GRE PROBLEM SET #1

GRANT HAATAJA
MATH 488 - MATHEMATICS CAPSTONE - FALL 2019
AUGUST 30, 2019

Problem 6. Evaluate $\displaystyle\int_0^{\pi/2} \frac{1}{1+\tan(x)}\,dx$

(1) $\frac{\pi}{2}$
(2) $\frac{\pi}{3}$
(3) $\frac{\pi}{4}$
(4) $\frac{\pi^2}{2}$
(5) $\frac{\pi^2}{4}$

Solution. To begin solving this problem, note that the expression is not possible to integrate in its current form. The first step we want to take is to multiply the top and bottom by the equivalent expressions, $\sec^2(x) = \tan^2(x) + 1$.

$$\int_0^{\pi/2} \frac{1}{1+\tan(x)} * \frac{\sec^2(x)}{\tan^2(x)+1}\,dx$$

$$\int_0^{\pi/2} \frac{\sec^2(x)}{(1+\tan(x))(\tan^2(x)+1)}\,dx$$

Next we perform a u-substitution. Let $u = \tan(x)$ and $du = \sec^2(x)$.

$$\int_0^{\pi/2} \frac{1}{(u+1)(u^2+1)}\,du$$

We now do partial fraction decomposition.

$$\frac{1}{(u+1)(u^2+1)} = \frac{A}{u+1} + \frac{Bu+C}{u^2+1}$$

Multiply both sides by $(u+1)(u^2+1)$.

$$1 = A(u^2+1) + (Bu+C)(u+1)$$

We can see that if we let $u = -1$, the B and C term will become zero, and we can solve to find $A = \frac{1}{2}$. Since we cannot plug in a real-number value for u that will make the A term become zero, we must expand the expression and equate coefficients.

$$1 = \frac{1}{2}u^2 + \frac{1}{2} + Bu^2 + Bu + Cu + C$$

$$1 = u^2(\frac{1}{2} + B) + u(B + C) + (C + \frac{1}{2})$$

$$\frac{1}{2} + B = 0\ ,\ B + C = 0\ ,\ C + \frac{1}{2} = 1$$

Thus, we can see that $B = -\frac{1}{2}$ and $C = \frac{1}{2}$. We now substitute the decomposed fraction form back into our integral to solve and simplify.

$$\int_0^{\pi/2} \left(\frac{\frac{1}{2}}{(u+1)} + \frac{(-\frac{1}{2}u + \frac{1}{2})}{(u^2+1)} \right)\ du$$

$$\int_0^{\pi/2} \left(\frac{1}{2(u+1)} + \frac{(1-u)}{2(u^2+1)} \right)\ du$$

$$\frac{1}{2}\int_0^{\pi/2} \left(\frac{1}{(u+1)} + \frac{(1-u)}{(u^2+1)} \right)\ du$$

$$\frac{1}{2}\int_0^{\pi/2} \frac{1}{u+1}\ du + \frac{1}{2}\int_0^{\pi/2} \frac{1-u}{u^2+1}\ du$$

$$\frac{1}{2}\int_0^{\pi/2} \frac{1}{u+1}\ du + \frac{1}{2}\int_0^{\pi/2} \frac{1}{u^2+1}\ du - \frac{1}{2}\int_0^{\pi/2} \frac{u}{u^2+1}\ du$$

$$= \left[\frac{1}{2}\ln{(u+1)} + \frac{1}{2}\arctan{(u)} - \frac{1}{4}\ln{(u^2+1)} \right]_0^{\pi/2}$$

Plug $\tan(x)$ back in for u.

$$= \left[\frac{1}{2}\ln{(\tan(x)+1)} + \frac{1}{2}\arctan{(\tan(x))} - \frac{1}{4}\ln{(\tan^2(x)+1)} \right]_0^{\pi/2}$$

$$= \left[\frac{1}{2}\ln{(\tan(x)+1)} - \frac{1}{4}\ln{(\sec^2(x))} + \frac{1}{2}x \right]_0^{\pi/2}$$

$$= \left[\frac{1}{2}\ln{(\tan(x)+1)} - \frac{1}{2}\ln{(\sec(x))} + \frac{1}{2}x \right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\ln(\tan(x)+1) - \ln(\sec(x)) + x\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\ln\left(\frac{\tan(x)+1}{\sec(x)}\right) + x\right]_0^{\pi/2}$$
$$= \frac{1}{2}\left[\ln(\sin(x)+\cos(x)) + x\right]_0^{\pi/2}$$

Finally, we plug the bounds in to evaluate the integral.

$$\frac{1}{2}\left[\ln\left(\sin\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{2}\right)\right) + \frac{\pi}{2}\right] - \frac{1}{2}\left[\ln(\sin(0)+\cos(0)) + 0\right]$$
$$= \frac{1}{2}\left[\ln(1) + \frac{\pi}{2}\right] - \frac{1}{2}\left[\ln(1)\right]$$
$$= \frac{1}{2} * \frac{\pi}{2}$$
$$= \frac{\pi}{4}$$

$\square$