Ucycles and Gucycles of Integer Compositions and Partitions

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September 4, 2024

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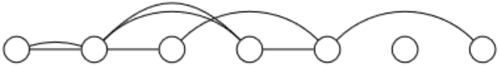

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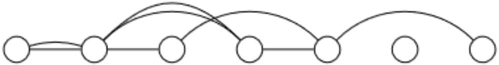

In this particular Gucycle, we consider windows of size 3, and read the number of edges between the third and first vertex, the number of edges between the first and second vertex, and the number of edges between the second and third vertex, to be the three numerical values of a particular string being represented.

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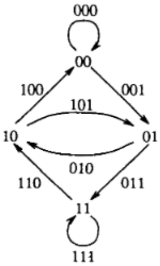

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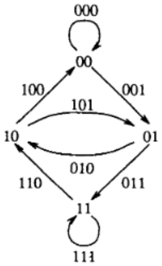

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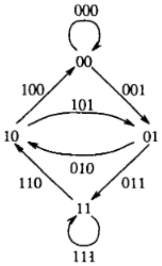

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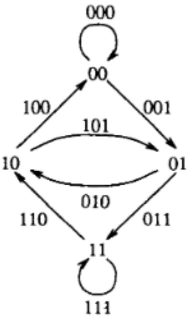

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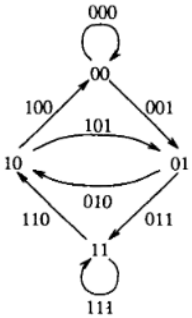

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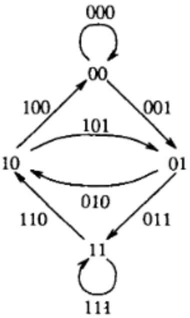

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Shifting the window to the right, we have that $(1, 3, 1, 2)$ can connect to $(3, 1, 2, 1)$, or $(3, 1, 3)$ in the transition digraph. Shifting to the left, we have that $(1, 3, 1, 2)$ can come from $(1, 1, 3, 1, 1)$, or $(2, 3, 1, 1)$.

Note that the edge labels are compositions of 8 ($(2, 3, 1, 2)$ and $(1, 1, 3, 1, 2)$ from the left, $(1, 3, 1, 3)$ and $(1, 3, 1, 2, 1)$ to the right), so any Eulerian Circuits will be Gucycles of Compositions of $(n + 1)$.

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Theorem

Let G_C be the transition digraph of integer compositions of any n. G_C is balanced, with each vertex having an in-degree and out-degree of 2. Additionally, G_C is strongly connected. Thus, G_C is Eulerian, meaning Gucycles of Integer Compositions exist for any n.

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$$11101000 \implies (4) \to (3,1) \to (2,2) \to (1,2,1) \to (2,1,1) \to (1,1,1,1) \to (1,1,2) \to (1,3) \to (4)$$

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Therefore, since we already know DeBruijn Sequences exist for binary strings of any length, Ucycles must exist for compositions of any $n \in \mathbb{N}$.

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- *Delete any vertex.* For any $p(n, t, d)$, there are d distinct ways to delete a vertex from any part. However, after Step 2 we have multiple different partitions of $(n+1)$ that could potentially have different values of d. Thus, we don't have a closed form for this.

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The best we can do is this:

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Let p be an integer partition graph with $d(p)$ distinct parts, and let α be the set of partitions of $(n+1)$ that are created after Step 2. We have then that the number of partitions that p is connected to is equal to

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By our definition of the edges, we have that each vertex has at least one loop to itself. However, since most of these loops would have the same labels as other loops, and loops don't really affect Eulerian Circuits, we choose to ignore all loops that don't have a unique edge label.

Thank you!

Funding for this presentation was granted by the Office of Undergraduate Research and Creative Activities

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