ISYE 3133 Team Project Part B

Amanda Nima
Erin Smith
Grant Cloud

Part B.1

1.

(a) $z = min\left(\frac{\sum_{j \in g} y_j}{\sum_{j \in g} d_j}\right) \quad \forall g \in G$

z is the fairness metric for the system. z is the minimum ratio of total demand satisfied over total energy demanded for all groups. This fairness metric follows the logic behind the common saying, "a chain is only as strong as its weakest link." Likewise, a solution is only as fair as its lowest group ratio. For example, in a scenario with two groups, if group one gets all their demand satisfied and group two gets no demand satisfied, to a human that would appear completely unfair. Thus, the metric is a zero for that solution. Similarly, if both groups get all their demand satisfied, that solution is completely fair, and thus the metric is a one.

While the energy flows for maximum demand satisfied have already been presented, it is necessary to present them here again in order to compare them to the flows seen later in this report. For the fairness metric determined by the existing LP model, the optimal flows were given in the following format:

energyFlow[i,j] = f
where f is the amount of flow along $x_{i,j}$ in the direction $i \rightarrow j$
energyFlow[1,4] = 25
energyFlow[1,5] = 29
energyFlow[1,13] = 5
...
energyFlow[13,28] = 0
energyFlow[17,11] = 5
energyFlow[17,29] = 0

The group ratios calculated from this optimal solution were given as:

Group 1 ratio: 1.0
Group 2 ratio: 0.9166666666666666
Group 3 ratio: 0.5
Group 4 ratio: 0.75
Group 5 ratio: 0.9090909090909091
Group 6 ratio: 0.5
Group 7 ratio: 0.0

Fairness Metric: 0.0
Max Flow is:103.0

This flow model gave a solution where the fairness metric is zero. This is due to group 7 receiving zero amount of the total energy, resulting in a ratio of zero. Because of the way the fairness metric is structured, it is only dependent on the lowest ratio of energy received over energy demanded that any one of the groups in the system receives. In addition, zero is the lowest possible score the fairness metric can give since it's not possible for a group or node to receive negative units of energy.

(b) A possible shortcoming of the metric is that it is structured around a ratio. Because it's a ratio, the metric only shows the percentage of the demand being met, rather than the individual units of energy that each group is receiving. For example, a situation could arise where one group is demanding a minimal amount of energy, such as 6 units, and are given 3. A second group could be demanding 1000 units of energy, and are given 500. In this solution the fairness metric would be 0.5 since both groups had half of their demand met. Now take a second situation where group one receives 3 units of energy to their 6 demanded and group two receives 100 units of energy to their 1000 demanded. The second metric would receive a lower fairness score (0.1) than the first situation. In this case the fairness score is much lower despite the actual difference between number of energy units supplied to the two groups being much closer. In fact, group one could argue that situation one in which group 2 receives 500 units of energy is much more unfair since group 1 only needs three more units to have complete demand satisfied.

Another example of a shortcoming in the fairness metric would be a situation where both groups demand 100 units of energy. In one case, both groups receive 50 units of energy, resulting in rating of 0.5 In the second case, one group receives 50 units of energy, while the other group receives 100 units of energy. This would also result in the same rating of 0.5. Comparing the two scenarios, it's apparent that the second case is less fair since one group is receiving half of the units of energy the other group is receiving (50 units versus 100 units) rather than reciprocating the first case in that both groups would receive the same number of energy units (75 units each). In the scenario where both groups receive 0.5 of their total demand, the ratio equals 0.5. Although, to a human it seems completely fair that both groups received the exact same amount of energy units.

(c) To formulate an LP model for deciding how power should be distributed in order to maximize the fairness metric, the following LP model was used:

Parameters:

G = the set of all groups of residents

$c_{i,j}$ = maximum capacity of each conduit, $\forall\ i,j \in A$

d_j = demand of node j $\forall\ j \in N$

A = the set of directed arcs in network, one for each conduit given by data

$\hat{A}$ = set of all arcs in each direction for each conduit

N = the set of all nodes in network given by data (i.e. $\{1,2,...,n\}$)

Decision Variables:

z = the minimum ratio of energy received over energy demanded for all groups

x_{ij} = flow along a conduit from node i to j, one variable for each $a \in \hat{A}$

y_j = amount of energy used at node j $\forall\ j \in N$

Objective Statement:

$max\ z$

Constraints:

1. $z \geq 0$ Non-Negativity Constraint

2. $z \leq \left(\frac{\sum_{j \in g} y_j}{\sum_{j \in g} d_j}\right)$ z = Min. ratio of all groups

3. $x_{ij} + x_{ji} \leq c_{ij}$ $\forall\ i,j \in A$ Capacity Constraint

4. $x_{ij} \geq 0 \quad \forall \ i,j \in \hat{A}$ — Non-Negativity Constraint

5. $y_1 = d_1$ — Source Node Meets Demand

6. $y_j \leq d_j \quad \forall \ j \in N$ — Max Energy Used

7. $\sum_{i \in \delta_j^-} x_{i,j} = y_j + \sum_{i \in \delta_j^+} x_{j,i}$ — Net Flow = 0

8. $\sum_{j \in N} y_j = y_1 + \sum_{j \in \delta_1^+} x_{1,j}$ — Energy is Conserved

This LP builds upon what was submitted for Part 1 of the project. The new LP has: added a new decision variable, changed the objective statement, added a new parameter, and added two new constraints. The new decision variable is z, the fairness metric, which is the minimum ratio of energy received over energy demanded for all groups. The new objective statement maximizes the new fairness metric, z. So in other words, it maximizes the minimum of all groups demand satisfied over energy demanded ratio. The added parameter G, is the set of all groups in residence on Deep Space 9. The new constraints include making sure that z, the new fairness metric, is not negative and z is less than or equal to the ratio of all groups. These new changes address the issue of fairness in energy distribution in the original LP model. It should also be noted that constraint #5 is needed in the LP model because there is no arc from node 1 to node 1, but energy is still supplied and used by node 1. While this may cause issues in alternate fairness metrics, this constraint does not impact this fairness metric.

Running this model returns data in the following format:

Group 1 ratio: 0.75
Group 2 ratio: 0.6298701298701298
Group 3 ratio: 0.6298701298701297
Group 4 ratio: 0.6298701298701305
Group 5 ratio: 0.6298701298701297
Group 6 ratio: 0.6298701298701298
Group 7 ratio: 0.6298701298701298

Fairness Metric: 0.6298701298701297
Fairness Flow is: 103.0

The flows to our optimal solution are:

energyFlow[1,4] = 25
energyFlow[1,5] = 29
energyFlow[1,13] = 5
...
energyFlow[13,28] = 3.41558
energyFlow[17,11] = 2.64935
energyFlow[17,29] = 15.3506

In comparison to the flows for the optimal solution given in Part B.1.1 (a), you'll notice that the flows along conduits for this max fairness metric solution are not all integer values. This is because rather than maximizing the flow out of each conduit as done in part A, the model is now maximizing the fairness metric. The model aims to push all the group ratios as close together as possible in order to maximize the metric, resulting in fractions of units of energy being given to some of the groups. This is important to note in case Starfleet doesn't have the capability to send energy along a conduit in non-integer values or the nodes don't have the capability to use energy in non-integer values.

Another observation to be taken from this data is that although the demand satisfied over energy demanded ratio for each group is almost equal, the units of energy being received by the different groups vary. This is an example where the metric falls short, since one group may receive vastly more energy units. A last note to be taken from these results is that while demand satisfied stays the same, the flows of energy from the source node to the other nodes vary in comparison to the flows of the original max flow problem. This shows that there are alternate optimal solutions to the original maximum flow problem.

2.

(a) In order to adapt the existing LP model to maximize the fairness metric, while still ensuring that the total demand satisfied is at least 95% of the maximum possible, the following constraint was added to the LP model.

New Constraint:

$\sum_{j \in \delta_1^+} x_{1,j} + y_1 = .95 * 103$ Energy Used = .95 * Max Possible Used

Adding this constraint was redundant and did not alter the fairness metric for the LP, because the model's optimal flow already satisfies the maximum possible demand. Because maximizing the fairness metric doesn't reduce the demand satisfied by the model, there is no trade-off between the fairness metric and demand satisfied. There would be no reason to reduce the total demand satisfied, since the metric cannot increase anymore. It is interesting to note that reducing the total demand satisfied may or may not decrease the fairness metric. If the total demand satisfied is reduced by one and the group having the lowest ratio receives one less energy unit, the fairness metric will decrease. Had another group received one less energy unit, and assuming that group's new ratio of energy demand to energy received is still greater than the minimum of all groups' ratios, the fairness metric would not change.

(b) In order to give Starfleet the option of using a trade-off curve to best decide how to distribute energy, the following changes to the LP model from part B.1.1 (c) have been made.

Removed:

z = the old fairness metric

constraints 1. and 2.

New Decision Variables:

$z_g = \left(\frac{\sum_{j \in g} y_j}{\sum_{j \in g} d_j}\right)$ $\forall g \in G$

$p = max(z_g)$ $\forall g \in G$

$q = min(z_g)$ $\forall\ g \in G$

$b =$ new fairness metric, equal to the difference between p and q

New Objective Statement:

$min\ b$

New Constraints:

$p \geq z_g$	$\forall\ g \in G$	Max Constraint
$q \leq z_g$	$\forall\ g \in G$	Min Constraint
$b = p - q$		Difference Constraint

This LP model includes a fairness metric that does have a trade off between demand satisfied and overall fairness. For this fairness metric, a lower value is considered more fair. If each group receives the same ratio of energy satisfied over energy demanded, then the fairness metric equals zero signalling that the solution is completely fair. If the maximum amount of energy available (103 units of energy) is distributed, then in certain situations, the units of energy that each group receives will result in greater variation and thus is deemed less fair. For example, if there were five groups all demanding 25 units from 103 units available, the obvious fair solution would be for each to receive 20 and not distribute the remaining 3. In this case, the model sacrifices maximizing demand satisfied in order to increase overall fairness. In contrast, if the model distributed the remaining 3 units in effort to increase the demand satisfied for some groups, the fairness metric would be higher.

Part B.2

1.

In order to help Starfleet determine which improvements should be made to maximize the total demand fulfilled after improvements, the following changes to LP model from part B.1.1 (c) have been made. This ILP model includes

Removed:

G = the set of all groups of residents

z = the old fairness metric

constraints 1. and 2.

New Decision Variables:

$b_{i,j}$ = number of bolts added to conduit $i,j\ \forall\ i,j \in A$

$d_{i,j}$ = binary variable. Equals 1 if $b_{i,j} \geq 1$

New Parameters:

$k_{i,j}$ = the max number of bolts that can be added to a conduit for every conduit $i,j\ \forall\ i,j \in A$

t = the number of engineering hours available

New Objective Statement:

$$max \sum_{j \in \delta_1^+} x_{1,j}$$

New Constraints:

$d_{i,j} \in \{0,1\}$	$\forall \; i,j \in A$	Binary Constraint
$\sum_{i,j \in A} (3d_{i,j} + b_{i,j}) \leq t$	$\forall \; i,j \in A$	Max Hours Used
$b_{i,j} \leq k_{i,j} \cdot d_{i,j}$	$\forall \; i,j \in A$	Bolts Added
$0 \leq x_{i,j} + x_{j,i} \leq c_{i,j} + b_{i,j}$		Non-negativity & Capacity Constraint
$b_{i,j} \in Z^{+}$	$\forall \; i,j \in A$	Integer Constraint on b

This ILP builds upon the original Part 1 LP. The major changes are the additions of the decision variables $b_{i,j}$, for the number of bolts added to a conduit and $d_{i,j}$, the binary variable. Because $b_{i,j}$ is the number of bolts added to a conduit, it is has to be a positive integer variable because you can not add half a bolt and you can not add a negative amount of bolts. In order for bolts to be added to a conduit, an engineer is needed to install them. It will take an engineer 3 hours to access and prepare the conduit and 1 hour per bolt being installed and there is a limited amount of hours for improvements. Because of this fixed time cost (in hours), the binary variable $d_{i,j}$ is needed. $d_{i,j}$ equals one if bolts are being added to conduit i,j and 0 otherwise. By multiplying the binary variable by the fixed cost which is 3 hours, and adding the number of bolts added to a single conduit, an accurate measure of how many engineering hours are needed to upgrade conduit i,j can be obtained. By summing the number of engineering hours needed for each conduit, an accurate measure of total engineering hours needed to upgrade all the conduits can be obtained.

Based on the model, the max possible demand satisfied is 154 energy units. This is eight energy units less than the total energy units demanded (162), meaning that satisfying all energy demanded is not feasible. The optimal solution to this ILP will output the number of bolts to be added to each conduit and the energy flows through each conduit once the bolts are added in order to meet the max possible demand satisfied. For example, using the sample outputs below, on conduit i,j from node 1 to 4, 14 bolts should be added and once the bolts are added, the energy flow from node 1 to node 4 will be 39.

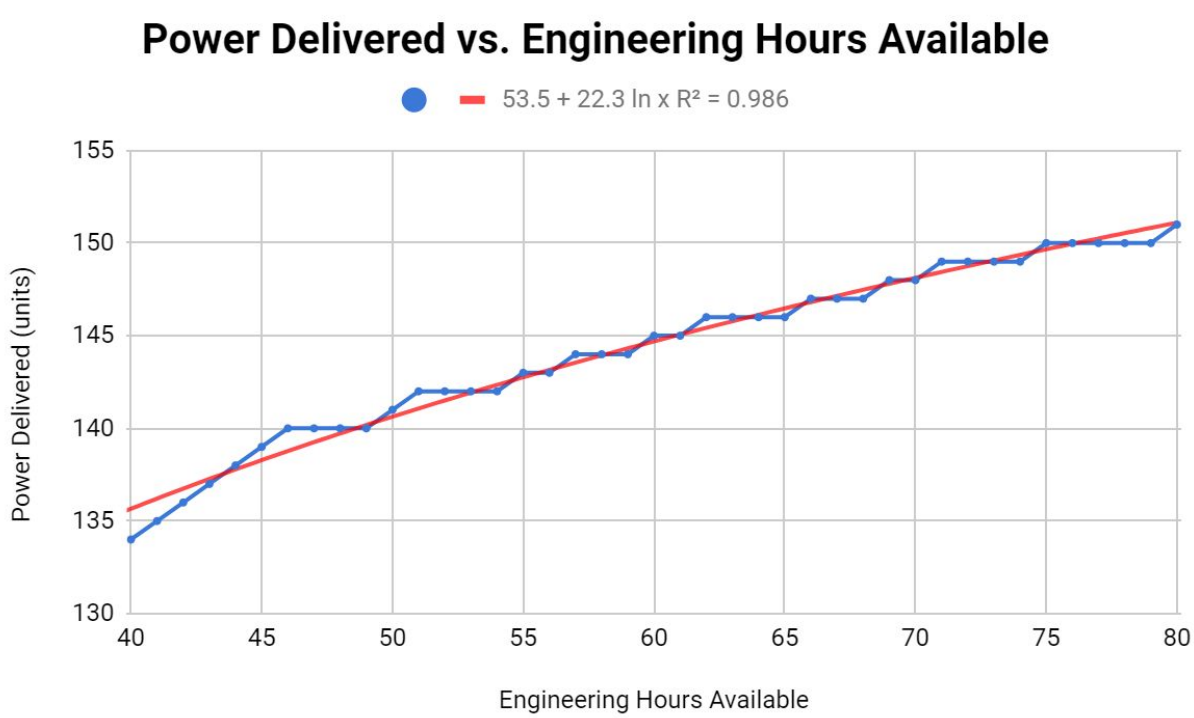
bolts[1,4] = 14	energyFlow[1,4] = 39
bolts[1,5] = 17	energyFlow[1,5] = 46
bolts[1,8] = 3	energyFlow[1,8] = 10
bolts[1,13] = 2	energyFlow[1,13] = 7
bolts[1,16] = 4	energyFlow[1,16] = 14
bolts[1,17] = 10	energyFlow[1,17] = 30
bolts[1,30] = 1	energyFlow[1,30] = 2

Max engineering hours possible: 387
Max possible demand satisfied: 154.0

In the solution to part B.2.1, the t parameter (the max number of engineering hours available) is set equal to 387 hours. 387 is the maximum number of hours that the engineering team can work on the conduits; this was calculated by summing three times the number of conduits plus the sum of all bolts that can be added to each conduit. By using 387 rather than a very large arbitrary value, the model can optimize faster, which would be crucial with very large data sets. The use of 387 is also to ensure the optimal solution is not constrained on the number of available engineering hours, since the goal was to determine the maximum possible demand satisfied once the conduit upgrades are implemented if the engineering team had unlimited hours to work on the system. The constraint is needed in the model for when a there is an actual time constraint on the number of hours available for the engineers to work, for example, part B.2.2.

2.

The graph provided below is the trade off curve between power delivered and engineering hours available. As you increase the number of engineering hours the engineering team can dedicate to increasing the capacity of conduits, the more total demand can be satisfied. To generate the trade-off curve, the ILP model was ran 41 times, once for each integer in the set $\{40,41,42\ldots,80\}$ where the integer passed in was t, the max engineering hours available. Then, the max demand satisfied for each t value was plotted against the t value. The blue line is a "step-wise" line that goes through each data point plotted, the red line is a logarithmic line of best fit. The blue line might be misleading in some cases where the increase in total demand satisfied actually increases on a non integer engineering hours available value. For example, the blue line shows that at 49 hours, 140 energy units of demand can be satisfied and at 50 hours, 141 energy units demand can be satisfied. In reality, 141 energy units of demand might be able to be satisfied at 49.5 hours, but the model only plots on integer hour values.



3.

Removed Parameters:

t = the number of engineering hours available

New Objective Value:

min t

New Decision Variables:

t = minimum number of engineering hours needed to max demand satisfied

Added Constraint:

$$\sum_{j \in N} y_j = 154$$

In order to calculate the minimum number of engineering hours needed to satisfy the maximum demand possible, a new decision variable, t, was added to the model. By minimizing t and adding a constraint that the total demand satisfied is equal to the maximum demand possible to satisfy (which was calculated in part B.2.1), the minimum engineering hours needed can be calculated. Based on the ILP model, the minimum number of engineering hours needed to maximize demand satisfied is 92 engineering hours.

Glossary for Part B.1.1 (c)

LP Model Variables and Parameters	Corresponding Gurobi Variable
G	groups
$c_{i,j}$	capacities[i , j]
d_j	demands[j]
A	arcs
Â	decArcs
N	nodes
z	z
$x_{i,j}$	x[i , j]
y_j	y[j]

Glossary for Part B.2.1

This glossary refers to the part B.1.1 (c) glossary with the following changes

Removed Variables and Parameters	Corresponding Gurobi Variable
G	groups
z	z
New Variables and Parameters	**Corresponding Gurobi Variable**
t	t
$k_{i,j}$	k[i , j]
$b_{i,j}$	b[i , j]
$d_{i,j}$	d[i , j]

Glossary for Part B.2.3

Refer to glossary for part B.2.1