

# Beginner Problem-Solving

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## Contents

<b>1 Induction</b>	<b>2</b>
<b>2 Pigeonhole Principle</b>	<b>7</b>
<b>3 Invariants and Monovariants</b>	<b>9</b>
<b>4 Fundamental Theorem of Algebra</b>	<b>12</b>
<b>5 Vieta's Formulas</b>	<b>14</b>
<b>6 Fundamental Theorem of Arithmetic</b>	<b>17</b>
<b>7 Greatest Common Divisor</b>	<b>19</b>
<b>8 Modular Arithmetic</b>	<b>23</b>
<b>9 Fermat's Little Theorem</b>	<b>26</b>
<b>10 Integer Polynomials</b>	<b>29</b>
<b>11 Intermediate Value Theorem</b>	<b>34</b>
<b>12 Taylor's Theorem</b>	<b>37</b>
<b>13 Mean Value Theorem</b>	<b>41</b>
<b>14 Riemann Sums</b>	<b>43</b>
<b>15 Inclusion-Exclusion Principle</b>	<b>45</b>
<b>16 bijections</b>	<b>47</b>
<b>17 Recurrence Relations</b>	<b>50</b>
<b>18 Recursive Counting</b>	<b>53</b>
<b>19 Standard Inequalities</b>	<b>56</b>
<b>20 Determinants</b>	<b>60</b>

## 1 Induction

1. Find and prove a formula for the sum of the first  $n$  odd numbers.

**Solution:** We claim that the sum of the first  $n$  odd numbers is

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

We will prove this by induction.

Base case: When  $n = 1$ , we have  $1 = 1^2$ , so the claim is true when  $n = 1$ .

Inductive step: Suppose the claim is true for some  $n \geq 1$ . That is, we have

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Then,

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2n - 1) + (2(n+1) - 1) \\ &= n^2 + (2(n+1) - 1) \\ &= n^2 + 2n + 1 \\ &= (n+1)^2. \end{aligned}$$

Thus, the claim is true for the  $n+1$  case. By induction, the claim is true for all positive integers  $n$ .  $\square$

2. Let  $(F_n)$  be the Fibonacci sequence, so  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$  for each  $n \geq 2$ . Prove that  $F_n \leq 2^n$  for all  $n \geq 0$ .

**Solution:** We will use strong induction on  $n$ .

Base cases: We have  $F_0 = 0 \leq 2^0$  and  $F_1 = 1 \leq 2^1$ .

Inductive step: Suppose we have  $F_k \leq 2^k$  for all  $k = 0, 1, \dots, n$  for some  $n \geq 1$ . Then we have

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &\leq 2^n + 2^{n-1} \\ &< 2^n + 2^n \\ &= 2^{n+1}. \end{aligned}$$

Thus,  $F_{n+1} \leq 2^{n+1}$ . By strong induction,  $F_n \leq 2^n$  for all  $n \geq 0$ .  $\square$

3. Define a sequence  $(b_n)$  by  $b_1 = 1$ , and for each  $n \geq 1$ ,  $b_{2n} = b_n$  and  $b_{2n+1} = b_n + 1$ . Prove that, for each positive integer  $n$ , the number of 1's in the binary representation of  $n$  is exactly  $b_n$ .

**Solution:** We will use strong induction on  $n$ .

Base cases: For  $n = 1$ , we have binary representation "1" and  $b_1 = 1$ .

Inductive step: Suppose  $b_k$  is the number of 1's in the binary representation of  $k$ , for each  $k = 1, 2, \dots, n$  for some  $n \geq 1$ . Then we have two possibilities for  $n + 1$ :

If  $n + 1$  is even, then write  $n + 1 = 2k$  for some integer  $k$ . Now, the binary representation of  $2k$  is the binary representation of  $k$  with an additional 0 at the end. Hence the number of 1's in the binary representation of  $2k$  is exactly the number of 1's in the binary representation of  $k$ , which by our inductive hypothesis is exactly  $b_k$ . Thus, since  $b_{n+1} = b_{2k} = b_k$ , then  $b_{n+1}$  gives the number of 1's in the binary representation of  $n + 1$ .

If  $n + 1$  is odd, then write  $n + 1 = 2k + 1$  for some integer  $k$ . Now, the binary representation of  $2k + 1$  is the binary representation of  $k$  with an additional 1 at the end. Hence the number of 1's in the binary representation of  $2k + 1$  is exactly one more than the number of 1's in the binary representation of  $k$ , which by our inductive hypothesis is exactly  $b_k$ . Thus, since  $b_{n+1} = b_{2k+1} = b_k + 1$ , then  $b_{n+1}$  gives the number of 1's in the binary representation of  $n + 1$ .

In both cases, the  $n + 1$  case of the problem is true. By strong induction, the problem is true for all positive integers  $n$ .  $\square$

4. For each positive integer  $n$ , let  $f(n) = 2 \cdot 7^n + 3 \cdot 5^n - 5$ . Show that  $f(n)$  is always a multiple of 24.

**Solution:** We will first prove the following claim: for each positive integer  $n$ ,  $4 \cdot 7^n + 20$  is a multiple of 24. We will do this by induction.

Base case: When  $n = 1$ , we have

$$4 \cdot 7^1 + 20 = 48 = 2(24)$$

so  $4 \cdot 7^n + 20$  is a multiple of 24 when  $n = 1$ .

Inductive step: Suppose  $4 \cdot 7^n + 20$  is a multiple of 24 for some positive integer  $n$ . Then we can write  $4 \cdot 7^n + 20 = 24k$  for some integer  $k$ . Then  $4 \cdot 7^n = 24k - 20$ , so

$$\begin{aligned} 4 \cdot 7^{n+1} + 20 &= 7(4 \cdot 7^n) + 20 \\ &= 7(24k - 20) + 20 \\ &= 168k - 120 \\ &= 24(7k - 5). \end{aligned}$$

Thus,  $4 \cdot 7^{n+1} + 20$  is a multiple of 24. By induction,  $4 \cdot 7^n + 20$  is a multiple of 24 for all positive integers  $n$ .

With our claim proven, we will proceed with induction on  $n$  for our original problem.

Base case: When  $n = 1$ , we have

$$f(1) = 2 \cdot 7^1 + 3 \cdot 5^1 - 5 = 24$$

so  $f(1)$  is a multiple of 24.

Inductive step: Suppose  $f(n)$  is a multiple of 24 for some positive integer  $n$ . Then we can write  $f(n) = 2 \cdot 7^n + 3 \cdot 5^n - 5 = 24k$  for some integer  $k$ . Then  $3 \cdot 5^n = 24k - 2 \cdot 7^n + 5$ , so

$$\begin{aligned} f(n+1) &= 2 \cdot 7^{n+1} + 3 \cdot 5^{n+1} - 5 \\ &= 14 \cdot 7^n + 5(3 \cdot 5^n) - 5 \\ &= 14 \cdot 7^n + 5(24k - 2 \cdot 7^n + 5) - 5 \\ &= 14 \cdot 7^n + 120k - 10 \cdot 7^n + 25 - 5 \\ &= 4 \cdot 7^n + 20 + 120k \end{aligned}$$

By our first claim,  $4 \cdot 7^n + 20$  is a multiple of 24. Additionally,  $120k = 24(5k)$  is also a multiple of 24. Thus, their sum is a multiple of 24, so  $f(n+1)$  is a multiple of 24. By induction,  $f(n)$  is a multiple of 24 for all positive integers  $n$ .  $\square$

5. Let  $f(n)$  be the number of regions which are formed by  $n$  lines in the plane, no two of which are parallel and no three meet at a single point. Hence  $f(1) = 2$ ,  $f(2) = 4$ , and  $f(3) = 7$ . Find and prove a closed-form expression for  $f(n)$ .

**Solution:** We claim that  $f(n) = \frac{n(n+1)}{2} + 1$ . We will use induction on  $n$ .

Base case: For  $n = 1$ , we have  $f(1) = 2 = \frac{1(1+1)}{2} + 1$ , so our claim is true for  $n = 1$ .

Inductive step: Suppose  $f(n) = \frac{n(n+1)}{2} + 1$  for some  $n \geq 1$ . Consider any  $n+1$  lines in the plane, no two of which are parallel and no three meet at a single point. Select one line and temporarily remove it. Then by our inductive hypothesis, the remaining  $n$  lines form  $f(n) = \frac{n(n+1)}{2} + 1$  regions in the plane. When we add back in our selected line, it will intersect the other  $n$  lines exactly once. Thus, it will split exactly  $n+1$  regions into two. Thus, there are  $n+1$  new regions formed, so

$$f(n+1) = f(n) + n + 1 = \frac{n(n+1)}{2} + 1 + n + 1 = \frac{(n+1)(n+2)}{2} + 1.$$

By induction,  $f(n) = \frac{n(n+1)}{2} + 1$  for all positive integers  $n$ .  $\square$

6. Let  $x_1, x_2, \dots, x_n$  be  $n$  positive numbers such that  $x_1 + x_2 + \dots + x_n = 1/2$ . Define  $f(x) = (1-x)/(1+x)$  for all  $x > 0$ . Prove that  $f(x_1)f(x_2)\cdots f(x_n) \geq 1/3$ .

**Solution:** We will use induction on  $n$ .

Base case: When  $n = 1$ , we have  $x_1 = 1/2$  so  $f(x_1) = (1 - 1/2)/(1 + 1/2) = 1/3$ .

Inductive step: Suppose  $f(x_1)f(x_2)\cdots f(x_n) \geq 1/3$  for positive  $x_1 + \dots + x_n = 1/2$  for some  $n \geq 1$ . Consider any  $n+1$  positive numbers  $y_1, \dots, y_{n+1}$  such that  $y_1 + \dots + y_{n+1} = 1/2$ . We can apply our inductive hypothesis on the  $n$  numbers  $y_1, \dots, y_{n-1}, (y_n + y_{n+1})$  (who have sum  $1/2$ ). Thus, we have

$$f(y_1)\cdots f(y_{n-1})f(y_n + y_{n+1}) \geq 1/3$$

If we are able to show that  $f(y_n)f(y_{n+1}) \geq f(y_n + y_{n+1})$ , then we will have finished our inductive step. We have:

$$\begin{aligned} & f(y_n)f(y_{n+1}) \geq f(y_n + y_{n+1}) \\ \iff & \frac{1-y_n}{1+y_n} \cdot \frac{1-y_{n+1}}{1+y_{n+1}} \geq \frac{1-y_n-y_{n+1}}{1+y_n+y_{n+1}} \\ \iff & (1-y_n)(1-y_{n+1})(1+y_n+y_{n+1}) \geq (1-y_n-y_{n+1})(1+y_n)(1+y_{n+1}) \\ \iff & 2y_n y_{n+1}(y_n + y_{n+1}) \geq 0 \end{aligned}$$

The final inequality is true since  $y_n, y_{n+1}$  are positive. Thus, our inductive step is finished. By induction, our result holds for all positive integers  $n$ .  $\square$

7. Suppose that you make a deal with the devil: whenever you want, you can trade a coin for any finite number of coins of lesser value. The catch is that you cannot earn coins other than by trading with the devil, you must pay the devil at least 1 cent per day, and when you run out of coins the devil gets your soul.

The possible coin denominations are 1 (penny), 5 (nickel), 10 (dime), and 25 cents (quarter). (For example, you can trade one quarter for a million dimes.) Can you start with a finite number of coins of your choosing and never lose your soul?

**Solution:** The result is true if we use any finite set of possible coin denominations  $c_1 < \dots < c_n$ . We will prove this general result. Suppose the largest denomination of coin we start with is  $c_k$ , for  $1 \leq k \leq n$ . We will induct on  $k$ .

Base case: For  $k = 1$ , we start with all coins of denomination  $c_1$ . But then we cannot make any trades with the devil, so we must eventually run out of coins and lose our soul.

Inductive step: Suppose that for any starting set of coins from  $c_1, \dots, c_k$  that we will eventually lose our soul (where  $k \geq 1$ ). Consider any starting set of coins from  $c_1, \dots, c_{k+1}$ .

We claim that we must eventually run out of coins of denomination  $c_{k+1}$ . For the sake of contradiction, suppose we can never lose our soul by holding on to some number of

coins of value  $c_{k+1}$ . Then by our inductive hypothesis, we will eventually run out of coins of denomination  $c_1, \dots, c_k$ . At this point, we must trade in at least one coin of value  $c_{k+1}$ , a contradiction. Thus, we will eventually run out of coins of value  $c_{k+1}$ . By our inductive hypothesis, we will also run out of coins of value  $c_1, \dots, c_k$ , at which point we must lose our soul. By induction, we must lose our soul for any starting set of coins of denomination  $c_1, \dots, c_n$ .  $\square$

## 2 Pigeonhole Principle

1. Let  $S$  be a subset of  $\{1, 2, 3, \dots, 2n\}$  with  $n + 1$  elements. Prove that  $S$  must contain a pair of consecutive integers.

**Solution:** Label  $n$  bins with the labels  $\{1, 2\}, \{3, 4\}, \dots, \{2n - 1, 2n\}$ . For each element in  $S$ , place it into the bin with that element on the label. Since there are  $n + 1$  elements that we are distributing among  $n$  bins, by the Pigeonhole Principle there must be some bin with at least two elements. These two elements will be consecutive.  $\square$

2. Let  $S$  be a subset of  $\{1, 2, 3, \dots, 2n\}$  with  $n + 1$  elements. Prove that  $S$  must contain a pair of integers that sum to exactly  $2n + 1$ .

**Solution:** Label  $n$  bins with the labels  $\{1, 2n\}, \{2, 2n - 1\}, \dots, \{n, n + 1\}$ . Note that the sums of the numbers on every bin is exactly  $2n + 1$ . For each element in  $S$ , place it into the bin with that element on the label. Since there are  $n + 1$  elements that we are distributing among  $n$  bins, by the Pigeonhole Principle there must be some bin with at least two elements. These two elements must then sum to  $2n + 1$ .  $\square$

3. Five points are placed inside a square of side length 1 unit. Prove that some two of them have distance at most  $\sqrt{2}/2$  units apart.

**Solution:** Cut the square into four smaller squares of side length  $1/2$ . Since we are placing five points among these four squares, by the Pigeonhole Principle one of these smaller squares must contain two points (either in its interior or on its border). The maximum distance of these two points is when they are on opposite corners of this square, which is  $\sqrt{2}/2$ .  $\square$

4. Suppose  $n$  is an odd positive integer and  $\{a_1, a_2, \dots, a_n\} = \{1, 2, \dots, n\}$ . Prove that  $(1 - a_1)(2 - a_2) \cdots (n - a_n)$  is even.

**Solution:** Write  $n = 2k - 1$  for some  $k \geq 1$ . For the sake of contradiction, suppose  $(1 - a_1) \cdots (n - a_n)$  is odd. Then each of the factors are odd. In particular,  $\{a_1, a_3, \dots, a_n\}$  is a set of  $k$  even numbers. But there are only  $k - 1$  even numbers in the set  $\{1, 2, \dots, 2k - 1\}$ , a contradiction. Thus,  $(1 - a_1) \cdots (n - a_n)$  is even.  $\square$

5. The integers  $1, 2, \dots, 10$  are written around a circle in some order. Prove that there are 3 adjacent numbers whose sum is at least 17.

**Solution:** For the sake of contradiction, suppose every 3 adjacent numbers sum to at most 16. Then if we sum all 10 sets of possible 3 adjacent numbers, we get  $S \leq 160$ . But each number  $\{1, 2, \dots, 10\}$  is included exactly 3 times in this sum. Thus,  $S = 3(1+2+\cdots+10) = 165$ , a contradiction. Thus, there must be some 3 adjacent numbers whose sum is at least 17.  $\square$

6. Let  $n$  be a positive integer. Prove that there exists a positive integer  $m$  whose decimal digits are just 1's and 0's and  $n$  divides  $m$ . (For example, with  $n = 367$  we have  $m = 1101 = 3 \cdot 367$ .)

**Solution:** Consider the  $n + 1$  integers  $1, 11, 111, \dots, (11\dots11)$  (where the last number contains  $n + 1$  1's). Since there are only  $n$  possible remainders after dividing by  $n$ , by the Pigeonhole Principle there exist some two of these  $n + 1$  integers with equal remainders after dividing by  $n$ . Then their (positive) difference is a multiple of  $n$  that contains only the digits 1 and 0.  $\square$

7. Let  $n$  be an odd positive integer. Prove that there exists a positive integer  $m$  such that  $n$  divides  $2^m - 1$ .

**Solution:** Consider the  $n + 1$  integers  $2^0, 2^1, \dots, 2^n$ . Since there are only  $n$  possible remainders after dividing by  $n$ , by the Pigeonhole Principle there exist some two of these  $n + 1$  integers with equal remainders after dividing by  $n$ . Suppose these two integers are  $2^i < 2^j$ . Then their difference  $2^j - 2^i = 2^i(2^{j-i} - 1)$  is divisible by  $n$ . But  $n$  is odd, so it is relatively prime to  $2^i$ . Thus,  $n$  divides  $2^{j-i} - 1$ .  $\square$

8. Consider the set of squares of side length 1 whose vertices are lattice points in the plane. Given any  $n$  of these squares, prove that there is a subset of at least  $n/4$  of them such that no two share any vertices.

**Solution:** Color the squares in the plane like a checkerboard, but alternatively color each row using black/white and red/blue. I.e., one row of squares is alternatively colored black and white, its adjacent rows are alternatively colored red and blue, and so on.

By the Pigeonhole Principle, of the  $n$  squares we choose there must be at least  $n/4$  of them that are all the same color. But by our coloring, no two squares of the same color can share any vertices. Thus, these  $n/4$  squares do not share any vertices.  $\square$

### 3 Invariants and Monovariants

1. Is it possible to partition  $\{1, 2, \dots, 1234\}$  into two sets  $A$  and  $B$  such that the sum of elements in  $A$  equals the sum of elements in  $B$ ? (A partition  $(A, B)$  of  $S$  satisfies  $A \cup B = S$  and  $A \cap B = \emptyset$ .)

**Solution:** It is not possible. Since  $A$  and  $B$  partition the set, the sum of the elements in both sets is invariant. Namely, if the sums of the elements in each set is  $s$ , then  $2s = 1 + 2 + \dots + 1234$ . But the sum on the right is odd since it contains  $1234/2 = 617$  odd numbers. Thus, no such partition exists.  $\square$

2. The numbers  $1, 2, \dots, 100$  are written on a chalkboard. You are allowed to erase two numbers, say  $a$  and  $b$ , and write the single number  $c = a + b - 1$  on the board. You repeat this process until only one number remains. Find this number.

**Solution:** The sum of the numbers on the board is a monovariant: after each erasure the sum decreases by exactly 1. Thus, after 99 erasures the sum of the elements will be  $(1 + 2 + \dots + 100) - 99 = 4951$ , which must be the final number on the board.  $\square$

3. The numbers  $1, 2, \dots, 100$  are written on a chalkboard. You are allowed to erase two numbers, say  $a$  and  $b$ , and write the single number  $c = ab + a + b$  on the board. You repeat this process until only one number remains. Find this number.

**Solution:** The product of one more than each number on the board is invariant after each erasure. Formally, suppose  $a_1, a_2, \dots, a_n$  are written on the board when we have  $n$  numbers, and define

$$P_n = (a_1 + 1)(a_2 + 1) \cdots (a_n + 1).$$

Suppose we erase  $a_1$  and  $a_2$ . Then

$$\begin{aligned} P_{n-1} &= (a_1 a_2 + a_1 + a_2 + 1)(a_3 + 1) \cdots (a_n + 1) \\ &= (a_1 + 1)(a_2 + 1) \cdots (a_n + 1) \\ &= P_n. \end{aligned}$$

Thus,  $P_{100} = P_{99} = \cdots = P_1$ , so the final number  $k$  satisfies

$$k + 1 = P_1 = P_{100} = (1 + 1)(2 + 1) \cdots (100 + 1) = 101!$$

Hence  $k = 101! - 1$ .  $\square$

4. Start with the set  $\{3, 4, 12\}$ . You are allowed to replace any two numbers, say  $a$  and  $b$ , with the new numbers  $0.8a + 0.6b$  and  $0.6a - 0.8b$ . Is it possible to eventually transform the set into  $\{6, 7, 8\}$ ?

**Solution:** It is not possible. The sum of the squares of the set is invariant under replacement. Formally, suppose  $\{a, b, c\}$  is transformed into  $\{0.8a + 0.6b, 0.6a - 0.8b, c\}$ . Then

$$\begin{aligned} & (0.8a + 0.6b)^2 + (0.6a - 0.8b)^2 + c^2 \\ &= (0.8a)^2 + 2(0.8)(0.6)ab + (0.6b)^2 + (0.6a)^2 - 2(0.6)(0.8)ab + (0.8b)^2 + c^2 \\ &= (0.8^2 + 0.6^2)a^2 + (0.6^2 + 0.8^2)b^2 + c^2 \\ &= a^2 + b^2 + c^2. \end{aligned}$$

Thus, since  $3^2 + 4^2 + 12^2 = 169 \neq 149 = 6^2 + 7^2 + 8^2$ , it is not possible to transform  $\{3, 4, 12\}$  into  $\{6, 7, 8\}$ .  $\square$

5. Seven squares of an  $8 \times 8$  grid are shaded. At each step, we shade in each unshaded square that has at least two shaded neighboring squares (horizontally or vertically). Is it possible for this process to end with the entire  $8 \times 8$  grid being shaded?

**Solution:** It is not possible. Call an edge a *border edge* if it separates a shaded square from an unshaded square or the board exterior. We claim that the number of border edges is a decreasing monovariant during this process.

To show this, consider an unshaded square that will become shaded. Then at least two of the four edges of this square must be border edges and at most two of the four edges are non-border edges. After shading this square, all of its border edges become non-border edges, and all of its non-border edges become border edges. Thus, at most two of the four edges become border edges and at least two of the four edges become non-border squares, so the number of border edges cannot increase.

Finally, when we begin with seven squares we have at most  $7 \cdot 4 = 28$  border edges, but a fully-shaded  $8 \times 8$  grid has 32 border edges. Thus, we cannot end the process with the entire  $8 \times 8$  grid being shaded.  $\square$

6. The integers  $1, 2, \dots, n$  are written down in order. A swap consists of interchanging the positions of some two distinct integers. Is it possible that, after exactly 2025 swaps, the numbers are back in order?

**Solution:** It is not possible. An *inversion* of a permutation  $(a_1, a_2, \dots, a_n)$  of  $(1, 2, \dots, n)$  is an ordered pair  $(i, j)$  such that  $i < j$  and  $a_i > a_j$ . We claim that the parity of the number of inversions is equal to the parity of the number of swaps.

To show this, suppose we swap some  $a_i$  and  $a_j$  with  $i < j$ . Then for  $k > j$ , the number of inversions with  $(i, k)$  and  $(j, k)$  remains constant. Similarly, for  $k < i$ , the number of inversions with  $(k, i)$  and  $(k, j)$  remains constant.

For  $i < k < j$ , any inversions with  $(i, k)$  and  $(k, j)$  become non-inversions with  $(k, i)$  and  $(j, k)$ , and any non-inversions with  $(i, k)$  and  $(k, j)$  become inversions with  $(k, i)$  and  $(j, k)$ . Note that there are  $j - i - 1$  possible values of  $k$ . Hence if there were  $n$  total inversions of

the type  $(i, k)$  or  $(k, j)$ , then there must have been  $2(j - i - 1) - n$  non-inversions. After the swap, we would then have  $2(j - i - 1) - n$  inversions, which has the same parity as  $n$ .

Finally, if  $(i, j)$  was an inversion then  $(j, i)$  is a non-inversion after the swap, and if  $(i, j)$  was a non-inversion then  $(j, i)$  is an inversion after the swap. This either decreases or increases the number of inversions by 1, and since the number of inversions among all other types of inversions maintains the same parity, the parity of the total number of inversions must flip. Since we start at 0 swaps with 0 inversions, then the parity of the number of swaps and the parity of the number of inversions must be equal for any number of swaps.

Since 2025 is odd, we must then have an odd number of inversions. In particular, we cannot have the numbers back in order since that would have 0 inversions.  $\square$

7. Starting with an ordered tuple of 4 integers, repeatedly perform the operation

$$(a, b, c, d) \rightarrow (|a - b|, |b - c|, |c - d|, |d - a|).$$

We say the process terminates if we eventually get the tuple  $(0, 0, 0, 0)$ . Must the process terminate in a finite number of operations for any choice of the starting tuple?

**Solution:** Yes, the process must eventually terminate. Denote the operation by  $f(a, b, c, d) = (|a - b|, |b - c|, |c - d|, |d - a|)$ , and  $n$  repeated applications of  $f$  as  $f^{(n)}$ . First, note that after one application of  $f$ , we will have a tuple of nonnegative integers. Thus, assume that our starting tuple is a tuple of nonnegative integers.

First, we claim that the maximum of the nonnegative integers in the tuple is a nonincreasing monovariant. Let  $M(a, b, c, d)$  denote the maximum value in the tuple  $(a, b, c, d)$ . Note that  $0 \leq x, y \leq M \Rightarrow |x - y| \leq M - 0 = M$ . Since  $0 \leq a, b, c, d \leq M(a, b, c, d)$ , we have  $|a - b|, |b - c|, |c - d|, |d - a| \leq M(a, b, c, d)$  and thus  $M(f(a, b, c, d)) \leq M(a, b, c, d)$ .

Next, we claim that the largest power of 2 that divides all elements in the tuple is an unbounded, nondecreasing monovariant. Note that

$$\begin{aligned} f(a, b, c, d) &= (|a - b|, |b - c|, |c - d|, |d - a|) \\ &\equiv (a + b, b + c, c + d, d + a) \pmod{2} \\ \Rightarrow f^{(2)}(a, b, c, d) &\equiv (a + 2b + c, b + 2c + d, c + 2d + a, d + 2a + b) \equiv (a + c, b + d, a + c, b + d) \\ \Rightarrow f^{(3)}(a, b, c, d) &\equiv (a + b + c + d, a + b + c + d, a + b + c + d, a + b + c + d) \\ \Rightarrow f^{(4)}(a, b, c, d) &\equiv (0, 0, 0, 0). \end{aligned}$$

Thus, all the elements of  $f^{(4)}$  are divisible by 2. Note that  $f(\lambda a, \lambda b, \lambda c, \lambda d) = \lambda f(a, b, c, d)$  for nonnegative integers  $\lambda, a, b, c, d$ . Thus, all the elements of  $f^{(4k)}$  are divisible by  $2^k$  for all  $k \geq 1$ .

Finally, define  $M_k = M(f^{(4k)}(a, b, c, d))$ . Since  $M_k$  is nonincreasing, we'll have  $2^K > M_K$  for sufficiently large  $K$ . But the only multiple of  $2^K$  in  $[0, M_K]$  is 0. Thus, we must have  $f^{(4K)}(a, b, c, d) = (0, 0, 0, 0)$ .  $\square$

## 4 Fundamental Theorem of Algebra

1. Let  $n$  be a positive integer. Determine all polynomials  $p(x)$  of degree at most  $n$  such that  $p(x) = 0$  has  $n + 1$  distinct solutions.

**Solution:** The only such polynomial is  $p(x) = 0$ . For any polynomial  $p$  of degree  $m \geq 1$ , the fundamental theorem of algebra states that  $p$  has at most  $m$  distinct roots. Thus, the only polynomial with more distinct roots than its degree is the constant zero polynomial  $p(x) = 0$ .  $\square$

2. Find all polynomials that have an infinite number of roots.

**Solution:** The only such polynomial is  $p(x) = 0$ . By the fundamental theorem of algebra, all non-constant polynomials of finite degree have a finite number of roots. Thus, the only polynomial with an infinite number of roots is the zero polynomial  $p(x) = 0$ .  $\square$

3. Let  $P(x)$  and  $Q(x)$  be real polynomials of degree  $n$  and  $m$ , respectively. Determine the maximum number of intersection points of the graphs of  $y = P(x)$  and  $y = Q(x)$ .

**Solution:** There can be at most  $\max(n, m)$  intersection points. Note that  $P$  and  $Q$  intersect at a point  $(x_0, y_0)$  if and only if  $P(x_0) = Q(x_0)$ . Define  $R(x) = P(x) - Q(x)$ . Then  $R$  is a polynomial of degree at most  $\max(n, m)$  whose roots are exactly the  $x$ -coordinates of the intersection points above. Thus, by the fundamental theorem of algebra, there can be at most  $\max(n, m)$  intersection points.  $\square$

4. Suppose  $P(x)$  is a real polynomial of degree 4 such that  $P(0) = P(1) = P(2) = P(3) = 1$  and  $P(4) = 0$ . Find  $P(5)$ .

**Solution:** Define  $Q(x) = P(x) - 1$ . Then  $Q$  is a polynomial of degree 4 with roots  $x = 0, 1, 2, 3$ . By the fundamental theorem of algebra, we can write  $Q(x) = a(x - 1)(x - 2)(x - 3)$  for some  $a \in \mathbb{R}$ . Since  $Q(4) = -1$ , we have  $a(-1)(-2)(-3)(-4) = -1 \Rightarrow a = -1/24$ . Thus,  $Q(x) = -x(x - 1)(x - 2)(x - 3)/24$ , so  $P(5) = Q(5) + 1 = -5(5 - 1)(5 - 2)(5 - 3)/24 + 1 = -4$ .  $\square$

5. Suppose  $P(x)$  is a real polynomial of degree 2024 such that  $P(n) = 1/n$  for  $n = 1, 2, \dots, 2025$ . Find  $P(2026)$ .

**Solution:** Define  $Q(x) = xP(x) - 1$ . Then  $Q(x)$  is a polynomial of degree 2025 with roots  $x = 1, 2, \dots, 2025$ . By the fundamental theorem of algebra, we can write  $Q(x) = a(x - 1)(x - 2) \cdots (x - 2025)$  for some  $a \in \mathbb{R}$ . Moreover, we know the constant term of  $Q$  is  $-1$  since  $Q(x) = xP(x) - 1$ . Thus,  $a(-1)(-2) \cdots (-2025) = -1 \Rightarrow a = 1/(2025!)$ . Since  $Q(2026) = (2026 - 1)(2026 - 2) \cdots (2026 - 2025)/(2025!) = 1$ , we have  $P(2026) = (Q(2026) + 1)/2026 = 1/1013$ .  $\square$

6. Determine all polynomials  $p(x)$  such that  $xp(x-1) = (x+1)p(x)$  for all real  $x$ .

**Solution:** The only such polynomial is  $p(x) = 0$ . Note that  $p(0) = 0$ ,  $p(1) = 1$ ,  $p(0)/2 = 0$ , and  $p(2) = 2p(1)/3 = 0$ . We claim  $p(n) = 0$  for all positive integers  $n$  by induction. The base case  $n = 1$  is shown above. For the inductive step, suppose  $p(n) = 0$  for some  $n \geq 1$ . Then  $p(n+1) = (n+1)p(n)/(n+2) = 0$ . Thus, by induction,  $p(n) = 0$  for all positive integers  $n$ , and in particular  $p$  is a polynomial with infinitely many roots. By the fundamental theorem of algebra,  $p(x) = 0$  identically.  $\square$

7. Determine all polynomials  $p(x)$  such that  $p(0) = 0$  and  $p(x^2 + 1) = p(x)^2 + 1$  for all real  $x$ .

**Solution:** The only such polynomial is  $p(x) = x$ . Note that  $p(1) = p(0^2 + 1) = p(0)^2 + 1 = 1$ ,  $p(2) = p(1^2 + 1) = p(1)^2 + 1 = 2$ ,  $p(5) = p(2^2 + 1) = p(2)^2 + 1 = 5$ , and  $p(26) = p(5^2 + 1) = p(5)^2 + 1 = 26$ .

Define the sequence  $(x_n)$  by  $x_1 = 1$  and  $x_n = x_{n-1}^2 + 1$  for  $n \geq 2$ . Note that  $x_n > x_{n-1}^2 \geq x_{n-1}$  since  $x_n \geq 1$  for all  $n$ . Thus,  $(x_n)$  is a strictly increasing sequence, and in particular its values are all distinct.

We claim that  $p(x_n) = x_n$  for all  $n \geq 1$  by induction. Our base case for  $n = 1$  holds as shown above. Now suppose  $p(x_n) = x_n$  for some  $n \geq 1$ . Then  $p(x_{n+1}) = p(x_n^2 + 1) = p(x_n)^2 + 1 = x_n^2 + 1 = x_{n+1}$ . Thus, our claim is proven by induction.

Finally,  $p(x) - x$  is a polynomial with infinitely many roots  $(x_n)$ . Thus, by the fundamental theorem of algebra, this polynomial must be the zero polynomial. Thus,  $p(x) = x$  for all  $x \in \mathbb{R}$ , and we can verify this polynomial satisfies  $p(0) = 0$  and  $p(x^2 + 1) = p(x)^2 + 1$  for all  $x \in \mathbb{R}$ .  $\square$

8. There exists a solution  $(a, b, c, d)$  to the system

$$\begin{aligned} a + 8b + 27c + 64d &= 1, \\ 8a + 27b + 64c + 125d &= 27, \\ 27a + 64b + 125c + 216d &= 125, \\ 64a + 125b + 216c + 343d &= 343. \end{aligned}$$

Determine the value of  $64a + 27b + 8c + d$ .

**Solution:** Define the polynomial  $p(x) = ax^3 + b(x+1)^3 + c(x+2)^3 + d(x+3)^3 - (2x-1)^3$ . Then we are given that  $p(1) = p(2) = p(3) = p(4) = 0$ . But  $p$  has degree at most 3 and at least 4 distinct roots, so by the fundamental theorem of algebra we must have  $p(x) = 0$  identically. Thus,  $p(-4) = -64a - 27b - 8c - d + 9^3 = 0 \Rightarrow 64a + 27b + 8c + d = 9^3 = 729$ .  $\square$

## 5 Vieta's Formulas

- Verify Vieta's Formulas for the polynomials  $p(x) = x^2 - 5x + 6$ ,  $q(x) = x^3 - 6x^2 + 11x - 6$ , and  $r(x) = x^4 - 4x^3 + 3x^2 + 4x - 4$ .

**Solution:** For the first polynomial, we have  $r_1 = 2$  and  $r_2 = 3$ . Thus,

$$r_1 + r_2 = 2 + 3 = 5 = -a_1$$

$$r_1 r_2 = 2 \cdot 3 = 6 = a_0$$

For the second polynomial, we have  $r_1 = 1$ ,  $r_2 = 2$ , and  $r_3 = 3$ . Thus,

$$r_1 + r_2 + r_3 = 1 + 2 + 3 = 6 = -a_2$$

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1 = 11 = a_1$$

$$r_1 r_2 r_3 = 1 \cdot 2 \cdot 3 = 6 = -a_0$$

For the third polynomial, we have  $r_1 = -1$ ,  $r_2 = 1$ ,  $r_3 = r_4 = 2$ . Thus,

$$r_1 + r_2 + r_3 + r_4 = -1 + 1 + 2 + 2 = 4 = -a_3$$

$$r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4 = -1 \cdot 1 + -1 \cdot 2 + -1 \cdot 2 + 1 \cdot 2 + 1 \cdot 2 + 2 \cdot 2 = 3 = a_2$$

$$r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2 = -1 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot -1 + 2 \cdot -1 \cdot 1 = -4 = -a_1$$

$$r_1 r_2 r_3 r_4 = -1 \cdot 1 \cdot 2 \cdot 2 = -4 = a_0$$

□

- It is given that two of the roots of  $p(x) = x^4 - 3x^3 - 4x^2 - 5x - 1$  sum to 4. Find the sum of the other two roots.

**Solution:** By Vieta's formulas, we know the sum of the roots is 3. Thus, the sum of the other two roots is  $3 - 4 = -1$ . □

- Let  $r_1, r_2, r_3$  be the roots of the polynomial  $x^3 - 4x^2 + 5x - 6$ . Compute  $1/r_1 + 1/r_2 + 1/r_3$  and  $1/(1+r_1) + 1/(1+r_2) + 1/(1+r_3)$ .

**Solution:** We have  $1/r_1 + 1/r_2 + 1/r_3 = (r_2 r_3 + r_3 r_1 + r_1 r_2)/(r_1 r_2 r_3) = 5/6$  by Vieta's. For the second quantity, we define  $q(x) = (x-1)^3 - 4(x-1)^2 + 5(x-1) - 6$ . Note that  $q$  has roots  $r_1 + 1 = s_1$ ,  $r_2 + 1 = s_2$ , and  $r_3 + 1 = s_3$ . We can expand  $q$  to find  $q(x) = x^3 - 7x^2 + 16x - 16$ , so by Vieta's we have  $1/(1+r_1) + 1/(1+r_2) + 1/(1+r_3) = 1/s_1 + 1/s_2 + 1/s_3 = (s_2 s_3 + s_3 s_1 + s_1 s_2)/(s_1 s_2 s_3) = 16/16 = 1$ . □

- Determine the sum and the product of the roots of the polynomial  $p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$  ( $a_n \neq 0$ ).

**Solution:** Since  $a_n \neq 0$ , we know that  $p(x)/a_n$  has the same roots as  $p(x)$ . Applying Vieta's Formulas to  $p(x)/a_n$ , we have  $r_1 + \dots + r_n = -a_{n-1}/a_n$  and  $r_1 \dots r_n = (-1)^n a_0/a_n$ . □

5. Find the sum of all the roots of the equation  $x^{2025} + (\frac{1}{2} - x)^{2025} = 0$ .

**Solution:** By the binomial theorem, we have

$$\begin{aligned} & x^{2025} + \left(\frac{1}{2} - x\right)^{2025} \\ &= x^{2025} - x^{2025} + 2025 \cdot (1/2) \cdot x^{2024} - \binom{2025}{2} \cdot (1/2)^2 \cdot x^{2023} + \dots + (1/2)^{2025} \\ &= \frac{2025}{2} x^{2024} - \frac{2025 \cdot 2024}{8} x^{2023} + \dots + 1/2^{2025} \end{aligned}$$

By Vieta's Formulas, the sum of the roots is  $\frac{2025 \cdot 2024}{8} / \frac{2025}{2} = 506$ .  $\square$

6. Three distinct points  $(1, 3)$ ,  $(2, 11)$ , and  $(a, b)$  all lie on a single line and all lie on the curve  $y = x^3 + x + 1$ . Find  $(a, b)$ .

**Solution:** The three distinct points all lie on the line  $y = 8x - 5$ . The intersection points of  $y = 8x - 5$  and  $y = x^3 + x + 1$  are given by the roots of  $(x^3 + x + 1) - (8x - 5) = x^3 - 7x + 6$ . By Vieta's Formulas, the sum of these roots is 0. Thus,  $1 + 2 + a = 0 \Rightarrow a = -3$ , so  $b = 8(-3) - 5 = -29$  and  $(a, b) = (-3, -29)$ .  $\square$

7. Three distinct points  $A$ ,  $B$ , and  $C$  all lie on a single line and all lie on the curve  $y = x^3 + x + 1$ . Given that  $A$  and  $B$  have the same distance to the  $y$ -axis, determine the coordinates of point  $C$ .

**Solution:** The intersection points of the single line and the given cubic are given by the roots of their difference. But the  $x^2$  coefficient of this difference will always be 0 (since the equation of a line has no  $x^2$  term). Thus, the sum of the  $x$ -coordinates of these intersection points must be 0. We are given that the sum of the  $x$ -coordinates of  $A$  and  $B$  are 0, so we must have  $C = (0, 1)$ .  $\square$

8. The curve  $y = x^3 - 20x + 24$  intersects a circle centered at  $(7, 24)$  at six distinct points  $(x_1, y_1), \dots, (x_6, y_6)$ . Find  $x_1 + \dots + x_6$ .

**Solution:** We can write the equation for the top half of the circle as  $y = 24 + \sqrt{r^2 - (x-7)^2}$  and the bottom half as  $y = 24 - \sqrt{r^2 - (x-7)^2}$  for some  $r > 0$ . For each of these equations, consider the intersections with the curve  $y = x^3 - 20x + 24$ . These intersections satisfy

$$\begin{aligned} & 24 \pm \sqrt{r^2 - (x-7)^2} = x^3 - 20x + 24 \\ & \Rightarrow (\pm \sqrt{r^2 - (x-7)^2})^2 = (x^3 - 20x)^2 \\ & \Rightarrow r^2 - (x-7)^2 = x^6 - 40x^4 + 400x^2 \\ & \Rightarrow x^6 - 40x^4 + 401x^2 - 14x + 49 - r^2 = 0 \end{aligned}$$

There are at most 6 solutions to this equation, which must be the 6 distinct intersection points. Thus, by Vieta's Formulas,  $x_1 + \dots + x_6 = 0$ .  $\square$

9. It's given that some two of the roots of the polynomial  $P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$  add to 4. Find all of the roots.

**Solution:** Let  $r_1, r_2, r_3, r_4$  be the roots of  $P$  and  $r_1 + r_2 = 4$ . By Vieta's Formulas, we have  $r_3 + r_4 = 6 - 4 = 2$ . Moreover,

$$\begin{aligned} 18 &= r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 \\ &= (r_1 + r_2)(r_3 + r_4) + r_1r_2 + r_3r_4 \\ \Rightarrow r_1r_2 + r_3r_4 &= 18 - 4 \cdot 2 = 10 \end{aligned}$$

and

$$\begin{aligned} 25 &= r_1r_2r_3r_4 \\ &= r_1r_2(10 - r_1r_2) \\ \Rightarrow r_1r_2 &= 5 = r_3r_4. \end{aligned}$$

Thus,  $r_1$  and  $r_2$  are the roots of  $x^2 - 4x + 5$  and  $r_3$  and  $r_4$  are the roots of  $x^2 - 2x + 5$ . By the quadratic formula, the roots are  $2 \pm i$  and  $1 \pm 2i$ .  $\square$

10. Let  $a, b, c$  be real numbers. Prove that  $a, b$ , and  $c$  are all nonnegative if and only if  $a + b + c$ ,  $ab + bc + ca$ , and  $abc$  are all nonnegative.

**Solution:** If  $a, b$ , and  $c$  are all nonnegative, then each of  $ab, bc, ca$ , and  $abc$  are nonnegative, so  $a + b + c, ab + bc + ca$ , and  $abc$  are all nonnegative. Now suppose  $a + b + c = s_2, ab + bc + ca = s_1$ , and  $abc = s_0$  are all nonnegative. Then the polynomial  $p(x) = x^3 - s_2x^2 + s_1x - s_0$  has roots  $a, b, c$ . Note that for  $x < 0$ , each term of  $p$  is negative, so  $p(x) < 0$ . Thus,  $p$  has no negative real roots, so  $a, b$ , and  $c$  are all nonnegative.  $\square$

## 6 Fundamental Theorem of Arithmetic

1. Find the smallest positive integer  $n$  such that  $n/2$  is a perfect square and  $n/3$  is a perfect cube.

**Solution:** Write  $n = 2^a 3^b 5^c \dots$  as a product of primes. For an integer to be a perfect square, all of the powers in its prime factorization must be even. Similarly, for an integer to be a perfect cube, all of the powers in its prime factorization must be a multiple of 3. Thus, we must have  $a - 1, b, c, \dots$  all be even and  $a, b - 1, c, \dots$  all be multiples of 3. We can take  $c, \dots$  to be 0, and we can find the smallest values of  $a$  and  $b$  to be  $a = 3$  and  $b = 4$ , giving us  $n = 2^3 \cdot 3^4 = 648$ .  $\square$

2. Let  $a$  and  $b$  be positive integers. Suppose there exists some prime  $p$  such that  $p \mid a$  but  $p \nmid b$ . Prove that  $a \nmid b$ .

**Solution:** We are given that  $p$  appears in the prime factorization of  $a$  but not  $b$ . For the sake of contradiction, suppose  $a$  divides  $b$ . Then there exists a positive integer  $k$  such that  $ak = b$ . Then  $p$  will appear in the prime factorization of the left side of the equation. Since prime factorizations are unique,  $p$  must appear in the prime factorization of  $b$ , a contradiction. Thus,  $a$  does not divide  $b$ .  $\square$

3. Suppose  $a$  and  $b$  are positive integers with prime factorizations  $a = p_1^{e_1} \cdots p_k^{e_k}$  and  $b = p_1^{f_1} \cdots p_k^{f_k}$ , where  $e_i, f_i$  are nonnegative integers. Prove that  $a \mid b$  if and only if  $e_i \leq f_i$  for all  $i = 1, 2, \dots, k$ .

**Solution:** Suppose  $a \mid b$ . Then there exists a positive integer  $n$  such that  $an = b$ . Rewrite  $p_1, \dots, p_j$  to include all primes in the factorizations of  $a$ ,  $b$ , and  $n$ . Write  $n = p_1^{g_1} \cdots p_j^{g_j}$ . Then we have  $an = b \Rightarrow p_1^{e_1+g_1} \cdots p_j^{e_j+g_j} = p_1^{f_1} \cdots p_j^{f_j}$ , and thus  $e_i \leq e_i + g_i = f_i$  for all  $i = 1, 2, \dots, j$ .

Now suppose  $e_i \leq f_i$  for all  $i = 1, 2, \dots, k$ . Then  $b/a = p_1^{f_1-e_1} \cdots p_j^{f_j-e_j}$ . Since  $f_i - e_i$  are all nonnegative integers, the product of primes is an integer. Thus,  $b/a$  is an integer, so  $a$  divides  $b$ .  $\square$

4. Let  $S$  be a set of 9 positive integers which are not divisible by any prime greater than 5. Prove that some two distinct integers in  $S$  have a product that is a perfect square.

**Solution:** We can write each element of  $S$  as  $2^{a_i} \cdot 3^{b_i} \cdot 5^{c_i}$  for nonnegative integers  $a_i, b_i, c_i$ . Note that there are  $2^3 = 8$  possible parities of the triple  $(a_i, b_i, c_i)$ . Since  $S$  contains  $9 > 8$  elements, by the Pigeonhole Principle some two elements of  $S$  have the same parity triple. Thus, their product will have all exponents to an even power, meaning their product is a perfect square.  $\square$

5. Let  $p$  be a prime number and  $n$  a positive integer. Find the largest integer  $k$  such that  $p^k$  divides  $n!$ . Your answer may involve the floor function ( $\lfloor x \rfloor$  is the largest integer not greater than  $x$ ).

**Solution:** Expand  $n!$  as the product of the prime factorizations of each of  $1, 2, \dots, n$ . We will count how many times  $p$  appears in each of these factorizations.

For some positive integer  $j$ , how many integers in  $\{1, 2, \dots, n\}$  are divisible by  $p^j$ ? This is precisely the set  $p^j, 2p^j, \dots, \lfloor \frac{n}{p^j} \rfloor p^j$ , so there are exactly  $\lfloor \frac{n}{p^j} \rfloor$  elements divisible by  $p^j$ .

Now, note that for any element divisible by  $p^j$ , it is counted in each of the sets containing  $\lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p^2} \rfloor, \dots, \lfloor \frac{n}{p^j} \rfloor$  elements. Thus, there are exactly  $\sum_{j=1}^{\infty} \lfloor \frac{n}{p^j} \rfloor$  factors of  $p$  in the prime factor expansion of  $n!$ .  $\square$

6. Find all positive integers  $n$  with an odd number of positive integer divisors.

**Solution:** Pair up the divisors of  $n$  as  $(a, b)$  if  $ab = n$ . Then we will have an odd number of divisors if and only if we have some pair that contains only one number. I.e., if  $a = b$ . This can happen if and only if  $n$  is a perfect square.  $\square$

7. Let  $n$  be a positive integer with prime factorization  $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ , where  $e_1, \dots, e_k$  are nonnegative integers. Find the number of positive divisors of  $n$  in terms of  $e_1, \dots, e_k$ .

**Solution:** Note that every divisor of  $n$  must be of the form  $p_1^{f_1} \cdots p_k^{f_k}$  for some nonnegative integers  $f_i \leq e_i$ . For each  $f_i$ , there are  $e_i + 1$  possible choices for its value (including  $f_i = 0$ ). Thus, there are  $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$  positive divisors of  $n$ .  $\square$

8. Let  $p$  and  $q$  be consecutive primes greater than 3 (i.e. there are no primes between  $p$  and  $q$ ). Prove that  $p + q$  has at least 5 positive divisors.

**Solution:** We have two cases.

Case 1:  $p + q$  is a power of 2. Then  $p + q \geq 5 + 7 = 12$ , so  $p + q = 2^k$  for some  $k \geq 4$ . Thus,  $p + q$  has  $k + 1 \geq 5$  positive divisors.

Case 2:  $p + q$  is not a power of 2. Then we can write  $p + q = 2^k m$  for some integer  $k \geq 1$  (since  $p + q$  is even) and odd integer  $m \geq 3$ . If  $k \geq 2$ , then  $p + q$  will have at least 6 divisors: 1, 2, 4,  $m$ ,  $2m$ , and  $4m$ . Thus, we can assume  $k = 1$ . We are left to show that  $m$  has at least 3 positive divisors.

For the sake of contradiction, suppose  $m$  has fewer than 3 positive divisors. Since  $m \geq 3$ ,  $m$  must have at least 2 positive divisors. But the only way for  $m$  to have 2 divisors is if  $m$  is prime. Then  $m = (p + q)/2$ , so  $m$  is a prime number between  $p$  and  $q$ , a contradiction. Thus,  $m$  cannot be prime, so  $m$  has at least 3 positive divisors.  $\square$

## 7 Greatest Common Divisor

- Let  $a > b$  be positive integers. Prove that  $\gcd(a, b) = \gcd(a - b, b)$ .

**Solution:** First, we will show that all common divisors of  $a - b$  and  $b$  are also common divisors of  $a$  and  $b$ . This will prove that  $\gcd(a - b, b) \leq \gcd(a, b)$ . Then we will show that  $\gcd(a, b)$  is a common divisor of  $a - b$  and  $b$ . This will prove that  $\gcd(a, b) \leq \gcd(a - b, b)$ . Thus, we must have  $\gcd(a, b) = \gcd(a - b, b)$ .

First, let  $d$  be a common divisor of  $a - b$  and  $b$ . Then  $d | (a - b)$  and  $d | b$ , so  $b/d = n$  is an integer and  $(a - b)/d = m$  is an integer. Hence  $a/d = (a - b)/d + b/d = m + n$  is an integer, so  $d | a$ . Thus,  $d$  is a common divisor of  $a$  and  $b$ .

Next, write  $g = \gcd(a, b)$ ,  $a = gx$ , and  $b = gy$  for some positive integers  $x$  and  $y$  (furthermore,  $x$  and  $y$  must be coprime, but we don't need that fact). Then  $a - b = g(x - y)$ , so  $g$  divides  $a - b$ . Thus,  $g$  is a common divisor of  $a - b$  and  $b$ .

Thus,  $\gcd(a, b) = \gcd(a - b, b)$ . □

- The Fibonacci sequence  $(F_n)$  satisfies  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 2$ . Prove that  $\gcd(F_n, F_{n+1}) = 1$  for all  $n \geq 1$ .

**Solution:** We will use induction on  $n$ . For our base case, we have  $\gcd(F_1, F_2) = \gcd(1, 1) = 1$ . For our inductive step, suppose  $\gcd(F_n, F_{n+1}) = 1$  for some  $n \geq 1$ . Then we have  $\gcd(F_{n+1}, F_{n+2}) = \gcd(F_{n+1}, F_{n+1} + F_n) = \gcd(F_{n+1}, F_{n+1} + F_n - F_{n+1}) = \gcd(F_{n+1}, F_n) = 1$ . Thus, by induction,  $\gcd(F_n, F_{n+1}) = 1$  for all  $n \geq 1$ . □

- Let  $n$  be a positive integer. Prove that  $21n + 4$  and  $14n + 3$  are coprime.

**Solution:** We have

$$\begin{aligned}\gcd(21n + 4, 14n + 3) &= \gcd(21n + 4 - (14n + 3), 14n + 3) \\&= \gcd(7n + 1, 14n + 3) \\&= \gcd(7n + 1, 14n + 3 - (7n + 1)) \\&= \gcd(7n + 1, 7n + 2) \\&= \gcd(7n + 1, 7n + 2 - (7n + 1)) \\&= \gcd(7n + 1, 1) \\&= 1.\end{aligned}$$

□

- Let  $a, b, c$  be positive integers such that  $a$  divides  $bc$ . If  $\gcd(a, b) = 1$ , prove that  $a$  divides  $c$ .

**Solution:** Since  $\gcd(a, b) = 1$ , the primes that appear in the prime factorizations of  $a$  and  $b$  must be disjoint. I.e., no prime that divides  $a$  can divide  $b$ , and no prime that divides  $b$  can divide  $a$ . But for  $a$  to divide  $bc$ , we must have the prime factorization of  $bc$  contain the prime factorization of  $a$ . These prime factors must come entirely from  $c$ , so we must have  $a$  divide  $c$ .  $\square$

5. Let  $p$  be a prime number. Prove that  $\sqrt{p}$  is irrational (i.e. there exist no positive integers  $n$  and  $m$  such that  $\sqrt{p} = n/m$ ).

**Solution:** For the sake of contradiction, suppose  $\sqrt{p} = n/m$  for coprime positive integers  $n$  and  $m$ . Squaring both sides and rearranging, we have  $p \cdot m^2 = n^2$ . Hence  $p$  divides  $n$ . Write  $n = pk$  for some positive integer  $k$ . Then  $p \cdot m^2 = p^2 \cdot k^2 \Rightarrow m^2 = p \cdot k^2$ . Hence  $p$  divides  $m$ . But  $m$  and  $n$  are coprime, a contradiction. Thus,  $\sqrt{p}$  must be irrational.  $\square$

6. Let  $p$  be a prime number and  $r$  a positive rational number that is not an integer. Prove that  $p^r$  is irrational.

**Solution:** Without loss of generality, we may assume  $r < 1$ . Otherwise, we can divide out integral powers of  $p$  without affecting the irrationality of  $p^r$ . Write  $r = a/b$  for some coprime positive integers  $a$  and  $b$  with  $b \geq 2$  and  $a < b$ .

For the sake of contradiction, suppose  $p^{a/b} = n/m$  for coprime positive integers  $n$  and  $m$ . Raising both sides to the  $b$  power and rearranging, we have  $p^a \cdot m^b = n^b$ . Hence  $p$  divides  $n$ . Write  $n = pk$  for some positive integer  $k$ . Then  $p^a \cdot m^b = p^b \cdot k^b \Rightarrow m^b = p^{b-a} \cdot k^b$ . Since  $b - a \geq 1$ ,  $p$  divides  $m$ . But  $m$  and  $n$  are coprime, a contradiction. Thus,  $p^r$  must be irrational.  $\square$

7. Let  $a$  and  $b$  be coprime positive integers. If  $ab$  is a perfect square, prove that  $a$  and  $b$  are each perfect squares.

**Solution:** Since  $ab$  is a perfect square, the primes in its prime factorization must all be raised to an even power. Since  $a$  and  $b$  are coprime, they share no primes in their prime factorizations. Thus, each prime that occurs in the prime factorization of  $a$  must have the same power as that prime in the prime factorization of  $ab$ , so  $a$  must have all of its primes in its prime factorization raised to an even power. Thus,  $a$  is a perfect square. Similarly,  $b$  must be a perfect square.  $\square$

8. Let  $a, b, n$  be positive integers satisfying  $n^2 < a < b < (n+1)^2$ . Prove that  $ab$  cannot be a perfect square.

**Solution:** Suppose  $\gcd(a, b) = g$ . Then  $a = gx$  and  $b = gy$  for coprime positive integers  $x$  and  $y$ .

For the sake of contradiction, suppose  $ab$  is a perfect square. Then  $g^2xy$  is a perfect square, and thus  $xy$  is a perfect square. Since  $x$  and  $y$  are coprime, they must each be a perfect square, say  $x = w^2$  and  $y = z^2$ . Then we have

$$\begin{aligned} n^2 &< a < b < (n+1)^2 \\ \Rightarrow n^2 &< gw^2 < gz^2 < (n+1)^2 \\ \Rightarrow n &< \sqrt{g}w < \sqrt{g}z < n+1 \\ \Rightarrow 0 &< \sqrt{g}z - \sqrt{g}w < (n+1) - n = 1 \\ \Rightarrow 0 &< z - w < 1/\sqrt{g} \leq 1. \end{aligned}$$

Thus,  $z$  and  $w$  are integers with  $0 < z - w < 1$ , a contradiction.  $\square$

9. The least common multiple of two positive integers  $a$  and  $b$  is the smallest positive integer that is a multiple of  $a$  and a multiple of  $b$ , and is denoted  $\text{lcm}(a, b)$ . Prove that  $\text{lcm}(a, b) = \frac{ab}{\gcd(a, b)}$ .

**Solution:** Let  $p_1, \dots, p_k$  be the primes that appear in the prime factorizations of  $a$  or  $b$ . Write  $a = p_1^{e_1} \cdots p_k^{e_k}$  and  $b = p_1^{f_1} \cdots p_k^{f_k}$ . Then we have  $\gcd(a, b) = p_1^{\min(e_1, f_1)} \cdots p_k^{\min(e_k, f_k)}$  and  $\text{lcm}(a, b) = p_1^{\max(e_1, f_1)} \cdots p_k^{\max(e_k, f_k)}$ , so

$$\gcd(a, b) \cdot \text{lcm}(a, b) = p_1^{\min(e_1, f_1) + \max(e_1, f_1)} \cdots p_k^{\min(e_k, f_k) + \max(e_k, f_k)} = p_1^{e_1 + f_1} \cdots p_k^{e_k + f_k} = ab,$$

$$\text{Thus, } \text{lcm}(a, b) = \frac{ab}{\gcd(a, b)}. \quad \square$$

10. For any positive integers  $a$  and  $b$ , prove that  $\gcd(a, b) + \text{lcm}(a, b) \geq a + b$ . When does equality occur?

**Solution:** Let  $\gcd(a, b) = g$ . Then  $a = gx$  and  $b = gy$  for coprime positive integers  $x$  and  $y$ , and  $\text{lcm}(a, b) = gxy$ . Thus,

$$\begin{aligned} \gcd(a, b) + \text{lcm}(a, b) &\geq a + b \\ \iff g + gxy &\geq gx + gy \\ \iff 1 + xy &\geq x + y \\ \iff 1 - x - y + xy &\geq 0 \\ \iff (x-1)(y-1) &\geq 0. \end{aligned}$$

Since  $x$  and  $y$  are positive integers, the inequality always holds. We get equality if and only if  $x = 1$  or  $y = 1 \iff a = g$  or  $b = g$ , i.e. if and only if  $a \mid b$  or  $b \mid a$ .  $\square$

11. Let  $a$  and  $b$  be coprime positive integers. Define  $c = a + b$ ,  $d = a^2 - ab + b^2$ . Prove that  $\gcd(c, d) = 1$  or  $3$ .

**Solution:** We have

$$\begin{aligned} \gcd(c, d) &= \gcd(c, d - c^2) \\ &= \gcd(a+b, -3ab) \\ &= \gcd(a+b, 3ab). \end{aligned}$$

Now, suppose  $k > 1$  divides  $a + b$ . Then we claim  $k$  does not divide  $ab$ . For the sake of contradiction, suppose  $k$  divides  $ab$ . Then some prime factor of  $k$ , say  $p$ , must divide  $a$  or  $b$ . Without loss of generality, suppose  $p$  divides  $a$ . Then  $p$  also divides  $(a + b) - a = b$ . But  $a$  and  $b$  are coprime, a contradiction. Thus, we have  $\gcd(a + b, 3ab) = \gcd(a + b, 3) = 1$  or 3.  $\square$

12. Let  $a, b, n$  be positive integers with  $n > 1$ . Prove that  $\gcd(n^a - 1, n^b - 1) = n^{\gcd(a, b)} - 1$ .

**Solution:** We will use the fact that  $n^k$  and  $n^j - 1$  are coprime for all positive integers  $k$  and  $j$ . Indeed, suppose some prime  $p$  divides  $n^k$ . Then  $p$  divides  $n$ , so  $p$  divides  $n^j$ , and  $p$  does not divide  $n^j - 1$ . So all primes in the factorization of  $n^k$  do not appear in the factorization of  $n^j - 1$ , so they are coprime.

Now, write  $\gcd(a, b) = g$ ,  $a = gx$ , and  $b = gy$  for coprime positive integers  $x$  and  $y$ . We will use strong induction on the value of  $s = x + y$ . For our base case of  $s = 2$ , we must have  $x = y = 1$ , so  $\gcd(n^g - 1, n^g - 1) = n^g - 1$ . For our inductive step, suppose  $\gcd(n^{gx} - 1, n^{gy} - 1) = n^g - 1$  for all  $x$  and  $y$  satisfying  $x + y \leq s$  for some  $s \geq 2$ . Suppose  $x + y = s + 1$  and without loss of generality assume  $x \geq y$ . Then

$$\begin{aligned}\gcd(n^{gx} - 1, n^{gy} - 1) &= \gcd(n^{gx} - 1 - (n^{gy} - 1), n^{gy} - 1) \\ &= \gcd(n^{gy}(n^{g(x-y)} - 1), n^{gy} - 1) \\ &= \gcd(n^{g(x-y)} - 1, n^{gy} - 1).\end{aligned}$$

Note that  $\gcd(x - y, y) = \gcd(x, y) = 1$  and  $(x - y) + y = x = s + 1 - y \leq s$ , so we can use our inductive hypothesis to get  $\gcd(n^{gx} - 1, n^{gy} - 1) = \gcd(n^{g(x-y)} - 1, n^{gy} - 1) = n^g - 1$ . Thus, by induction the result holds for all  $x$  and  $y$ , and since  $g$  was arbitrary the result holds for all positive integers  $a$  and  $b$ .  $\square$

## 8 Modular Arithmetic

- Calculate the remainder when  $1111 \cdot 2222 \cdot 3333 \cdot 4444$  is divided by 5.

**Solution:** Note that

$$\begin{aligned} 1111 &= 1110 + 1 = 222 \cdot 5 + 1 \\ \Rightarrow 1111 &\equiv 1 \pmod{5}. \end{aligned}$$

From this we have

$$\begin{aligned} 2222 &\equiv 2 \pmod{5} \\ 3333 &\equiv 3 \pmod{5} \\ 4444 &\equiv 4 \pmod{5} \end{aligned}$$

Thus,

$$1111 \cdot 2222 \cdot 3333 \cdot 4444 \equiv 1 \cdot 2 \cdot 3 \cdot 4 \equiv 24 \equiv 4 \pmod{5}$$

so the product has a remainder of 4 when divided by 5.  $\square$

- Determine which of the following numbers are multiples of 11: (i)  $21^5 - 12^5$ , (ii)  $21^{50} - 12^{50}$ , (iii)  $21^{21^{50}} - 12^{12^{50}}$ .

**Solution:** Note that  $21 \equiv -1 \pmod{11}$  and  $12 \equiv 1 \pmod{11}$ . Thus,

$$\begin{aligned} 21^5 - 12^5 &\equiv (-1)^5 - 1^5 \equiv -1 - 1 \equiv -2 \pmod{11} \\ 21^{50} - 12^{50} &\equiv (-1)^{50} - 1^{50} \equiv 1 - 1 \equiv 0 \pmod{11} \\ 21^{21^{50}} - 12^{12^{50}} &\equiv (-1)^{21^{50}} - 1^{12^{50}} \equiv -1 - 1 \equiv -2 \pmod{11} \end{aligned}$$

(since  $21^{50}$  is odd for the last line). Thus, only  $21^{50} - 12^{50}$  is a multiple of 11.  $\square$

- Find the units digit of  $9^{8^7}$  and  $7^{8^9}$ .

**Solution:** Finding the units digit is the same as finding the remainder when dividing by 10, which is the same as taking modulo 10. Since  $9^{8^7} \equiv (-1)^{8^7} \equiv 1 \pmod{10}$ , the units digit of  $9^{8^7}$  is 1, and since  $7^{8^9} \equiv 7^{2^{27}} \equiv 49^{2^6} \equiv (-1)^{2^6} \equiv 1 \pmod{10}$ , the units digit of  $7^{8^9}$  is also 1.  $\square$

- Let  $P$  be the product of all positive integers less than 100 that are not multiples of 5. Find the units digit of  $P$ .

**Solution:** Taking modulo 10, we have

$$\begin{aligned} P &\equiv (1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8 \cdot 9)^{10} \\ &\equiv (1 \cdot 2 \cdot 3 \cdot 4 \cdot (-4) \cdot (-3) \cdot (-2) \cdot (-1))^{10} \\ &\equiv (1 \cdot 2 \cdot 3 \cdot 4)^{20} \\ &\equiv 4^{20} \\ &\equiv (4^5)^4 \\ &\equiv 4^4 \\ &\equiv 6 \pmod{10} \end{aligned}$$

So the units digit of  $P$  is 6. □

5. Let  $S(n)$  denote the sum of the digits of the positive integer  $n$ . Prove that  $n \equiv S(n) \pmod{9}$ .

**Solution:** Write  $n = \overline{d_k \cdots d_0}$  for some digits  $d_0, \dots, d_k$ . Then  $S(n) = d_0 + \cdots + d_k$  and  $n = d_0 + 10d_1 + \cdots + 10^k d_k$ . Since  $10^i \equiv 1^i \equiv 1 \pmod{9}$  for all  $i \geq 0$ , we have

$$\begin{aligned} n &\equiv d_0 + 10d_1 + \cdots + 10^k d_k \\ &\equiv d_0 + d_1 + \cdots + d_k \\ &\equiv S(n) \pmod{9}. \end{aligned}$$

□

6. Prove that  $1 \cdot 3 \cdot 5 \cdots 2025 + 2 \cdot 4 \cdot 6 \cdots 2026$  is divisible by 2027.

**Solution:** We have

$$\begin{aligned} 1 \cdot 3 \cdot 5 \cdots 2025 + 2 \cdot 4 \cdot 6 \cdots 2026 &\\ \equiv 1 \cdot 3 \cdot 5 \cdots 2025 + (-2025) \cdot (-2023) \cdot (-2021) \cdots (-1) &\\ \equiv 1 \cdot 3 \cdot 5 \cdots 2025 + (-1)^{1013} \cdot 2025 \cdot 2023 \cdot 2021 \cdots 1 &\\ \equiv 1 \cdot 3 \cdot 5 \cdots 2025 - 2025 \cdot 2023 \cdot 2021 \cdots 1 &\\ \equiv 0 \pmod{2027} & \end{aligned}$$

Thus,  $1 \cdot 3 \cdot 5 \cdots 2025 + 2 \cdot 4 \cdot 6 \cdots 2026$  is divisible by 2027. □

7. Find all pairs of integers  $(x, y)$  such that  $x^2 = 4y + 3$ .

**Solution:** We claim there are no solutions. For the sake of contradiction, suppose  $x^2 = 4y + 3$  for some integers  $x, y$ . Then taking modulo 4 yields  $x^2 \equiv 3 \pmod{4}$ . But the only possibilities for  $x \pmod{4}$  are  $x \equiv 0, 1, 2, 3$ , which give us  $x^2 \equiv 0, 1, 0, 1 \pmod{4}$ . Thus, we have  $x^2 \not\equiv 3 \pmod{4}$ , a contradiction. □

8. Find the smallest positive integer  $n$  such that 77 divides  $n^2 - 70n + 89$ .

**Solution:** We'll find the positive integers such that 7 divides  $n^2 - 70n + 89$  and 11 divides  $n^2 - 70n + 89$ , then take the intersection.

First, we have  $n^2 - 70n + 89 \equiv n^2 - 9 \equiv (n+3)(n-3) \pmod{7}$ . Thus,  $n^2 - 70n + 89 \equiv 0 \pmod{7} \iff n \equiv \pm 3 \pmod{7}$ , so  $n \in \{3, 4, 10, 11, 17, 18, \dots\}$ .

Next, we have  $n^2 - 70n + 89 \equiv n^2 + 7n + 12 \equiv (n+3)(n+4) \pmod{11}$ . Thus,  $n^2 - 70n + 89 \equiv 0 \pmod{11} \iff n \equiv -3, -4 \pmod{11}$ , so  $n \in \{7, 8, 18, \dots\}$ .

Finally, the smallest positive integer  $n$  such that 77 divides  $n^2 - 70n + 89$  is 18. □

9. Find the largest integer  $k$  such that there exist increasing arithmetic sequences of integers  $(a_n)$  and  $(b_n)$  satisfying  $a_0 = b_0 = 1$  and  $a_k b_k = 1000$ .

**Solution:** Let  $a_n = an + 1$  and  $b_n = bn + 1$  for some positive integers  $a$  and  $b$ . Then  $a_k b_k = (ak + 1)(bk + 1) = 1000$ . Taking this mod  $k$ , we have  $1 \equiv 1000 \pmod{k}$ , so  $k \mid 999 = 3^3 \cdot 37$ . Now, since  $(a_n)$  and  $(b_n)$  are increasing sequences of positive integers, we have  $a_n \geq n$  and  $b_n \geq n$ . Thus,  $a_k b_k \geq k \cdot k \Rightarrow k^2 \leq 1000 \Rightarrow k \leq 31$ , so  $k \mid 3^3$ .

Without loss of generality, assume  $a \leq b$ . Checking  $k = 27$ , we have  $1000 = (27a + 1)(27b + 1) \leq (27a + 1)^2 \Rightarrow 27a + 1 \leq 31 \Rightarrow a = 1$ . Then  $28(27b + 1) = 1000$ , but 28 does not divide 1000, so  $k = 27$  does not work. Checking  $k = 9$ , we find  $a = 1, b = 11$  works. Thus,  $k = 9$  is the largest such integer.  $\square$

## 9 Fermat's Little Theorem

- Find the remainder when  $2^{121}$  is divided by 11.

**Solution:** By Fermat's Little Theorem, we have

$$2^{121} \equiv 2 \cdot (2^{10})^{12} \equiv 2 \cdot (1)^{12} \equiv 2 \pmod{11}.$$

So  $2^{121}$  has a remainder of 2 when divided by 11.  $\square$

- Find all primes  $p$  such that  $p$  divides  $29^p + 1$ .

**Solution:** Note that  $p$  divides  $29^p + 1$  if and only if  $29^p + 1 \equiv 0 \pmod{p}$ . If  $p = 29$ , then  $29^p + 1 \equiv 1 \not\equiv 0 \pmod{p}$ , so 29 doesn't work. Now assume  $p \neq 29$ . By Fermat's Little Theorem, we have

$$0 \equiv 29^p + 1 \equiv 29 + 1 \equiv 30 \pmod{p}.$$

Hence  $p \mid 30$ , so  $p = 2, 3, 5$ . We can check that all of these primes satisfy.  $\square$

- Let  $p$  be prime. Prove that  $p$  divides  $2^{2p+1} - 2^{p+2}$ .

**Solution:** Note that  $2^{2p+1} - 2^{p+2}$  is always even, so  $p = 2$  works. Now assume  $p \geq 3$ . By Fermat's Little Theorem, we have

$$2^{2p+1} - 2^{p+2} \equiv 2 \cdot (2^p)^2 - 2^2 \cdot 2^p \equiv 2 \cdot 2^2 - 2^2 \cdot 2 \equiv 0 \pmod{p}.$$

$\square$

- Prove that there are infinitely many integers  $n \geq 2$  such that  $n \mid (2^n - 4)$ .

**Solution:** Let  $n = 2p$  for any odd prime  $p$ . Then  $2^{2p} - 4$  is even, so  $2 \mid 2^{2p} - 4$ , and by Fermat's Little Theorem,  $2^{2p} - 4 \equiv 2^2 - 4 \equiv 0 \pmod{p}$ , so  $p$  divides  $2^{2p} - 4$ . Thus, since 2 and  $p$  are coprime, we must have  $2p \mid 2^{2p} - 4$ . Since there are infinitely many (odd) primes  $p$ , there are infinitely many integers  $n = 2p$ .  $\square$

- Define  $a_1 = 4$  and  $a_n = 4^{a_{n-1}}$  for  $n \geq 2$ . Find the remainder when  $a_{2025}$  is divided by 7.

**Solution:** We have

$$a_{2025} \equiv 4^{a_{2024}} \pmod{7}.$$

In order to use Fermat's Little Theorem, we need to reduce  $a_{2024}$  modulo  $7 - 1 = 6$ . Clearly  $a_n$  is always even, and  $a_n \equiv 4^{a_{n-1}} \equiv 1^{a_{n-1}} \equiv 1 \pmod{3}$ , so  $a_n \equiv 4 \pmod{6}$ . Thus,

$$a_{2025} \equiv 4^{a_{2024}} \equiv 4^4 \equiv 4 \pmod{7}.$$

$\square$

6. Let  $a$  and  $b$  be integers and  $p$  be an odd prime. Suppose  $p \mid (a^p + b^p)$ . Prove that  $p^2 \mid (a^p + b^p)$ .

**Solution:** Since  $p$  is odd, we can use the identity

$$a^p + b^p = (a+b)(a^{p-1} - a^{p-2}b + a^{p-3}b^2 - \cdots + a^2b^{p-3} - ab^{p-2} + b^{p-1}).$$

First, we're given  $a^p + b^p \equiv 0 \pmod{p}$ , so  $a^p \equiv -b^p \pmod{p}$ . By Fermat's Little Theorem, we have  $a \equiv -b \pmod{p}$ , so  $a+b \equiv 0 \pmod{p}$ . Thus, the first factor of  $a^p + b^p$  is divisible by  $p$ .

Next, we claim that  $p$  divides  $a^{p-1} - a^{p-2}b + a^{p-3}b^2 - \cdots + a^2b^{p-3} - ab^{p-2} + b^{p-1}$ . Indeed, since  $b \equiv -a \pmod{p}$ ,

$$\begin{aligned} & a^{p-1} - a^{p-2}b + a^{p-3}b^2 - \cdots + a^2b^{p-3} - ab^{p-2} + b^{p-1} \\ & \equiv a^{p-1} - a^{p-2}(-a) + a^{p-3}(-a)^2 - \cdots + a^2(-a)^{p-3} - a(-a)^{p-2} + (-a)^{p-1} \\ & \equiv a^{p-1} + a^{p-1} + a^{p-1} \dots \\ & \equiv pa^{p-1} \\ & \equiv 0 \pmod{p}. \end{aligned}$$

Thus, both factors of  $a^p + b^p$  are divisible by  $p$ , so  $a^p + b^p$  is divisible by  $p^2$ .  $\square$

7. Let  $n$  be a nonnegative integer. Prove that  $2^{2^{6n+2}} + 3$  is a multiple of 19.

**Solution:** As in problem 5, we aim to reduce the exponent modulo 18. Note that  $2^{6n+2}$  is always even, and  $2^{6n+2} \equiv 4 \cdot 8^{2n} \equiv 4 \cdot (-1)^{2n} \equiv 4 \pmod{9}$ . Thus,  $2^{6n+2} \equiv 4 \pmod{18}$ , so by Fermat's Little Theorem,

$$2^{2^{6n+2}} + 3 \equiv 2^4 + 3 \equiv 19 \equiv 0 \pmod{19}.$$

Thus,  $2^{2^{6n+2}} + 3$  is a multiple of 19.  $\square$

8. Let  $a$  and  $b$  be integers. Define  $c = ab(a^{60} - b^{60})$ . Prove that the first six prime numbers divide  $c$ , and find two other prime factors of  $c$ .

**Solution:** Note that if  $k \mid 60$ , then  $a^k - b^k \mid a^{60} - b^{60}$  and thus  $ab(a^k - b^k)$  divides  $c$ . Now, if  $k+1 = p$  for some prime  $p$ , then by Fermat's Little Theorem we have

$$ab(a^k - b^k) \equiv a^p b - ab^p \equiv ab - ab \equiv 0 \pmod{p}$$

and thus  $p$  divides  $c$ . Our possible values of  $k$  are the divisors of 60. These include  $k = 1, 2, 4, 6, 10, 12$ , which correspond to the first six primes  $p = 2, 3, 5, 7, 11, 13$  all dividing  $c$ . Additionally, we can take  $k = 30, 60$  to see that the primes  $p = 31, 61$  also divide  $c$ .  $\square$

9. Prove that the sequence  $2^n - 3$ ,  $n \geq 2$ , has an infinite subsequence whose terms are pairwise coprime.

**Solution:** We will construct such a subsequence  $a_n$ . Let  $a_0 = 2^3 - 3 = 5$ . For each  $n \geq 1$ , let  $S_n = \{p_1, \dots, p_k\}$  be the set of all primes that divide at least one of the integers  $a_0, \dots, a_{n-1}$ . We can construct  $a_n$  to be coprime to each prime in  $S_n$  and thus coprime to each of  $a_0, \dots, a_{n-1}$ . By Fermat's Little Theorem, we know that

$$2^{(p_1-1)\cdots(p_k-1)} \equiv 2^0 \equiv 1 \pmod{p_i}$$

for all  $p_i \in S_n$ . Thus,  $p_i$  divides  $2^{(p_1-1)\cdots(p_k-1)} - 1$ . Set  $a_n = 2^{(p_1-1)\cdots(p_k-1)} - 3$ . Then  $p_\ell$  divides  $a_n + 2$ , and since  $p_\ell$  is odd (since  $2^n - 3$  is odd), we know  $p_\ell$  does not divide  $a_n$ . Thus,  $a_n$  is coprime to  $a_0, \dots, a_{n-1}$ , and by construction  $(a_n)$  is an infinite sequence that is pairwise coprime.  $\square$

## 10 Integer Polynomials

- Find all rational roots of  $3x^3 - 5x^2 + 5x - 2$ .

**Solution:** Let  $p(x) = 3x^3 - 5x^2 + 5x - 2$ . By the rational root theorem, all possible rational roots of  $p$  are  $\pm 1, \pm 2, \pm 1/3, \pm 2/3$ . Checking, we have  $p(2/3) = 0$ . By polynomial division,  $p(x) = (3x - 2)(x^2 - x + 1)$ . Since  $(-1)^2 - 4(1)(1) = -3 < 0$ , the other two roots of  $p$  are imaginary. Thus,  $2/3$  is the only rational root of  $p$ .  $\square$

- Does there exist an integer polynomial  $p$  with  $p(1) = 1$  and  $p(10) = 11$ ?

**Solution:** There cannot exist such an integer polynomial  $p$ . For the sake of contradiction, suppose such a  $p$  exists. Then we must have  $(10 - 1) \mid (p(10) - p(1))$ , or  $9 \mid 10$ , a contradiction.  $\square$

- Let  $p(x)$  be a monic integer polynomial (an integer polynomial with leading coefficient 1). Suppose  $p(r) = 0$  for some rational number  $r$ . Prove that  $r$  is an integer.

**Solution:** By the rational root theorem, if  $r = a/b$  is a rational root of  $p$ , then  $b \mid 1$ . Thus,  $b = \pm 1$ , so  $r$  is an integer.  $\square$

- Find all rational numbers  $x$  such that  $3x^3 + x^2 + \frac{2}{x}$  is an integer.

**Solution:** Suppose  $3x^3 + x^2 + \frac{2}{x} = k$  for some integer  $k$ . Then we have  $3x^4 + x^3 - kx + 2 = 0$ , so  $x$  is a rational root of  $p(r) = 3r^4 + r^3 - kr + 2$ . By the rational root theorem, we can write  $x = a/b$  for  $b \mid 3$  and  $a \mid 2$ . In particular, we have  $\frac{2}{x} = \frac{2b}{a}$  an integer. Thus, we just need  $3x^3 + x^2$  to be an integer.

If  $x$  is an integer, then  $x = \pm 1, \pm 2$ , which all work. If  $x$  is not an integer, then  $x = \pm 1/3, \pm 2/3$ . These yield  $3x^3 + x^2 = (\pm 1 + 1)/9, (\pm 8 + 4)/9$ , of which only  $x = -1/3$  gives an integer. Thus, the only solutions are  $x \in \{\pm 1, \pm 2, -1/3\}$ .  $\square$

- Let  $p(x)$  be an integer polynomial. Set  $n = p(1) - 2p(50) + p(99)$ . Find an odd prime that is a factor of  $n$  for all choices of  $p$ .

**Solution:** For any integer polynomial  $p$ , we have  $49 \mid (p(99) - p(50))$  and  $49 \mid (p(50) - p(1))$ . Thus  $49 \mid (p(99) - p(50)) - (p(50) - p(1)) = p(1) - 2p(50) + p(99)$ , so 7 divides  $n$ .  $\square$

- Let  $p(x)$  be an integer polynomial. Suppose  $p(0)$  and  $p(1)$  are both odd. Prove that  $p$  has no integer roots.

**Solution:** Write  $p(x) = a_n x^n + \cdots + a_1 x + a_0$ . We are given that  $p(0) = a_0$  is odd and  $p(1) = a_n + \cdots + a_1 + a_0$  is odd.

Suppose  $k$  is an even integer. Then

$$\begin{aligned} p(k) &\equiv a_n k^n + \cdots + a_1 k + a_0 \\ &\equiv a_0 \\ &\equiv 1 \pmod{2}. \end{aligned}$$

In particular,  $p(k) \neq 0$ . Now suppose  $k$  is odd. Then

$$\begin{aligned} p(k) &\equiv a_n k^n + \cdots + a_1 k + a_0 \\ &\equiv a_n + \cdots + a_1 + a_0 \\ &\equiv 1 \pmod{2}. \end{aligned}$$

In particular,  $p(k) \neq 0$ . In both cases,  $k$  cannot be a root of  $p$ , so  $p$  has no integer roots.  $\square$

7. Let  $p(x)$  be an integer polynomial. Suppose there exists a positive integer  $n$  such that none of  $p(0), p(1), \dots, p(n-1)$  are divisible by  $n$ . Prove that  $p$  has no integer roots.

**Solution:** We will use the fact that for any integer polynomial  $p$  and positive integer  $n$ , we have

$$a \equiv b \pmod{n} \Rightarrow p(a) \equiv p(b) \pmod{n}.$$

This follows from the integer addition/multiplication properties of modular arithmetic, since

$$\begin{aligned} p(a) &= a_n a^n + a_{n-1} a^{n-1} + \cdots + a_1 a + a_0 \\ &\equiv a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0 \\ &= p(b) \pmod{n}. \end{aligned}$$

Now, for the sake of contradiction suppose  $p$  has an integer root  $k$ . Then  $p(k) \equiv 0 \pmod{n}$ , and there exists a  $k_0 \in \{0, 1, \dots, n-1\}$  such that  $k \equiv k_0 \pmod{n}$ . Thus,  $p(k_0) \equiv p(k) \equiv 0 \pmod{n}$ , so  $p(k_0)$  is divisible by  $n$ , a contradiction.  $\square$

8. Let  $p(x)$  be an integer polynomial. Suppose there exist three distinct integers  $a, b, c$  such that  $p(a) = p(b) = p(c) = 1$ . Prove that  $p(x)$  has no integer roots.

**Solution:** For the sake of contradiction, suppose  $p$  has an integer root  $k$ . Then we have

$$(k-a) \mid (p(k) - p(a)) = -1$$

$$(k-b) \mid (p(k) - p(b)) = -1$$

$$(k-c) \mid (p(k) - p(c)) = -1$$

Thus,  $(k-a), (k-b), (k-c) \in \{1, -1\}$ . By the Pigeonhole Principle, two of these numbers must be equal, say  $k-a$  and  $k-b$ . But then  $a=b$ , a contradiction since  $a, b, c$  are distinct integers. Thus,  $p$  cannot have an integer root.  $\square$

9. Let  $p(x)$  be a monic integer polynomial of degree 4. Suppose  $p$  has four distinct integer roots  $a, b, c, d$ , and  $k$  is an integer such that  $p(k) = 4$ . Prove that  $k = (a + b + c + d)/4$ .

**Solution:** Since  $p(x)$  is monic, has degree four, and has four integer roots, we can factor  $p(x) = (x - a)(x - b)(x - c)(x - d)$ . Then  $p(k) = (k - a)(k - b)(k - c)(k - d) = 4$ . Each of the four factors on the left hand side are distinct integers, so they must each divide 4.

If any factor has magnitude 4, then the remaining three factors must have magnitude 1. But by Pigeonhole Principle two of these factors must have equal value, contradicting  $a, b, c, d$  being distinct.

If more than two factors have a magnitude of 1, then by Pigeonhole Principle some two must have equal value. Similarly if more than two factors have a magnitude of 2, then by Pigeonhole Principle some two must have equal value. Thus, exactly two factors must have magnitude 1 and two must have magnitude 2.

Without loss of generality, we can set  $k - a = 1$ ,  $k - b = -1$ ,  $k - c = 2$ , and  $k - d = -2$ . Then adding these equations yields  $4k - a - b - c - d = 0 \Rightarrow k = (a + b + c + d)/4$ .  $\square$

10. Let  $p(x)$  be a monic integer polynomial. Suppose there exist distinct integers  $a, b, c, d$  such that  $p(a) = p(b) = p(c) = p(d) = 5$ . Prove that there is no integer  $k$  such that  $p(k) = 8$ .

**Solution 1:** Define the monic integer polynomial  $q(x) = p(x) - 5$ . Then  $q$  has integer roots  $a, b, c, d$ . We want to show that  $q(k) = 3$  has no integer solutions.

For the sake of contradiction, suppose  $q(k) = 3$  for some integer  $k$ . We can factor  $q(x) = (x - a)(x - b)(x - c)(x - d)r(x)$  for some integer polynomial  $r$ . Thus,  $q(k) = (k - a)(k - b)(k - c)(k - d)r(k) = 3$ . Note each of the five factors on the left hand side are integers, so each of  $(k - a), (k - b), (k - c), (k - d)$  are distinct divisors of 3. But 3 has only 4 divisors, namely  $\pm 1$  and  $\pm 3$ .

Thus, without loss of generality we have  $k - a = 1$ ,  $k - b = -1$ ,  $k - c = 3$ , and  $k - d = -3$ . But then  $r(k) = 3 / [(k - a)(k - b)(k - c)(k - d)] = 1/3$ , a contradiction since  $r(k)$  is an integer.  $\square$

**Solution 2:** For the sake of contradiction, suppose  $p(k) = 8$  for some integer  $k$ . Then we have

$$(k - r) | (p(k) - p(r)) = 3$$

for  $r = a, b, c, d$ . Thus,  $(k - a), (k - b), (k - c), (k - d)$  must be distinct divisors of 3, and proceed as in solution 1.  $\square$

11. Find the largest integer  $k$  such that there exist  $k + 1$  distinct integers  $n_0, n_1, \dots, n_k$  and a monic integer polynomial  $p(x)$  such that  $p(n_0) = 4050$  and  $p(n_1) = \dots = p(n_k) = 2025$ . Find a  $p$  for which this  $k$  is achieved.

**Solution:** We can write  $p(x) = (x - n_1) \cdots (x - n_k)q(x) + 2025$  for some integer polynomial  $q$ . Then  $p(n_0) = 4050 \Rightarrow (n_0 - n_1) \cdots (n_0 - n_k)q(n_0) = 2025 = 3^4 5^2$ . In particular,  $(n_0 - n_1), \dots, (n_0 - n_k)$  are distinct divisors of  $3^4 5^2$  with product of magnitude at most  $3^4 5^2$ .

We claim that the largest number of such divisors is 7, which is achieved with  $3^4 5^2 = (1)(-1)(3)(-3)(5)(-5)(9)$ . Indeed, the smallest magnitude product of 8 distinct divisors of  $3^4 5^2$  is

$$|(1)(-1)(3)(-3)(5)(-5)(9)(-9)| = 18225 > 2025.$$

Thus,  $k = 7$  is maximal, and can be achieved with

$$p(x) = (x^2 - 1)(x^2 - 9)(x^2 - 25)(x - 9) + 2025$$

where  $p(0) = 4050$  and  $p(m) = 2025$  for  $m \in \{\pm 1, \pm 3, \pm 5, 9\}$ .  $\square$

12. Find all integer polynomials  $p(x)$  so that  $p(n)$  is prime for all integers  $n$ .

**Solution:** The only such polynomials are constant polynomials  $p(x) = c$  for some prime number  $c$ .

Suppose  $p(x)$  is an integer polynomial such that  $p(n)$  is prime for all integers  $n$ . Set  $p(0) = c$  for some prime number  $c$ . Then for any integer  $k$ , we have

$$(kc - 0) \mid (p(kc) - p(0)) \Rightarrow kc \mid p(kc) - c.$$

In particular, we have  $c \mid p(kc)$ . But  $p(kc)$  must be prime, so we have  $p(kc) = c$  for all integers  $k$ . Define the polynomial  $q(x) = p(x) - c$ . Then  $q$  has infinitely many roots  $kc$  ( $k \in \mathbb{Z}$ ), so  $q$  must be the zero polynomial. Thus,  $p(x) = c$  for all  $x$ .  $\square$

13. Find all integer polynomials  $p(x)$  so that  $p(n)$  divides  $p(n+1)$  for all positive integers  $n$ .

**Solution:** The only such polynomials are nonzero constant polynomials. We may assume  $p(n)$  is nonzero for all positive integers  $n$ . Set  $a_n = p(n+1)/p(n)$  for each positive integer  $n$ . Then  $(a_n)$  is a sequence of integers. But for any polynomial  $p$ , the limit of  $p(n+1)/p(n)$  as  $n \rightarrow \infty$  is 1. Thus,  $(a_n) \rightarrow 1$ , so there exists a positive integer  $n_0$  such that  $a_n = 1$  for all  $n \geq n_0$ . Thus,  $p(n_0) = p(n_0+1) = p(n_0+2) = \dots$ . Defining the polynomial  $q(x) = p(x) - p(n_0)$ , we see  $q$  has infinitely many roots  $x = n_0, n_0+1, n_0+2, \dots$ , and thus must be the zero polynomial. Hence  $p(x) = p(n_0)$  is a constant polynomial, and we can check that any constant polynomial satisfies.  $\square$

14. Find all integer polynomials  $p(x)$  so that  $p(n+1) + p(n-1) = 2p(n)$  for all integers  $n$ .

**Solution:** The only such polynomials are polynomials of degree at most 1. Define the polynomials  $q(x) = p(x+1) - p(x)$  and  $r(x) = q(x+1) - q(x)$ . We are given that  $q(n+1) = q(n)$  and thus  $r(n) = 0$  for all integers  $n$ . Thus,  $r$  is the zero polynomial, so  $q$  is a constant polynomial and  $p$  is a linear polynomial. We can check that all linear integer polynomials  $p$  satisfy.  $\square$

15. Let  $\{x\}$  denote the fractional part of a real number  $x$ . For example,  $\{7\} = 0$ ,  $\{3.14\} = 0.14$ , and  $\{-9.8\} = 0.2$ . Find all real numbers  $r$  that satisfy  $\{r\} = \{r^2\} = \{r^3\}$ .

**Solution:** The only such real numbers  $r$  are the integers. First, note that if  $r$  is an integer we have  $\{r\} = \{r^2\} = \{r^3\} = 0$ . For the sake of contradiction, suppose  $r$  is not an integer. Set  $n = r^2 - r$  and  $m = r^3 - r^2$ . Since  $\{r\} = \{r^2\} = \{r^3\}$ ,  $n$  and  $m$  are integers and  $m = r^3 - r^2 = r(r^2 - r) = rn$ . If  $n = 0$ , then  $r = 0, 1$ , a contradiction. Otherwise,  $r = m/n$  must be rational. By the rational root theorem on  $x^2 - x - n = 0$  (of which  $r$  is a root),  $r$  must be an integer, a contradiction.  $\square$

## 11 Intermediate Value Theorem

1. Prove that the equation  $\tan x = 1 - x$  has a real solution.

**Solution:** It is equivalent to show that the function  $f(x) = \tan x + x - 1$  has a real root. Note that  $f$  is continuous on the interval  $(-\pi/2, \pi/2)$ . Since  $f(0) = -1$  and  $f(\pi/4) = \pi/4$ , and  $-1 < 0 < \pi/4$ , by IVT there exists a number  $c \in (0, \pi/4)$  such that  $f(c) = 0$ .  $\square$

2. Let  $p(x)$  be a real polynomial of odd degree. Prove that  $p$  has at least one real root.

**Solution:** Since  $p(x)$  has a real root if and only if  $-p(x)$  has a real root, we can assume without loss of generality that  $p(x)$  has a positive leading coefficient. Then  $\lim_{x \rightarrow \infty} p(x) = +\infty$  and  $\lim_{x \rightarrow -\infty} p(x) = -\infty$ , so there exist numbers  $a, b$  such that  $p(a) > 0$  and  $p(b) < 0$ . Since  $p$  is continuous on  $\mathbb{R}$ , by IVT  $p$  must have a real root on  $(a, b)$ .  $\square$

3. Let  $p(x)$  be a real cubic polynomial. Suppose  $p$  has at least two distinct real roots. Prove that all roots of  $p$  are real.

**Solution:** If the third root of  $p$  is equal to one of the two other real roots, then we are done. Thus, we can assume all three roots of  $p$  are distinct.

Suppose  $x_1 < x_2$  are the given real roots of  $p$ , and assume  $p$  has no real root on  $(x_1, x_2)$ . Without loss of generality, assume  $p$  is positive on  $(x_1, x_2)$ . Then  $p$  is negative in an interval immediately to the left and right of the interval  $(x_1, x_2)$ . But since  $p$  has odd degree, it has opposite signs as  $x \rightarrow \infty$  and  $x \rightarrow -\infty$ . Then IVT guarantees a real root either to the left or to the right of the interval  $(x_1, x_2)$ . Since  $p$  has at most three roots, all of the roots of  $p$  must be real.  $\square$

4. Suppose  $f$  is continuous and satisfies  $f(0) > 0$  and  $f(2) < 4$ . Prove that the equation  $f(x) = x^2$  has a real solution.

**Solution:** Let  $g(x) = f(x) - x^2$ . It is equivalent to show that  $g$  has a real root. Note that  $g$  is continuous with  $g(0) = f(0) > 0$  and  $g(2) = f(2) - 4 < 0$ . Thus, by IVT  $g$  has a real root.  $\square$

5. Given  $n$  points  $x_1, \dots, x_n \in [0, 1]$ , show there exists a point  $y \in [0, 1]$  such that the average of the  $n$  distances from  $y$  to  $x_1, \dots, x_n$  is exactly  $1/2$ .

**Solution:** Define the function

$$f(y) = \frac{1}{n} \sum_{i=1}^n |y - x_i|$$

for  $y \in [0, 1]$ . Note that  $f$  is continuous since it is the finite sum of continuous functions. Further, we have  $f(0) = (x_1 + \dots + x_n)/n$  and  $f(1) = (n - x_1 - \dots - x_n)/n = 1 - f(0)$ .

If  $f(0) = 1/2$ , then we can take  $y = 0$ . If  $f(0) > 1/2$ , then  $f(1) < 1 - 1/2 = 1/2$ , so by the IVT there exists a  $c \in (0, 1)$  such that  $f(c) = 1/2$ . Similarly, if  $f(0) < 1/2$ , then  $f(1) > 1 - 1/2 = 1/2$ , so by the IVT there exists a  $c \in (0, 1)$  such that  $f(c) = 1/2$ .  $\square$

6. Let  $f$  be continuous on  $\mathbb{R}$  with  $f(x)f(f(x)) = 1$  for all  $x \in \mathbb{R}$ . Given that  $f(1000) = 999$ , find  $f(500)$ .

**Solution:** Note that  $f(1000)f(f(1000)) = 1 \Rightarrow f(999) = 1/999$ . Since  $f$  is continuous and  $999 > 500 > 1/999$ , by the IVT there exists a  $c \in (999, 1000)$  such that  $f(c) = 500$ . Thus  $f(c)f(f(c)) = 1 \Rightarrow f(500) = 1/500$ .  $\square$

7. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous with  $f(0) = f(1)$ . Prove that the equation  $f(x) = f(x + 1/2025)$  has a solution on the interval  $[0, 2024/2025]$ .

**Solution:** Define  $g(x) = f(x) - f(x + 1/2025)$  for  $x \in [0, 2024/2025]$ . We are left to show that  $g$  has a real root. Note  $g$  is continuous and

$$\begin{aligned} & g(0) + g(1/2025) + \cdots + g(2024/2025) \\ &= f(0) - f(1/2025) + f(1/2025) - f(2/2025) + \cdots + f(2024/2025) - f(1) \\ &= f(0) - f(1) = 0. \end{aligned}$$

Then  $g(0), \dots, g(2024/2025)$  cannot all be positive or all be negative, so at least one is zero or two have opposite signs, which by IVT means  $g$  has a root.  $\square$

8. Let  $\ell$  be a line in the plane and  $P$  a polygon in the plane. Prove that  $P$  can be divided into two polygons of equal area by a line parallel to  $\ell$ .

**Solution:** Without loss of generality, assume  $\ell$  is a vertical line. Let  $\ell_0$  denote a line parallel to  $\ell$  that is entirely to the left of  $P$ . Similarly, let  $\ell_1$  denote a line parallel to  $\ell$  that is entirely to the right of  $P$ . Let  $\ell(t)$  denote the line obtained by shifting  $\ell_0$  a fraction  $t$  of the distance to  $\ell_1$ . Then  $\ell(0) = \ell_0$  and  $\ell(1) = \ell_1$ .

Let  $f(t)$  denote the area of  $P$  to the left of  $\ell(t)$  minus the area of  $P$  to the right of  $\ell(t)$ . Then  $f$  is a continuous function with  $f(0) = (\text{Area of } P) > 0$  and  $f(1) = -f(0) < 0$ , so by IVT there exists a  $c \in (0, 1)$  such that  $f(c) = 0$ . Hence  $\ell(c)$  splits  $P$  into two polygons of equal area.  $\square$

9. Prove that any convex polygon can be divided by a line into two polygons of equal area and of equal perimeter.

**Solution:** Let  $P_0$  be a point on the perimeter of the convex polygon. Let  $P(t)$  denote the point a fraction of  $t$  around the perimeter of the polygon from  $P_0$ , so  $P(0) = P(1) = P_0$ . For  $t \in [0, 1/2]$ , let  $\ell(t)$  denote the line connecting points  $P(t)$  and  $P(t + 1/2)$ . Note that  $\ell(t)$  divides the polygon into two polygons of equal perimeter.

Let  $f(t)$  denote the difference in areas of the polygon divided by  $\ell(t)$ . Then  $f(0) = -f(1/2)$ , so by the IVT there exists a  $c \in [0, 1/2]$  such that  $f(c) = 0$ . Thus,  $\ell(c)$  divides the polygon into two polygons of equal area and of equal perimeter.  $\square$

10. Let  $p(x)$  be a polynomial with odd degree greater than 1. Let  $Q$  be a point in the plane. Prove that there exists a line through  $Q$  that is tangent to the graph  $y = p(x)$  at some point.

**Solution:** By shifting  $p$ , we may assume without loss of generality that  $Q = (0, 0)$ . We are left to show there exists a slope  $m$  and a point  $x_0$  such that  $p(x_0) = mx_0$  and  $p'(x_0) = m$ , or equivalently that  $p(x) - xp'(x) = 0$  has a real solution.

Set  $p(x) = a_n x^n + \dots + a_0$ . Then  $xp'(x) = n a_n x^n + \dots + x a_1$ , so  $p(x) - xp'(x) = a_n(n-1)x^n + \dots$ . Thus,  $p(x) - xp'(x)$  is a polynomial of odd degree, which by IVT is guaranteed to have a real root.  $\square$

## 12 Taylor's Theorem

- Find the Taylor Series of  $\sin x$  and  $\cos x$  around the point  $a = 0$ .

**Solution:** We can show by induction that

$$\begin{aligned}\frac{d^n}{dx^n} \sin x &= \sin x, n \equiv 0 \pmod{4} \\ &= \cos x, n \equiv 1 \pmod{4} \\ &= -\sin x, n \equiv 2 \pmod{4} \\ &= -\cos x, n \equiv 3 \pmod{4}\end{aligned}$$

and

$$\begin{aligned}\frac{d^n}{dx^n} \cos x &= \cos x, n \equiv 0 \pmod{4} \\ &= -\sin x, n \equiv 1 \pmod{4} \\ &= -\cos x, n \equiv 2 \pmod{4} \\ &= \sin x, n \equiv 3 \pmod{4}.\end{aligned}$$

Note that all terms with  $\sin x$  will vanish when we plug in  $a = 0$ . Thus,

$$\begin{aligned}\sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ \cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\end{aligned}$$

□

- Find the Taylor Series of  $e^x$  around the point  $a = 0$ .

**Solution:** We can show by induction that  $\frac{d^n}{dx^n} e^x = e^x$ . Thus,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

□

- Find the Taylor Series of  $\ln(1 - x)$  around the point  $a = 0$ .

**Solution:** We will show by induction that  $\frac{d^n}{dx^n} \ln(1 - x) = -(n-1)!/(1-x)^n$  for all  $n \geq 1$ . For our base case we have

$$\frac{d}{dx} \ln(1 - x) = \frac{-1}{1 - x}$$

and for our inductive step we have

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}} \ln(1-x) &= \frac{d}{dx} \frac{d^n}{dx^n} \ln(1-x) \\
&= \frac{d}{dx} \frac{-(n-1)!}{(1-x)^n} \\
&= \frac{-(n-1)! \cdot -n}{(1-x)^{n+1}} \cdot -1 \\
&= \frac{-n!}{(1-x)^{n+1}}.
\end{aligned}$$

Thus,

$$\ln(1-x) = \sum_{k=1}^{\infty} \frac{-(k-1)!}{(1-0)^k} \cdot \frac{x^k}{k!} = \sum_{k=1}^{\infty} \frac{-x^k}{k} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$$

□

4. Find the first three terms of the Taylor Series expansion of  $\sqrt{1-x}$  around the point  $a=0$ .

**Solution:** We have

$$\begin{aligned}
\frac{d}{dx} \sqrt{1-x} &= \frac{-1}{2\sqrt{1-x}} \\
\frac{d^2}{dx^2} \sqrt{1-x} &= \frac{-1}{4\sqrt{(1-x)^3}}.
\end{aligned}$$

Thus, the first three terms of the Taylor Series around  $a=0$  are

$$\begin{aligned}
\sqrt{1-x} &= \sqrt{1-0} + \frac{-1}{2\sqrt{1-0}} \cdot \frac{x}{1!} + \frac{-1}{4\sqrt{(1-0)^3}} \cdot \frac{x^2}{2!} + \dots \\
&= 1 - \frac{x}{2} - \frac{x^2}{8} - \dots
\end{aligned}$$

□

5. Using Taylor's Theorem with the first two terms of  $\sin x$ , prove that

$$x - \frac{x^3}{6} \leq \sin x \leq \frac{24x - 4x^3}{24 - x^4}$$

for all  $x \in [0, \pi/2]$ .

**Solution:** By Taylor's Theorem, for  $x \in [0, \pi/2]$  we have

$$\sin x = x - x^3/3! + \sin(c)x^4/4!$$

for some  $c \in (0, x)$ . Since  $\sin(c) \geq 0$  for  $c \in (0, x)$  we have

$$\sin x \geq x - x^3/3!$$

for  $x \in [0, \pi/2]$ . Furthermore,  $\sin x$  is increasing on  $(0, \pi/2)$ . Since  $0 < c < x$ , we have

$$\begin{aligned}\sin x &\leq x - x^3/3! + \sin(x)x^4/4! \\ &\Rightarrow (1 - x^4/24)\sin x \leq x - x^3/6 \\ &\Rightarrow \sin x \leq \frac{x - x^3/6}{1 - x^4/24} = \frac{24x - 4x^3}{24 - x^4}\end{aligned}$$

since  $x^4 \leq (\pi/2)^4 < 2^4 < 24$ .

□

6. Prove that

$$\frac{2}{3} \leq \int_0^1 e^{-x^2} dx \leq \frac{23}{30}.$$

**Solution:** By Taylor's Theorem on  $e^x$  with  $a = 0$ , we have

$$e^x = 1 + x + \frac{x^2 e^{c_1}}{2!} = 1 + x + \frac{x^2}{2!} + \frac{x^3 e^{c_2}}{3!}$$

for some  $c_1, c_2 \in (0, x)$ . Thus,

$$e^{-x^2} = 1 - x^2 + \frac{x^4 e^{c_1}}{2!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6 e^{c_2}}{3!}$$

for some  $c_1, c_2 \in (-x^2, 0)$ . Since  $e^x$  is positive and increasing, we must have

$$1 - x^2 \leq e^{-x^2} \leq 1 - x^2 + \frac{x^4}{2}.$$

Since  $0 \leq 1 - x^2$  for  $x \in [0, 1]$ , we can integrate all parts of the inequality to get

$$\begin{aligned}\int_0^1 1 - x^2 dx &\leq \int_0^1 e^{-x^2} dx \leq \int_0^1 1 - x^2 + \frac{x^4}{2} dx \\ &\Rightarrow \frac{2}{3} \leq \int_0^1 e^{-x^2} dx \leq \frac{23}{30}.\end{aligned}$$

□

7. Find a quadratic polynomial  $p(x)$  such that for all  $|x| \leq 1$ ,

$$\left| p(x) - \frac{1}{x-4} \right| < 0.01.$$

**Solution:** By Taylor's Theorem with  $a = 0$ , for all  $|x| \leq 1$  we have

$$\frac{1}{x-4} = -\frac{1}{4} - \frac{x}{16} - \frac{x^2}{64} - \frac{x^3}{256} + \frac{x^4}{(c-4)^5}$$

for some  $c$  between 0 and  $x$ . Take  $p(x) = -1/4 - x/16 - x^2/64$ . Then we have

$$\begin{aligned}
\left| p(x) - \frac{1}{x-4} \right| &= \left| \frac{x^3}{256} - \frac{x^4}{(c-4)^5} \right| \\
&\leq \frac{|x|^3}{256} + \frac{|x|^4}{(4-c)^5} \\
&\leq \frac{1}{256} + \frac{1}{3^5} \\
&< \frac{1}{200} + \frac{1}{200} \\
&= 0.01.
\end{aligned}$$

□

8. Find the 5<sup>th</sup> digit after the decimal point of  $\sqrt{11,111,111}$ .

**Solution:** Note that  $11111111 = (10^8 - 1)/9$ . Thus,

$$\sqrt{11111111} = \sqrt{(10^8 - 1)/9} = (10^4/3)\sqrt{1 - 1/10^8}.$$

By Taylor's Theorem for  $\sqrt{1-x}$  around  $a=0$ , we have

$$\sqrt{1 - 1/10^8} = 1 - \frac{1}{2 \cdot 10^8} - \frac{1}{\sqrt{(1-c)^3}} \cdot \frac{1}{8 \cdot 10^{16}}$$

for some  $c \in (0, 10^{-8})$ . Thus, taking  $c=0$  and  $c=3/4$  (since the remainder term is decreasing in  $c$ ) yields

$$\begin{aligned}
1 - \frac{1}{2 \cdot 10^8} - \frac{1}{10^{16}} &\leq \sqrt{1 - 1/10^8} \leq 1 - \frac{1}{2 \cdot 10^8} - \frac{1}{8 \cdot 10^{16}} \\
\Rightarrow \frac{10^4}{3} - \frac{1}{6 \cdot 10^4} - \frac{1}{3 \cdot 10^{12}} &\leq \frac{10^4}{3} \sqrt{1 - 1/10^8} \leq \frac{10^4}{3} - \frac{1}{6 \cdot 10^4} - \frac{1}{24 \cdot 10^{12}}.
\end{aligned}$$

The fifth digit after the decimal of  $N$  is the units digit of  $\lfloor 10^5 N \rfloor$ . We have

$$\begin{aligned}
\left\lfloor \frac{10^9}{3} - \frac{10}{6} - \frac{1}{3 \cdot 10^7} \right\rfloor &\leq \left\lfloor \frac{10^9}{3} \sqrt{1 - 1/10^8} \right\rfloor \leq \left\lfloor \frac{10^9}{3} - \frac{10}{6} - \frac{1}{24 \cdot 10^7} \right\rfloor \\
\Rightarrow \left\lfloor \frac{10^9 - 5}{3} - \frac{1}{3 \cdot 10^7} \right\rfloor &\leq \left\lfloor \frac{10^9}{3} \sqrt{1 - 1/10^8} \right\rfloor \leq \left\lfloor \frac{10^9 - 5}{3} - \frac{1}{24 \cdot 10^7} \right\rfloor.
\end{aligned}$$

Since the fractional part of  $(10^9 - 5)/3$  is  $2/3 > 1/(3 \cdot 10^7) > 1/(24 \cdot 10^7)$ , we have

$$\begin{aligned}
\left\lfloor \frac{10^9 - 5}{3} \right\rfloor &\leq \left\lfloor \frac{10^9}{3} \sqrt{1 - 1/10^8} \right\rfloor \leq \left\lfloor \frac{10^9 - 5}{3} \right\rfloor \\
\Rightarrow \left\lfloor \frac{10^9}{3} \sqrt{1 - 1/10^8} \right\rfloor &= \left\lfloor \frac{10^9 - 5}{3} \right\rfloor = \frac{10^9 - 7}{3}.
\end{aligned}$$

We are left to find the units digit of  $(10^9 - 7)/3$ , which we can do by taking modulo 10:

$$(10^9 - 7)/3 \equiv -7 \cdot 3^{-1} \equiv -7 \cdot 7 \equiv 1 \pmod{10}.$$

Thus, the fifth digit after the decimal of  $\sqrt{11,111,111}$  is 1. □

## 13 Mean Value Theorem

- Let  $f$  be a continuous function such that  $\int_a^b f(x) dx = 0$  for some  $a < b$ . Prove that  $f$  has at least one real root.

**Solution:** By the mean value theorem,  $\exists c \in (a, b)$  such that  $f(c) = \int_a^b f(x) dx = 0$ . □

- Let  $f$  be a continuously differentiable function such that  $f'$  has no real roots. Prove that  $f$  has at most one real root.

**Solution:** For the sake of contradiction, suppose  $f$  has at least two real roots  $x_1 < x_2$ . Then by the mean value theorem, there exists a  $c \in (x_1, x_2)$  such that  $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$ , a contradiction. □

- Suppose  $f$  is a differentiable function with  $n$  distinct real roots. Prove that  $f'$  has at least  $n - 1$  real roots.

**Solution:** Let  $f$  have roots  $x_1 < \dots < x_n$ . For each  $i = 1, \dots, n-1$ , by the mean value theorem there exists a  $c_i \in (x_i, x_{i+1})$  such that  $f'(c_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = 0$ . Thus,  $f'$  has at least  $n - 1$  distinct real roots. □

- Let  $f$  be a differentiable function. Given that  $|f'(x)| \leq 1$  for all  $x$ , prove that  $|f(x) - f(y)| \leq |x - y|$  for all  $x, y$ .

**Solution:** Without loss of generality, let  $y < x$  be fixed. Then by the mean value theorem there exists a  $c \in (y, x)$  such that

$$\begin{aligned} f'(c) &= \frac{f(x) - f(y)}{x - y} \\ \Rightarrow \frac{|f(x) - f(y)|}{|x - y|} &= |f'(c)| \leq 1 \\ \Rightarrow |f(x) - f(y)| &\leq |x - y|. \end{aligned}$$

□

- Let  $f$  be a function on  $[a, b]$  such that  $f''$  is continuous on the interval  $(a, b)$ . Let  $\ell$  denote the line segment with endpoints  $(a, f(a))$  and  $(b, f(b))$ . Given that  $\ell$  intersects the graph of  $f$  on the interval  $(a, b)$ , prove that  $f''$  has a real root.

**Solution:** Note that  $\ell$  is given by the linear polynomial  $g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$ . If  $\ell$  intersects  $f$  on  $(a, b)$ , then the equation  $f(x) - g(x)$  has three real roots: two at  $x = a$  and  $x = b$ , and one more at some  $c \in (a, b)$ . By problem 3, this means  $\frac{d}{dx}(f(x) - g(x))$  has at least two real roots and  $\frac{d^2}{dx^2}(f(x) - g(x))$  has at least one real root. Since  $g''(x) = 0$ , this means  $f''(x)$  has a real root. □

6. Let  $a_0, a_1, \dots, a_n$  be real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0.$$

Prove there exists a real number  $x$  such that  $a_0 + a_1x + \cdots + a_nx^n = 0$ .

**Solution:** Define  $p(x) = a_0 + a_1x + \cdots + a_nx^n$ . We want to show that  $p$  has a real root. Note that

$$\begin{aligned} \int_0^1 p(x) dx &= \int_0^1 (a_0 + a_1x + \cdots + a_nx^n) dx \\ &= \left[ \frac{a_0x}{1} + \frac{a_1x^2}{2} + \cdots + \frac{a_nx^{n+1}}{n+1} \right]_0^1 \\ &= \frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} \\ &= 0. \end{aligned}$$

Thus, by problem 1,  $p$  must have a root on  $(0, 1)$ . □

7. Let  $f : [a, b] \rightarrow [a, b]$  be differentiable such that  $f(a) = b$  and  $f(b) = a$ . Prove that there exist  $\alpha, \beta \in (a, b)$  such that  $\alpha \neq \beta$  and  $f'(\alpha)f'(\beta) = 1$ .

**Solution:** Define the differentiable function  $g(x) = f(f(x)) - x$  on  $[a, b]$ . Then  $g(a) = g(b) = 0$  and  $g'(x) = f'(f(x))f'(x) - 1$ . By the mean value theorem, there exists a  $c \in (a, b)$  such that  $g'(c) = 0 \Rightarrow f'(f(c))f'(c) = 1$ . If  $f(c) \neq c$ , then we can take  $\alpha = c$  and  $\beta = f(c)$ .

Otherwise,  $f(c) = c$ , so by the mean value theorem there exists an  $\alpha \in (a, c)$  such that  $f'(\alpha) = \frac{f(c)-f(a)}{c-a}$  and  $\beta \in (c, b)$  such that  $f'(\beta) = \frac{f(b)-f(c)}{b-c}$ . Hence  $\alpha \neq \beta$  and

$$\begin{aligned} f'(\alpha)f'(\beta) &= \frac{f(c)-f(a)}{c-a} \cdot \frac{f(b)-f(c)}{b-c} \\ &= \frac{c-b}{c-a} \cdot \frac{a-c}{b-c} \\ &= 1. \end{aligned}$$

□

## 14 Riemann Sums

1. Let  $H_n$  denote the  $n^{\text{th}}$  harmonic number:

$$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Prove that  $\lim_{n \rightarrow \infty} H_n / \ln(n) = 1$ .

**Solution:** Note that  $H_n - 1$  is an underestimate of the integral  $\int_1^n \frac{1}{x} dx$  and an overestimate of the integral  $\int_2^{n+1} \frac{1}{x} dx$ . Thus, we have

$$\begin{aligned} \int_2^{n+1} \frac{1}{x} dx &\leq H_n - 1 \leq \int_1^n \frac{1}{x} dx \\ \Rightarrow \ln(n+1) - \ln(2) &\leq H_n - 1 \leq \ln(n) \\ \Rightarrow \frac{\ln(n+1) - \ln(2) + 1}{\ln(n)} &\leq \frac{H_n}{\ln(n)} \leq \frac{\ln(n) + 1}{\ln(n)}. \end{aligned}$$

Now, we have  $\ln(n+1)/\ln(n) \rightarrow n/(n+1) \rightarrow 1$  as  $n \rightarrow \infty$  by L'Hopital, and thus by the squeeze theorem we have  $H_n/\ln(n) \rightarrow 1$ .  $\square$

2. For each positive integer  $n$ , let

$$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}.$$

Evaluate  $\lim_{n \rightarrow \infty} S_n / \sqrt{n}$ .

**Solution:** Note that  $S_n - 1$  is an underestimate of the integral  $\int_1^n \frac{1}{\sqrt{x}} dx$  and an overestimate of the integral  $\int_2^{n+1} \frac{1}{\sqrt{x}} dx$ . Thus, we have

$$\begin{aligned} \int_2^{n+1} \frac{1}{\sqrt{x}} dx &\leq S_n - 1 \leq \int_1^n \frac{1}{\sqrt{x}} dx \\ \Rightarrow 2\sqrt{n+1} - 2\sqrt{2} &\leq S_n - 1 \leq 2\sqrt{n} - 2\sqrt{1} \\ \Rightarrow \frac{2\sqrt{n+1} - 2\sqrt{2} + 1}{\sqrt{n}} &\leq \frac{S_n}{\sqrt{n}} \leq \frac{2\sqrt{n} - 1}{\sqrt{n}}. \end{aligned}$$

Now, we have  $2\sqrt{n+1}/\sqrt{n} \rightarrow 2$  as  $n \rightarrow \infty$ , and thus by the squeeze theorem we have  $S_n/\sqrt{n} \rightarrow 2$ .  $\square$

3. Prove one form of Stirling's approximation:

$$\lim_{n \rightarrow \infty} \frac{\ln(n!)}{n \ln(n/e)} = 1.$$

**Solution:** Let  $S_n = \ln(n!) = \ln(2) + \ln(3) + \dots + \ln(n)$  for all  $n \geq 2$ . Note that  $S_n$  is an overestimate of the integral  $\int_1^n \ln(x) dx$  and an underestimate of the integral  $\int_2^{n+1} \ln(x) dx$ . Thus, we have

$$\int_1^n \ln(x) dx \leq S_n \leq \int_2^{n+1} \ln(x) dx.$$

We can use integration by parts to get  $\int \ln(x) dx = x(\ln(x) - 1) + C = x\ln(x/e) + C$ , so we have

$$\begin{aligned} n\ln(n/e) - \ln(1/e) &\leq S_n \leq (n+1)\ln((n+1)/e) - 2\ln(2/e) \\ \Rightarrow 1 - \frac{\ln(1/e)}{n\ln(n/e)} &\leq \frac{S_n}{n\ln(n/e)} \leq \frac{(n+1)\ln((n+1)/e)}{n\ln(n/e)} - \frac{2\ln(2/e)}{n\ln(n/e)} \end{aligned}$$

We are left to show that  $\frac{(n+1)\ln((n+1)/e)}{n\ln(n/e)} \rightarrow 1$ , which is true since  $(n+1)/n \rightarrow 1$  and  $\ln(n+1)/\ln(n) \rightarrow 1$ .  $\square$

4. For each positive integer  $n$ , let

$$S_n = \ln\left(\sqrt[n]{1^1 \cdot 2^2 \cdots n^n}\right) - \ln(\sqrt{n}).$$

Find  $\lim_{n \rightarrow \infty} S_n$ .

**Solution:** By log properties, we have

$$S_n = \frac{1\ln(1) + 2\ln(2) + \dots + n\ln(n)}{n^2} - \frac{\ln(n)}{2}.$$

Define  $T_n = 2\ln(2) + 3\ln(3) + \dots + n\ln(n)$  for all  $n \geq 2$ . Note that  $T_n$  is an overestimate of the integral  $\int_1^n x\ln(x) dx$  and an underestimate of the integral  $\int_2^{n+1} x\ln(x) dx$ . Thus, we have

$$\int_1^n x\ln(x) dx \leq T_n \leq \int_2^{n+1} x\ln(x) dx.$$

We can use integration by parts to get  $\int x\ln(x) dx = x^2\ln(x)/2 - x^2/4 + C$ , so we have

$$\begin{aligned} \frac{2n^2\ln(n) - n^2 + 1}{4} &\leq T_n \leq \frac{2(n+1)^2\ln(n+1) - (n+1)^2 - 4\ln 2 + 4}{4} \\ \Rightarrow \frac{2n^2\ln(n) - n^2 + 1}{4n^2} &\leq \frac{T_n}{n^2} \leq \frac{2(n+1)^2\ln(n+1) - (n+1)^2 - 4\ln 2 + 4}{4n^2} \\ \Rightarrow \frac{2n^2\ln(n) - n^2 + 1 - 2n^2\ln(n)}{4n^2} &\leq \frac{T_n}{n^2} - \frac{\ln(n)}{2} \\ \leq \frac{2(n+1)^2\ln(n+1) - (n+1)^2 - 4\ln 2 + 4 - 2n^2\ln(n)}{4n^2} & \\ \Rightarrow \frac{-1}{4} + \frac{1}{4n^2} \leq S_n &\leq \frac{2(n+1)^2\ln(n+1) - (n+1)^2 - 4\ln 2 + 4 - 2n^2\ln(n)}{4n^2} \\ \leq \frac{-1}{4} + \frac{(n+1)^2\ln(n+1) - n^2\ln(n)}{2n^2} + \frac{1 - \ln 2}{n^2} & \end{aligned}$$

Note that

$$\frac{(n+1)^2\ln(n+1) - n^2\ln(n)}{2n^2} = \frac{\ln(n+1) - \ln(n)}{2} + \frac{(2n+1)\ln(n+1)}{2n^2} \rightarrow 0$$

as  $n \rightarrow \infty$ , and thus by the squeeze theorem we have  $S_n \rightarrow -1/4$ .  $\square$

## 15 Inclusion-Exclusion Principle

- How many positive integers  $n \leq 1000$  are divisible by 2 or 5?

**Solution:** Let  $A$  be the set of positive integers  $n \leq 1000$  that are divisible by 2 and  $B$  be the set of positive integers  $n \leq 1000$  that are divisible by 5. We are asked to find  $|A \cup B| = |A| + |B| - |A \cap B|$ . First,  $|A| = 1000/2 = 500$  and  $|B| = 1000/5 = 200$ . Now,  $A \cap B$  is the set of all positive integers  $n \leq 1000$  that are divisible by 10. Thus,  $|A \cap B| = 1000/10 = 100$ , so  $|A \cup B| = 500 + 200 - 100 = 600$ .  $\square$

- Find the number of subsets of  $\{1, 2, \dots, 300\}$  that are subsets of neither  $\{1, 2, \dots, 200\}$  nor  $\{101, 102, \dots, 300\}$ .

**Solution:** We will find the number of subsets of  $S = \{1, 2, \dots, 300\}$  that are subsets of  $T = \{1, 2, \dots, 200\}$  or  $U = \{101, 102, \dots, 300\}$  and subtract this from the number of subsets of  $S$ .

By the inclusion-exclusion principle, this is the number of subsets of  $T$  plus the number of subsets of  $U$  minus the number of subsets of  $T \cap U$ , or  $2^{|T|} + 2^{|U|} - 2^{|T \cap U|} = 2 \cdot 2^{200} - 2^{100}$ . Thus, our final answer is  $2^{300} - 2^{201} + 2^{100}$ .  $\square$

- Find a formula for the number of elements in the set  $A \cup B \cup C$ , where  $A, B, C$  are finite sets.

**Solution:** We have

$$\begin{aligned} & |A \cup (B \cup C)| \\ &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\ &= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |(A \cap B) \cap (A \cap C)| \\ &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|. \end{aligned}$$

$\square$

- Find the number of permutations of  $(1, 2, \dots, n)$  such that the  $i^{\text{th}}$  number in the permutation is not  $i$  for all  $i = 1, 2, \dots, n$ .

**Solution:** Such a permutation is called a derangement. We will count the number of permutations that are not a derangement and subtract this from the total number of permutations  $n!$ .

Let  $A_i$  be the set of permutations that have  $i$  in position  $i$ . Then we are asked to find  $n! - |A_1 \cup \dots \cup A_n|$ . By the inclusion-exclusion principle, this is

$$n! - \sum_{i=1}^n S_i (-1)^{i-1}$$

where

$$S_i = \sum_{k_1 < \dots < k_i} |A_{k_1} \cap \dots \cap A_{k_i}|.$$

Now,  $A_{k_1} \cap \dots \cap A_{k_i}$  is the set of permutations with  $k_1$  in position  $k_1, \dots, k_i$  in position  $k_i$ . Thus,  $|A_{k_1} \cap \dots \cap A_{k_i}| = (n - i)!$ . Furthermore, there are  $\binom{n}{i}$  summands in the expression of  $S_i$ , so we have

$$S_i = \binom{n}{i} (n - i)! = \frac{n!}{i!}.$$

Our final answer is thus

$$\begin{aligned} n! - \sum_{i=1}^n \frac{n!}{i!} (-1)^{i-1} \\ = n! \sum_{i=0}^n \frac{(-1)^i}{i!}. \end{aligned}$$

□

## 16 Bijections

1. A path to a point  $P$  in the plane is a sequence of moves one unit up or one unit to the right starting from the origin and ending at  $P$ . Find the number of paths to  $(10, 10)$ .

**Solution:** We form a bijection between the set of such paths  $A$  and the set of permutations of 10  $R$  symbols and 10  $U$  symbols  $B$ . For each path in  $A$ , form a permutation by walking the path from the origin to  $(10, 10)$  and writing an  $R$  when a step to the right is taken and a  $U$  when a step upward is taken.

We'll justify why our mapping is a bijection. First, consider any path in  $A$ . We must have exactly 10 moves to the right and 10 moves upward in order to reach the point  $(10, 10)$ . Thus, the resulting permutation is unique and must be in  $B$ . Now, consider any such permutation in  $B$ . Since it contains 10  $R$  and 10  $U$  symbols, it is mapped to by a unique path that starts at the origin and ends at  $(10, 10)$ , i.e. a path in  $A$ . Thus, our mapping is bijective.

We are left to count the number of permutations of 10  $R$  and 10  $U$  symbols, which is  $20!/(10! \cdot 10!) = \binom{20}{10}$ .  $\square$

2. Find the number of paths to  $(10, 10)$  that do not pass through the point  $(5, 5)$ .

**Solution:** Note that the number of paths not through  $(5, 5)$  is the total number of paths minus the paths that do pass through  $(5, 5)$ . There are  $\binom{20}{10}$  paths from question 1, and by similar reasoning there are  $\binom{10}{5}$  paths from the origin to  $(5, 5)$  and  $\binom{10}{5}$  paths from  $(5, 5)$  to  $(10, 10)$ . Thus, there are  $\binom{20}{10} - \binom{10}{5}^2$  paths to  $(10, 10)$  that do not contain  $(5, 5)$ .  $\square$

3. Find the number of solutions  $(x, y, z)$  of nonnegative integers to the equation  $x+y+z=100$ .

**Solution:** We form a bijection between the set  $A$  of such solutions  $(x, y, z)$  and the set  $B$  of ways to distribute 100 balls to 3 distinct boxes. For each solution  $(x, y, z)$  in  $A$ , distribute the balls so that the first box contains  $x$  balls, the second contains  $y$  balls, and the third contains  $z$  balls.

We'll justify why our mapping is a bijection. First, consider any tuple  $(x, y, z)$  in  $A$ . Since  $x, y, z$  are nonnegative, we can place  $x, y, z$  balls into their respective boxes, and since  $x+y+z=100$ , we will be placing 100 balls total. Thus, each solution  $(x, y, z)$  corresponds to a unique distribution in  $B$ . Now, consider any distribution in  $B$ . Since each box contains a nonnegative number of balls, and the total number of balls is 100, this distribution corresponds to a unique tuple in  $A$ . Thus, our mapping is bijective.

We are left to count the number of ways to distribute 100 balls into 3 boxes, which is  $\binom{100+3-1}{3-1} = \binom{102}{2} = 5151$ .  $\square$

4. Find the number of solutions  $(x, y, z)$  of nonnegative integers to the equation  $x+y+z \leq 100$ .

**Solution:** We form a bijection between the set  $A$  of such solutions  $(x, y, z)$  and the set  $B$  of ways to distribute 100 balls to 4 distinct boxes. For each solution  $(x, y, z)$  in  $A$ , distribute the balls so that the first box contains  $x$  balls, the second contains  $y$  balls, the third contains  $z$  balls, and the fourth contains  $100 - x - y - z$  balls.

We'll justify why our mapping is a bijection. First, consider any tuple  $(x, y, z)$  in  $A$ . Since  $x, y, z$  are nonnegative, we can place  $x, y, z$  balls into their respective boxes, and since  $x + y + z \leq 100$ , we can place  $100 - x - y - z \geq 0$  balls into the fourth box. We will be placing 100 balls total, so each solution  $(x, y, z)$  corresponds to a unique distribution in  $B$ . Now, consider any distribution in  $B$ . Since each box contains a nonnegative number of balls, and the first three boxes contain at most 100 balls combined, this distribution corresponds to a unique tuple in  $A$ . Thus, our mapping is bijective.

We are left to count the number of ways to distribute 100 balls into 4 boxes, which is  $\binom{100+4-1}{4-1} = \binom{103}{3}$ .  $\square$

5. Let  $S$  be the set of all positive integers whose digits (in base 10) form a strictly monotonic sequence of nonzero digits. For example, 1, 1456 and 921 are in  $S$ , but 4556, 8123, and 3210 are not in  $S$ . Determine the sum of all elements in  $S$ .

**Solution:** Let  $S_n$  denote the sum of all  $n$ -digit positive integers whose digits form a strictly monotonic sequence of nonzero digits. First, the sum of all positive 1-digit numbers is  $S_1 = 45$ . For  $n = 2, \dots, 9$ , let  $A_n$  be the set of positive integers with  $n$  strictly increasing nonzero digits and  $B_n$  the set of positive integers with  $n$  strictly decreasing nonzero digits.

For each  $n$ , let  $f_n(a) : A_n \rightarrow B_n$  map each digit  $d$  in  $a$  to the digit  $10 - d$ . (For example,  $f_4(2459) = 8651$ .) Since  $a$  has strictly increasing nonzero digits, the result of  $f_n(a)$  will be an  $n$ -digit integer with strictly decreasing nonzero digits. Note that  $f_n$  is a bijection since  $f_n(k_1) = f_n(k_2) \Rightarrow k_1 = k_2$  and  $f_n(f_n(a)) = a$  for all  $a \in A_n$ . Furthermore,  $a + f_n(a) = 10 + 10^2 + \dots + 10^n = \frac{10}{9}(10^n - 1)$  for all  $a \in A_n$ . Thus, we have

$$\begin{aligned} S_n &= \sum_{a \in A_n} a + \sum_{b \in B_n} b \\ &= \sum_{a \in A_n} (a + f(a)) \\ &= \sum_{a \in A_n} \frac{10}{9}(10^n - 1) \\ &= \frac{10}{9}(10^n - 1) \cdot |A_n|. \end{aligned}$$

Now, to calculate  $|A_n|$ , note that any strictly increasing  $n$ -digit integer is given uniquely by the set of its digits. That is, once we choose the set of digits in our number, there is only

one way to order them to make a strictly increasing  $n$ -digit number. Thus,  $|A_n| = \binom{9}{n}$ , so we have

$$S_n = \frac{10}{9}(10^n - 1)\binom{9}{n},$$

and our final answer is

$$\begin{aligned} & \sum_{n=1}^9 S_n \\ &= 45 + \sum_{n=2}^9 \frac{10}{9}(10^n - 1)\binom{9}{n} \\ &= 45 + \frac{10}{9} \sum_{n=2}^9 10^n \binom{9}{n} - \frac{10}{9} \sum_{n=2}^9 \binom{9}{n} \\ &= 45 + \frac{10}{9} [(10+1)^9 - 10 \cdot 9 - 1] - \frac{10}{9} [(1+1)^9 - 9 - 1] \\ &= \frac{10}{9}(11^9 - 2^9) - 45 \end{aligned}$$

where we used the binomial theorem from the third to the fourth line.  $\square$

6. Let  $S$  be the set of all  $3 \times 3$  matrices whose elements are all positive integers less than 10. Find the sum of the determinants of the matrices in  $S$ .

**Solution:** Partition  $S$  into three sets  $A, B, C$  such that:

- (a) The matrices in  $A$  all have the integer made from the first column greater than the integer made from the second column
- (b) The matrices in  $B$  all have the integer made from the first column less than the integer made from the second column
- (c) The matrices in  $C$  all have the integer made from the first column equal to the integer made from the second column

Note that the determinant of every matrix in  $C$  is zero. Now, consider the map  $f : A \rightarrow B$  where we swap the first and second columns. This map is a bijection because every matrix in  $A$  is mapped to a unique matrix in  $B$ , and every matrix in  $B$  is mapped to by a unique matrix in  $A$ . Thus, we have

$$\begin{aligned} \sum_{M \in S} d(M) &= \sum_{M \in A} d(M) + \sum_{M \in B} d(M) + \sum_{M \in C} d(M) \\ &= \sum_{M \in A} d(M) + \sum_{M \in B} d(M) \\ &= \sum_{M \in A} d(M) + d(f(M)) \\ &= \sum_{M \in A} d(M) - d(M) \\ &= 0. \end{aligned}$$

$\square$

## 17 Recurrence Relations

- The Fibonacci sequence is defined by  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$  for all  $n \geq 2$ . Find a closed-form expression for  $F_n$ .

**Solution:** The characteristic polynomial of  $(F_n)$  is  $p(x) = x^2 - x - 1$ , which has roots  $(1 \pm \sqrt{5})/2$ . Thus, we have

$$F_n = c_1 \left( \frac{1+\sqrt{5}}{2} \right)^n + c_2 \left( \frac{1-\sqrt{5}}{2} \right)^n$$

for some  $c_1, c_2$ . Plugging in  $F_0 = 0$  and  $F_1 = 1$  yields  $c_1 + c_2 = 0$  and  $(c_1 + c_2)/2 + \sqrt{5}(c_1 - c_2)/2 = 1$ , which we can solve to find  $c_1 = 1/\sqrt{5}$  and  $c_2 = -1/\sqrt{5}$ , so

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n.$$

(This is known as Binet's Formula.) □

- The Pell numbers are defined by  $P_0 = 0$ ,  $P_1 = 1$ , and  $P_n = 2P_{n-1} + P_{n-2}$  for all  $n \geq 2$ . Find positive real numbers  $r$  and  $L$  such that  $\lim_{n \rightarrow \infty} P_n/r^n = L$ .

**Solution:** The characteristic polynomial of  $(P_n)$  is  $p(x) = x^2 - 2x - 1$ , which has roots  $1 \pm \sqrt{2}$ . Thus, we have

$$P_n = c_1(1 + \sqrt{2})^n + c_2(1 - \sqrt{2})^n$$

for some  $c_1, c_2$ . Plugging in  $P_0 = 0$  and  $P_1 = 1$  yields  $c_1 + c_2 = 0$  and  $(c_1 + c_2) + \sqrt{2}(c_1 - c_2) = 1$ , which we can solve to find  $c_1 = 1/(2\sqrt{2})$  and  $c_2 = -1/(2\sqrt{2})$ , so

$$P_n = \frac{(1 + \sqrt{2})^n}{2\sqrt{2}} - \frac{(1 - \sqrt{2})^n}{2\sqrt{2}}.$$

Now,  $|(1 - \sqrt{2})| < 1$ , so  $\frac{(1 - \sqrt{2})^n}{2\sqrt{2}} \rightarrow 0$  as  $n \rightarrow \infty$ . Thus, we may choose  $r = 1 + \sqrt{2}$  and  $L = 1/(2\sqrt{2})$ . □

- Define the sequence  $(a_n)$  by  $a_0 = 0$  and  $a_n = 2a_{n-1} + 1$  for all  $n \geq 1$ . Find a closed-form expression for  $a_n$ .

**Solution:** We can calculate  $a_1 = 1$ ,  $a_2 = 3$ ,  $a_3 = 7$ ,  $a_4 = 15$ ,  $a_5 = 31$ , and  $a_6 = 63$ . At this point, we can guess that  $a_n = 2^n - 1$ . We will prove this by induction. Our base case is given as  $a_0 = 2^0 - 1 = 0$ . Suppose  $a_n = 2^n - 1$  for some  $n \geq 0$ . Then

$$a_{n+1} = 2a_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1.$$

This proves our claim by induction. □

4. Define the sequence  $(a_n)$  by  $a_1 = 1/2$  and  $a_n = na_{n-1}/(2n-2)$  for all  $n \geq 2$ . Find a closed-form expression for  $a_n$ .

**Solution:** We can rearrange the relation into

$$\frac{a_n}{n} = \frac{1}{2} \cdot \frac{a_{n-1}}{n-1}.$$

This motivates us to look at the sequence  $b_n = a_n/n$ , which satisfies  $b_1 = 1/2$  and  $b_n = b_{n-1}/2$  for  $n \geq 2$ . By induction, we can show that  $b_n = 1/2^n$ , and thus  $a_n = nb_n = n/2^n$ .  $\square$

5. Define the sequence of integers  $(a_n)$  by  $a_0 = 0$ ,  $a_1 = 1$ , and  $a_n = 5a_{n-1} - 6a_{n-2}$  for all  $n \geq 2$ . Prove that if  $n \mid m$ , then  $a_n \mid a_m$ .

**Solution:** The characteristic polynomial of  $(a_n)$  is  $p(x) = x^2 - 5x + 6$ , which has roots  $x = 2, 3$ . Thus, we have

$$a_n = c_1 2^n + c_2 3^n$$

for all  $n \geq 0$ . Plugging in  $a_0 = 0$  and  $a_1 = 1$  yields  $c_1 = -1$  and  $c_2 = 1$ , so

$$a_n = 3^n - 2^n$$

for all  $n \geq 0$ . Note that, for all integers  $x$  and  $y$ ,  $(x-y) \mid (x^k - y^k)$  for all  $k \geq 0$ . Thus,  $(3^n - 2^n) \mid (3^{nk} - 2^{nk})$  for all  $k \geq 0$ . It follows that  $a_n \mid a_{nk}$  for all  $k \geq 0$ .  $\square$

6. Let  $a_0, a_1, \dots$  be a sequence of positive integers such that  $2a_n = 5a_{n-1} - 2a_{n-2}$  for all  $n \geq 2$ . Find the smallest value of  $a_{2025}$  over all possible sequences.

**Solution:** The characteristic polynomial of  $(a_n)$  is  $p(x) = x^2 - 5x/2 + 1$ , which has roots  $x = 2, 1/2$ . Thus, we have

$$a_n = c_1 2^n + c_2 / 2^n$$

for some numbers  $c_1$  and  $c_2$ . Now,  $a_0$  and  $a_1$  are positive integers, and thus  $c_1 + c_2$  and  $2c_1 + c_2/2$  are positive integers. It follows that  $2(c_1 + c_2) - (2c_1 + c_2/2) = 3c_2/2$  is an integer, so  $c_2 = 2k/3$  for some integer  $k$ . It also follows that  $(c_1 + c_2) - 2(2c_1 + c_2/2) = -3c_1$  is an integer, so  $c_1 = j/3$  for some integer  $j$ .

Thus, we have  $a_n = (j2^n + 2k/2^n)/3$  for all  $n \geq 0$ . In particular, we must have  $j2^n + 2k/2^n$  be a positive integer and thus  $k/2^{n-1}$  be an integer for all  $n \geq 0$ . But this can only happen if  $k = 0 \Rightarrow c_2 = 0$ . Thus, we have  $a_n = c2^n$  for some positive integer  $c$ . It follows that  $a_{2025} = c2^{2025} \geq 2^{2025}$ , which is achieved with  $a_0 = 1$  and  $a_1 = 2$  yielding  $a_n = 2^n$ .  $\square$

7. Find a closed-form expression for  $a_n$  given that  $a_0 = a_1 = 2$  and  $a_n = a_{n-1} + 2a_{n-2} + 5 - 2n$  for all  $n \geq 2$ .

**Solution:** If we had  $a_n = a_{n-1} + 2a_{n-2}$  with  $a_0 = a_1 = 2$ , we can solve the recursive relation to find  $a_n = c_1 2^n + c_2 (-1)^n$  for some  $c_1, c_2$ . Call this the homogeneous solution. We are left to find the particular solution  $a_n - a_{n-1} - 2a_{n-2} = 5 - 2n$ . Since we have a linear term on

the right, we can guess  $a_n = An + B$  for some constants  $A$  and  $B$ . Indeed, we can plus this into the recursive relation:

$$An + B - A(n-1) - B - 2A(n-2) - 2B = 5 - 2n.$$

Since this holds for all  $n \geq 2$ , we need the constant and first degree coefficients of each side to be equal. Thus,  $A = 1$  and  $B = 0$ . We are left with

$$a_n = c_1 2^n + c_2 (-1)^n + n$$

for some  $c_1, c_2$ , which  $a_0 = a_1 = 2$  yields  $c_1 = c_2 = 1$ . Thus,  $a_n = 2^n + (-1)^n + n$  for all  $n \geq 0$ .

□

## 18 Recursive Counting

- Find the number of ways to tile a  $1 \times 2025$  grid with  $1 \times 1$  and  $1 \times 2$  tiles.

**Solution:** Let  $a_n$  denote the number of ways to tile a  $1 \times n$  grid with  $1 \times 1$  and  $1 \times 2$  tiles. Then  $a_1 = 1$  and  $a_2 = 2$ . For  $n \geq 3$ , consider the last grid square. It can either be tiled with a  $1 \times 1$  tile, which leaves  $n - 1$  tiles, or it can be tiled with a  $1 \times 2$  tile, which leaves  $n - 2$  tiles. Thus,  $a_n = a_{n-1} + a_{n-2}$ . Given  $a_1 = 1$  and  $a_2 = 2$ , we find that  $a_n = F_{n+1}$ , where  $F_n$  is the  $n^{\text{th}}$  Fibonacci number. Thus, there are  $a_{2025} = F_{2026}$  ways to tile a  $1 \times 2025$  grid.  $\square$

- There are 2025 balls ordered in a line. Find the number of ways to color the balls, each either red, green, or blue, such that no two adjacent balls are the same color.

**Solution:** Let  $a_n$  denote the number of ways to color  $n$  balls in a line, each either red, green, or blue, such that no two adjacent balls are the same color. Then  $a_1 = 3$  and  $a_2 = 3 \cdot 2$ . For  $n \geq 3$ , consider the color of the last ball. Given any such coloring with  $n$  balls, the first  $n - 1$  balls must have no adjacent balls of the same color, and there are 2 choices of color left for the last ball. Hence  $a_n \leq 2a_{n-1}$ . Given any such coloring with  $n - 1$  balls, we can color the last ball one of two colors to produce a coloring with  $n$  balls. Hence  $a_n \geq 2a_{n-1}$ , and thus  $a_n = 2a_{n-1}$ . By induction we have  $a_n = 3 \cdot 2^{n-1}$ , so  $a_{2025} = 3 \cdot 2^{2024}$ .  $\square$

- Let  $(2 + \sqrt{3})^{2n-1} = a_n + b_n\sqrt{3}$  for integers  $a_n$  and  $b_n$  ( $n \geq 1$ ). Express  $a_n$  in terms of  $n$ .

**Solution:** Note that for  $n \geq 2$ ,

$$\begin{aligned} a_n + b_n\sqrt{3} &= (2 + \sqrt{3})^{2n-1} = (7 + 4\sqrt{3})(2 + \sqrt{3})^{2n-3} = (7 + 4\sqrt{3})(a_{n-1} + b_{n-1}\sqrt{3}) \\ &\Rightarrow a_n + b_n\sqrt{3} = (7a_{n-1} + 12b_{n-1}) + \sqrt{3}(4a_{n-1} + 7b_{n-1}) \end{aligned}$$

since  $(a_n), (b_n)$  are integer sequences, it follows that

$$\begin{aligned} a_n &= 7a_{n-1} + 12b_{n-1}, \\ b_n &= 4a_{n-1} + 7b_{n-1}. \end{aligned}$$

Solving for  $b_{n-1}$  in the first equation and substituting into the second equation yields

$$a_n = 14a_{n-1} - a_{n-2}$$

for all  $n \geq 3$ . We can solve this recurrence relation to get

$$a_n = c_1(7 + 4\sqrt{3})^n + c_2(7 - 4\sqrt{3})^n$$

for some  $c_1, c_2$ . Substituting  $a_1 = 2$  and  $a_2 = 26$  yields  $c_1 = \frac{1}{2(2+\sqrt{3})}$  and  $c_2 = \frac{1}{2(2-\sqrt{3})}$ . Thus,

$$\begin{aligned} a_n &= [(7 + 4\sqrt{3})^n / (2 + \sqrt{3}) + (7 - 4\sqrt{3})^n / (2 - \sqrt{3})] / 2 \\ &= [(2 + \sqrt{3})^{2n} / (2 + \sqrt{3}) + (2 - \sqrt{3})^{2n} / (2 - \sqrt{3})] / 2 \\ &= [(2 + \sqrt{3})^{2n-1} + (2 - \sqrt{3})^{2n-1}] / 2 \end{aligned}$$

for all  $n \geq 1$ .  $\square$

4. Prove the identity  $F_{2n} = F_n^2 + F_{n-1}^2$ , where  $F_n$  is the  $n^{\text{th}}$  Fibonacci number. (Hint: Tile a  $1 \times 2n$  grid with  $1 \times 1$  and  $1 \times 2$  tiles, and consider whether the  $n$  and  $n+1$  grid squares are connected or not.)

**Solution:** From problem 1, we know there are  $F_{2n}$  ways to tile a  $1 \times 2n$  grid with  $1 \times 1$  and  $1 \times 2$  tiles. Now consider the  $n$  and  $n+1$  grid squares. If they are covered with a single  $1 \times 2$  tile, then we have two  $1 \times (n-1)$  grids left to tile. We can do this in  $F_{n-1}^2$  ways. Otherwise, we can tile the first  $1 \times n$  grid and the second  $1 \times n$  grid separately. We can do this in  $F_n^2$  ways. Thus,  $F_{2n} = F_n^2 + F_{n-1}^2$ .  $\square$

5. Find the number of sequences of length 2025 whose elements are from  $\{0, 1, 2, 3, 4\}$  and every pair of adjacent elements differ by exactly one.

**Solution:** Let  $s_n$  be the number of such sequences of length  $n$ . Let  $a_n, b_n, c_n$  denote the number of sequences of length  $n$  that end in 0, 1, 2, respectively. Then we have  $s_n = 2a_n + 2b_n + c_n$  since we can replace all elements  $s$  in a sequence by  $4 - s$  and still have a valid sequence. Now, we have  $a_n = b_{n-1}$ ,  $b_n = a_{n-1} + c_{n-1}$ , and  $c_n = 2b_{n-1}$ . Thus,  $b_n = b_{n-2} + 2b_{n-2} = 3b_{n-2}$ . Using  $b_1 = 1$  and  $b_2 = 2$ , we have  $b_{2k-1} = 3^{k-1}$  and  $b_{2k} = 2 \cdot 3^{k-1}$ . Thus,  $s_n = 2b_{n-1} + 2b_n + 2b_{n-1} = 2b_n + 4b_{n-1}$ , so

$$s_{2025} = 2b_{2025} + 4b_{2024} = 2 \cdot 3^{1012} + 8 \cdot 3^{1011} = 14 \cdot 3^{1011}.$$

$\square$

6. There are 2025 balls ordered in a circle. Find the number of ways to color the balls, each either red, green, or blue, such that no two adjacent balls are the same color.

**Solution:** Let  $a_n$  denote the number of ways to color  $n$  balls in a circle, each either red, green, or blue, such that no two adjacent balls are the same color. For convenience, number the balls 1 through  $n$  around the circle. Consider the set of colorings of  $n$  balls without the condition of balls 1 and  $n$  being different colors. This set is in bijection with the set of colorings of  $n$  balls in a line which no adjacent balls having the same color. From problem 2, there are  $3 \cdot 2^{n-1}$  such colorings.

We can partition the above set of colorings into two sets: the colorings with balls 1 and  $n$  having the same color and the colorings with balls 1 and  $n$  having different color.

If balls 1 and  $n$  have the same color, we can form a bijection to the set of colorings of  $n-1$  balls in a circle by considering ball 1 and ball  $n$  the same ball. This gives us  $a_{n-1}$  such colorings.

If balls 1 and  $n$  have different color, we have  $a_n$  colorings by definition. Thus, we have

$$3 \cdot 2^{n-1} = a_{n-1} + a_n.$$

This recurrence has homogeneous solution  $a_n = c_1(-1)^n$  and particular solution  $a_n = 2^n$ . Thus, with  $a_3 = 6$ , we have  $a_n = 2(-1)^n + 2^n$  for  $n \geq 3$ , so  $a_{2025} = 2^{2025} - 2$ .  $\square$

7. For each integer  $n \geq 3$ , let  $F(n)$  denote the number of ways to write  $2n$  as the sum of three distinct positive integers, without regard to order. For example, since

$$10 = 7 + 2 + 1 = 6 + 3 + 1 = 5 + 4 + 1 = 5 + 3 + 2,$$

we have  $F(5) = 4$ . Prove that  $F(n)$  is odd if and only if  $3 \mid n$ .

**Solution:** Let  $S(n)$  denote the set of solutions  $(a, b, c)$  of positive integers to  $a + b + c = 2n$  with  $a > b > c$ . Then  $F(n) = |S(n)|$ . Partition  $S(n)$  into three sets:  $S_1(n)$  the set with  $c = 1$ ,  $S_2(n)$  the set with  $c = 2$ , and  $S_3(n)$  the set with  $c \geq 3$ .

- For  $(a, b, c) \in S_1(n)$ , we have  $a + b = 2n - 1$ . We can see that  $(a, b) = (2n - 3, 2), (2n - 4, 3), \dots, (n, n - 1)$  are all of the solutions, so  $|S_1(n)| = n - 2$ .
- For  $(a, b, c) \in S_2(n)$ , we have  $a + b = 2n - 2$ . We can see that  $(a, b) = (2n - 5, 3), (2n - 6, 4), \dots, (n, n - 2)$  are all of the solutions, so  $|S_2(n)| = n - 4$ .
- For  $(a, b, c) \in S_3(n)$ , we have  $2n - 6 = (a - 2) + (b - 2) + (c - 2) = x + y + z$ , where  $x > y > z$  are positive integers. Hence  $|S_3(n)| = |S(n - 3)|$ .

Finally, we have  $|S(n)| = |S_1(n)| + |S_2(n)| + |S_3(n)| = 2n - 6 + |S(n - 3)|$ , so  $|S(n)| \equiv |S(n - 3)| \pmod{2}$ , i.e.  $F(n) \equiv F(n - 3) \pmod{2}$ . Since  $F(3) = 1$ ,  $F(4) = 2$ , and  $F(5) = 4$ , by induction we have  $F(n)$  odd if and only if  $n \equiv 0 \pmod{3}$ .  $\square$

## 19 Standard Inequalities

1. Prove the  $n = 2$  case for AM-GM: For nonnegative numbers  $a$  and  $b$ , prove that  $\frac{a+b}{2} \geq \sqrt{ab}$ .

**Solution:** We have

$$\begin{aligned}\frac{a+b}{2} &\geq \sqrt{ab} \\ \iff a - 2\sqrt{ab} + b &\geq 0 \\ \iff (\sqrt{a} - \sqrt{b})^2 &\geq 0.\end{aligned}$$

We have equality if and only if  $\sqrt{a} = \sqrt{b} \iff a = b$ . □

2. Let  $x_1, \dots, x_n$  be positive numbers. Find the minimum value of

$$(x_1 + \dots + x_n) \left( \frac{1}{x_1} + \dots + \frac{1}{x_n} \right).$$

**Solution:** By Cauchy-Schwarz, we have

$$(x_1 + \dots + x_n) \left( \frac{1}{x_1} + \dots + \frac{1}{x_n} \right) \geq \left( \sqrt{x_1} \cdot \frac{1}{\sqrt{x_1}} + \dots + \sqrt{x_n} \cdot \frac{1}{\sqrt{x_n}} \right)^2 = n^2.$$

We can achieve this minimum if and only if  $x_1 = \dots = x_n$ . □

3. Positive integers are placed at every lattice point in the plane so that each is the average of its four nearest neighbors. Given that 2025 is placed at the origin, find the number placed at the point (2025, 2025).

**Solution:** Let  $m$  be the smallest positive integer placed in the plane. Then the four numbers nearest to  $m$ , say  $a, b, c, d$ , satisfy  $4m = a + b + c + d \geq m + m + m + m = 4m$ . Thus, we must have  $a = b = c = d = m$ . By induction outward from  $m$ , we find that all integers in the plane must be equal to  $m$ . Thus, if a 2025 is at the origin, then 2025 must be at the point (2025, 2025). □

4. Let  $a, b$ , and  $c$  be real numbers. Prove that  $a^2 + b^2 + c^2 \geq ab + bc + ca$ , and determine when equality holds.

**Solution:** We have

$$\begin{aligned}(a-b)^2 + (b-c)^2 + (c-a)^2 &\geq 0 \\ \iff 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca &\geq 0 \\ \iff a^2 + b^2 + c^2 &\geq ab + bc + ca.\end{aligned}$$

Equality holds if and only if  $a = b = c$ . □

5. Let  $a$ ,  $b$ , and  $c$  be positive numbers. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c},$$

and determine when equality holds.

**Solution:** By AM-GM, we have

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} \geq 2\sqrt{\frac{a^2}{b^2} \cdot \frac{b^2}{c^2}} = 2\frac{a}{c}.$$

Similarly,

$$\frac{b^2}{c^2} + \frac{c^2}{a^2} \geq 2\frac{b}{a}, \quad \frac{c^2}{a^2} + \frac{a^2}{b^2} \geq 2\frac{c}{b}.$$

Adding all three inequalities together yields

$$\begin{aligned} 2\frac{a^2}{b^2} + 2\frac{b^2}{c^2} + 2\frac{c^2}{a^2} &\geq 2\frac{b}{a} + 2\frac{c}{b} + 2\frac{a}{c} \\ \Rightarrow \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} &\geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}. \end{aligned}$$

Equality holds if and only if  $\frac{a^2}{b^2} = \frac{b^2}{c^2} = \frac{c^2}{a^2}$ , i.e.  $a = b = c$ .  $\square$

6. Let  $f(x) = 3 \sin x + 4 \cos x$ . Find the maximum possible value of  $f$  over all real numbers  $x$ .

**Solution:** By Cauchy-Schwarz, we have

$$(3 \sin x + 4 \cos x)^2 \leq (3^2 + 4^2)(\sin^2 x + \cos^2 x) = 25.$$

Equality holds if and only if  $\sin x = 3k$  and  $\cos x = 4k$  for some number  $k$ . We can solve for  $k$  with  $(3k)^2 + (4k)^2 = 1 \Rightarrow k = \pm 1/5$ . Thus, we have the maximum value  $f(x) = 5$  achieved when  $x = \arcsin(3/5)$ .  $\square$

7. Let  $A$ ,  $B$ , and  $C$  be the three interior angles of a triangle. Prove that  $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$ .

**Solution:** Note that  $A + B + C = \pi$  and  $\sin x$  is a concave function for  $x \in [0, \pi]$ . Thus, by Jensen's inequality,

$$\begin{aligned} \frac{\sin A + \sin B + \sin C}{3} &\leq \sin\left(\frac{A+B+C}{3}\right) = \sin(\pi/3) = \sqrt{3}/2 \\ \Rightarrow \sin A + \sin B + \sin C &\leq \frac{3\sqrt{3}}{2}. \end{aligned}$$

We can note that equality occurs if  $A = B = C = \pi/3$ , i.e. our triangle is equilateral.  $\square$

8. Use Jensen's inequality to prove AM-GM for all  $n \geq 1$ .

**Solution:** First, if any of  $a_1, \dots, a_n$  are zero, then AM-GM holds since the RHS is zero of the LHS is nonnegative. Now, let  $a_1, \dots, a_n$  be positive numbers. Since  $\ln x$  is increasing on  $(0, \infty)$ , we have

$$\begin{aligned} \frac{a_1 + \dots + a_n}{n} &\geq \sqrt[n]{a_1 \cdots a_n} > 0 \\ \iff \ln\left(\frac{a_1 + \dots + a_n}{n}\right) &\geq \ln\left(\sqrt[n]{a_1 \cdots a_n}\right) \\ \iff \ln\left(\frac{a_1 + \dots + a_n}{n}\right) &\geq \frac{\ln a_1 + \dots + \ln a_n}{n} \end{aligned}$$

Since  $\ln x$  is concave on  $(0, \infty)$ , the last inequality holds by Jensen's. Thus, the first inequality holds.  $\square$

9. Let  $x, y$ , and  $z$  be complex numbers with  $x+y+z=0$ . Prove that  $|x|+|y|+|z|\geq 2\max\{|x|, |y|, |z|\}$ , and determine when equality holds.

**Solution:** Without loss of generality, assume  $|x| \geq |y| \geq |z|$ . Then we need to prove

$$\begin{aligned} |x| + |y| + |z| &\geq 2|x| \\ \iff |y| + |z| &\geq |x| = |y + z| \end{aligned}$$

which holds by the triangle inequality. Equality holds if and only if  $\arg y = \arg z$ , i.e. if and only if  $(x, y, z) = (-(c+1)w, w, cw)$  for some real  $c > 0$  and complex  $w$ .  $\square$

10. Let  $z$  and  $w$  be complex numbers. Prove that  $||z|-|w|| \leq |z-w|$ .

**Solution:** Without loss of generality assume  $|z| \geq |w|$ . Then we are left to show that  $|z| - |w| \leq |z-w|$ , or equivalently  $|z| \leq |z-w| + |w|$ , which holds by the triangle inequality.  $\square$

11. Let  $n$  be a positive integer. Given  $2n-1$  real numbers  $x_1, \dots, x_{2n-1}$ , let

$$f(x) = |x - x_1| + \dots + |x - x_{2n-1}|.$$

Find a real number  $x_0$  such that  $f(x_0) \leq f(x)$  for all real  $x$ .

**Solution:** Without loss of generality, assume  $x_1 \leq x_2 \leq \dots \leq x_{2n-1}$ . Consider  $x_0 = x_n$ . Then we have

$$\begin{aligned} f(x_0) &= |x_n - x_1| + \dots + |x_n - x_n| + |x_n - x_{n+1}| + \dots \\ &= (x_n - x_1) + \dots + (x_n - x_n) + (x_{n+1} - x_n) + \dots \\ &= x_{n+1} + \dots + x_{2n-1} - x_1 - \dots - x_{n-1}. \end{aligned}$$

To show this is a minimum, note that for all  $i = 1, \dots, n-1$ ,

$$|x - x_{n-i}| + |x - x_{n+i}| \geq |(x - x_{n-i}) + (x_{n+i} - x)| = x_{n+i} - x_{n-i}.$$

Summing over all  $i$  yields  $f(x) \geq f(x_0)$ .  $\square$

12. Suppose  $a, b, c, d, e$  are real numbers that satisfy  $a+b+c+d+e=8$  and  $a^2+b^2+c^2+d^2+e^2=16$ . Find the maximum possible value of  $a$ .

**Solution:** We have  $b+c+d+e=8-a$  and  $b^2+c^2+d^2+e^2=16-a^2$ . By Cauchy-Schwarz, we have

$$\begin{aligned}(b \cdot 1 + c \cdot 1 + d \cdot 1 + e \cdot 1)^2 &\leq (b^2 + c^2 + d^2 + e^2)(1^2 + 1^2 + 1^2 + 1^2) \\ \Rightarrow (8-a)^2 &\leq 4(16-a^2) \\ \Rightarrow a(5a-16) &\leq 0.\end{aligned}$$

Thus,  $a=16/5$  is the maximum value, which is achieved with  $b=c=d=e=6/5$ .  $\square$

13. Given  $a_1+\cdots+a_n=n$  ( $a_i \in \mathbb{R}$ ), prove that  $a_1^4+\cdots+a_n^4 \geq n$ .

**Solution:** By Cauchy-Schwarz, we have

$$\begin{aligned}(a_1 \cdot 1 + \cdots + a_n \cdot 1)^2 &\leq (a_1^2 + \cdots + a_n^2)(1^2 + \cdots + 1^2) \\ \Rightarrow a_1^2 + \cdots + a_n^2 &\geq n^2/n = n.\end{aligned}$$

By Cauchy-Schwarz again,

$$\begin{aligned}(a_1^2 \cdot 1 + \cdots + a_n^2 \cdot 1)^2 &\leq (a_1^4 + \cdots + a_n^4)(1^2 + \cdots + 1^2) \\ \Rightarrow a_1^4 + \cdots + a_n^4 &\geq (a_1^2 + \cdots + a_n^2)^2/n \geq n^2/n = n.\end{aligned}$$

$\square$

14. Let  $p(x)=x^{2025}-20x^{25}-25x^{20}-2025$ . Given that  $p$  has exactly one positive root  $r$ , prove that for any real or complex root  $s$ , we have  $|s| \leq r$ .

**Solution:** We have

$$\begin{aligned}p(s) &= s^{2025} - 20s^{25} - 25s^{20} - 2025 = 0 \\ \Rightarrow s^{2025} &= 20s^{25} + 25s^{20} + 2025 \\ \Rightarrow |s^{2025}| &= |20s^{25} + 25s^{20} + 2025| \leq |20s^{25}| + |25s^{20}| + |2025| \\ \Rightarrow |s|^{2025} &\leq 20|s|^{25} + 25|s|^{20} + 2025 \\ \Rightarrow |s|^{2025} - 20|s|^{25} - 25|s|^{20} - 2025 &\leq 0 \\ \Rightarrow p(|s|) &\leq 0.\end{aligned}$$

Since  $p$  has only one positive root  $r$ , and since  $p(0) < 0$  and  $p(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , we have  $p(x) \leq 0$  for  $x \geq 0$  if and only if  $x \in [0, r]$ . Thus,  $p(|s|) \leq 0 \Rightarrow |s| \leq r$ .  $\square$

## 20 Determinants

- Find the determinant of the  $n \times n$  matrix all of whose entries are 1.

**Solution:** For  $n = 1$  the determinant is 1. For  $n \geq 2$ , the columns of this matrix are linearly dependent, so the determinant is 0.  $\square$

- Suppose  $A$  is an  $n \times n$  matrix such that the elements of each row of  $A$  sum to 0. Prove that  $\det(A) = 0$ .

**Solution 1:** Let  $\vec{u}$  be the  $n \times 1$  column vector of all 1's. Since each row sums to 0,  $\vec{r}_i \cdot \vec{u} = 0$  for all rows  $\vec{r}_i$  of  $A$ . Thus,  $A\vec{u} = \vec{0}$ , so  $A\vec{v} = \vec{0}$  has a nonzero solution, meaning  $\det(A) = 0$ .  $\square$

**Solution 2:** Let  $\vec{c}_i$  be the  $i^{\text{th}}$  column of  $A$ . Then  $\sum_{i=1}^n \vec{c}_i = \vec{0}$  since each row of  $A$  sums to zero. Thus, the columns of  $A$  are linearly dependent, so  $\det(A) = 0$ .  $\square$

- A square matrix  $A$  is skew-symmetric if  $A^T = -A$ . Does there exist an invertible  $2025 \times 2025$  skew-symmetric matrix?

**Solution:** No such matrix exists. We have

$$\det(A) = \det(A^T) = \det(-A) = (-1)^{2025} \det(A) = -\det(A).$$

Hence  $\det(A) = 0$ , so  $A$  cannot be invertible.  $\square$

- Determine the value of  $k$  so that the following matrix is not invertible:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & k \end{pmatrix}$$

**Solution:** By cofactor expansion along the first row, the determinant is

$$\begin{aligned} 1 \begin{vmatrix} 2 & 3 \\ 3 & k \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 1 & k \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \\ = (2k - 9) - (k - 3) + (3 - 2) \\ = k - 5. \end{aligned}$$

A matrix is not invertible if and only if its determinant is 0, so we need  $k = 5$ .  $\square$

- Let  $A$  be an invertible matrix such that  $I + A$  is invertible. Prove that  $I + A^{-1}$  is invertible.

**Solution:** We have

$$\det(I + A^{-1}) = \det((A + I)A^{-1}) = \det(A + I)\det(A^{-1}) = \det(A + I)/\det(A) \neq 0$$

since  $\det(A + I), \det(A) \neq 0$ . Thus,  $I + A^{-1}$  is invertible.  $\square$

6. A square matrix  $M$  is upper triangular if all of the entries below the main diagonal are 0. Prove  $\det(M)$  is the product of the diagonal entries of  $M$ .

**Solution:** We will induct on the size of the matrix. For any  $1 \times 1$  matrix, it is upper triangular and its determinant is the product of its diagonal entries, which is just the single entry.

Now, assume the determinant of any  $n \times n$  upper triangular matrix is the product of the diagonal entries for some  $n \geq 1$ . Consider some  $(n+1) \times (n+1)$  upper triangular matrix  $M$  with diagonal entries  $m_1, \dots, m_{n+1}$ . Then using cofactor expansion along the first column (whose entries are  $m_1$  followed by  $n$  zeroes), we have

$$\det(M) = m_1 \det(M')$$

where  $M'$  is an  $n \times n$  upper triangular matrix with diagonal entries  $m_2, \dots, m_{n+1}$ . By our inductive hypothesis,  $\det(M') = m_2 \cdots m_{n+1}$ . Thus,  $\det(M) = m_1 m_2 \cdots m_{n+1}$ , so by induction the determinant of any upper triangular matrix is the product of its diagonal entries.  $\square$

7. Suppose  $A$  and  $B$  are  $n \times n$  matrices with  $AB = BA$ . Prove that

$$\det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A^2 - B^2).$$

**Solution:** Subtract the second  $n$  columns from the first  $n$  columns, then add the first  $n$  rows to the second  $n$  rows to get

$$\det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det \begin{pmatrix} A-B & B \\ B-A & A \end{pmatrix} = \det \begin{pmatrix} A-B & B \\ O & A+B \end{pmatrix}.$$

Thus,

$$\begin{aligned} \det \begin{pmatrix} A & B \\ B & A \end{pmatrix} &= \det(A-B) \det(A+B) \\ &= \det((A-B)(A+B)) \\ &= \det(A^2 + AB - BA - B^2) \\ &= \det(A^2 - B^2). \end{aligned}$$

$\square$

8. Let  $A$  be an  $n \times n$  matrix whose entries are integers between 0 and 9, inclusive. Let  $k_i$  be the integer that results from concatenating the entries of row  $i$ . Suppose there exists an integer  $d$  such that  $d \mid k_i$  for all  $i = 1, \dots, n$ . Prove that  $d \mid \det(A)$ .

**Solution:** Let  $c_1, \dots, c_n$  be the columns of  $A$  indexed from right to left. For each  $i = 2, \dots, n$ , add  $10^{i-1} c_i$  to column  $c_1$ . This doesn't change the determinant and transforms column  $c_1 = (k_1 \ k_2 \ \cdots \ k_n)^T$ . Then by cofactor expansion on  $c_1$ , we have

$$\det(A) = \sum_{i=1}^n (-1)^{i-1} k_i \det(A_i)$$

where  $A_i$  is the matrix formed by removing the  $i^{\text{th}}$  row and first column from  $A$ . In particular,  $A_i$  has integer entries, so  $\det(A_i)$  is an integer. Then  $d$  divides each (integer) term in the sum, so  $d$  divides  $\det(A)$ .  $\square$

9. Find the determinant of the following  $n \times n$  matrix:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{pmatrix}$$

where entry  $a_{ij}$  is  $\max(i, j)$ .

**Solution:** Index the columns  $c_1, \dots, c_n$  from left to right. Performing the operations  $c_i \rightarrow c_i - c_{i-1}$  ( $i = 2, \dots, n$ ) doesn't change the determinant and gives the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 0 & 1 & \cdots & 1 \\ 3 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

By cofactor expansion along the bottom row, the determinant is equal to  $(-1)^{n-1} n \det(M)$ , where  $M$  is the  $(n-1) \times (n-1)$  matrix with 1's in and above the diagonal and 0's below the diagonal. Since  $M$  is upper-triangular, its determinant is the product of its diagonal entries. Thus, our original determinant has value  $(-1)^{n-1} n$ .  $\square$

10. Find the determinant of the  $n \times n$  matrix whose diagonal entries are 0 and all other entries are 1.

**Solution 1:** Let  $n \geq 2$ . Let  $\vec{c}_i$  be the  $i^{\text{th}}$  column of the matrix. Then performing the operations  $\vec{c}_i \rightarrow \vec{c}_i - \vec{c}_{i+1}$  for each  $i = 1, \dots, n-1$  doesn't change the determinant and gives us the matrix

$$\begin{pmatrix} -1 & 0 & 0 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 1 \\ 0 & 1 & -1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where the  $i^{\text{th}}$  column is  $\vec{d}_i = -\vec{e}_i + \vec{e}_{i+1}$  for  $i = 1, \dots, n-1$  and  $\vec{d}_n = \vec{e}_1 + \dots + \vec{e}_{n-1}$ . Let  $r_i$  be the  $i^{\text{th}}$  row of this new matrix. Then performing the operations  $\vec{r}_i \rightarrow \vec{r}_i + \vec{r}_{i-1}$  for each  $i = 2, \dots, n$  doesn't change the determinant and gives us the matrix

$$\begin{pmatrix} -1 & 0 & 0 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & 2 \\ 0 & 0 & -1 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \end{pmatrix}$$

which is upper-triangular with diagonal entries  $-1$  for the first  $n-1$  entries and  $n-1$  as the last entry. Thus, the determinant is  $(-1)^{n-1}(n-1)$  (which also works for the  $n=1$  case).  $\square$

**Solution 2:** Let  $J_n$  be the  $n \times n$  matrix of all 1's. Then the given matrix is  $J_n - I_n$ . Let  $\vec{u}$  be the  $n \times 1$  column vector of all 1's and  $\vec{e}_i$  be the  $i^{\text{th}}$  standard basis vector of  $\mathbb{R}^n$ . Then one can verify that  $J_n$  has eigenvalues  $n$  with eigenvector  $\vec{u}$  and 0 with eigenvectors  $n\vec{e}_i - \vec{u}$  for  $i = 1, \dots, n-1$ . (Refer to the first solution to problem 12 for a similar proof). Thus,  $J_n - I_n$  has eigenvalues  $n-1$  with multiplicity 1 and  $-1$  with multiplicity  $n-1$ , so  $\det(J_n - I_n) = (-1)^{n-1}(n-1)$ .  $\square$

11. Find the max possible determinant of a  $3 \times 3$  matrix whose entries are  $\pm 1$ .

**Solution:** The max possible determinant is 4. Let  $A = (a_{ij})$  be a  $3 \times 3$  matrix with  $a_{ij} \in \{-1, 1\}$ . Then by cofactor expansion, we have

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

In particular,  $\det(A)$  must be an integer. By the triangle inequality,

$$|\det(A)| \leq |a_{11}a_{22}a_{33}| + |a_{11}a_{23}a_{32}| + |a_{12}a_{21}a_{33}| + |a_{12}a_{23}a_{31}| + |a_{13}a_{21}a_{32}| + |a_{13}a_{22}a_{31}| = 6.$$

Thus,  $\det(A) \leq 6$ . For the sake of contradiction, suppose  $\det(A) = 6$ . Then each of the six terms in the expansion of  $\det(A)$  must be positive. In particular, there must be an even number of  $-1$ 's among each of  $\{a_{11}a_{22}a_{33}\}$ ,  $\{a_{12}a_{23}a_{31}\}$ , and  $\{a_{13}a_{21}a_{32}\}$ , and an odd number of  $-1$ 's among each of  $\{a_{11}a_{23}a_{32}\}$ ,  $\{a_{12}a_{21}a_{33}\}$ , and  $\{a_{13}a_{22}a_{31}\}$ . Thus, there are an odd number of  $-1$ 's (counting duplicates) among all six sets. But each entry of  $A$  appears exactly twice among all six sets, so there must be an even number of  $-1$ 's (counting duplicates) among all six sets, a contradiction.

Furthermore,  $a_{ij} \equiv 1 \pmod{2}$  for all  $i, j$ , so  $\det(A) \equiv 0 \pmod{2}$ . Thus,  $\det(A) \leq 4$ , and we can achieve  $\det(A) = 4$  with

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$\square$

12. Let  $n \geq 2$  be an integer and  $x$  a real number. Find the determinant of the  $n \times n$  matrix whose diagonal entries are 1 and all other entries are  $x$ .

**Solution 1:** Let  $A_n$  be the given matrix. We claim that  $\det(A_n) = (1-x)^{n-1}(nx-x+1)$ . We'll prove this by showing  $A_n$  has eigenvalues  $(1-x)$  (with multiplicity  $n-1$ ) and  $nx-x+1$  (with multiplicity 1), whose product is exactly the determinant on  $A_n$ .

Let  $\vec{u}$  be the  $n \times 1$  column vector of all 1's and  $\vec{e}_i$  be the  $i^{\text{th}}$  standard basis vector of  $\mathbb{R}^n$ . Then

$$A_n \vec{u} = (nx-x+1) \vec{u},$$

so  $nx - x + 1$  is an eigenvalue with eigenvector  $\vec{u}$ . Now, let  $\vec{v}_i = n\vec{e}_i - \vec{u}$  for  $i = 1, \dots, n-1$ . Then

$$\begin{aligned} A_n \vec{v}_i &= A_n(n\vec{e}_i - \vec{u}) \\ &= nA_n\vec{e}_i - A_n\vec{u} \\ &= n(x\vec{u} + (1-x)\vec{e}_i) - (nx - x + 1)\vec{u} \\ &= n(1-x)\vec{e}_i - (1-x)\vec{u} \\ &= (1-x)\vec{v}_i, \end{aligned}$$

so  $1-x$  is an eigenvalue with eigenvectors  $\vec{v}_1, \dots, \vec{v}_{n-1}$ . We're left to show that  $\vec{u}, \vec{v}_1, \dots, \vec{v}_{n-1}$  are linearly independent. Set  $\vec{w}_i = (\vec{v}_i + \vec{u})/n$  for  $i = 1, \dots, n-1$  and  $\vec{w}_n = \vec{u} - \vec{w}_1 - \dots - \vec{w}_{n-1}$ . Then it is equivalent to show that  $\vec{w}_1, \dots, \vec{w}_n$  are linearly independent, which follows since  $\vec{w}_i = \vec{e}_i$  for all  $i = 1, \dots, n$ .  $\square$

**Solution 2:** Let  $J_n$  be the  $n \times n$  matrix of all 1's. Then the given matrix is  $xJ_n + (1-x)I_n$ . Let  $\vec{u}$  be the  $n \times 1$  column vector of all 1's and  $\vec{e}_i$  be the  $i^{\text{th}}$  standard basis vector of  $\mathbb{R}^n$ . Then one can verify that  $J_n$  has eigenvalues  $n$  with eigenvector  $\vec{u}$  and 0 with eigenvectors  $n\vec{e}_i - \vec{u}$  for  $i = 1, \dots, n-1$ . (Refer to solution 1 for a similar proof). Thus,  $xJ_n + (1-x)I_n$  has eigenvalues  $xn + (1-x)$  with multiplicity 1 and  $(1-x)$  with multiplicity  $n-1$ , so the determinant is  $(1-x)^{n-1}(nx - x + 1)$ .  $\square$