Fubini Rankings and the Hunt for the Lucky Statistic

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For example, (1,1,3,2,5) is a parking function with outcome (1,2,3,4,5), but (2,2,3,4,5) is not a parking function.

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Definition

If α is a parking function, we call a car $c \in \alpha$ *lucky* if it is in its preferred space. We call lucky(α) the number of lucky cars in α, or the *lucky statistic*. We call the set of length-n parking functions PF_n.

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Gessel and Seo gave the following generating function for the lucky car statistic on parking functions:

$$L_{PF_n}(q) = \sum_{\alpha \in \mathrm{PF}n} q^{\text{lucky}(\alpha)} = q \prod_{i=1}^{n-1} (i + (n - i + 1)q).$$

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Our work has been expanding the amount of knowledge we have on the lucky statistic of these parking functions, both by exploring parking functions with fixed lucky statistics, and fixed lucky sets.

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Notation/Things We Know

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3. $L'_{X_n}(1) = \sum_{k=1}^{n} k(f_X(n,k))$ *gives the total number of lucky cars in* $X_n \subseteq \mathrm{PF}_n$.

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3. $L'_{X_n}(1) = \sum_{k=1}^{n} k(f_X(n,k))$ *gives the total number of lucky cars in* $X_n \subseteq \mathrm{PF}_n$.
4. *Therefore,* $\mathbb{E}(\text{lucky}(\alpha \in X_n)) = \frac{L'_{X_n}(1)}{L_{X_n}(1)}$

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Basically, imagine a Fubini ranking as an acceptable outcome of a race; therefore, 1134 is acceptable, but 1124 is not, since the next person to finish after the first two would be third. Fubini rankings are permutation-invariant, meaning rearrangements of Fubini rankings are still Fubini rankings.

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Theorem (Bradt et al.)

The set of Fubini Rankings of length n, which we call FR_n*, is a subset of* PF_n.

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n	Lucky Polynomial
1	$1q$
2	$1q + 2q^2$
3	$1q + 6q^2 + 6q^3$
4	$1q + 14q^2 + 36q^3 + 24q^4$
5	$1q + 30q^2 + 150q^3 + 240q^4 + 120q^5$

OEIS 019538: Generating Functions for lucky cars in Fubini rankings for $0 \leq n \leq 5$.

In terms of the Twelvefold Way, a Fubini ranking corresponds to the number of ways n labeled balls can be placed in k labeled urns such that none are empty, giving our first form for $f_{\mathrm{FR}}(n, k)$:

$$f_{\mathrm{FR}}(n, k) = k!S(n, k).$$

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We also know that $|\mathrm{FR}_n| = L_{FR_n}(1)$ is equal to the nth Fubini number.

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Theorem

An alternate form for $f_{\mathrm{FR}}(n,k)$ is as follows:

$$f_{\mathrm{FR}}(n,k) = \sum_{(c_1,c_2,\ldots,c_k)\vDash n} \binom{n}{c_1, c_2, \ldots, c_k} = \sum_{(c_1,c_2,\ldots,c_k)\vDash n} \frac{n!}{c_1!c_2!\cdots c_k!}.$$

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Corollary

The exponential generating function for the number of Fubini rankings of length n with k lucky cars is given by

$$\sum_{n\geq 0}\sum_{k\geq 0} f_{\mathrm{FR}}(n,k)q^k\frac{x^n}{n!} = \frac{1}{1-e^{x-1}q}.$$

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Definition

We denote the set of weakly increasing parking functions with an up arrow, as in $\mathrm{PF}_n^{\uparrow}$, and in general, for $X_n \subseteq \mathrm{PF}_n$, $X_n^{\uparrow} \subseteq \mathrm{PF}_n^{\uparrow}$ is the set of weakly increasing parking functions in X_n. Necessarily, $X_n^{\uparrow} \subseteq X_n$.

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Lemma

$$f^{\uparrow}_{\mathrm{FR}}(n, k) = \binom{n-1}{k-1}$$

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The lucky polynomial for weakly increasing Fubini rankings of length n is given by

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Corollary

$$\mathbb{E}[\text{lucky}(\alpha \in \mathrm{FR}_n^{\uparrow})] = \frac{1}{2^{n-1}} \sum_{k=1}^{n} k \cdot \binom{n-1}{k-1} = \frac{n+1}{2}$$

ℓ-interval Parking Functions

Definition

ℓ-interval parking functions are a subset of PF_n in which a car can only park ℓ spaces ahead of its preferred spot, which we denote $\mathrm{PF}_n(\ell)$. When $\ell = 1$, the resultant set of parking functions is called the *unit interval parking functions*, or UPF_n.

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Bradt et al. have shown that there is a bijection between Fubini rankings and unit interval parking functions, and their intersection is non-trivial.

Unit Interval Parking Functions

Definition (Hadaway)

Given a Fubini ranking $\alpha = (a_1, a_2, \ldots, a_n)$, if $a_{i_1}, a_{i_2}, \ldots, a_{i_j}$ is the full set of entries tied at rank x, the map $\psi : \mathrm{FR}_n \to \mathrm{UPF}_n$, takes

$$a_{i_1}, a_{i_2}, \ldots, a_{i_j} \mapsto x, x, x+1, x+2, \ldots, x+j-1.$$

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2. $L_{\mathrm{UPF}_n}(q) = L_{\mathrm{FR}_n}(q) = \sum_{k=1}^{n} k!S(n,k)q^k$
3. $\mathbb{E}[\text{lucky}(\alpha \in \mathrm{UPF}_n)] = \mathbb{E}[\text{lucky}(\alpha \in \mathrm{FR}_n)] = \frac{1}{|\mathrm{FR}_n|} \sum_{k=1}^{n} k \cdot k!S(n,k)$

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4. $f^{\uparrow}_{\mathrm{UPF}}(n,k) = f^{\uparrow}_{\mathrm{FR}}(n,k) = \binom{n-1}{k-1}$
5. $L^{\uparrow}_{\mathrm{UPF}_n}(q) = L^{\uparrow}_{\mathrm{FR}_n}(q) = q(q+1)^{n-1}$
6. $\mathbb{E}[\text{lucky}(\alpha \in \mathrm{UPF}^{\uparrow}_n)] = \mathbb{E}[\text{lucky}(\alpha \in \mathrm{FR}^{\uparrow}_n)] = \frac{n+1}{2}$

Unit Interval ∩ Fubini Rankings

Lemma

$$f_{\mathrm{UFR}}(n,k) = \frac{n!}{2^{n-k}}\binom{k}{n-k}; f^{\uparrow}_{\mathrm{UFR}}(n,k) = \binom{k}{n-k}$$

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Lemma (Elder, et al.)

The number of weakly increasing unit Fubini rankings of length n is

$$|\mathrm{UFR}^{\uparrow}_{n}| = F_{n+1},$$

where F_{n+1} denotes the $(n+1)$th Fibonacci number which follows the recursion relation $F_{n+1} = F_n + F_{n-1}$ with seed values $F_1 = F_2 = 1$.

A Result for $FR_n(\ell = 2)$!

Theorem

Let $g(n,k)$ denote the number of duet Fubini rankings, (i.e. $\mathrm{FR}_n(\ell = 2)$), with $n-k$ distinct ranks. Define $k = 2i + j$. Then, $g(n,k) =$

$$\underbrace{\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{\lfloor k/2 \rfloor}{i} \binom{n-k}{j}}_{1} \times \underbrace{(n-2j-3i)!}_{2} \times \underbrace{\prod_{c_1=0}^{i-1} \binom{n-3c_1}{3} \times \prod_{c_2=0}^{j-1} \binom{n-2c_2-3i}{2}}_{3}. \tag{1}$$

The summation on the left simplifies to $\binom{n-k}{j} \cdot 2^{\lfloor \frac{k}{2} \rfloor}$.

A Result for $FR_n(\ell = 2)$!

Theorem

Let $g(n,k)$ denote the number of duet Fubini rankings, (i.e. $\mathrm{FR}_n(\ell = 2)$), with $n-k$ distinct ranks. Define $k = 2i + j$. Then, $g(n,k) =$

$$\underbrace{\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{\lfloor k/2 \rfloor}{i} \binom{n-k}{j}}_{1} \times \underbrace{(n-2j-3i)!}_{2} \times \underbrace{\prod_{c_1=0}^{i-1} \binom{n-3c_1}{3} \times \prod_{c_2=0}^{j-1} \binom{n-2c_2-3i}{2}}_{3}. \tag{1}$$

The summation on the left simplifies to $\binom{n-k}{j} \cdot 2^{\lfloor \frac{k}{2} \rfloor}$. We've also generalized and proven that each ℓ-interval parking function must have at least $\lceil \frac{n}{\ell+1} \rceil$ lucky cars.

Fixed Sets of Lucky Cars

Suppose we fix a set of k lucky cars such that all other cars are unlucky. We denote this $\mathrm{FR}_n(\mathcal{C})$ with $\mathcal{C} = \{c_1, c_i, \ldots, c_k\}$. Then $|\mathrm{Lucky}_{FR_n}(\mathcal{C})|$ denotes the number of Fubini rankings with the fixed lucky set $\mathcal{C}$.

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- the first car in any parking function is lucky,
- one of the lucky cars, not necessarily car 1, must prefer spot 1, or spot 1 will not be filled under the classic parking rule,
- in Fubini rankings, a lucky car must be the first to prefer its rank.

Example of Fixed Lucky Set

Example

Suppose we want to enumerate all the Fubini rankings of length 4 such that car 3 is lucky. Then we have,

- $\alpha_1 = \boxed{1}\ 1\ \boxed{3}\ 3$
- $\alpha_2 = \boxed{3}\ 3\ \boxed{1}\ 1$
- $\alpha_3 = \boxed{1}\ 1\ \boxed{4}\ 1$
- $\alpha_4 = \boxed{2}\ 2\ \boxed{1}\ 2$

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Fixed Sets of Lucky Cars in Fubini Rankings

Lemma

Let $\mathrm{FR}_n(\mathcal{C})$ *such that* $\mathcal{C} = \{1, i, n\}$ *with* $i \in [2, n-1]$. *Then the number of Fubini rankings for all values of* i *in this set is,*

$$|\mathsf{Lucky}_{\mathrm{FR}}(\{1, i, n\})| = 3 \sum_{x=1}^{n-2} 2^x. \tag{2}$$

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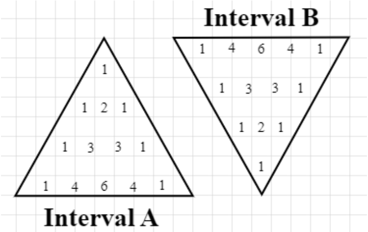

$$\alpha = \underbrace{u}_{a_1} \longleftarrow \text{Interval A} \longrightarrow \underbrace{v}_{a_i} \longleftarrow \text{Interval B} \longrightarrow \underbrace{w}_{a_n}.$$

Example

Let $n = 5$. We know that one of u, v, w will equal 1, and that $|A| + |B| = (n - 3)$.

$$\begin{aligned}
\alpha_{i=2} &= \boxed{a_1}\ \boxed{a_2}\ a_3\ a_4\ \boxed{a_n} \\
\alpha_{i=3} &= \boxed{a_1}\ a_2\ \boxed{a_3}\ a_4\ \boxed{a_n} \\
\alpha_{i=4} &= \boxed{a_1}\ a_2\ a_3\ \boxed{a_4}\ \boxed{a_n}
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Continuing our example for $n = 5$, with $i = 2$: $\alpha_{i=2} = \boxed{a_1}\ \boxed{a_2}\ a_3\ a_4\ \boxed{a_n}$.

- $u = 1$; (12225), (13332), (14115), (14143), (14413), (13315), (13135), (15114).
- $v = 1$; (21225), (31135), (31315), (31332), (41115), (41143), (41413), (51114).
- $w = 1$; (23331), (24241), (24421), (25221), (32331), (42241), (42421), (52221).

$\rightarrow$ We have $3(8) = 3(2^3)$.

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Continuing our example for $n = 5$, with $i = 3$: $\alpha_{i=3} = \boxed{a_1}\ a_2\ \boxed{a_3}\ a_4\ \boxed{a_n}$.

- $u = 1$; (11335), (11443), (11514), (11415).
- $v = 1$; (22125), (33115), (33132), (44113).
- $w = 1$; (22441), (22521), (33231), (44221).

$\rightarrow$ We have $3(4) = 3(2^2)$.

$$\alpha = \underbrace{u}_{a_1} \longleftarrow \text{Interval A} \longrightarrow \underbrace{v}_{a_i} \longleftarrow \text{Interval B} \longrightarrow \underbrace{w}_{a_n}. \qquad (5)$$

Continuing our example for $n = 5$, with $i = 4$: $\alpha_{i=4} = \boxed{a_1}\ a_2\ a_3\ \boxed{a_4}\ \boxed{a_n}$.

- $u = 1$; (11145), (11154).
- $v = 1$; (22215), (33312).
- $w = 1$; (22251), (33321).

$\rightarrow$ We have $3(2) = 3(2^1)$.

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$\rightarrow$ We have $3(2) = 3(2^1)$.

$\Rightarrow$ Summing over $i = 2, 3, 4$, and noting that for $n = 5$, $(n-2) = 3$, we have $3(2^1 + 2^2 + 2^3) = \sum_{x=1}^{3} 2^x$, as desired.

Fixed Sets in Fubini Rankings & Pascal's Triangle

$$|\mathrm{Lucky}_{\mathrm{FR}}(\{1,i,n\})| = 3\sum_{x=1}^{n-2} 2^x. \tag{6}$$

Row	Coefficients	Sum
0	1	
1	1 1	$2\binom{n-2}{n-2} = 2$
2	1 2 1	$\vdots$
3	1 3 3 1	$2\binom{n-2}{n-5}$
4	1 4 6 4 1	$\vdots$
5	1 5 10 10 5 1	$\vdots$
$\vdots$	...	$\vdots$
$n-2$	...	$2\binom{n-2}{1} = 2(n-2)$

Corollary for $\mathcal{C} = \{1, i, n\}$

Corollary

Let $\mathrm{FR}_n(\mathcal{C})$ such that $\mathcal{C} = \{1, i, n\}$ for $n \geq (i+1)$ with $i \in [2, n-1]$. Then the number of Fubini rankings for any given i in this set is,

$$|\mathrm{Lucky}_{\mathrm{FR}_n}(\{1, i, n\})| = 3(2^{n-i}). \tag{7}$$

Conjecture $|\mathrm{Lucky}_{\mathrm{FR}_n}(\{1, i_2, i_3, n\})| = 3^2 \sum_{x=4}^{n} 2^x$.

Further Research

- Complete counting through fixed sets for Fubini Rankings, Unit Fubinis, and ℓ-Fubinis, producing generalized equations for specific index values.
- Calculate the Lucky polynomial for ℓ-Fubinis in general, possibly through recursive counting backwards from the highest possible ℓ value, $(n-1)$, to the lowest, $\ell = 0$, the permutations.
- Calculate the Lucky polynomial for ℓ-Interval parking functions.

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- Calculate the Lucky polynomial for ℓ-Fubinis in general, possibly through recursive counting backwards from the highest possible ℓ value, $(n-1)$, to the lowest, $\ell = 0$, the permutations.
- Calculate the Lucky polynomial for ℓ-Interval parking functions.
- Study connections to Hopf Algebras.

Further Research: $FB_n(\ell)$, $\ell = 1$, a.k.a. Unit Fubinis

n							
1	$1q$						
2	$1q$	$+2q^2$					
3		$6q^2$	$+6q^3$				
4		$6q^2$	$+36q^3$	$+24q^4$			
5			$90q^3$	$+240q^4$	$+120q^5$		
6			$90q^3$	$+1080q^4$	$+1800q^5$	$+720q^6$	
7				$2520q^4$	$+12600q^5$	$+15120q^6$	$+5040q^7$
8				$2520q^4$	$+50400q^5$	$+151200q^6$	$+141120q^7$

Theorem

Zero Coefficient Theorem. A parking function that is both Fubini and ℓ-interval must have at least $\lceil \frac{n}{\ell+1} \rceil$ lucky cars.

- The coefficients of the "twins" follow the sequence A000680 on OEIS: $a(2k-1) = a(2k) = \frac{(2k)!}{2^k}$.

Further Research: $FR_n(\ell)$, $\ell = 2$

n							
1	$1q$						
2	$1q$	$+2q^2$					
3	$1q$	$6q^2$	$+6q^3$				
4		$14q^2$	$+36q^3$	$+24q^4$			
5		$20q^2$	$130q^3$	$+240q^4$	$+120q^5$		
6		$20q^2$	$450q^3$	$+1560q^4$	$+1800q^5$	$+720q^6$	
7			$840q^3$	$7560q^4$	$+16800q^5$	$+15120q^6$	$+5040q^7$
8			$1680q^3$	$23800q^4$	$+114240q^5$	$+--q^6$	$+141120q^7$
9			$1680q^3$	$907200q^4$	$+--q^5$	$+--q^6$	$+--q^7$
10				$218400q^4$	$+--q^5$	$+--q^6$	$+--q^7$
11				$369600q^4$	$+--q^5$	$+--q^6$	$+--q^7$
12				$369600q^4$	$+--q^5$	$+--q^6$	$+--q^7$

- The coefficients of the two identical "triplets" follow the sequence A014606 on OEIS: $a(3k-1) = a(3k) = \frac{3k!}{6^k}$.

Further Research: $FR_n(\ell)$, $\ell = 3$

n							
1	$1q$						
2	$1q$	$+2q^2$					
3	$1q$	$6q^2$	$+6q^3$				
4	$1q$	$14q^2$	$+36q^3$	$+24q^4$			
5		$30q^2$	$+150q^3$	$+240q^4$	$+120q^5$		
6		$50q^2$	$+270q^3$	$+1200q^4$	$+1800q^5$	$+720q^6$	
7		$70q^2$	$+1260q^3$	$+10400q^4$	$+--q^5$	$+--q^6$	
8		$70q^2$	$+3360q^3$	$+--q^4$	$+--q^5$	$+--q^6$	$+141120q^7$

- There are "quadruplets", with identical twins. The equation for the twin coefficients is $a(4k-1) = a(4k) = \frac{(4k)!}{24^k}$, corresponding to OEIS sequence A014608, adjusted for double entries.

Further Research: $FR_n(\ell)$, $\ell = 3$

n							
1	$1q$						
2	$1q$	$+2q^2$					
3	$1q$	$6q^2$	$+6q^3$				
4	$1q$	$14q^2$	$+36q^3$	$+24q^4$			
5		$30q^2$	$+150q^3$	$+240q^4$	$+120q^5$		
6		$50q^2$	$+270q^3$	$+1200q^4$	$+1800q^5$	$+720q^6$	
7		$70q^2$	$+1260q^3$	$+10400q^4$	$+--q^5$	$+--q^6$	
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- We conjecture that for $\mathrm{FR}_n(\ell)$, there are double coefficients satisfying $a((\ell+1)k-1) = a((\ell+1)k) = \frac{((\ell+1)k)!}{((\ell+1)!)^k}$.

Further Research: $FR_n(\ell)$, $\ell = 3$

n							
1	$1q$						
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6		$50q^2$	$+270q^3$	$+1200q^4$	$+1800q^5$	$+720q^6$	
7		$70q^2$	$+1260q^3$	$+10400q^4$	$+--q^5$	$+--q^6$	
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- We conjecture that for $\mathrm{FR}_n(\ell)$, there are double coefficients satisfying $a((\ell+1)k-1) = a((\ell+1)k) = \frac{((\ell+1)k)!}{((\ell+1)!)^k}$.
- Sequences A000680, A014606, and A014608 appear in their singleton form in a paper by Thibon and Novelli that connects m-permutations, $(m+1)$-ary trees, and m-parking functions through Hopf algebras.

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Thank you!