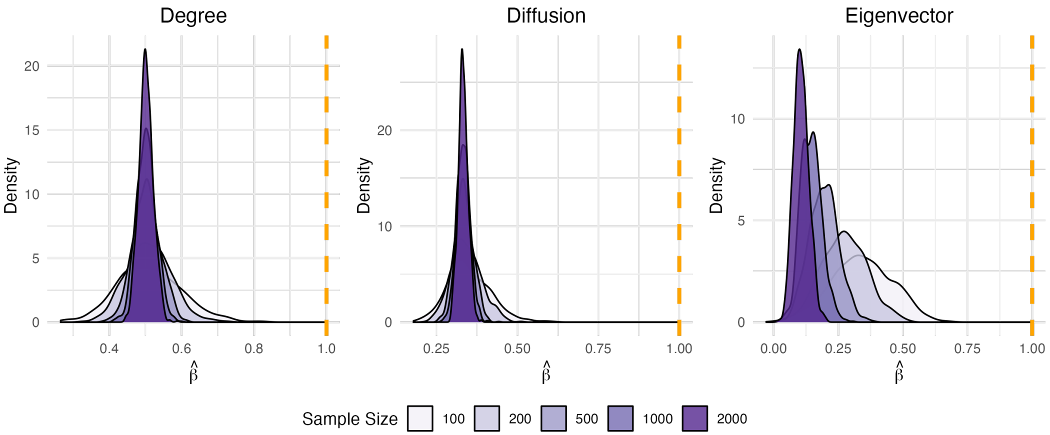


Figure 1: Distribution of $\hat{\beta}^{(d)}$ for $p_n = 1/n$. For $\hat{\beta}^{(\infty)}$, $a_n = \sqrt{n}$. $\beta = 1$ (orange dashed line).

Consistency with True Networks

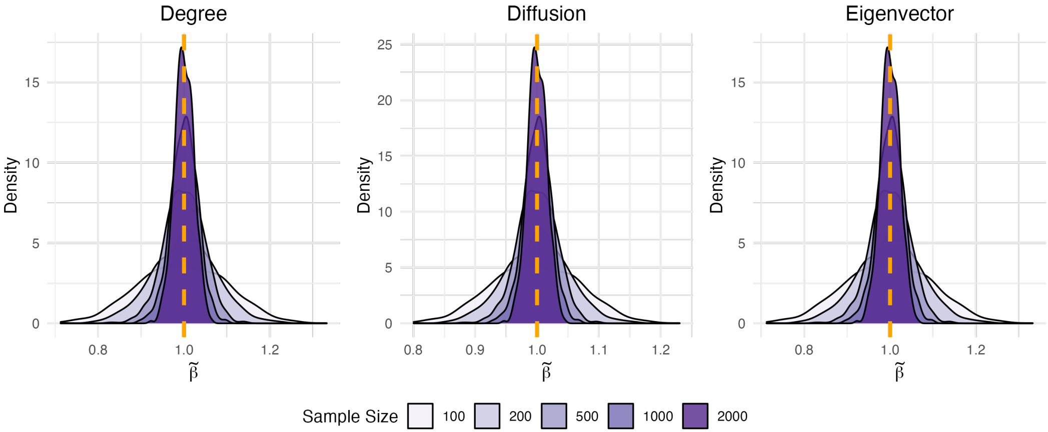


Figure 2: Distribution of $\tilde{\beta}^{(d)}$ for $p_n = 1/n$. For $\tilde{\beta}^{(\infty)}$, $a_n = \sqrt{n}$. $\beta = 1$ (orange dashed line).

Bias correction is effective

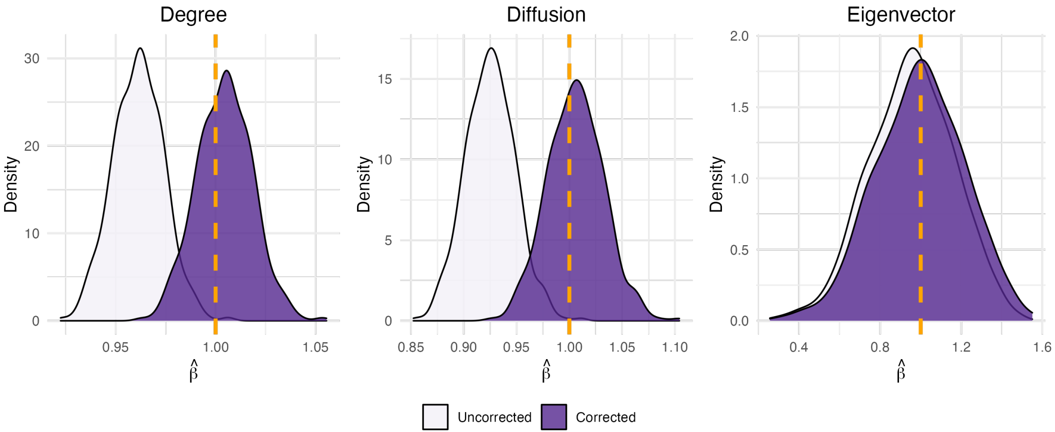


Figure 3: Distributions of $\hat{\beta}^{(d)}$ and their bias corrected versions $\check{\beta}^{(d)}$ for $p_n = 1/\sqrt{n}$, $n = 500$, $a_n = \sqrt{\lambda_1(\hat{A})}$. $\beta = 1$ (orange dashed line).

Distributional theory is accurate

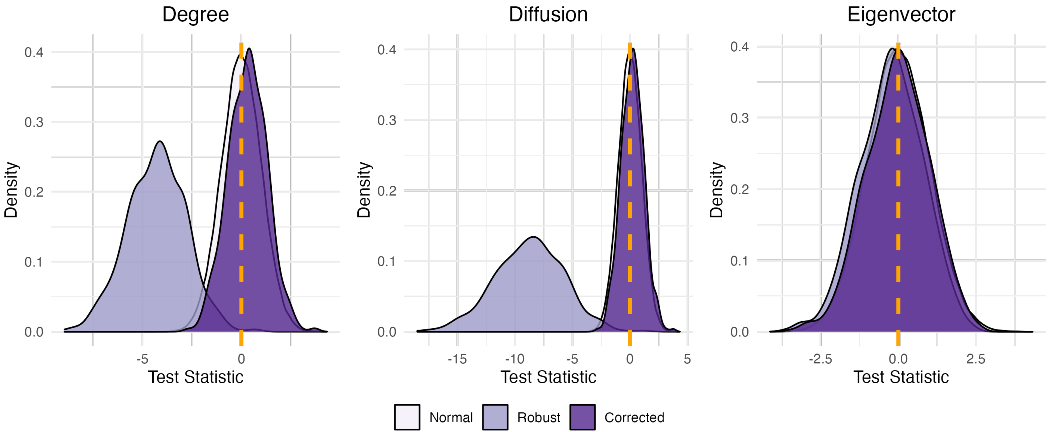


Figure 4: Distribution of the centered and scaled test statistics. Robust refers to tests based on t-statistic with robust (hc) standard errors. $p_n = 1/\sqrt{n}$, $n = 500$, $a_n = \sqrt{\lambda_1(\hat{A})}$.

Adjusted tests are better

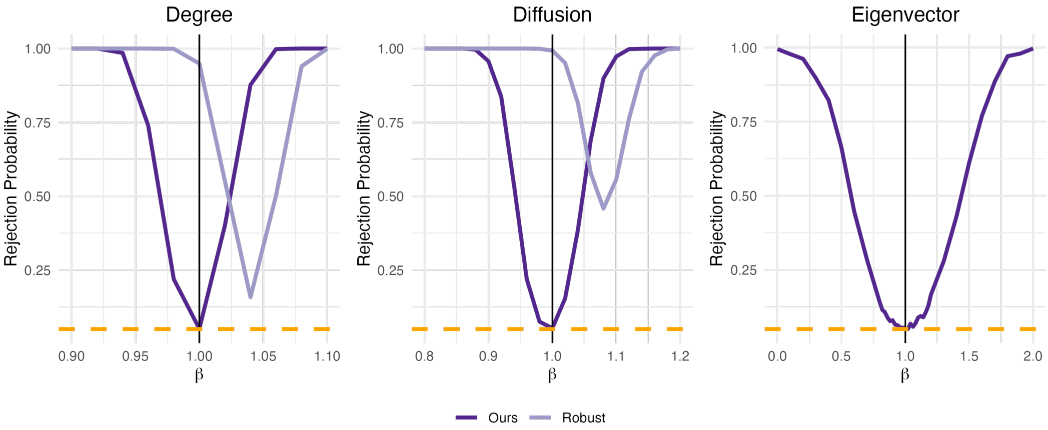


Figure 5: Power of the two-sided test of $H_0 : \beta = 1$ under various alternatives. Test at 5% level of significance (orange dashed line). $p_n = 1/\sqrt{n}$, $n = 500$, $a_n = \sqrt{\lambda_1(\hat{A})}$.

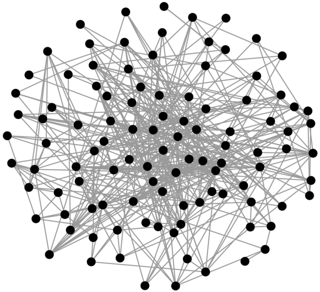
Empirical Demonstration

Empirical Demonstration

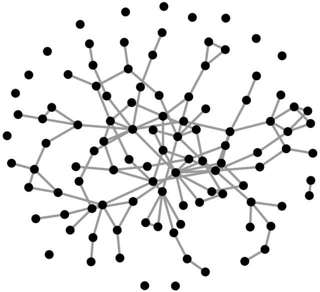
- De Weerdt and Dercon (2006): want to know if informal insurance can help consumption smoothing
- Village of 119 households in Nyakatoke, Tanzania
- Regress variance in food expenditure on centrality in network

Empirical Demonstration

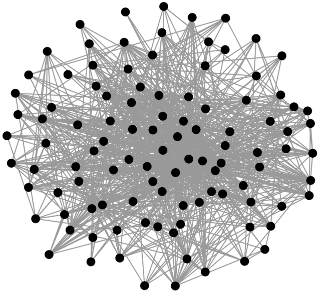
Unilateral Social



Bilateral Social



Unilateral Financial



$(n = 119)$	Mean	Median	Min	Max
Unilateral Social	8.02	7	1	31
Bilateral Social	2.30	2	0	10
Unilateral Financial	16.53	14	3	79

Table 3: Degree distributions of various networks in Nyakatoke

Empirical Demonstration

		Estimate	p-value	Atten.	Bias Corr.
Unilateral Social	Degree	-1064	0.67	0.91	-1172
	Diffusion	-4274	0.77	1.00	-4290
	Eigenvector	-12353	0.86	0.91	-13548
Bilateral Social	Degree	-11604	0.06	0.74	-15592
	Diffusion	-23672	0.16	0.95	-24883
	Eigenvector	-10543	0.93	0.78	-13434
Unilateral Financial	Degree	-412	0.70	0.96	-429
	Diffusion	-4559	0.74	1.00	-4561
	Eigenvector	-15040	0.77	0.96	-15699

Table 4: Regression results. For diffusion, $\delta = 1/\sqrt{\lambda_1(\hat{A})}$, $T = 2$. For eigenvector, $a_n = \sqrt{\lambda_1(\hat{A})}$.

▸ Conf. Int

Conclusion

Conclusion

1. Show that OLS can become inconsistent with sparse, proxy networks
    - Characterize threshold at which inconsistency occurs
    - Show that eigenvector is less robust than degree and diffusion
    - Comparing significance involves both economic *and* statistical properties
    - Rule-of-Thumb for sparsity regime

2. Distributional theory with sparse, proxy networks
    - Even when consistent, OLS estimators are asymptotically biased
    - Asymptotic bias can be large relative to variance
    - Slower rate of convergence than reflected by robust standard errors
    - Usual confidence intervals and tests may not be valid

3. Novel bias correction and inference methods

Thank you!

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Regularized Eigenvectors

Regularized Eigenvectors

- Suppose $d = np_n$ is known. Define:

$$w_i := \min\left\{\frac{2d}{C_i^{(1)}(\hat{A})}, 1\right\}$$

- w_i is the ratio by which the degree of i exceeds $2d$.

- Let the regularized matrix $\tilde{A}$ be defined as follows:

$$\tilde{A}_{ij} = \sqrt{w_i w_j} \cdot \hat{A}_{ij}$$

- $\tilde{A}$ is the adjacency matrix in which we down-weight the links of high-degree agents so that degree is windsorized at $2d$.

- Le et al. (2017) show that this regularized matrix concentrates to A in spectral norm even under sparsity.

Proposition 1 (Regularized Eigenvector)

Suppose $a_n \to \infty$. The linear regressions of Y on $C^{(\infty)}(\tilde{A})$ is consistent if and only if

$$p_n \gg n^{-1} .$$

Additional Simulations

Scaling of Eigenvector Matters

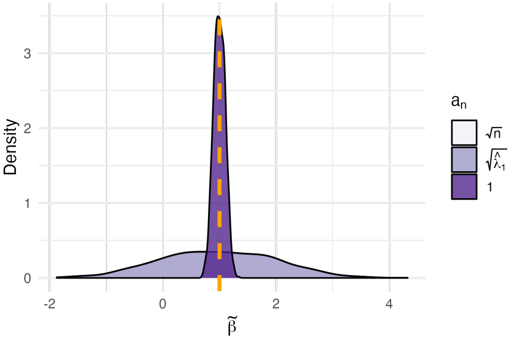

Figure 6: Distribution of $\tilde{\beta}^{(\infty)}$ for $n = 100$, $p_n = 1/n$ under various a_n. $\beta = 1$ (orange dashed line).

Eigenvector is more sensitive to sparsity than Degree and Diffusion

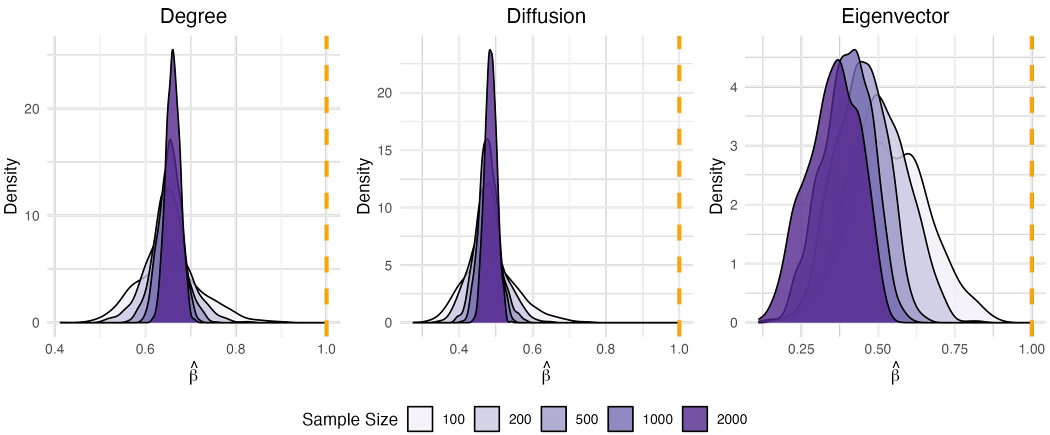

Figure 7: Distribution of $\hat{\beta}^{(d)}$ for $p_n = n^{-1}\sqrt{\log n / \log\log n}$. For $\tilde{\beta}^{(\infty)}$, $a_n = \sqrt{n}$. $\beta = 1$ (orange dashed line).

Size of $H_0 : \beta = 1$

p_n	Statistic		Sample Size				
			100	200	500	1000	2000
0.1	Degree	Ours	0.055	0.052	0.067	0.062	0.065
		Robust	0.656	0.673	0.690	0.668	0.674
	Diffusion	Ours	0.049	0.053	0.064	0.059	0.060
		Robust	0.889	0.894	0.887	0.871	0.898
	Eigenvector		0.045	0.043	0.037	0.056	0.044
$n^{-1/3}$	Degree	Ours	0.066	0.065	0.067	0.058	0.065
		Robust	0.330	0.450	0.573	0.705	0.783
	Diffusion	Ours	0.080	0.070	0.074	0.057	0.064
		Robust	0.645	0.734	0.813	0.888	0.934
	Eigenvector		0.045	0.042	0.051	0.042	0.058
$n^{-1/2}$	Degree	Ours	0.072	0.049	0.051	0.037	0.062
		Robust	0.659	0.801	0.949	0.993	0.999
	Diffusion	Ours	0.071	0.045	0.053	0.037	0.059
		Robust	0.881	0.948	0.993	1.000	1.000
	Eigenvector		0.077	0.045	0.050	0.050	0.047

Power of $H_0 : \beta = 0$ under the alternative $H_1 : \beta = 1$

p_n	Statistic	Sample Size				
		100	200	500	1000	2000
0.1	Degree - Robust	1.000	1.000	1.000	1.000	1.000
	Diffusion - Robust	1.000	1.000	1.000	1.000	1.000
	Eigenvector	0.845	0.995	1.000	1.000	1.000
$n^{-1/3}$	Degree - Robust	1.000	1.000	1.000	1.000	1.000
	Diffusion - Robust	1.000	1.000	1.000	1.000	1.000
	Eigenvector	0.998	1.000	1.000	1.000	1.000
$n^{-1/2}$	Degree - Robust	1.000	1.000	1.000	1.000	1.000
	Diffusion - Robust	1.000	1.000	1.000	1.000	1.000
	Eigenvector	0.832	0.947	0.994	1.000	1.000

Table 6: Power of 5% level two-sided tests of $H_0 : \beta = 0$ when $\beta = 1$. Under this H_0, the our test statistics is the usual t-statistic with robust (heteroskedasticity-consistent) standard errors.

Local Asymptotics

- Similar in spirit to modeling:
    - Correlation of weak instruments and endogenous variables decaying to 0 (e.g. Staiger and Stock 1997).
    - Power of tests using local alternatives (Pitman drift, see e.g. Rothenberg 1984).
    - Local to unity asymptotics for time series (e.g. Chan and Wei 1987)

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Weak Ties Theory

- Granovetter (1973): Weak ties which are more numerous are key drivers of outcomes
  - Weak ties: A_{ij} is small
  - Numerous: most A_{ij} non-zero ($O(n^2)$)

- Job referrals in Newton, MA:
  - Most recent job changers found jobs through friends "marginally included in the current network of contacts".
  - *"It is remarkable that people receive crucial information from individuals whose very existence they have forgotten."*
- Other examples: innovation (e.g. Reagans and Zuckerman 2001), economic development (e.g. Eagle et al. 2010), job referrals (e.g. Rajkumar et al. 2022).

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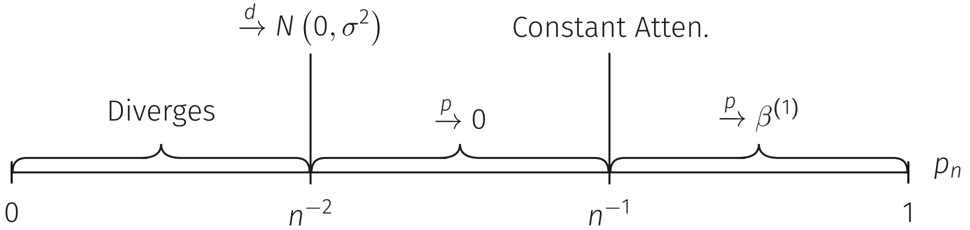
All Phase Transitions in $\hat{\beta}^{(1)}$



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Rule of Thumb for Consistency

Rule of Thumb

(a) Treat $\hat{\beta}^{(1)}$ and $\hat{\beta}^{(T)}$ as consistent if there exists a giant component with at least $N/2$ nodes.

(b) Treat $\hat{\beta}^{(\infty)}$ as consistent if the network is fully connected.

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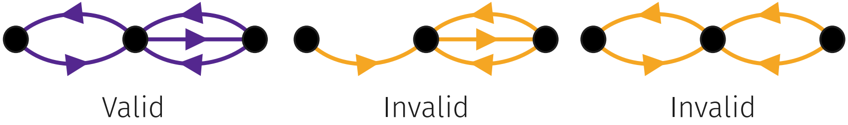
Bias Correction

- Bias correction reduces to the problem of $\iota' A^t \iota$ at a sufficiently fast rate.
- Plug-in $\iota' \hat{A}^t \iota$ is consistent, but does not converge fast enough
- Main Problem:

$$E\left[\hat{A}^t_{ij} \mid U\right] = A_{ij} \neq A^t_{ij}$$

- Difference between the two relates to the number of paths of a given length on a line in which each edge is traversed at least twice.



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Assumption 1 (Rank R Graphon)

Suppose f has rank $R < \infty$:

$$f(u,v) = \sum_{r=1}^{R} \tilde{\lambda}_r \phi_r(u)\phi_r(v) \quad , \tag{3}$$

where $\|\phi_r\| = 1$ for all $r \in [R]$ and if $r \neq s$,

$$\int_{[0,1]} \phi_r(u)\phi_s(u)du = 0 \ .$$

Furthermore, suppose that

$$\Delta_{\min} = \min_{1 \geq r \geq R-1} \left| \tilde{\lambda}_r - \tilde{\lambda}_{r+1} \right| > 0$$

Low Rank Assumption

- The rank assumption means the networks have "structure" (Chatterjee 2015).
- Many popular network models are low rank
  - Stochastic Block Model (Holland et al. 1983)
  - Random Dot Product Graphs (Young and Scheinerman 2007)
- Also common in the matrix completion literature (e.g. Candès and Tao 2010, Negahban and Wainwright 2012, Athey et al. 2021).

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Empirical Demonstration

		90%	95%	99%
Degree	Robust	(-19500, ∞)	(-21700, ∞)	(-25900, ∞)
	Ours	(-18800, ∞)	(-20000, ∞)	(-22700, ∞)
Diffusion	Robust	(-45000, ∞)	(-51000, ∞)	(-62400, ∞)
	Ours	(-25200, ∞)	(-25300, ∞)	(-25500, ∞)

Table 7: One-sided confidence intervals for degree and diffusion in Bilateral Social network.

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