Section 16.5 Homework

1-21 odd

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$
$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

1-7

Find the curl and divergence of the vector field.

1

$$\mathbf{F}(x, y, z) = \langle xy^2z^2, x^2yz^2, x^2y^2z^2 \rangle$$

Solution:

$$\nabla \cdot \mathbf{F} = y^2z^2 + x^2z^2 + x^2y^2$$
$$\nabla \times \mathbf{F} = \langle 2x^2yz^2 - 2x^2yz, 2xy^2z - 2xy^2z, 2xyz^2 - 2xyz^2 \rangle = \mathbf{0}$$

3

$$\mathbf{F}(x, y, z) = \langle xye^z, 0, yze^x \rangle$$

Solution:

$$\nabla \cdot \mathbf{F} = ye^z + ye^x$$
$$\nabla \times \mathbf{F} = \langle ze^x, xye^z - yze^x, -xe^z \rangle$$

5

$$\mathbf{F}(x, y, z) = \begin{bmatrix} \frac{\sqrt{x}}{1+z} & \frac{\sqrt{y}}{1+x} & \frac{\sqrt{z}}{1+y} \end{bmatrix}$$

Solution:

$$\nabla \cdot \mathbf{F} = \frac{1}{2\sqrt{x}(1+z)} + \frac{1}{2\sqrt{y}(1+x)} + \frac{1}{2\sqrt{z}(1+y)}$$
$$\nabla \times \mathbf{F} = \begin{bmatrix} -\sqrt{z}(1+y)^{-2} & -\sqrt{x}(1+z)^{-2} & -\sqrt{y}(1+x)^{-2} \end{bmatrix}$$

7

$$\mathbf{F}(x, y, z) = \langle e^x \sin(y), e^y \sin(z), e^z \sin(x) \rangle$$

Solution

$$\nabla \cdot \mathbf{F} = \mathbf{F}$$
$$\nabla \times \mathbf{F} = \langle -e^y \cos(z), -e^z \cos(x), -e^x \cos(y) \rangle$$

9 We are given that the z component is 0, and from the picture it appears the x component of all vectors is also 0 (and so are the partials of P and R). Q (the y component) is shrinking as y increases, signifying a negative first partial of Q and thus a negative divergence ($\nabla \cdot \mathbf{F} = 0 + Q_y + 0$). The curl is $\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$ If $P(x, y, z) = R(x, y, z) = 0$, then the partials of those functions are 0. Q is only dependent on y and thus the partials $Q_x = Q_z = 0$. So, the curl is $\langle 0-0, 0-0, 0-0 \rangle = \mathbf{0}$.

11 As with (9), $R(x, y, z) = 0$. There appears to be no vertical (y) component to any vectors, so we can assume $Q(x, y, z) = Q_y(x, y, z) = 0$. The horizontal component of the vectors $P(x, y, z)$ varies with y: it increases as y increases (so P_y is positive). There does not appear to be any dependence of P on x, so $P_x = 0$. The divergence is then 0 ($P_x + Q_y + R_z = 0$). The curl is $\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$, but $R_x = R_y = Q_z = Q_x = P_x = 0$, so the resulting vector is $\langle 0, 0, -P_y \rangle$. Since P_y is positive, the curl is pointing in the negative z direction.

13-17

Determine if the field is conservative. If it is, find a potential function.

13 $\mathbf{F}(x, y, z) = y^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2 z^2 \mathbf{k}$

- **Solution**: A conservtive vector field has a curl vector of $\mathbf{0}$ for a simply connected open region (continuous partials and all that too). The function $\mathbf{F}$ in question here has simply polynomial componenent functions so is continuous everywhere and has continuous first partials. Testing the curl:

$$\text{curl}(\mathbf{F}) = \langle 6xyz^2 - 6xyz^2, 3y^2z^2 - 3y^2z^2, 2yz^3 - 2yz^3 \rangle = \mathbf{0}$$

- The curl is $\mathbf{0}$ so $\mathbf{F}$ is conservative. To find the potential function f:

$$\begin{aligned} f &= \int P(x, y, z)dx = \int y^2 z^3 dx = xy^2z^3 + g(y, z) \\ f_y &= Q \\ 2xyz^3 + g_y(y, z) &= 2xyz^3 \to g_y(y, z) = 0 \\ g(y, z) = \int (0) dz &= h(z) \\ f &= xy^2z^3 + h(z) \\ 3xy^2z^2 + h'(z) &= 3xy^2z^2 \to h(z) = C \end{aligned}$$

- The general solution is $f(x, y, z) = xy^2z^3 + C$

15 $\mathbf{F}(x, y, z) = \langle z\cos(y), xz\sin(y), x\cos(y) \rangle$

- **Solution** P, Q, R are continuous and have continuous first partials. The curl is

$$\text{curl}(\mathbf{F}) = \langle -x\sin(y) - x\sin(y), \cos(y) - \cos(y), z\sin(y) - (-z\sin(y)) \rangle = \langle -2x\sin(y), 0, 2z\sin(y) \rangle$$

- The curl is a nonzero vector and therefore the field is not conservative.

17 $\mathbf{F}(x, y, z) = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$

- **Solution** Continuous checks: confirmed. Curl?

$$\nabla \times \mathbf{F} = \langle xe^{yz} - xyze^{yz} - (xe^{yz} - xyze^{yz}), ye^{yz} - ye^{yz}, ze^{yz} - ze^{yz} \rangle = \mathbf{0}$$

- Field is conservative. Find f:

$$\begin{aligned} f &= \int e^{yz}dx = xe^{yz} + g(y,z) \\ f_y = xze^{yz} + g'(y,z) &= xze^{yz} = Q \\ &\to g'(y,z) = 0 \qquad g(y,z) = h(z) \\ f &= xe^{yz} + h(z) \\ f_z = xye^{yz} + h'(z) &= xye^{yz} = R \\ &\to h'(z) = 0 \qquad h(z) = C \end{aligned}$$

- The general solution is $f(x,y,z) = xe^{yz} + C$

19 Is there a vector field $\mathbf{F}$ on $\mathbb{R}^3$ such that $\text{curl}(\mathbf{F}) = \langle x\sin(y), \cos(y), z - xy\rangle$?

- **Solution**: We know that $\nabla \cdot \nabla \times \mathbf{F} = 0$:

$$\nabla \cdot \nabla \times \mathbf{F} = \frac{\partial}{\partial x}(x\sin(y)) + \frac{\partial}{\partial y}(\cos(y)) + \frac{\partial}{\partial z}(z - xy) = \sin(y) - \sin(y) + 1 \neq 0$$

  No, there does not exist such a function $\mathbf{F}$.

21 Show that any vector field of the form

$$\mathbf{F}(x,y,z) = \langle f(x), g(y), h(z)\rangle$$

where f, g, h are differentiable, is irrotational.

- **Solution** Irrotational is synonymous with 0 curl. The curl of $\mathbf{F}$ will have components equal to 0 because when computing curl, partials with respect to variables other than the "position variable" of component vectors are taken. In the case of $\mathbf{F}$, where the component functions are only dependent on their "position variable", any other partials will be 0 and thus the curl will be $\mathbf{0}$.