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A note on the convergence rate of MCMC for robust Bayesian multivariate linear regression with proper priors

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The multivariate linear regression model with errors from a scale mixture of Gaussian densities yields a complex likelihood function. Combining this likelihood with any nontrivial prior distribution leads to a highly intractable posterior density. If a *conditionally conjugate* prior is used, then there is a well known and easy-to-implement data augmentation (DA) algorithm available for exploring the posterior. Hobert et al recently showed that, under an *improper* conditionally conjugate prior (and weak regularity conditions), the Markov chain that drives the DA algorithm converges at a geometric rate. Unfortunately, the model studied by Hobert et al can only be used in situations where the X matrix has full column rank. In this note, analogous convergence rate results are established for a *proper* conditionally conjugate prior. An important advantage of using a proper prior is that, not only is the X matrix allowed to be column rank deficient, but it can also have more columns than rows, that is, our model is applicable in cases where $p > n$. This is an important extension in the era of big data.

KEYWORDS

data augmentation algorithm, drift condition, geometric ergodicity, heavy-tailed distribution, scale mixture

1 | INTRODUCTION

Let $Y_1, Y_2, \dots, Y_n$ be independent d-dimensional random vectors from the multivariate linear regression model

$$Y_i = \beta^T x_i + \Sigma^{\frac{1}{2}} \varepsilon_i, \tag{1}$$

where x_i is a $p \times 1$ vector of known covariates associated with Y_i, β is a $p \times d$ matrix of unknown regression coefficients, and Σ is an unknown $d \times d$ positive definite scale matrix. The random vectors $\varepsilon_1, \dots, \varepsilon_n$ are iid d-dimensional errors from a density taking the form

$$f_h(\varepsilon) = \int_{\mathbb{R}_+} \frac{u^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}} \exp\left\{-\frac{u}{2}\varepsilon^T\varepsilon\right\} h(u)\, du,$$

where $\mathbb{R}_+ := (0, \infty)$, and $h : \mathbb{R}_+ \to [0, \infty)$ is the density function of some positive random variable. We shall refer to h as a *mixing density*. Heavy-tailed error densities can be produced by choosing h with appropriate behavior near the origin.[1-3]

Some typical choices for h are the gamma, inverse gamma, generalized inverse Gaussian, and log-normal densities. However, in principle, h can be taken to be any density on the positive half-line. We assume throughout that

$$\int_0^\infty u^{\frac{d}{2}} h(u)\, du < \infty,$$

and we refer to this as "condition $\mathcal{M}$." As we shall see, this condition is required for the existence of the data augmentation (DA) algorithm.

Denote the n equations in (1) collectively as

$$Y = X\beta + \varepsilon\, \Sigma^{\frac{1}{2}}, \tag{2}$$

where Y is the $n \times d$ matrix whose ith row is Y_i^T, X is the $n \times p$ matrix whose ith row is x_i^T, and ε represents the $n \times d$ matrix whose ith row is ε_i^T. Also, let y and y_i denote the observed values of Y and Y_i, respectively. The joint density of the data from model (2) is given by

$$f(y|\beta,\Sigma) = \prod_{i=1}^n \left[\int_0^\infty \frac{u^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left\{ -\frac{u}{2} \left(y_i - \beta^T x_i\right)^T \Sigma^{-1} \left(y_i - \beta^T x_i\right) \right\} h(u)\, du \right].$$

Consider a Bayesian analysis with a proper conditionally conjugate prior that is defined sequentially as follows: $\omega(\beta,\Sigma) = \omega(\beta|\Sigma)\omega(\Sigma)$, where $\beta|\Sigma \sim \mathrm{N}_{p,d}(\theta, A, \Sigma)$ and $\Sigma \sim \mathrm{IW}_d(\nu, \Theta)$. Here, $\mathrm{N}_{p,d}$ and IW_d denote the matrix normal and inverse Wishart distributions, respectively, and the associated densities are defined in the Appendix. The hyperparameters are θ (a $p \times d$ matrix), A (a $p \times p$ positive definite matrix), $\nu > d - 1$, and Θ (a $d \times d$ positive definite matrix). This prior is standard in multivariate regression settings and is often used in conjunction with model (2)[4,chap8]. The highly intractable posterior density that results from this model is, of course, given by

$$\pi(\beta,\Sigma|y) = \frac{f(y|\beta,\Sigma)\ \omega(\beta,\Sigma)}{m(y)},$$

where

$$m(y) = \int_{\mathcal{S}_d} \int_{\mathbb{R}^{p\times d}} f(y|\beta,\Sigma)\ \omega(\beta,\Sigma)\ d\beta\ d\Sigma,$$

and $\mathcal{S}_d \subset \mathbb{R}^{\frac{d(d+1)}{2}}$ denotes the space of $d \times d$ positive definite matrices. Note that there are no restrictions on X in our model. In particular, X is allowed to have more columns than rows.

We now introduce the latent data that is used to construct the DA algorithm. Conditional on (β,Σ), let $\{(Y_i, Z_i)\}_{i=1}^n$ be independent pairs such that $Y_i|Z_i,\beta,\Sigma \sim \mathrm{N}_d(\beta^T x_i, \Sigma/Z_i)$, and $Z_i|\beta,\Sigma \sim h$. Let $f(y,z|\beta,\Sigma)$ denote the joint density of Y and $Z := (Z_1,\dots,Z_n)$ given (β,Σ). It is easy to see that

$$\int_{\mathbb{R}_+^n} f(y,z|\beta,\Sigma)\ dz = f(y|\beta,\Sigma).$$

If we now define the so-called complete data posterior density $\pi : \mathbb{R}^{p\times d} \times \mathcal{S}_d \times \mathbb{R}_+^n \to [0,\infty)$ as

$$\pi(\beta,\Sigma,z|y) = \frac{f(y,z|\beta,\Sigma)\omega(\beta,\Sigma)}{m(y)},$$

then it is clear that

$$\int_{\mathbb{R}_+^n} \pi(\beta,\Sigma,z|y)\ dz = \pi(\beta,\Sigma|y),$$

which is our target posterior. The DA algorithm simulates a Markov chain, $\Phi = \{(\beta_m,\Sigma_m)\}_{m=0}^\infty$, with state space $\mathsf{X} := \mathbb{R}^{p\times d} \times \mathcal{S}_d$, by alternating between draws from $\pi(z|\beta,\Sigma,y)$ and $\pi(\beta,\Sigma|z,y)$. Two important facts about the conditional density $\pi(z|\beta, \Sigma, y)$: (1) It does not depend on the prior and (2) it is a product of n univariate densities. We now describe its form in more detail. As in Hobert et al[5] (HJK&Q), we define a parametric family of univariate density functions indexed by $s \geq 0$ as follows:

$$\psi(u; s) = b(s) u^{\frac{d}{2}} e^{-\frac{su}{2}} h(u),$$

where $b^{-1}(s) = \int_0^\infty v^{\frac{d}{2}} e^{-\frac{sv}{2}} h(v)\, dv$. Using this notation, we can see that

$$\pi(z|\beta, \Sigma, y) = \prod_{i=1}^{n} \psi(z_i; r_i),$$

where $r_i = (\beta^T x_i - y_i)^T \Sigma^{-1} (\beta^T x_i - y_i)$. We must be able to draw from $\psi(\cdot; s)$ in order to run the DA algorithm. When h is a density from a standard parametric family, ψ is often standard as well. For example, when h is gamma, ψ is also gamma, and when h is inverse gamma, ψ is generalized inverse Gaussian. If ψ is not a standard density, it can often be efficiently sampled using a rejection sampler with h as the candidate. Because the prior on (β, Σ) is conditionally conjugate, the density $\pi(\beta, \Sigma|z, y)$ takes the same form as the prior, that is, $\pi(\beta|\Sigma, z, y)$ is matrix normal, and $\pi(\Sigma|z, y)$ is inverse Wishart.

In order to formally state the DA algorithm, we need to introduce a bit more notation. For $z = (z_1, \ldots, z_n) \in \mathbb{R}_+^n$, let Q be an $n \times n$ diagonal matrix whose ith diagonal element is z_i^{-1}. Also, define $\Omega = (X^T Q^{-1} X + A^{-1})^{-1}$ and $\mu = (X^T Q^{-1} X + A^{-1})^{-1} (X^T Q^{-1} y + A^{-1}\theta)$. We can now formally state the DA algorithm. If the current state of the DA Markov chain is $(\beta_m, \Sigma_m) = (\beta, \Sigma)$, then we simulate the new state, $(\beta_{m+1}, \Sigma_{m+1})$, using the following three-step procedure.

Iteration $m + 1$ of the DA algorithm:

1. Draw $\{Z_i\}_{i=1}^n$ independently with $Z_i \sim \psi\left(\cdot\, ; \left(\beta^T x_i - y_i\right)^T \Sigma^{-1} \left(\beta^T x_i - y_i\right)\right)$, and call the result $z = (z_1, \ldots, z_n)$.
2. Draw
$$\Sigma_{m+1} \sim \mathrm{IW}_d \left(n + \nu, \left(\Theta^{-1} + \theta^T A^{-1} \theta + y^T Q^{-1} y - \mu^T \Omega^{-1} \mu\right)^{-1}\right).$$
3. Draw $\beta_{m+1} \sim \mathrm{N}_{p,d} \left(\mu, \Omega, \Sigma_{m+1}\right)$.

HJK&Q considered the same likelihood function, but a different prior on (β, Σ). In particular, they used an improper conditionally conjugate prior that takes the form $\omega^*(\beta, \Sigma) \propto |\Sigma|^{-a}\, I_{S_d}(\Sigma)$, where a is a hyperparameter. Taking $a = (d + 1)/2$ yields the independence Jeffreys prior, which is a standard default prior for multivariate location scale problems. Let Λ denote the $n \times (p + d)$ matrix $(X : y)$. That is, Λ is the matrix that results when the $n \times d$ matrix y is appended to the right of the $n \times p$ matrix X. Under the prior ω^*, the following conditions are *necessary* for propriety:

($N1$) $\mathrm{rank}(\Lambda) = p + d$;
($N2$) $n > p + 2d - 2a$.

Clearly, ($N1$) cannot hold unless X has full column rank. This obviously rules out cases in which $p > n$.

Not surprisingly, HJK&Q's DA algorithm is quite similar to ours. Let $\Omega_* = (X^T Q^{-1} X)^{-1}$ and $\mu_* = (X^T Q^{-1} X)^{-1} X^T Q^{-1} y$. If the current state of HJK&Q's DA Markov chain is $(\beta_m^*, \Sigma_m^*) = (\beta, \Sigma)$, then we simulate the new state, $(\beta_{m+1}^*, \Sigma_{m+1}^*)$, using the following three-step procedure.

Iteration $m + 1$ of HJK&Q's DA algorithm:

1. Draw $\{Z_i\}_{i=1}^n$ independently with $Z_i \sim \psi\left(\cdot\, ; \left(\beta^T x_i - y_i\right)^T \Sigma^{-1} \left(\beta^T x_i - y_i\right)\right)$, and call the result $z = (z_1, \ldots, z_n)$.
2. Draw
$$\Sigma_{m+1}^* \sim \mathrm{IW}_d \left(n - p + 2a - d - 1, \left(y^T Q^{-1} y - \mu_*^T \Omega_*^{-1} \mu_*\right)^{-1}\right).$$
3. Draw $\beta_{m+1}^* \sim \mathrm{N}_{p,d} \left(\mu_*, \Omega_*, \Sigma_{m+1}^*\right)$.

Under conditions ($N1$) and ($N2$), this algorithm is well defined.

2 | THE MAIN RESULT

The Markov transition density (Mtd) of Φ is given by

$$k(\beta, \Sigma | \tilde{\beta}, \tilde{\Sigma}) = \int_{\mathbb{R}_+^n} \pi(\beta, \Sigma | z, y) \pi(z | \tilde{\beta}, \tilde{\Sigma}, y) \, dz.$$

It is easy to show that Φ is Harris ergodic (irreducible, aperiodic, and positive Harris recurrent), and that $\pi(\beta, \Sigma | y)$ is the stationary density. The chain Φ is geometrically ergodic if there exist $M : \mathsf{X} \to [0, \infty)$ and $\rho \in [0, 1)$ such that, for all $m \in \mathbb{N}$,

$$\int_{S_d} \int_{\mathbb{R}^{p \times d}} \left| k^m(\beta, \Sigma | \tilde{\beta}, \tilde{\Sigma}) - \pi(\beta, \Sigma | y) \right| \, d\beta \, d\Sigma \leq M(\tilde{\beta}, \tilde{\Sigma}) \rho^m,$$

where k^m is the m-step Mtd. The importance of using geometrically ergodic Markov chains in Markov chain Monte Carlo has been well documented.[6]

In order to state our main result concerning the convergence rate of Φ, we must introduce several classes of mixing densities that were defined in HJK&Q. Let $\mathcal{Z}$ denote the set of mixing densities that are zero near the origin, that is, $h \in \mathcal{Z}$ if there exists $\delta > 0$ such that $h(u) = 0$ for all $u \in (0, \delta)$. Similarly, let $\mathcal{P}$ denote the set of mixing densities that are strictly positive near the origin, that is, $h \in \mathcal{P}$ if there exists $\delta > 0$ such that $h(u) > 0$ for all $u \in (0, \delta)$. If $h \in \mathcal{P}$ and there exists a $c > -1$ such that

$$\lim_{u \to 0} \frac{h(u)}{u^c} \in \mathbb{R}_+,$$

then we say that h is *polynomial near the origin with power c*. Finally, if $h \in \mathcal{P}$ and, for every $c > 0$, there exists an $\eta_c > 0$ such that the ratio $\frac{h(u)}{u^c}$ is strictly increasing in $(0, \eta_c)$, then we say that h is *faster than polynomial near the origin*. HJK&Q demonstrated that every mixing density that is a member of a standard parametric family is either polynomial near the origin, or faster than polynomial near the origin. Here is our main result.

Proposition 1. *Let $h : \mathbb{R}_+ \to [0, \infty)$ be a mixing density that satisfies condition $\mathcal{M}$. If any one of the following conditions holds, then the DA Markov chain Φ is geometrically ergodic.*

1. *$h \in \mathcal{Z}$.*
2. *h is faster than polynomial near the origin.*
3. *h is polynomial near the origin with power $c > \frac{n+\nu}{2}$.*

Proposition 1 is an extension of the following result.

Theorem 1. *[HJK&Q] Assume that (N1) and (N2) hold. Let $h : \mathbb{R}_+ \to [0, \infty)$ be a mixing density that satisfies condition $\mathcal{M}$. If any one of the following conditions holds, then the posterior distribution is proper and the DA Markov chain $\Phi^* = \{(\beta_m^*, \Sigma_m^*)\}_{m=0}^{\infty}$ is geometrically ergodic.*

1. *$h \in \mathcal{Z}$.*
2. *h is faster than polynomial near the origin.*
3. *h is polynomial near the origin with power $c > \frac{n-p+2a-d-1}{2}$.*

We reiterate that HJK&Q's model cannot be used unless X has full column rank.

3 | PROOF OF PROPOSITION 1

Recall that the Mtd of Φ is given by

$$k(\beta, \Sigma | \tilde{\beta}, \tilde{\Sigma}) = \int_{\mathbb{R}_+^n} \pi(\beta, \Sigma | z, y) \pi(z | \tilde{\beta}, \tilde{\Sigma}, y) \, dz.$$

The Mtd of the DA Markov chain studied in HJK&Q takes the form

$$k^*(\beta,\Sigma|\tilde\beta,\tilde\Sigma) = \int_{\mathbb{R}^n_+} \pi^*(\beta,\Sigma|z,y)\pi(z|\tilde\beta,\tilde\Sigma,y)\,dz.$$

Note that $\pi(z|\beta,\Sigma,y)$ appears in both integrands. (Recall that this density does not depend on the prior.) Because the improper prior is conditionally conjugate, $\pi^*(\beta,\Sigma|z,y)$ has the same form as $\pi(\beta,\Sigma|z,y)$, that is, it is the product of a matrix normal and an inverse Wishart. Due to the similarities between k and k^*, we are able to reuse many of the calculations in HJK&Q's proof of Theorem 1.

As in HJK&Q, we prove our result by establishing drift and minorization conditions with the following drift function:

$$V(\beta,\Sigma) = \sum_{i=1}^n (y_i - \beta^T x_i)^T \Sigma^{-1} (y_i - \beta^T x_i).$$

For an introduction to this method, see Jones and Hobert.[7] The minorization condition follows immediately from a calculation in HJK&Q. Indeed, fix $l > 0$ and define

$$B_l = \{(\beta,\Sigma) : V(\beta,\Sigma) \le l\}.$$

HJK&Q construct $\epsilon \in (0,1)$ and a density function $\hat f : \mathbb{R}^n_+ \to [0,\infty)$ such that, for all $(\tilde\beta,\tilde\Sigma) \in B_l$,

$$\pi(z|\tilde\beta,\tilde\Sigma,y) \ge \epsilon \hat f(z).$$

Thus, for all $(\tilde\beta,\tilde\Sigma) \in B_l$, we have

$$k\left(\beta,\Sigma|\tilde\beta,\tilde\Sigma\right) = \int_{\mathbb{R}^n_+} \pi(\beta,\Sigma|z,y)\;\pi(z|\tilde\beta,\tilde\Sigma,y)\,dz \ge \epsilon \int_{\mathbb{R}^n_+} \pi(\beta,\Sigma|z,y)\,\hat f(z)\,dz = \epsilon f^*(\beta,\Sigma).$$

This is the required minorization condition. Now we move on to the drift condition. We must show that there exists $\lambda \in [0,1)$ and $L < \infty$ such that

$$\int_{\mathsf{X}} V(\beta,\Sigma) k(\beta,\Sigma|\tilde\beta,\tilde\Sigma)\,d\beta\,d\Sigma \le \lambda V(\tilde\beta,\tilde\Sigma) + L,$$

for all $(\tilde\beta,\tilde\Sigma) \in \mathsf{X}$. Note that

$$\begin{aligned}
&\int_{\mathsf{X}} V(\beta,\Sigma) k(\beta,\Sigma|\tilde\beta,\tilde\Sigma)\,d\beta\,d\Sigma \\
&= \int_{\mathbb{R}^n_+} \left[\int_{\mathsf{X}} V(\beta,\Sigma)\pi(\beta,\Sigma|z,y)\,d\beta\,d\Sigma\right] \pi(z|\tilde\beta,\tilde\Sigma,y)\,dz \\
&= \int_{\mathbb{R}^n_+} \left[\int_{S_d} \left[\int_{\mathbb{R}^{p\times d}} V(\beta,\Sigma)\pi(\beta|\Sigma,z,y)\,d\beta\right] \pi(\Sigma|z,y)\,d\Sigma\right] \pi(z|\tilde\beta,\tilde\Sigma,y)\,dz.
\end{aligned} \tag{3}$$

We attack (3) using an argument similar to one found in Roy and Hobert.[8] The innermost integral can be expressed as

$$E\left[\sum_{i=1}^n y_i^T \Sigma^{-1} y_i - 2\sum_{i=1}^n x_i^T \beta \Sigma^{-1} y_i + \sum_{i=1}^n x_i^T \beta \Sigma^{-1} \beta^T x_i | \Sigma, z, y\right],$$

where $\beta|\Sigma,z,y \sim \mathrm{N}_{p,d}(\mu,\Omega,\Sigma)$. Standard results for the matrix normal distribution,[9,chap17] imply that

$$E(\beta\Sigma^{-1}\beta^T|\Sigma,z,y) = d\Omega + \mu\Sigma^{-1}\mu^T.$$

It follows that

$$\int_{\mathbb{R}^{p\times d}} V(\beta,\Sigma)\pi(\beta|\Sigma,z,y)\,d\beta = \sum_{i=1}^{n} y_i^T\Sigma^{-1}y_i - 2\sum_{i=1}^{n} x_i^T\mu\Sigma^{-1}y_i + \sum_{i=1}^{n} x_i^T\left[d\Omega + \mu\Sigma^{-1}\mu^T\right]x_i$$

$$= \sum_{i=1}^{n}(y_i-\mu^Tx_i)^T\Sigma^{-1}(y_i-\mu^Tx_i) + d\sum_{i=1}^{n} x_i^T\Omega x_i.$$

Now recall that $\Sigma|z,y \sim \mathrm{IW}_d\left(n+\nu, \left(\Theta^{-1}+\theta^TA^{-1}\theta+y^TQ^{-1}y-\mu^T\Omega^{-1}\mu\right)^{-1}\right)$. Hence, we have

$$E(\Sigma^{-1}|z,y) = (n+\nu)\left(\Theta^{-1}+\theta^TA^{-1}\theta+y^TQ^{-1}y-\mu^T\Omega^{-1}\mu\right)^{-1}.$$

Therefore,

$$\int_{\mathcal{S}_d}\left[\int_{\mathbb{R}^{p\times d}} V(\beta,\Sigma)\pi(\beta|\Sigma,z,y)\,d\beta\right]\pi(\Sigma|z,y)\,d\Sigma$$

$$= \sum_{i=1}^{n}(y_i-\mu^Tx_i)^T E(\Sigma^{-1}|z,y)(y_i-\mu^Tx_i) + d\sum_{i=1}^{n} x_i^T\Omega x_i$$

$$\le (n+\nu)\sum_{i=1}^{n}(y_i-\mu^Tx_i)^T\left(\theta^TA^{-1}\theta+y^TQ^{-1}y-\mu^T\Omega^{-1}\mu\right)^{-1}(y_i-\mu^Tx_i) + d\sum_{i=1}^{n} x_i^T\Omega x_i,$$

where we have used the fact that Θ, and hence Θ^{-1}, is positive definite. Note that we were able to compute the first two conditional expectations exactly. Unfortunately, we are not able to compute the outer-most expectation in closed form. Instead, we will compute the expectation of a simple upper bound. First, observe that

$$\begin{aligned}
&y^TQ^{-1}y-\mu^T\Omega^{-1}\mu+\theta^TA^{-1}\theta \\
&\quad = y^TQ^{-1}y+\mu^T\Omega^{-1}\mu-2\mu^T\Omega^{-1}\mu+\theta^TA^{-1}\theta \\
&\quad = \sum_{i=1}^{n} z_iy_iy_i^T + \mu^T(X^TQ^{-1}X+A^{-1})\mu - 2\mu^T(X^TQ^{-1}y+A^{-1}\theta)+\theta^TA^{-1}\theta \\
&\quad = \sum_{i=1}^{n} z_iy_iy_i^T + \mu^T\left(\sum_{i=1}^{n} z_ix_ix_i^T\right)\mu - 2\mu^T\sum_{i=1}^{n} z_ix_iy_i^T + \mu^TA^{-1}\mu - 2\mu^TA^{-1}\theta+\theta^TA^{-1}\theta \\
&\quad = \sum_{i=1}^{n} z_i(y_i-\mu^Tx_i)(y_i-\mu^Tx_i)^T + (\mu-\theta)^TA^{-1}(\mu-\theta).
\end{aligned}$$

It follows that

$$\begin{aligned}
&(y_i-\mu^Tx_i)^T\left(y^TQ^{-1}y-\mu^T\Omega^{-1}\mu+\theta^TA^{-1}\theta\right)^{-1}(y_i-\mu^Tx_i) \\
&= \frac{1}{z_i}(y_i-\mu^Tx_i)^T\left(\sum_{j=1}^{n}\frac{z_j}{z_i}(y_j-\mu^Tx_j)(y_j-\mu^Tx_j)^T + \frac{1}{z_i}(\mu-\theta)^TA^{-1}(\mu-\theta)\right)^{-1}(y_i-\mu^Tx_i).
\end{aligned}$$

Now,

$$\frac{1}{z_i}(y^TQ^{-1}y-\mu^T\Omega^{-1}\mu+\theta^TA^{-1}\theta) = \sum_{j=1}^{n}\frac{z_j}{z_i}(y_j-\mu^Tx_j)(y_j-\mu^Tx_j)^T + \frac{1}{z_i}(\mu-\theta)^TA^{-1}(\mu-\theta),$$

is positive definite, and

$$\begin{aligned}\sum_{j=1}^{n}\frac{z_j}{z_i}(y_j-\mu^T x_j)(y_j-\mu^T x_j)^T+\frac{1}{z_i}(\mu-\theta)^T A^{-1}(\mu-\theta)-(y_i-\mu^T x_i)(y_i-\mu^T x_i)^T\\=\sum_{j\neq i}\frac{z_j}{z_i}(y_j-\mu^T x_j)(y_j-\mu^T x_j)^T+\frac{1}{z_i}(\mu-\theta)^T A^{-1}(\mu-\theta),\end{aligned}$$

is positive semidefinite. It then follows from lemma 3 in Roy and Hobert[8] that

$$(y_i-\mu^T x_i)^T(y^T Q^{-1}y-\mu^T\Omega^{-1}\mu+\theta^T A^{-1}\theta)^{-1}(y_i-\mu^T x_i)\leq\frac{1}{z_i}.$$

Therefore,

$$(n+v)\sum_{i=1}^{n}(y_i-\mu^T x_i)^T(y^T Q^{-1}y-\mu^T\Omega^{-1}\mu+\theta^T A^{-1}\theta)^{-1}(y_i-\mu^T x_i)\leq(n+v)\sum_{i=1}^{n}\frac{1}{z_i}.$$

We now focus on bounding the term $d\sum_{i=1}^{n}x_i^T\Omega x_i$. Since the matrix A is positive definite,

$$\frac{1}{z_i}X^T Q^{-1}X+\frac{1}{z_i}A^{-1}=\sum_{j=1}^{n}\frac{z_j}{z_i}x_j x_j^T+\frac{1}{z_i}A^{-1}$$

is positive definite. Similarly,

$$\frac{1}{z_i}X^T Q^{-1}X+\frac{1}{z_i}A^{-1}-x_i x_i^T=\sum_{j\neq i}\frac{z_j}{z_i}x_j x_j^T+\frac{1}{z_i}A^{-1}$$

is also positive definite. Another application of lemma 3 from Roy and Hobert[8] yields

$$z_i x_i^T\Omega x_i=x_i^T\left(\frac{1}{z_i}X^T Q^{-1}X+\frac{1}{z_i}A^{-1}\right)^{-1}x_i=x_i^T\left(\sum_{j=1}^{n}\frac{z_j}{z_i}x_j x_j^T+\frac{1}{z_i}A^{-1}\right)^{-1}x_i\leq 1.$$

Thus,

$$d\sum_{i=1}^{n}x_i^T\Omega x_i\leq d\sum_{i=1}^{n}\frac{1}{z_i}.$$

Putting all of this together, we have

$$\int_{S_d}\left[\int_{\mathbb{R}^{p\times d}}V(\beta,\Sigma)\pi(\beta|\Sigma,z,y)\,d\beta\right]\pi(\Sigma|z,y)\,d\Sigma\leq(n+v+d)\sum_{i=1}^{n}\frac{1}{z_i},$$

and hence

$$\begin{aligned}\int_{\mathsf{X}}V(\beta,\Sigma)k(\beta,\Sigma|\tilde\beta,\tilde\Sigma)\,d\beta\,d\Sigma&\leq(n+v+d)\int_{\mathbb{R}_+^n}\left(\sum_{i=1}^{n}z_i^{-1}\right)\pi(z|\tilde\beta,\tilde\Sigma,y)\,dz\\&=(n+v+d)\sum_{i=1}^{n}b(\tilde r_i)\int_0^\infty u^{\frac{d-2}{2}}e^{-\frac{\tilde r_i u}{2}}h(u)\,du,\end{aligned}$$

where $\tilde r_i=(y_i-\tilde\beta^T x_i)^T\tilde\Sigma^{-1}(y_i-\tilde\beta^T x_i)$. Now, in conjunction with our assumptions about h, the results in section 4 of HJK&Q imply the existence of $\lambda\in[0,\frac{1}{n+v+d})$ and $L\in\mathbb{R}$ such that

$$\frac{\int_0^\infty u^{\frac{d-2}{2}}e^{-\frac{su}{2}}h(u)du}{\int_0^\infty u^{\frac{d}{2}}e^{-\frac{su}{2}}h(u)du}\leq\lambda s+L$$

for every $s \geq 0$. Therefore, we have

$$\begin{aligned}\int_{\mathsf{X}} V(\beta,\Sigma)k(\beta,\Sigma|\tilde{\beta},\tilde{\Sigma})\, d\beta\, d\Sigma &\leq (n+\nu+d)\left(\lambda\sum_{i=1}^{n}\tilde{r}_i + nL\right)\\ &= \lambda(n+\nu+d)V(\tilde{\beta},\tilde{\Sigma}) + (n+\nu+d)nL\\ &=:\ \lambda' V(\tilde{\beta},\tilde{\Sigma}) + L',\end{aligned}$$

where $\lambda' = \lambda(n+\nu+d) \in [0,1)$ and $L' = (n+\nu+d)nL \in \mathbb{R}$. Hence, the drift condition has been established. Since the minorization condition holds for all $l > 0$, the proof is complete.

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APPENDIX A. MATRIX NORMAL AND INVERSE WISHART DENSITIES

Matrix Normal Distribution Suppose Z is an $r \times c$ random matrix with density

$$f_Z(z) = \frac{1}{(2\pi)^{\frac{rc}{2}} |A|^{\frac{c}{2}} |B|^{\frac{r}{2}}} \exp\left[-\frac{1}{2}\mathrm{tr}\left\{A^{-1}(z-\theta)B^{-1}(z-\theta)^T\right\}\right],$$

where θ is an $r \times c$ matrix, and A and B are $r \times r$ and $c \times c$ positive definite matrices. Then Z is said to have a *matrix normal distribution* and we denote this by $Z \sim N_{r,c}(\theta, A, B)$,[9,chap17].

Inverse Wishart Distribution Suppose W is an $r \times r$ random positive definite matrix with density

$$f_W(w) = \frac{|w|^{-\frac{\nu+r+1}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\Theta^{-1}w^{-1}\right)\right\}}{2^{\frac{\nu r}{2}} \pi^{\frac{r(r-1)}{4}} |\Theta|^{\frac{\nu}{2}} \prod_{i=1}^{r} \Gamma\left(\frac{1}{2}(\nu+1-i)\right)} I_{S_r}(w),$$

where $\nu > r-1$ and Θ is an $r \times r$ positive definite matrix. Then W is said to have an *inverse Wishart distribution* and this is denoted by $W \sim \mathrm{IW}_r(\nu, \Theta)$.