Take Home Quiz 10

1 Let C be the top half of the unit circle, oriented *clockwise*. Let $\mathbf{F}(x, y) =< y, x^2 >$ be a force field. Calculate the work done by $\mathbf{F}$ acting on a particle that moves along C. Assume force is measured in Newtons and distance in meters.

- **Solution**: C can be parameterized by $\mathbf{r}(t) =< \cos(t), \sin(t) >$ for $t \in [0, \pi]$. Since $\mathbf{F}$ is not conservative, we have to calculate the integral manually. $\mathbf{F}(\mathbf{r}(t)) =< \sin(t), \cos^2(t) >$ and $d\mathbf{r}(t) =< -\sin(t), \cos(t) >$ dt.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(\mathbf{t})dt \\
&= \int_0^{2\pi} < \sin(t), \cos^2(t) > \cdot < -\sin(t), \cos(t) > dt \\
&= \int_0^{2\pi} (-\sin^2(t) + \cos^3(t))dt \\
&= \int_0^{2\pi} \cos^3(t)dt - \int_0^{2\pi} \sin^2(t)dt
\end{aligned}$$

- For the first integral, we split the cubic power of cosine to get $\cos^2(t) = 1 - \sin^2(t)$. For the second integral, we can derive a double angle relation for $\sin^2(t)$:

$$\begin{aligned}
e^{2ti} &= (\cos(t) + i\sin(t))(\cos(t) + i\sin(t)) \\
&= \cos^2(t) - \sin^2(t) + 2i\sin(t)\cos(t) \\
\cos(2t) &= \mathrm{Re}(e^{2ti}) = \cos^2(t) - \sin^2(t) \\
\cos(2t) &= (1 - \sin^2(t)) - \sin^2(t) = 1 - 2\sin^2(t) \\
\sin^2(t) &= \frac{1 - \cos(2t)}{2}
\end{aligned}$$

Back to our integrals:

$$\begin{aligned}
\int_0^{2\pi} \cos^3(t)dt - \int_0^{2\pi} \sin^2(t)dt = \int_0^{2\pi} (1 - \sin^2(t))\cos(t)dt - \int_0^{2\pi} \frac{1 - \cos(2t)}{2}dt \\
u = \sin(t) \qquad du = \cos(t)dt \\
u(0) = 0 \quad u(2\pi) = 0 \\
= 0 - \frac{1}{2}[t - \frac{1}{2}\sin(2t)]_0^{2\pi} = \pi \quad \mathrm{N}\cdot\mathrm{m}
\end{aligned}$$

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$$\mathbf{F} =< \frac{2x}{y} - e^x, -\frac{x^2}{y^2} + 3y >$$

- Draw the largest simply connected region D that contains the point (1,5) on which $\mathbf{F}$ is defined.
- Verify that $\mathbf{F}$ is conservative on D.
- Find a potential function f on D

Solution

- The vector field is undefined for the line $y = 0$. A simply connected region must not contain any holes and must be in one complete piece; therefore D must be all points (x, y) such that $y > 0$ (above the x axis).

$$\mathcal{D} = \{(x, y)| \quad y > 0\}$$

- Testing if $\mathbf{F}$ is conservative (for vector fields in $\mathbb{R}^2$ this can be thought of as finding the 'curl' and verifying it is 0, although not really).

$$P_y = -\frac{2x}{y^2}$$
$$Q_x = -\frac{2x}{y^2}$$

$\mathbf{F}$ is conservative on the domain $\mathcal{D}$. To find the potential function f, I think integrating $P(x, y)$ against x is the easier way to start:

$$\begin{aligned} f = \int P(x,y)dx &= \frac{x^2}{y} - e^x + g(y) \\ f_y &= Q(x,y) \\ -\frac{x^2}{y^2} + g'(y) &= -\frac{x^2}{y^2} + 3y \\ g'(y) = 3y &\rightarrow g(y) = \frac{3}{2}y^2 + C \end{aligned}$$

The general solution is

$$f(x, y) = \frac{x^2}{y} + \frac{3}{2}y^2 - e^x + C$$

Any particular solution is a potential function that satisfies $\mathbf{F}(x, y) = \nabla f(x, y)$