Rose Conjecture: Total Acquisition in Diameter Two Graphs

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Outline

- Background
- Past Attempts
- Probability Approach
- Induction Approach
- Edge Swap Algorithm
- Locking Algorithm
- Digraph Approach
- Next Steps

Total Acquisition and Diameter

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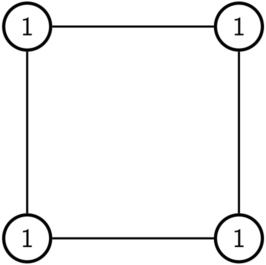
Definition

The total acquisition number denoted as $a_t(G)$ is the minimum number of vertices with nonzero weight after a total acquisition process has been performed.

Total Acquisition Example - C_4



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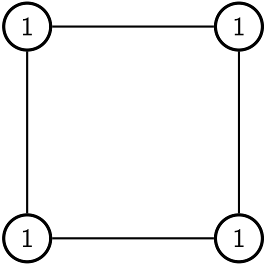

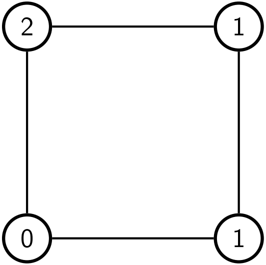

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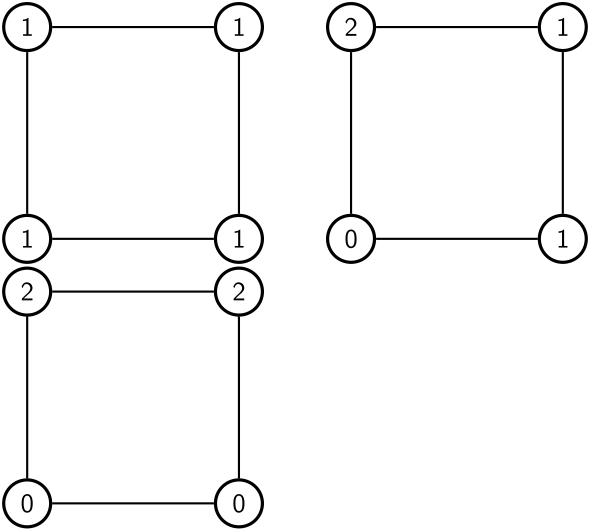

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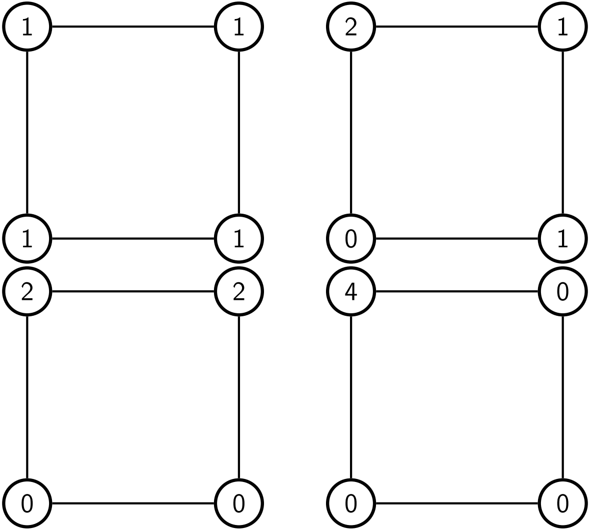

Background

- LeSaulnier, et al. proved in 2013 that $a_t(G) \leq 32 \ln(n) \ln(\ln(n))$ for any diameter two graph, and if it contains no four-cycles, $a_t(G)$ must be equal to 1 or 2.

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- To date, no diameter 2 graph is known with $a_t(G) > 2$.
- In 2017, Eastham, Kay, McCarty and Spencer proved in an unpublished work from this REU (detailing work that took place in 2015) that $a_t(G)$ for all diameter 2 graphs is at most 4 if every n-regular bipartite multigraph with n vertices in each partition contains a subgraph such that, for every integer between one and n, there exists a vertex in each partition whose degree is that integer.

The Rose Conjecture

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Each n-regular bipartite multigraph with n vertices in each partition, which we call a Rose Graph, contains a subgraph such that, for every integer between one and n, there exists a vertex in each partition whose degree is that integer.

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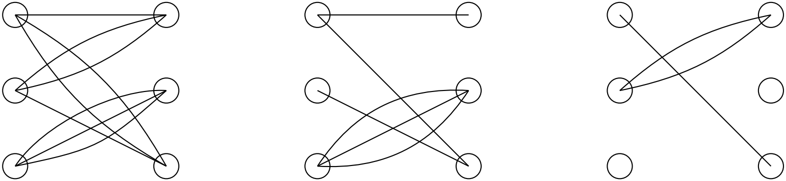
Rose Conjecture

Each $n \times n$ matrix A with non-negative integer entries in which each row and column has entries summing to n, can be decomposed into the sum of two matrices, $B + C$, where B and C both have non-negative integer entries, B has row and column sums equal to $\{1, 2, \ldots, n\}$ and C has row and column sums equal to $\{0, 1, \ldots, n-1\}$.

Example

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



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- Counting Argument + Compatibility
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In the spring of 2024, Robert Beeler and Grant Shirley begun working on a different edge swap algorithm, which we have adopted during this attempt.

FKG Inequality

The Harris inequality, a specific adaptation of the Fortuin–Kasteleyn–Ginibre inequality, is a correlation inequality used in statistical mechanics and probabilistic combinatorics that is particularly important in the probabilistic method and random graphs, which we are now attempting to implement in the probabilistic approach to the Rose Conjecture.

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Lemma

Each $n \times n$ k-regular bipartite multigraph has k distinct perfect matchings, for any non-negative integers n and k.

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So far, we've seen that it works for each connected graph we've tried it on; the next steps would be showing that it works on disconnected graphs and proving that an alternating path always exists.

Locking Rows and Columns

- **Locking a row/column** Locking a row or column in a matrix means that cells in that row/column may not have their values changed for the remainder of the algorithmic process.
- **Free Cell** A free cell is one which has both its column and row unlocked.

Example of the Algorithm

$$\begin{bmatrix} 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 4 & 1 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 2 & 0 & 0 & 0 & 3 \\ 2 & 3 & 0 & 0 & 0 \end{bmatrix}$$

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\begin{array}{cc}
\begin{bmatrix}
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0 & 0 & 0 & 3 & 0 \\
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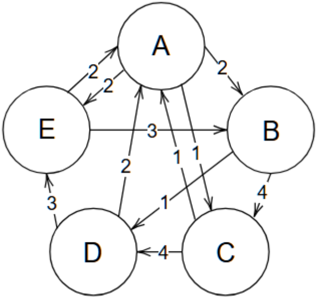
0 & 1 & 0 & 0 & 0
\end{bmatrix}
&
\begin{matrix} 4 \\ 5 \\ 3 \\ 2 \\ 1 \end{matrix}
\\
\begin{matrix} 1 & 2 & 5 & 4 & 3 \end{matrix} &
\end{array}
$$

Defining the Analogy

Matrix/Digraph Conversion is the process of converting an $n \times n$ matrix of nonnegative integers to a degree n digraph or vice versa.



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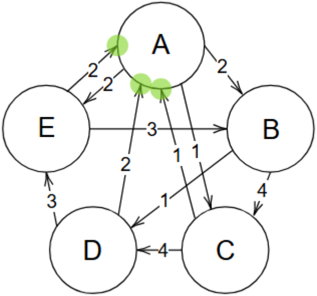

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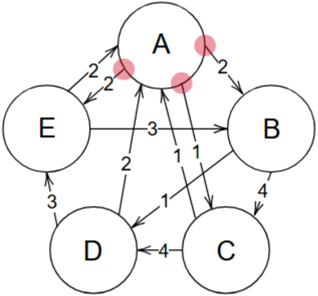

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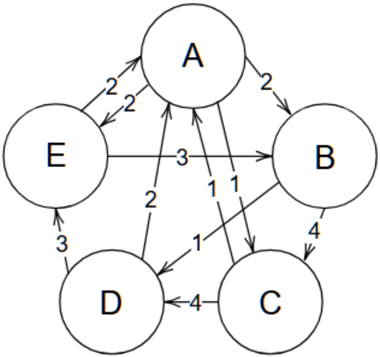

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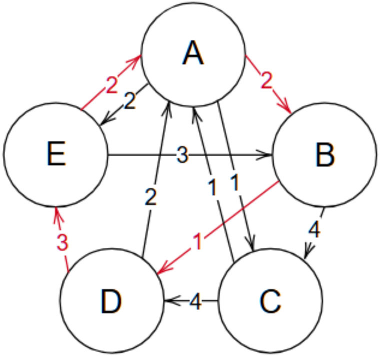

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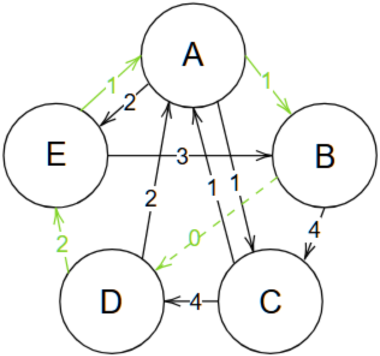

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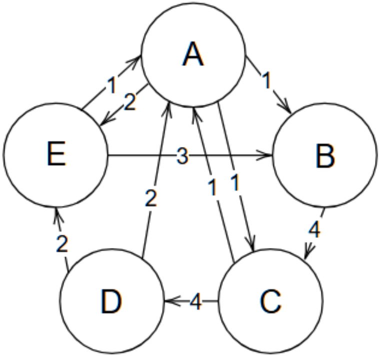

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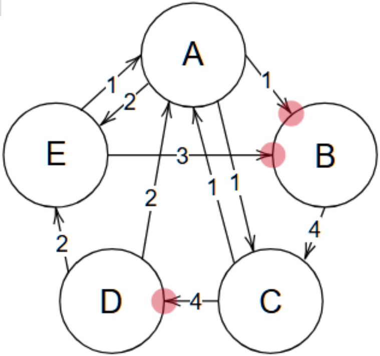

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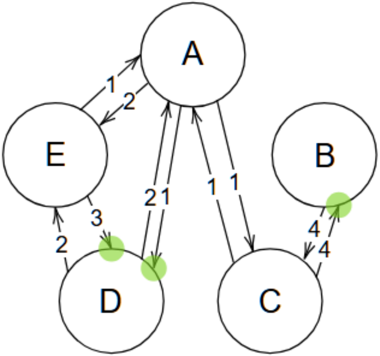

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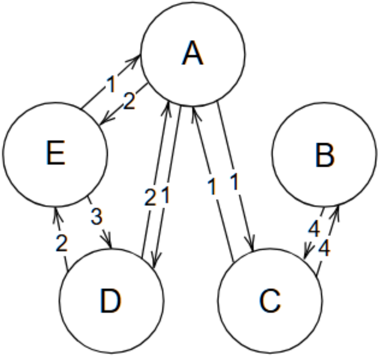

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Current Status

We conjecture that an algorithm could be produced by this method. It is a very short proof to show that this implies the Rose Conjecture.

Proof.

Assume the Weighted Digraph Conjecture. This conjecture declares that a Solution Digraph exists for any given Rose Digraph. Using Matrix/Digraph Conversion, we will revert these digraphs into a Rose Matrix and Solution Matrix. Using cell subtraction, you can find a third matrix, C such that $A = B + C$. Thus, if the Weighted Digraph Conjecture is true, so is the Rose Conjecture. $\square$

Next Steps

- Generalize the Janson/FKG approach to all n in order to prove that a solution exists
- Continue to refine algorithmic approaches in order to learn more about this class of matrices and graphs
- Explore further implications of the problem

References

A lot of the literature we've used in this project is unpublished, however we do have the following:

- Alon, Noga, and Joel H. Spencer. The Probabilistic Method. Wiley, 2016.
- C. M. Fortuin, J. Ginibre, P. W. Kasteleyn. "Correlation inequalities on some partially ordered sets." Communications in Mathematical Physics, 22(2) 89-103 1971.
- LeSaulnier, Timothy D., et al. “Total acquisition in graphs.” SIAM Journal on Discrete Mathematics, vol. 27, no. 4, Jan. 2013, pp. 1800–1819, https://doi.org/10.1137/110856186.

Questions?