Introduction to Algorithms
CSI 3344
Homework 1

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January 30, 2019

Prepare a LaTex file with each problem beginning on a fresh page. Use the Upload Site to submit your solution. These problems are a review of material from earlier courses that will serve us well in CSI 3344:

1. In each of the following situations, indicate whether $f(n) \in O(g(n))$, or $f(n) \in \Omega(\text{g(n)})$, or both (in which case $f(n) \in \Theta(\text{g(n)})$).

f(n)	g(n)	Answer
$n - 100$	$n - 200$	$f(n) \in \Theta(g(n))$
$n^{\frac{1}{2}}$	$n^{\frac{2}{3}}$	$f(n) \in O(g(n))$
$100n + log(n)$	$n + (log(n))^2$	$f(n) \in \Theta(g(n))$
$n\ log(n)$	$10n\ log(10n)$	$f(n) \in \Theta(g(n))$
$log(2n)$	$log(3n)$	$f(n) \in \Theta(g(n))$
$10log(n)$	$log(n^2)$	$f(n) \in \Theta(g(n))$
$n^{1.01}$	$nlog^2n$	$f(n) \in \Omega(g(n))$
$\frac{n^2}{log(n)}$	$n(log(n))^2$	$f(n) \in \Omega(g(n))$
$n^{0.1}$	$(log(n))^{10}$	$f(n) \in \Omega(g(n))$
$(log\ n)^{log\ n}$	$\frac{n}{log\ n}$	$f(n) \in \Omega(g(n))$
$\sqrt{n}$	$(log\ n)^3$	$f(n) \in \Omega(g(n))$
$n^{\frac{1}{2}}$	$5\ log_2\ n$	$f(n) \in O(g(n))$
$n\ 2^n$	3^n	$f(n) \in O(g(n))$
2^n	$2^{(n+1)}$	$f(n) \in \Theta(g(n))$
$n!$	2^n	$f(n) \in \Omega(g(n))$
$(log\ n)^{log\ n}$	$2^{(log_2\ n)^2}$	$f(n) \in O(g(n))$
$\sum_{i=1}^{n} i^k$	n^{k+1}	$f(n) \in \Theta(g(n))$

2. Geometric series.

   Note: We will use the geometric series formula: $S(n) = \frac{a-ar^{n+1}}{1-r}$.

   (a) Give a simple upper bound on $1+2+4+\ldots+2^n$. Conclude that this sum is $O(2^n)$.

   - This can be written as $\sum_{i=0}^{n} 2^i = 2^{n+1}-1$. We can ignore the constant and from the above table we know that $2^{n+1} \in O(2^n)$.

   (b) Do the same for $1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}$ and conclude that the sum is O(1).

   - This can be written as $\sum_{n=0}^{\infty}(\frac{1}{2})^n = \frac{\frac{1}{2}}{1-\frac{1}{2}}$ using the geometric series formula. In this case, $a=1, r=\frac{1}{2}$.
   - $\sum_{n=0}^{\infty}(\frac{1}{2})^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$. Thus the series converges to 1. $S \in O(1)$.

   (c) By considering a more general series $S(n) = 1+c+c^2+\ldots+c^n$, establish the following very useful rule: in big-O terms, the sum of a geometric series is simply:
   Note: In this case, the common ratio r = c.

   i. the first term if the series is strictly decreasing,
      - The series is strictly decreasing if $c < 1$.
      - Using the formula above we have $S(n) = \frac{1-c^{n+1}}{1-c}$.
        If $c < 1$, then $1-c < 1-c^{n+1} < 1$. Dividing by $1-c$ gives $1 < \frac{1-c^{n+1}}{1-c} < \frac{1}{1-c}$.
        $1 < S(n) < \frac{1}{1-c}$.
      - Thus, $S(n) \in \Theta(1)$, where 1 is the first term.

   ii. the last term if the series is strictly increasing,
      - The series is strictly increasing if $c > 1$,
      - Using the formula above we have $S(n) = \frac{1-c^{n+1}}{1-c} = \frac{c^{n+1}-1}{c-1}$.
        If $c > 1$, then $c^n < c^{n+1}-1 < c^{n+1}$. Divide by 1-c gives $\frac{1}{1-c} * c^n < S(n) < \frac{c}{1-c} * c^n$
      - Thus, $S(n) \in \Theta(c^n)$, where c^n is the last term.

   iii. the number of terms if the series is unchanging.
      - if $c = 1$, then $S(n) = 1+1+1+...+1 = 1+n$.
        If $S(n) = n+1$, then $S(n) \in \Theta(n)$.

3. Proving by induction. We would like to establish the following formula for the sum of the first n odd numbers: $1+3+5+...+(2n-1)=n^2$. A nice way to do this is by induction. Let S(n) be the statement above. **We want to prove** $\sum_{i=1}^{n}(2n-1)=n^2$.

   An inductive proof would have the following steps:

   (a) Show that S(1) is true.

      i. Let $n=1$. Then $\sum_{i=1}^{1}(2(1)-1)=1^2$.
      That gives $(2(1)-1)=1^2$.
      $(2-1)=1^2$.
      $1=1$. Thus, it is true for the base case.

   (b) Assume that if S(1),...,S(k) are true,

   (c) then show S(k+1) is true.

      i. If it is true for S(1)...S(k), then $\sum_{i=1}^{k}(2k-1)=k^2$.

      ii. Let n = k + 1. Then we have, $\sum_{i=1}^{k+1}(2(k+1)-1)=(k+1)^2$.

      $\sum_{i=1}^{k+1}(2k+2-1)=k^2+2k+1$.

      $\sum_{i=1}^{k+1}(2k+1)=k^2+2k+1$.

      $3+5+7+..+2k-1+2(k+1)-1=k^2+2k+1$.

      $\sum_{i=1}^{k}(2k-1)+2(k+1)-1=k^2+2k+1$.

      $k^2+2(k+1)-1=k^2+2k+1$. (Inductive hypothesis)

      $k^2+2k+1=k^2+2k+1$. Thus, it is true for the base case and inductive hypothesis. QED.

4. Practice with $big-O$ and Ω. For some fixed positive integer c, consider the summation $S(n) = 1^c + 2^c + 3^c + \ldots + n^c$.

   (a) Show that S(n) is $O(n^{c+1})$. Hint: There are n terms in the series, and each is at most ...?

   (b) Show that S(n) is $\Omega(n^{c+1})$. Hint: Look just at the second half of the series.

- $S(n) = \sum_{i=1}^{n} i^c = \frac{1}{c+1}n^{c+1}$ by definition. Let $f(n) = \frac{1}{c+1}n^{c+1}$ and $g(n) = n^{c+1}$.

  Using L'Hopital's rule: $\lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \frac{n^c}{(c+1)n^c}$. Drop constant and get 1. Thus,

  $f(n) \in \Theta(n^{c+1})$. Since f(n) = S(n), $S(n) \in \Theta(n^{c+1})$.

  By definition of Θ, if $S(n) \in \Theta(n^{c+1})$, then $S(n) \in O(n^{c+1})$ and $S(n) \in \Omega(n^{c+1})$

5. Logarithms (base two). Recall the definition of logarithm base two: saying $p = log_2 m$ is the same as saying $m = 2^p$. In this class, we will typically write log to mean log_2.

(a) How many bits are needed to write down a positive integer n? Give your answer in big-O notation, as a function of n, where n is the length of the number.

Number	Number of bits
0	0
1	1
2	2
3	2
4	3
5	3
6	3
7	3
8	4
.	4
.	4
16	5
.	5
.	5
32	6
.	6
.	6
64	7
.	7
.	7

Largest Number with n digits	n	Number of bits
9	1	4
99	2	7
999	3	10
9999	4	14

Thus, it takes $O(\lceil nlog_2(10) \rceil)$, where $log_2(10) \approx 3.3$.

For example, from the right table, if $n = 4$, the number of bits required is $\lceil 4 * 3.3 \rceil = 14$.

(b) How many times does the following piece of code print "hello"? Assume n is an integer, and that division rounds down to the nearest integer. Give your answer in big-O form, as a function of n.

```
while n > 1:
   print "hello"
   n := n/2
```

n	Number of prints
0	0
1	0
2	1
3	1
4	2
5	2
6	2
7	2
8	3
9	3
10	3
11	3
12	3
13	3
14	3
15	3
16	4

Looks like $O(log_2(n))$