MA153 Mathematical Modeling & Introduction to Differential Equations
Board Sheet # 4 Solutions
Lesson 34-36: Homogeneous Linear Systems

1. Find a general solution of the system $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

(a) $\begin{pmatrix} 1 & 3 \\ 12 & 1 \end{pmatrix}$ $\lambda_1 = 7 \leftrightarrow \vec{K}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = -5 \leftrightarrow \vec{K}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\boxed{\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-5t}}$$

(b) $\begin{pmatrix} -1 & 1 \\ 8 & 1 \end{pmatrix}$ $\lambda_1 = 3 \leftrightarrow \vec{K}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\lambda_2 = -3 \leftrightarrow \vec{K}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\boxed{\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t}}$$

(c) $\begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$ $\lambda_1 = \lambda_2 = -1 \rightarrow \vec{K} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{P} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\boxed{\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} \right]}$$

(d) $\begin{pmatrix} -1 & -2 \\ 8 & -1 \end{pmatrix}$ $\lambda = -1 \pm 4i \leftrightarrow \vec{K} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$

$$\boxed{\begin{aligned} \vec{X}(t) = c_1 e^{t} &\left[\begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos 4t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 4t \right] \\ + c_2 e^{t} &\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 4t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin 4t \right] \end{aligned}}$$

2. Find the solution to the given system that satisfies the given initial conditions.

(a) $\mathbf{X}'(t) = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \mathbf{X}(t)$, $\quad \mathbf{X}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda_1 = -2 \leftrightarrow \vec{K}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\lambda_2 = 4 \leftrightarrow \vec{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{X}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

$\vec{X}(0) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -c_1 + c_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$c_1 = -1, c_2 = 2$

$$\boxed{\vec{X}(t) = -\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}}$$

(b) $\mathbf{X}'(t) = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} \mathbf{X}(t)$, $\quad \mathbf{X}(0) = \begin{pmatrix} -10 \\ -6 \end{pmatrix}$

$\lambda_1 = 3 \leftrightarrow \vec{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = 4 \leftrightarrow \vec{K}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$

$\vec{X}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} c_1 + 3c_2 \\ c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \end{pmatrix}$

$c_1 = 2, c_2 = -4$

$$\boxed{\vec{X}(t) = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} - 4 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}}$$

(c) $\mathbf{X}'(t) = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{X}(t)$, $\quad \mathbf{X}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\lambda = -2 \pm i \leftrightarrow \vec{K} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i$

$\vec{X}(t) = c_1 e^{-2t} \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right]$

$+ c_2 e^{-2t} \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t \right]$

$\vec{X}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 \\ -c_1 - c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$c_1 = -1, c_2 = 1$

$$\boxed{\begin{aligned} \vec{X}(t) = -e^{-2t} &\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right] \\ + e^{-2t} &\left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t \right] \end{aligned}}$$

3. Find the eigenvector and eigenvalues to the following matrices.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$

$\lambda_1 = 1 \Leftrightarrow \vec{K}_1 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$

$\lambda_2 = 2 \Leftrightarrow \vec{K}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\lambda_3 = 5 \leftrightarrow \vec{K}_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{pmatrix}$

$\lambda_1 = -10 \Leftrightarrow \vec{K}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_2 = \lambda_3 = 5$

$\vec{K}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\vec{K}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{pmatrix}$

$\lambda_1 = 2 \Leftrightarrow \vec{K}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 2+i$, $\lambda_3 = 2-i$

$\begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} i$ $\begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i$