Beginner Problem-Solving

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1 Induction

> Suppose $P(n)$ is a mathematical statement with n a positive integer. If $P(n_0)$ is true, and $P(n) \Rightarrow P(n+1)$ is true for all $n \geq n_0$, then $P(n)$ is true for all $n \geq n_0$.

1. Find and prove a formula for the sum of the first n odd numbers.
2. Let (F_n) be the Fibonacci sequence, so $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for each $n \geq 2$. Prove that $F_n \leq 2^n$ for all $n \geq 0$.
3. Define a sequence (b_n) by $b_1 = 1$, and for each $n \geq 1$, $b_{2n} = b_n$ and $b_{2n+1} = b_n + 1$. Prove that, for each positive integer n, the number of 1's in the binary representation of n is exactly b_n.
4. For each positive integer n, let $f(n) = 2 \cdot 7^n + 3 \cdot 5^n - 5$. Show that $f(n)$ is always a multiple of 24.
5. Let $f(n)$ be the number of regions which are formed by n lines in the plane, no two of which are parallel and no three meet at a single point. Hence $f(1) = 2$, $f(2) = 4$, and $f(3) = 7$. Find and prove a closed-form expression for $f(n)$.
6. Let $x_1, x_2, \ldots, x_n$ be n positive numbers such that $x_1 + x_2 + \cdots + x_n = 1/2$. Define $f(x) = (1-x)/(1+x)$ for all $x > 0$. Prove that $f(x_1)f(x_2)\cdots f(x_n) \geq 1/3$.
7. Suppose that you make a deal with the devil: whenever you want, you can trade a coin for any finite number of coins of lesser value. The catch is that you cannot earn coins other than by trading with the devil, you must pay the devil at least 1 cent per day, and when you run out of coins the devil gets your soul.

   The possible coin denominations are 1 (penny), 5 (nickel), 10 (dime), and 25 cents (quarter). (For example, you can trade one quarter for a million dimes.) Can you start with a finite number of coins of your choosing and never lose your soul?

2 Pigeonhole Principle

> Given $n+1$ balls distributed among n bins, at least one of the bins must contain at least 2 balls.
>
> More generally, given $nk+1$ balls distributed among n bins, at least one of the bins must contain at least $k+1$ balls.

1. Let S be a subset of $\{1,2,3,\ldots,2n\}$ with $n+1$ elements. Prove that S must contain a pair of consecutive integers.
2. With the same setup as the previous problem, prove that S must contain a pair of integers that sum to exactly $2n+1$.
3. Five points are placed inside a square of side length 1 unit. Prove that some two of them have distance at most $\sqrt{2}/2$ units apart.
4. Suppose n is an odd positive integer and $\{a_1, a_2, \ldots, a_n\} = \{1,2,\ldots,n\}$. Prove that $(1-a_1)(2-a_2)\cdots(n-a_n)$ is even.
5. The integers $1,2,\ldots,10$ are written around a circle in some order. Prove that there are 3 adjacent numbers whose sum is at least 17.
6. Let n be a positive integer. Prove that there exists a positive integer m whose decimal digits are just 1s and 0s and n divides m. (For example, with $n = 367$ we have $m = 1101 = 3\cdot 367$.)
7. Let n be an odd positive integer. Prove that there exists a positive integer m such that n divides $2^m - 1$.
8. Consider the set of squares of side length 1 whose vertices are lattice points in the plane. Given any n of these squares, prove that there is a subset of at least $n/4$ of them such that no two share any vertices.

3 Invariants and Monovariants

An invariant is a quantity that does not change after some operation, or a quantity that does not depend on some choice.

Similarly, a monovariant is a quantity that only increases (or only decreases) after some operation.

1. Is it possible to partition $\{1, 2, \ldots, 1234\}$ into two sets A and B such that the sum of elements in A equals the sum of elements in B? (A partition (A, B) of S satisfies $A \cup B = S$ and $A \cap B = \emptyset$.)

2. The numbers $1, 2, \ldots, 100$ are written on a chalkboard. You are allowed to erase two numbers, say a and b, and write the single number $c = a + b - 1$ on the board. You repeat this process until only one number remains. Find this number.

3. Same setup as the previous problem, except $c = ab + a + b$.

4. Start with the set $\{3, 4, 12\}$. You are allowed to replace any two numbers, say a and b, with the new numbers $0.8a + 0.6b$ and $0.6a - 0.8b$. Is it possible to eventually transform the set into $\{6, 7, 8\}$?

5. Seven squares of an 8×8 grid are shaded. At each step, we shade in each unshaded square that has at least two shaded neighboring squares (horizontally or vertically). Is it possible for this process to end with the entire 8×8 grid being shaded?

6. The integers $1, 2, \ldots, n$ are written down in order. A swap consists of interchanging the positions of some two distinct integers. Is it possible that, after exactly 2025 swaps, the numbers are back in order?

7. Starting with an ordered tuple of 4 integers, repeatedly perform the operation
$$(a, b, c, d) \to (|a - b|, |b - c|, |c - d|, |d - a|).$$
We say the process terminates if we eventually get the tuple $(0, 0, 0, 0)$. Must the process terminate in a finite number of operations for any choice of the starting tuple?

4 Fundamental Theorem of Algebra

Let $p(x)$ be a polynomial of degree $n \geq 1$ with real coefficients. Then there is a unique collection of (not necessarily distinct) complex numbers $r_1, r_2, \ldots, r_n$ and a real number a such that

$$p(x) = a(x - r_1)(x - r_2)\cdots(x - r_n).$$

1. Let n be a positive integer. Determine all polynomials $p(x)$ of degree at most n such that $p(x) = 0$ has $n+1$ distinct solutions.
2. Find all polynomials that have an infinite number of roots.
3. Let $P(x)$ and $Q(x)$ be real polynomials of degree n and m, respectively. Determine the maximum number of intersection points of the graphs of $y = P(x)$ and $y = Q(x)$.
4. Suppose $P(x)$ is a real polynomial of degree 4 such that $P(0) = P(1) = P(2) = P(3) = 1$ and $P(4) = 0$. Find $P(5)$.
5. Suppose $P(x)$ is a real polynomial of degree 2024 such that $P(n) = 1/n$ for $n = 1, 2, \ldots, 2025$. Find $P(2026)$.
6. Determine all polynomials $p(x)$ such that $xp(x-1) = (x+1)p(x)$ for all real x.
7. Determine all polynomials $p(x)$ such that $p(0) = 0$ and $p(x^2+1) = p(x)^2 + 1$ for all real x.
8. There exists a solution (a, b, c, d) to the system

$$\begin{aligned} a + 8b + 27c + 64d &= 1, \\ 8a + 27b + 64c + 125d &= 27, \\ 27a + 64b + 125c + 216d &= 125, \\ 64a + 125b + 216c + 343d &= 343. \end{aligned}$$

   Determine the value of $64a + 27b + 8c + d$.

5 Vieta's Formulas

Let $p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ be a monic polynomial of degree n with roots $r_1, r_2, \ldots, r_n$. Then

$$r_1 + r_2 + \cdots + r_n = -a_{n-1}$$

$$r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n = a_{n-2}$$

$$\vdots$$

$$r_1 r_2 \cdots r_n = (-1)^n a_0.$$

1. Verify Vieta's Formulas for the polynomials $p(x) = x^2 - 5x + 6$, $q(x) = x^3 - 6x^2 + 11x - 6$, and $r(x) = x^4 - 4x^3 + 3x^2 + 4x - 4$.
2. It is given that two of the roots of $p(x) = x^4 - 3x^3 - 4x^2 - 5x - 1$ sum to 4. Find the sum of the other two roots.
3. Let r_1, r_2, r_3 be the roots of the polynomial $x^3 - 4x^2 + 5x - 6$. Compute $1/r_1 + 1/r_2 + 1/r_3$ and $1/(1+r_1) + 1/(1+r_2) + 1/(1+r_3)$.
4. Determine the sum and the product of the roots of the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ $(a_n \neq 0)$.
5. Find the sum of all the roots of the equation $x^{2025} + (\frac{1}{2} - x)^{2025} = 0$.
6. Three distinct points $(1,3)$, $(2,11)$, and (a,b) all lie on a single line and all lie on the curve $y = x^3 + x + 1$. Find (a,b).
7. Three distinct points A, B, and C all lie on a single line and all lie on the curve $y = x^3 + x + 1$. Given that A and B have the same distance to the y-axis, determine the coordinates of point C.
8. The curve $y = x^3 - 20x + 24$ intersects a circle centered at $(7,24)$ at six distinct points $(x_1, y_1), \ldots, (x_6, y_6)$. Find $x_1 + \cdots + x_6$.
9. It's given that some two of the roots of the polynomial $P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$ add to 4. Find all of the roots.
10. Let a, b, c be real numbers. Prove that a, b, and c are all nonnegative if and only if $a + b + c$, $ab + bc + ca$, and abc are all nonnegative.

6 Fundamental Theorem of Arithmetic

> Every positive integer greater than 1 has a unique factorization into prime numbers (up to rearrangement). A prime number is a positive integer greater than 1 that is divisible by only 1 and itself.

1. Find the smallest positive integer n such that $n/2$ is a perfect square and $n/3$ is a perfect cube.
2. Let a and b be positive integers. Suppose there exists some prime p such that $p \mid a$ but $p \nmid b$. Prove that $a \nmid b$.
3. Suppose a and b are positive integers with prime factorizations $a = p_1^{e_1} \cdots p_k^{e_k}$ and $b = p_1^{f_1} \cdots p_k^{f_k}$, where e_i, f_i are nonnegative integers. Prove that $a \mid b$ if and only if $e_i \le f_i$ for all $i = 1, 2, \ldots, k$.
4. Let S be a set of 9 positive integers which are not divisible by any prime greater than 5. Prove that some two distinct integers in S have a product that is a perfect square.
5. Let p be a prime number and n a positive integer. Find the largest integer k such that p^k divides $n!$. Your answer may involve the floor function ($\lfloor x \rfloor$ is the largest integer not greater than x).
6. Find all positive integers n with an odd number of positive integer divisors.
7. Let n be a positive integer with prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where $e_1, \ldots, e_k$ are nonnegative integers. Find the number of positive divisors of n in terms of $e_1, \ldots, e_k$.
8. Let p and q be consecutive primes greater than 3 (i.e. there are no primes between p and q). Prove that $p + q$ has at least 5 positive divisors.

7 Greatest Common Divisor

> Let a and b be positive integers. Then $\gcd(a,b)$ is the largest integer that divides both a and b. If $\gcd(a,b)=1$, we say a and b are *coprime* (or *relatively prime*).

1. Let $a > b$ be positive integers. Prove that $\gcd(a,b) = \gcd(a-b,b)$.
2. The Fibonacci sequence (F_n) satisfies $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that $\gcd(F_n, F_{n+1}) = 1$ for all $n \geq 1$.
3. Let n be a positive integer. Prove that $21n+4$ and $14n+3$ are coprime.
4. Let a,b,c be positive integers such that a divides bc. If $\gcd(a,b)=1$, prove that a divides c.
5. Let p be a prime number. Prove that $\sqrt{p}$ is irrational (i.e. there exist no positive integers n and m such that $\sqrt{p} = n/m$).
6. Let p be a prime number and r a positive rational number that is not an integer. Prove that p^r is irrational.
7. Let a and b be coprime positive integers. If ab is a perfect square, prove that a and b are each perfect squares.
8. Let a,b,n be positive integers satisfying $n^2 < a < b < (n+1)^2$. Prove that ab cannot be a perfect square.
9. The least common multiple of two positive integers a and b is the smallest positive integer that is a multiple of a and a multiple of b, and is denoted $\operatorname{lcm}(a,b)$. Prove that $\operatorname{lcm}(a,b) = \frac{ab}{\gcd(a,b)}$.
10. For any positive integers a and b, prove that $\gcd(a,b) + \operatorname{lcm}(a,b) \geq a+b$. When does equality occur?
11. Let a and b be coprime positive integers. Define $c = a+b$, $d = a^2 - ab + b^2$. Prove that $\gcd(c,d) = 1$ or 3.
12. Let a,b,n be positive integers with $n > 1$. Prove that $\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1$.

8 Modular Arithmetic

Let n be a positive integer. We say the integer a is congruent to the integer b modulo n, written $a \equiv b \pmod{n}$, if a and b have the same remainder when divided by n.

We typically use modular arithmetic to determine if some integer k is a multiple of n, or $k \equiv 0 \pmod{n}$.

Modular arithmetic allows us to take a lot of shortcuts when determining divisibility. For example, if $a \equiv x \pmod{n}$ and $b \equiv y \pmod{n}$, then $a + b \equiv x + y \pmod{n}$, $ab \equiv xy \pmod{n}$, and $a^b \equiv x^b \pmod{n}$. Note that $a^b \not\equiv x^y \pmod{n}$, as we will see with Fermat's Little Theorem.

1. Calculate the remainder when $1111 \cdot 2222 \cdot 3333 \cdot 4444$ is divided by 5.
2. Determine which of the following numbers are multiples of 11: **(i)** $21^5 - 12^5$, **(ii)** $21^{50} - 12^{50}$, **(iii)** $21^{21^{50}} - 12^{12^{50}}$.
3. Find the units digit of 9^{8^7} and 7^{8^9}.
4. Let P be the product of all positive integers less than 100 that are not multiples of 5. Find the units digit of P.
5. Let $S(n)$ denote the sum of the digits of the positive integer n. Prove that $n \equiv S(n) \pmod{9}$.
6. Prove that $1 \cdot 3 \cdot 5 \cdots 2025 + 2 \cdot 4 \cdot 6 \cdots 2026$ is divisible by 2027.
7. Find all pairs of integers (x, y) such that $x^2 = 4y + 3$.
8. Find the smallest positive integer n such that 77 divides $n^2 - 70n + 89$.
9. Find the largest integer k such that there exist increasing arithmetic sequences of integers (a_n) and (b_n) satisfying $a_0 = b_0 = 1$ and $a_k b_k = 1000$.

9 Fermat's Little Theorem

Let p be a prime and a an integer not divisible by p. Then

$$a^{p-1} \equiv 1 \pmod{p}.$$

This theorem allows us to reduce the exponent in modular expressions.

1. Find the remainder when 2^{121} is divided by 11.
2. Find all primes p such that p divides $29^p + 1$.
3. Let p be prime. Prove that p divides $2^{2p+1} - 2^{p+2}$.
4. Prove that there are infinitely many integers $n \geq 2$ such that $n \mid (2^n - 4)$.
5. Define $a_1 = 4$ and $a_n = 4^{a_{n-1}}$ for $n \geq 2$. Find the remainder when a_{2025} is divided by 7.
6. Let a and b be integers and p be an odd prime. Suppose $p \mid (a^p + b^p)$. Prove that $p^2 \mid (a^p + b^p)$.
7. Let n be a nonnegative integer. Prove that $2^{2^{6n+2}} + 3$ is a multiple of 19.
8. Let a and b be integers. Define $c = ab(a^{60} - b^{60})$. Prove that the first six prime numbers divide c, and find two other prime factors of c.
9. Prove that the sequence $2^n - 3$, $n \geq 2$, has an infinite subsequence whose terms are pairwise coprime.

10 Integer Polynomials

We call a polynomial an *integer polynomial* if all of its coefficients are integers. These kinds of polynomials inherit all the nice properties of real polynomials, plus some additional properties thanks to number theory.

Factoring: If $f(x)$ is an integer polynomial and $f(k) = 0$ for some integer k, then there exists a unique integer polynomial $g(x)$ such that

$$f(x) = (x - k)g(x).$$

Rational Root Theorem: Let $f(x) = a_n x^n + \cdots + a_0$ be an integer polynomial with $a_n \neq 0$. If f has a rational root p/q where p and q are coprime, then $p \mid a_0$ and $q \mid a_n$.

Divisibility Property: Let f be an integer polynomial and a, b be integers. Then $(a - b) \mid (f(a) - f(b))$.

1. Find all rational roots of $3x^3 - 5x^2 + 5x - 2$.
2. Does there exist an integer polynomial p with $p(1) = 1$ and $p(10) = 11$?
3. Let $p(x)$ be a monic integer polynomial (an integer polynomial with leading coefficient 1). Suppose $p(r) = 0$ for some rational number r. Prove that r is an integer.
4. Find all rational numbers x such that $3x^3 + x^2 + \frac{2}{x}$ is an integer.
5. Let $p(x)$ be an integer polynomial. Set $n = p(1) - 2p(50) + p(99)$. Find an odd prime that is a factor of n for all choices of p.
6. Let $p(x)$ be an integer polynomial. Suppose $p(0)$ and $p(1)$ are both odd. Prove that p has no integer roots.
7. Let $p(x)$ be an integer polynomial. Suppose there exists a positive integer n such that none of $p(0), p(1), \ldots, p(n-1)$ are divisible by n. Prove that p has no integer roots.
8. Let $p(x)$ be an integer polynomial. Suppose there exist three distinct integers a, b, c such that $p(a) = p(b) = p(c) = 1$. Prove that $p(x)$ has no integer roots.

9. Let $p(x)$ be a monic integer polynomial of degree 4. Suppose p has four distinct integer roots a, b, c, d, and k is an integer such that $p(k) = 4$. Prove that $k = (a + b + c + d)/4$.

10. Let $p(x)$ be a monic integer polynomial. Suppose there exist distinct integers a, b, c, d such that $p(a) = p(b) = p(c) = p(d) = 5$. Prove that there is no integer k such that $p(k) = 8$.

11. Find the largest integer k such that there exist $k+1$ distinct integers $n_0, n_1, \ldots, n_k$ and a monic integer polynomial $p(x)$ such that $p(n_0) = 4050$ and $p(n_1) = \cdots = p(n_k) = 2025$. Find a p for which this k is achieved.

12. Find all integer polynomials $p(x)$ so that $p(n)$ is prime for all integers n.

13. Find all integer polynomials $p(x)$ so that $p(n)$ divides $p(n+1)$ for all positive integers n.

14. Find all integer polynomials $p(x)$ so that $p(n+1) + p(n-1) = 2p(n)$ for all integers n.

15. Let $\{x\}$ denote the fractional part of a real number x. For example, $\{7\} = 0$, $\{3.14\} = 0.14$, and $\{-9.8\} = 0.2$. Find all real numbers r that satisfy $\{r\} = \{r^2\} = \{r^3\}$.

11 Intermediate Value Theorem

> Let f be a continuous function on the interval $[a, b]$. Then for any number L between $f(a)$ and $f(b)$, there exists a number $c \in [a, b]$ such that $f(c) = L$. In other words, a continuous function attains every value between any two of its points.

1. Prove that the equation $\tan x = 1 - x$ has a real solution.
2. Let $p(x)$ be a real polynomial of odd degree. Prove that p has at least one real root.
3. Let $p(x)$ be a real cubic polynomial. Suppose p has at least two distinct real roots. Prove that all roots of p are real.
4. Suppose f is continuous and satisfies $f(0) > 0$ and $f(2) < 4$. Prove that the equation $f(x) = x^2$ has a real solution.
5. Given n points $x_1, \ldots, x_n \in [0, 1]$, show there exists a point $y \in [0, 1]$ such that the average of the n distances from y to $x_1, \ldots, x_n$ is exactly $1/2$.
6. Let f be continuous on $\mathbb{R}$ with $f(x)f(f(x)) = 1$ for all $x \in \mathbb{R}$. Given that $f(1000) = 999$, find $f(500)$.
7. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$. Prove that the equation $f(x) = f(x + 1/2025)$ has a solution on the interval $[0, 2024/2025]$.
8. Let ℓ be a line in the plane and P a polygon in the plane. Prove that P can be divided into two polygons of equal area by a line parallel to ℓ.
9. Prove that any convex polygon can be divided by a line into two polygons of equal area and of equal perimeter.
10. Let $p(x)$ be a polynomial with odd degree greater than 1. Let Q be a point in the plane. Prove that there exists a line through Q that is tangent to the graph $y = p(x)$ at some point.

12 Taylor's Theorem

Let n be a positive integer and a a real number. If a function f is defined on some open interval I around a and $f^{(n+1)}$ exists on I, then for all x in I, $x \neq a$, there exists a number c between x and a such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Note that the value of c depends on the value of x. If f is infinitely differentiable at a and the remainder term converges to 0, then we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

1. Find the Taylor Series of $\sin x$ and $\cos x$ around the point $a = 0$.
2. Find the Taylor Series of e^x around the point $a = 0$.
3. Find the Taylor Series of $\ln(1-x)$ around the point $a = 0$.
4. Find the first three terms of the Taylor Series expansion of $\sqrt{1-x}$ around the point $a = 0$.
5. Using Taylor's Theorem with the first two terms of $\sin x$, prove that
$$x - \frac{x^3}{6} \leq \sin x \leq \frac{24x - 4x^3}{24 - x^4}$$
for all $x \in [0, \pi/2]$.
6. Prove that
$$\frac{2}{3} \leq \int_0^1 e^{-x^2}\,dx \leq \frac{23}{30}.$$
7. Find a quadratic polynomial $p(x)$ such that for all $|x| \leq 1$,
$$\left| p(x) - \frac{1}{x-4} \right| < 0.01.$$
8. Find the 5^{th} digit after the decimal point of $\sqrt{11,111,111}$.

13 Mean Value Theorem

Let f be a continuous function on the interval $[a,b]$ and differentiable on (a,b). Then there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or equivalently

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

1. Let f be a continuous function such that $\int_a^b f(x)\,dx = 0$ for some $a < b$. Prove that f has at least one real root.
2. Let f be a continuously differentiable function such that f' has no real roots. Prove that f has at most one real root.
3. Suppose f is a differentiable function with n distinct real roots. Prove that f' has at least $n-1$ real roots.
4. Let f be a differentiable function. Given that $|f'(x)| \le 1$ for all x, prove that $|f(x)-f(y)| \le |x-y|$ for all x, y.
5. Let f be a function on $[a,b]$ such that f'' is continuous on the interval (a,b). Let ℓ denote the line segment with endpoints $(a, f(a))$ and $(b, f(b))$. Given that ℓ intersects the graph of f on the interval (a,b), prove that f'' has a real root.
6. Let $a_0, a_1, \ldots, a_n$ be real numbers satisfying
$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0.$$
Prove there exists a real number x such that $a_0 + a_1 x + \cdots + a_n x^n = 0$.
7. Let f be differentiable on an interval $[a,b]$. Given $f(a) = b$ and $f(b) = a$, prove that there exist $\alpha, \beta \in (a,b)$ such that $\alpha \neq \beta$ and $f'(\alpha)f'(\beta) = 1$.

14 Riemann Sums

Let f be a continuous, increasing function on an interval I. Then the left Riemann sum is an underestimate and the right Riemann sum is an overestimate of the integral of f over I.

An important application of this is estimating sums with integrals. Suppose f is a continuous, increasing function. Let $S_n = f(2)+f(3)+\cdots+f(n)$. Then S_n is a left Riemann sum of the integral over $[2, n+1]$ and a right Riemann sum of the integral over $[1, n]$. Thus,

$$\int_1^n f(x)\,dx \le S_n \le \int_2^{n+1} f(x)\,dx.$$

This can be used to find the asymptotic behavior of S_n.

Note: For decreasing functions, left Riemann sums are overestimates and right Riemann sums are underestimates, so the final inequality signs will be flipped.

1. Let H_n denote the n^{th} harmonic number:
$$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$
Prove that $\lim_{n\to\infty} H_n/\ln(n) = 1$.
2. For each positive integer n, let
$$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}.$$
Evaluate $\lim_{n\to\infty} S_n/\sqrt{n}$.
3. Prove one form of Stirling's approximation:
$$\lim_{n\to\infty} \frac{\ln(n!)}{n\ln(n/e)} = 1.$$
4. For each positive integer n, let
$$S_n = \ln\left(\sqrt[n^2]{1^1\cdot 2^2\cdots n^n}\right) - \ln(\sqrt{n}).$$
Find $\lim_{n\to\infty} S_n$.

Basic Combinatorial Results

1. The number of ways to order n balls of different colors is $n!$.
2. The number of ways to order $n+m$ balls, where n are red and m are blue, is $\frac{(n+m)!}{n!m!}$.
3. The number of ways to order $n+m+k$ balls, where n are red, m are blue, and k are green, is $\frac{(n+m+k)!}{n!m!k!}$.
4. The number of ways to choose a k-element subset from an n-element set is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
5. The number of ways to distribute n identical balls into k distinct boxes is $\binom{n+k-1}{k-1}$.

15 Inclusion-Exclusion Principle

Let A and B be finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

where $|S|$ denotes the number of elements in a set S.

1. How many positive integers $n \leq 1000$ are divisible by 2 or 5?
2. Find the number of subsets of $\{1, 2, \dots, 300\}$ that are subsets of neither $\{1, 2, \dots, 200\}$ nor $\{101, 102, \dots, 300\}$.
3. Find a formula for the number of elements in the set $A \cup B \cup C$, where A, B, C are finite sets.
4. Find the number of permutations of $(1, 2, \dots, n)$ such that the i^{th} number in the permutation is not i for all $i = 1, 2, \dots, n$.

16 Bijections

A bijection is a function $f : A \to B$ that is injective (one-to-one) and surjective (onto). In other words, f maps each element of A to a unique element of B (injective) and each element of B has an element of A mapped to it (surjective).

An important consequence of having a bijection $f : A \to B$ is that $|A| = |B|$. Thus, if we wish to count the elements of A, we can instead find a bijection from A to B and then count the elements of B.

Another important usage of bijections is pairing up elements. Suppose we want to calculate $\sum_{s\in S} g(s)$ for some set S. We may be able to partition S into two sets A and B and "pair up" the elements of A and B by finding a bijection $f : A \to B$. If f is chosen well, we may be able to easily calculate $\sum_{s\in S} g(s) = \sum_{a\in A}[g(a) + g(f(a))]$.

1. A path to a point P in the plane is a sequence of moves one unit up or one unit to the right starting from the origin and ending at P. Find the number of paths to $(10, 10)$.
2. Find the number of paths to $(10, 10)$ that do not pass through the point $(5, 5)$.
3. Find the number of solutions (x, y, z) of nonnegative integers to the equation $x + y + z = 100$.
4. Find the number of solutions (x, y, z) of nonnegative integers to the equation $x + y + z \leq 100$.
5. Let S be the set of all positive integers whose digits (in base 10) form a strictly monotonic sequence of nonzero digits. For example, 1, 1456 and 921 are in S, but 4556, 8123, and 3210 are not in S. Determine the sum of all elements in S.
6. Let S be the set of all 3×3 matrices whose elements are all positive integers less than 10. Find the sum of the determinants of the matrices in S.

17 Recurrence Relations

Given a recurrence relation $a_{n+2} + \alpha a_{n+1} + \beta a_n = 0$, its characteristic polynomial is $p(x) = x^2 + \alpha x + \beta$. If r_1, r_2 are the distinct roots of p, then the closed-form expression for a_n is

$$a_n = c_1 r_1^n + c_2 r_2^n$$

for some c_1, c_2. We can solve for c_1, c_2 if we're given initial conditions of the recurrence relation.

Relations not of the form above can be solved by finding a pattern and proving it with induction.

1. The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Find a closed-form expression for F_n.
2. The Pell numbers are defined by $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for all $n \geq 2$. Find positive real numbers r and L such that $\lim_{n\to\infty} P_n/r^n = L$.
3. Define the sequence (a_n) by $a_0 = 0$ and $a_n = 2a_{n-1} + 1$ for all $n \geq 1$. Find a closed-form expression for a_n.
4. Define the sequence (a_n) by $a_1 = 1/2$ and $a_n = na_{n-1}/(2n-2)$ for all $n \geq 2$. Find a closed-form expression for a_n.
5. Define the sequence of integers (a_n) by $a_0 = 0$, $a_1 = 1$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for all $n \geq 2$. Prove that if $n \mid m$, then $a_n \mid a_m$.
6. Let $a_0, a_1, \ldots$ be a sequence of positive integers such that $2a_n = 5a_{n-1} - 2a_{n-2}$ for all $n \geq 2$. Find the smallest value of a_{2025} over all possible sequences.
7. Find a closed-form expression for a_n given that $a_0 = a_1 = 2$ and $a_n = a_{n-1} + 2a_{n-2} + 5 - 2n$ for all $n \geq 2$.

18 Recursive Counting

When we are asked to count the number of ways a_n to do something with n objects, we can often do so in terms of $a_{n-1}, a_{n-2}, \ldots$. This gives us a recurrence relation, which we can manipulate however we need.

Typically, when counting a_n, we split into cases based on the n^{th} object. For example, suppose we wanted to count the number of subsets of $\{1, 2, \ldots, n\}$, denoted s_n. We can partition all possible subsets into those containing n and those not containing n. In both cases, there are s_{n-1} such subsets (why?), so we have the relation $s_n = 2s_{n-1}$. With $s_1 = 2$, we can solve $s_n = 2^n$.

1. Find the number of ways to tile a 1×2025 grid with 1×1 and 1×2 tiles.
2. There are 2025 balls ordered in a line. Find the number of ways to color the balls, each either red, green, or blue, such that no two adjacent balls are the same color.
3. Let $(2+\sqrt{3})^{2n-1} = a_n + b_n\sqrt{3}$ for integers a_n and b_n $(n \geq 1)$. Express a_n in terms of n.
4. Prove the identity $F_{2n} = F_n^2 + F_{n-1}^2$, where F_n is the n^{th} Fibonacci number. (Hint: Tile a $1 \times 2n$ grid with 1×1 and 1×2 tiles, and consider whether the n and $n+1$ grid squares are connected or not.)
5. Find the number of sequences of length 2025 whose elements are from $\{0, 1, 2, 3, 4\}$ and every pair of adjacent elements differ by exactly one.
6. There are 2025 balls ordered in a circle. Find the number of ways to color the balls, each either red, green, or blue, such that no two adjacent balls are the same color.
7. For each integer $n \geq 3$, let $F(n)$ denote the number of ways to write $2n$ as the sum of three distinct positive integers, without regard to order. For example, since

$$10 = 7+2+1 = 6+3+1 = 5+4+1 = 5+3+2,$$

we have $F(5) = 4$. Prove that $F(n)$ is odd if and only if $3 \mid n$.

19 Standard Inequalities

1. **(Square)** For $x \in \mathbb{R}$, $x^2 \geq 0$, with equality if and only if $x = 0$.

2. **(AM-GM)** For nonnegative real numbers $a_1, \dots, a_n$,
$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n},$$
with equality if and only if $a_1 = \cdots = a_n$.

3. **(Cauchy-Schwarz)** For real numbers $a_1, b_1, \dots, a_n, b_n$,
$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2),$$
with equality if and only if there exists a k such that $a_i = k b_i$ for all $i = 1, \dots, n$.

4. **(Jensen)** For any real-valued convex function $f(x)$ (i.e. $f''(x) \geq 0$ if f is twice differentiable) and real numbers $x_1, \dots, x_n$,
$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}$$
with equality if (but not only if) $x_1 = \cdots = x_n$. If f is concave (i.e. $f''(x) \leq 0$ if f is twice differentiable), then the inequality sign is flipped.

5. **(Triangle)** Let $z_1, \dots, z_n$ be complex numbers. Then
$$|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|,$$
with equality if and only if $\arg z_1 = \cdots = \arg z_n$.

1. Prove the $n = 2$ case for AM-GM: For nonnegative numbers a and b, prove that $\frac{a+b}{2} \geq \sqrt{ab}$.

2. Let $x_1, \dots, x_n$ be positive numbers. Find the minimum value of
$$(x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right).$$

3. Positive integers are placed at every lattice point in the plane so that each is the average of its four nearest neighbors. Given that 2025 is placed at the origin, find the number placed at the point $(2025, 2025)$.

4. Let a, b, and c be real numbers. Prove that $a^2 + b^2 + c^2 \ge ab + bc + ca$, and determine when equality holds.

5. Let a, b, and c be positive numbers. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \ge \frac{b}{a} + \frac{c}{b} + \frac{a}{c},$$

and determine when equality holds.

6. Let $f(x) = 3\sin x + 4\cos x$. Find the maximum possible value of f over all real numbers x.

7. Let A, B, and C be the three interior angles of a triangle. Prove that $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$.

8. Use Jensen's inequality to prove AM-GM for all $n \ge 1$.

9. Let x, y, and z be complex numbers with $x + y + z = 0$. Prove that $|x| + |y| + |z| \ge 2\max\{|x|, |y|, |z|\}$, and determine when equality holds.

10. Let z and w be complex numbers. Prove that $||z| - |w|| \le |z - w|$.

11. Let n be a positive integer. Given $2n - 1$ real numbers $x_1, \ldots, x_{2n-1}$, let

$$f(x) = |x - x_1| + \cdots + |x - x_{2n-1}|.$$

Find a real number x_0 such that $f(x_0) \le f(x)$ for all real x.

12. Suppose a, b, c, d, e are real numbers that satisfy $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$. Find the maximum possible value of a.

13. Given $a_1 + \cdots + a_n = n$ ($a_i \in \mathbb{R}$), prove that $a_1^4 + \cdots + a_n^4 \ge n$.

14. Let $p(x) = x^{2025} - 20x^{25} - 25x^{20} - 2025$. Given that p has exactly one positive root r, prove that for any real or complex root s, we have $|s| \le r$.

20 Determinants

1. **(Properties of the Determinant)** Let A and D be square matrices:
   - $\det(AD) = \det(A)\det(D)$
   - $\det(A^T) = \det(A)$
   - If A is invertible, $\det(A^{-1}) = 1/\det(A)$
   - $\det\begin{pmatrix} A & B \\ O & D \end{pmatrix} = \det(A)\det(D)$
   - If A is invertible, $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B)$
2. **(Invertible Matrix Theorem)** The following are equivalent:
   - $\det(A) \neq 0$
   - A is invertible
   - The rows/columns of A are linearly independent
   - The equation $A\vec{v} = \vec{0}$ has only one solution, $\vec{v} = \vec{0}$
3. **(Elementary Row Operations)** The three elementary row operations have the following effects on the determinant:
   - Swapping two rows flips the sign of the determinant
   - Scaling a row by k scales the determinant by k
   - Adding a multiple of one row to another row doesn't change the determinant

   The word "row" can be replaced by the word "column".
4. **(Eigenvalues)** Suppose A is an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $\det(A) = \lambda_1 \cdots \lambda_n$.
5. **(Cofactor Expansion)** Know what this is.

1. Find the determinant of the $n \times n$ matrix all of whose entries are 1.
2. Suppose A is an $n \times n$ matrix such that the elements of each row of A sum to 0. Prove that $\det(A) = 0$.

3. A square matrix A is skew-symmetric if $A^T = -A$. Does there exist an invertible 2025×2025 skew-symmetric matrix?

4. Determine the value of k so that the following matrix is not invertible:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & k \end{pmatrix}$$

5. Let A be an invertible matrix such that $I + A$ is invertible. Prove that $I + A^{-1}$ is invertible.

6. A square matrix M is upper triangular if all of the entries below the main diagonal are 0. Prove $\det(M)$ is the product of the diagonal entries of M.

7. Suppose A and B are $n \times n$ matrices with $AB = BA$. Prove that

$$\det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A^2 - B^2).$$

8. Let A be an $n \times n$ matrix whose entries are integers between 0 and 9, inclusive. Let k_i be the integer that results from concatenating the entries of row i. Suppose there exists an integer d such that $d \mid k_i$ for all $i = 1, \ldots, n$. Prove that $d \mid \det(A)$.

9. Find the determinant of the following $n \times n$ matrix:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{pmatrix}$$

where entry a_{ij} is $\max(i, j)$.

10. Find the determinant of the $n \times n$ matrix whose diagonal entries are 0 and all other entries are 1.

11. Find the max possible determinant of a 3×3 matrix whose entries are ± 1.

12. Let $n \geq 2$ be an integer and x a real number. Find the determinant of the $n \times n$ matrix whose diagonal entries are 1 and all other entries are x.