Formalized Accelerando: An Extension of Rhythmic Techniques in Nancarrow's Acceleration Canons

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> "[A]cceleration canon does not have the kind of clean structural implications Nancarrow found in the straight tempo canon . . . Perceptually fascinating, it did not lead to the kind of compositional richness he found elsewhere." (Gann 1995, 163)

Of his fifty-one studies for player piano, Nancarrow wrote only three canons based on large-scale continuous acceleration:[1] Studies 21, 22, and 27.[2] In his book on Nancarrow's music, Kyle Gann offers the above statement as an explanation as to why Nancarrow never returned to the acceleration canon after the last of these. I am inclined to agree with Gann's statement, with the following modification: it was Nancarrow's conception of accelerando, focusing on duration rather than tempo, which limited its compositional value, rather than any limitations inherent in large-scale accelerandi. The purpose of this paper is to explore the nature of Nancarrow's conception of accelerando, and the manner in which his approach to acceleration canons limited the structural elements which proved so useful in the straight tempo canons. Having identified the problems with this approach, I will consider a generalized model of acceleration, borrowing techniques from calculus. The latter portion of the paper will consider examples which exploit the potentials of this more general approach.

Nancarrow's Conception of Accelerando

Before considering Nancarrow's approach to acceleration, it will be useful to briefly summarize the primary structural elements of straight tempo canons—canons in which each voice remains in a constant tempo for large passages and the relationship of tempi between voices is not 1:1. Whereas in canons in which the voices are in a 1:1 tempo relationship the elapsed time between entries of the canonic line (which Gann refers to as the echo distance) remains constant, in tempo canons this distance is continuously varying. The echo distance increases when faster voices are ahead of slower voices in the canon line, and decreases when the situation is reversed. A special feature of tempo canons is the moment at which the echo distance is reduced to zero, referred to by Gann as the convergence point. Convergence points often serve as a structural climax, whether occurring at the beginning, ending, in the middle, or at numerous moments in between. By comparing the change in the echo distance over time, it is possible for the listener to determine if the canon is approaching or receding from a convergence point, and, thus, to locate any given moment within the overall formal design of the canon. As an aid to perceiving the echo distance, Nancarrow often provides what may be called salient markers—gestures which the listener can easily perceive and retain, and compare the elapsed time between their appearance in one voice and another. These gestures often take the form of texturally distinct figures such as trills, glissandi, staccato chord punctuations, etc. Convergence points are often emphasized by special markers. A noteworthy example is the rising chromatic glissandi in Study No. 36 which signal the approach of the convergence point, itself marked by trills in all voices, and the corresponding descending glissandi which mark the subsequent departure.

In the acceleration studies, Nancarrow employs two different types of accelerando, which Gann (pp. 146–8) refers to as arithmetic and geometric. In an arithmetic acceleration, a constant is subtracted (or added in the case of a ritardando) to each successive duration so that the duration for the n^{th} beat[4] is:

$$d_n = d_0 - cn, \tag{1}$$

where d_0 is the initial duration, and c is the constant. This type of acceleration works well for relatively brief passages, but is not conducive to a smooth acceleration occurring over a long period of time. In fact, the acceleration only becomes perceivable once the ratio of successive durations becomes sufficiently large. Thus when arithmetic acceleration is employed for longer sequences, the listener perceives a more or less steady tempo for the majority of the passage, followed by a brief and quite rapid acceleration. Taking the ratios between successive durations in the series as a measure of discrete changes in tempo (for instance 150:149, 149:148, . . . , 6:5, 5:4, 4:3, 3:2, 2:1), we can see that not only does the series accelerate or decelerate throughout, but that the *rate* of acceleration increases or decreases throughout as well, by noting that the ratios become consistently larger or smaller in either direction.

In order to solve this problem, Nancarrow employs a geometric acceleration, in which successive durations differ, not by a constant length, but by a constant ratio designated by percentage. Under this type of acceleration, with a percentage change of x%, each duration is exactly $1 + {}^{x}/_{100}$ times as long as the next duration (or $(1 + {}^{x}/_{100})^{-1}$ for a ritardando). The duration for the n^{th} beat, d_n, is:

$$d_n = d_0 r^n, \tag{2}$$

where d_0 is the initial duration, and r is the ratio between any b_n and b_{n-1}. For example, an initial duration of 2” and a percent change of 5% would yield the duration sequence 2”, 1.905”, 1.814”, 1.728”, 1.645”, etc., with each duration being $r = 1.05\text{-}1 \approx 0.952$ as long as the previous duration.

Although geometric acceleration is smoother and more conducive to large-scale accelerandi than arithmetic acceleration, there is still an upward limit on the length for which it is useful, since the rate of acceleration continues to increase throughout (for positive acceleration).[5] It may seem that the rate of acceleration is constant in a geometric acceleration since the ratio between successive durations is always the same, but this is not the case. The reason is that acceleration is not being defined with respect to an unchanging unit of time, but with respect to a unit beat which is itself becoming progressively shorter or longer. It is as if gravitational acceleration were a function of the distance an object has fallen rather than the elapsed time since the object first began to fall.[6] This is an example of a positive feedback loop, whereby acceleration varies in conjunction with tempo. Thus, a positive acceleration leads to an increase in tempo, which in turn leads to an increase in the rate of acceleration, and so on. (Of course the same type of behavior occurs for negative acceleration.) Due to this positive feedback between acceleration and tempo, both arithmetic and geometric accelerations lead to similar gestures—a relatively slow increase in tempo for most of the gesture which then explodes into a flurry of notes at the end, or the reverse gesture which begins with a flurry. This is evident upon examining the rapid increase in the number of beats per unit of time as the formal limit of the accelerando is approached. Example 1 shows the number of beats occurring per second beginning at 15” from the onset of an accelerando with an initial duration of 1” and a ratio of 5%.

time span	number of beats
15”-16”	~3.7
16”-17”	~4.6
17”-18”	~5.9
18”-19”	~8.3
19”-20”	~14.2
20”-21”	~503.9
21”-22”	∞

EXAMPLE 1: NUMBER OF BEATS PER SECOND IN A GEOMETRIC ACCELERATION WITH AN INITIAL DURATION OF ONE SECOND AND A PERCENT ACCELERATION OF 5%.

There are other problems with this methodology stemming from the choice of a recursive operation on duration as the means of acceleration, rather than a continuous function of tempo. One such problem concerns subdivisions of the beat. A conception of geometric acceleration expressed by the duration series above does not allow for the possibility of subdivisions. We can evaluate the duration from beat 1.5 to 2.5, since the span is exactly one beat, but not the precise placement of the beginning of this duration. (The beat cannot be simply divided into a number of equal parts either, since that would make the accelerando uneven.) In the two canons described by percentages rather than tempo ratios, Studies Nos. 22 and 27, the canonic line does not contain any subdivisions of the beat. A rhythmic structure which allows a variety of rhythmic values (often ranging from long, sustained chords to blindingly fast runs within a single study) is one of the primary means in Nancarrow's music for the rhythmic differentiation necessary to establish rhythmic groupings which serve as salient markers. These salient markers render the echo distance and, thus, the approach to or departure from a convergence point, audible. Without these markers, rhythmic groupings will be tend to be weak, and echo distance and convergence points will tend not be structurally important elements. Whereas the straight tempo canon implies a variety of structures, Nancarrow's conception of the acceleration canon *prescribes* the basic rhythmic gesture. Gann's statement that the "acceleration canon is less conducive to making" (Gann 1995, 163) convergence point an important issue is quite valid for this particular method of acceleration, but not for the more general model to be discussed shortly.

The problem is solved by formalizing geometric acceleration to express the elapsed time from the beginning of the acceleration at which the i^{th} beat occurs:

$$t_i = \sum_{n=1}^{i} dr^{n-1} . \tag{3}$$

Recalling the closed-form formula

$$\sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r} , \tag{4}$$

the equation for Nancarrow's geometric acceleration becomes

$$t_i = \sum_{n=1}^{i} dr^{n-1} = \frac{d(1-r^i)}{1-r} . \tag{5}$$

In this equation, any subdivision may be specified, such as the moment halfway between beats 1 and 2 by setting $i = 1.5$.

Another difficulty is determining the tempo of any voice at a specific time, since tempo must be estimated from duration, and duration is determined by the number of beats which have elapsed by the desired point in time. Thus, there is little, if any, control over instantaneous tempo relationships. One exception is the second canon of Study No. 27, in which the final note of every voice begins and ends in synchrony, so that the accelerations for each voice in this canon converge on the same tempo. (See Gann, 163) This suggests a different type of structural convergence where the acceleration converges to a predetermined tempo or tempo relationship (such as 3:2, 6:5, or any other ratio). However, Nancarrow's continuous accelerations only converge on one of two extremes—infinitely fast or infinitely slow. In two of the examples at the end of this paper, convergence on a tempo or tempo relationship is an important structural element.

It is no surprise that Nancarrow's experiments with acceleration canons proved short lived. The lack of rhythmic variety due to the positive feedback between acceleration and tempo, rhythmic flexibility to establish salient markers arising from a diminished variety of rhythmic values (due in large part to an absence of subdivisions), and control of tempo ratios are all inherent in his conception of accelerando and limit the compositional value of this technique. I must emphasize that my intent is not to criticize Nancarrow. Far from it! It was my fascination with his music and admiration for his technique which first led me to consider these questions. Had a composer of Nancarrow's ingenuity worked with a less constrictive view of accelerando, he certainly would have been able to exploit the resulting structural potentials to great effect. In order to explore the structural possibilities suggested, but left untouched, by Nancarrow's experiments in formalized accelerando, it will prove useful to consider a more general model of acceleration.

A Generalized Model of Accelerando

The first step in proceeding toward a generalized view of accelerando is to express acceleration as a function with respect to time, rather than a series of durations. Nancarrow's geometric acceleration becomes (setting $f(t)=i$, the variable for elapsed beats in equation 5):

$$r^i = 1 - \frac{t(1-r)}{d},$$

$$f(t) = i = \log_r\left(1 - \frac{t(1-r)}{d}\right), \text{ or}$$

$$f(t) = \frac{\ln\left(1 - \frac{t(1-r)}{d}\right)}{\ln r}. \qquad (6)$$

Evaluating the above function for any value of t yields the number of beats which will have elapsed over the given amount of time. Expressed in this manner, it becomes obvious that this particular acceleration is only one out of an infinite number of variable tempo designs. Any function of time may be used to designate the number of elapsed beats over time, as long as for any succession of time points, $t_0 < t_1 < t_2$, $f(t_0) < f(t_1) < f(t_2)$. (This is to avoid marking off negative beats over time.) However, there is still no direct control over tempo or acceleration, which is often much more intuitive than elapsed beats.

In order to determine the *instantaneous* tempo at any given time, we must find the slope of the function for elapsed beats at that precise moment, which is equivalent to taking the first order derivative. For example, consider the function graphed in Example 2a, $f(t)=\upsilon t$, which increases from 0 to υ elapsed beats over the duration of one minute, yielding a tempo of beats/minute. This equation is of the standard form $y = mx + b$, in which m represents the slope of the line and is equivalent to the tempo. If the slope of the line is increased, the tempo is increased. Likewise, acceleration is derived by finding the slope of a function of tempo with respect to time. For functions in which the slope is not a constant, the slope at any given point is obtained by finding the tangent to the curve at the given point, as demonstrated in Example 2b.[7]

For a concrete example, consider $f(t) = 30t^2 + 60t$ as a function of beats over time measured in minutes. The first order derivative of this function,

$$\frac{df}{dt},$$

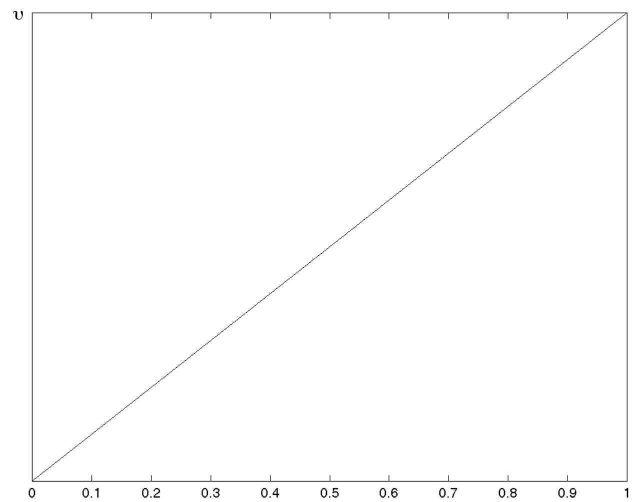
is $60t + 60$ so that the tempo for this accelerando is 60 beats/minute at $t = 0$, 120 beats/minute at $t = 1$, 180 beats/minute at $t = 2$, and so forth. The second order derivative of $f(t)$,

$$\frac{d^2f}{dt^2},$$

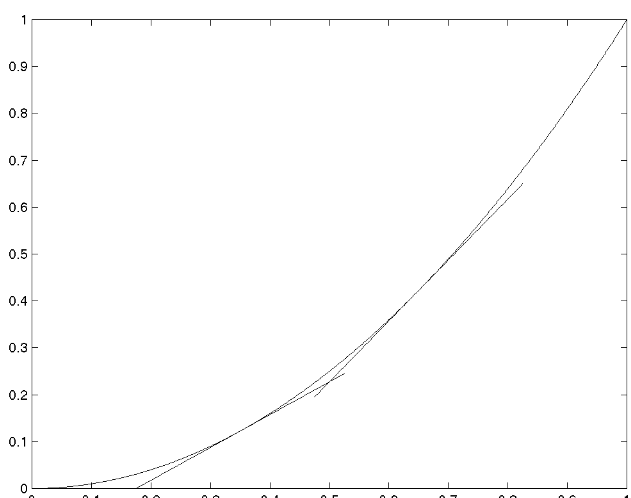
is 60, yielding a constant acceleration of 60 beats per minute per minute. (see Example 3)

If we wish to begin with a function for tempo, we can discover the number of elapsed beats at any given time by taking the area under the curve, which is equivalent to taking the definite integral. If the relevance of taking the area under a tempo curve is not immediately obvious, the following examples should help. Consider the situation in which a tempo of 60 beats per minute lasts for exactly two minutes, $g_1(t) = 60$. How many beats have elapsed? Obviously, the answer is 120. Now, consider a tempo which accelerates in a linear fashion from 60 to 120 beats per minute over exactly two minutes, $g_2(t) = 30t + 60$, and answer the same question as before. Since the average tempo is 90, one may assume,

(a)



(b)

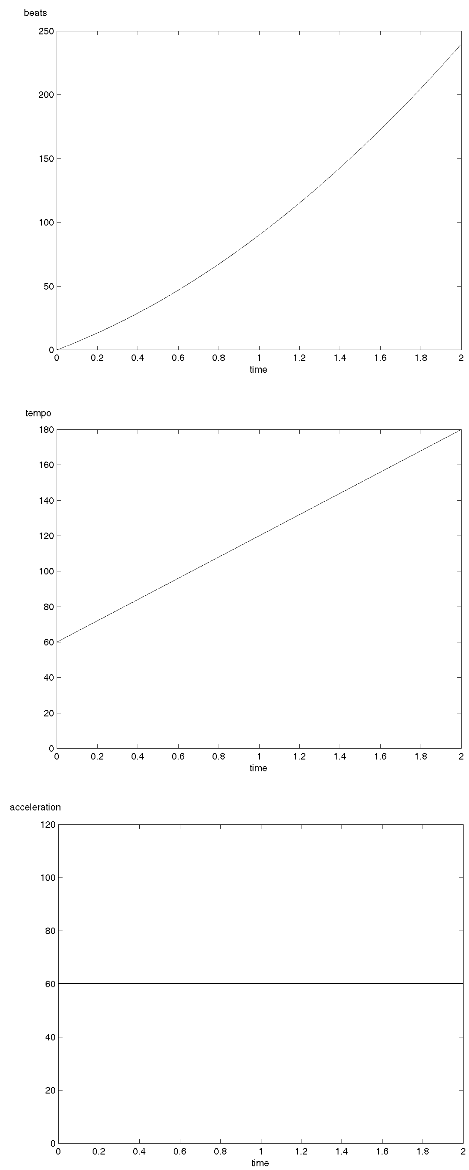


EXAMPLE 2: GRAPHS OF HYPOTHETICAL TEMPO FUNCTIONS.

$f(t)$

$\dfrac{df}{dt}$

$\dfrac{d^2f}{dt^2}$



EXAMPLE 3: GRAPHS OF FUNCTIONS FOR ELAPSED BEATS, TEMPO, AND ACCELERATION WITH RESPECT TO TIME.

rightly, that the answer is the same as if the tempo were a constant 90 beats per minute for two minutes, resulting in a total of 180 beats. However, we could have solved both problems by graphing the appropriate function and taking the area bounded by the x- and y-axes, the function $g_1(t)$ or $g_2(t)$, and the vertical line $x = 2$. For $g_1(t)$ the bounded region is a rectangle with the area given by the height, 60, times the base, 2. For $g_2(t)$ the bounded region is a trapezoid with the area given by the average of the two heights, 90, times the base, 2. While it is not necessary to use these geometric models for intuitively obvious situations, it is necessary when considering more complicated tempo functions. Using similar reasoning, it can be shown that instantaneous tempo may also be found by taking the definite integral of a function for acceleration. For example, an acceleration which decreases in a linear manner from 120 to 0 beats per minute per minute over the course of 1 minute, $h(t) = -120t + 120$, would yield the tempo function $g(t) = -60t^2 + 120t + C_1$ and the function for elapsed beats $f(t) = -20t^3 + 60t^2 + C_1 t + C_2$. C_1 represents the initial tempo, and C_2 represents when the voice begins with respect to $t = 0$ (i.e., if a voice enters 30 seconds into a piece, then C_2 would be .5).[8]

For the remainder of this paper, functions of elapsed beats, tempo, and acceleration will be designated by $b(t)$, $mm(t)$, and $acc(t)$, respectively. Summarizing the above relationships between these functions, we have:

$$\frac{d^2 b}{dt^2} = \frac{dmm}{dt} = acc(t) \text{, and} \tag{9}$$

$$\iint acc(t) = \int mm(t) = b(t) \text{.}^9 \tag{10}$$

Since the equation for Nancarrow's geometric accelerando was originally conceived in terms of duration, equations of instantaneous tempo and acceleration for this particular accelerando are not particularly intuitive or useful.[10] However, these relations allow one to begin working with tempo or acceleration, managing tight control over relations between simultaneous tempi, and then proceed to functions of elapsed beats in order to find the precise attack and release points. The remainder of this paper will focus on three examples which exploit this potential.

Canon

Canon is a work for midi-controlled acoustic piano which explores the simultaneous presentation of multiple interpretations of an accelerando.

The work consists of three voices, or strata, which begin at M.M. 30 and accelerate to an ending tempo of M.M. 180 over a duration of precisely two minutes. Each voice accelerates independently according to the following tempo functions:[11]

$$mm_1(t) = 150\sin^a\left(\frac{\pi}{4}t\right) + 30,$$

$$mm_2(t) = 75t + 30, \text{ and}$$

$$mm_3(t) = -150\left(\cos^a\left(\frac{\pi}{4}t\right) - 1\right) + 30, \quad (11)$$

where a=0.982828. (Example 4)

The specific tempo functions were chosen for three reasons: 1) to achieve a very gradual change in the rate of acceleration (rendered by trigonometric functions); 2) to provide a mirror structure between the upper and lower strata; and 3) to yield a total number of elapsed beats in each voice so that all three were an integer multiple of the same number (achieved by warping the trigonometric function with an appropriate value for the variable exponent). The last condition allows all three voices to cycle through a repeating forty-two beat theme (Example 5), with the highest voice stating the canon line exactly six times, the middle voice five, and the bass voice four:

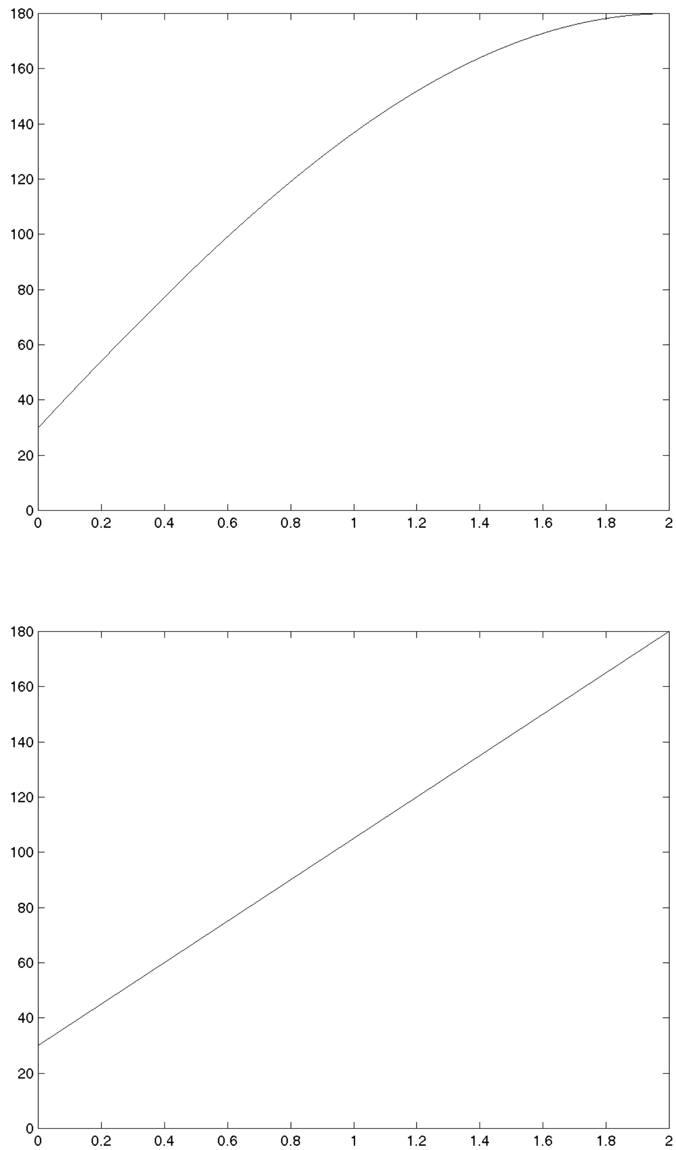
$$b_1(t) = \int_0^2 mm_1(t) = 6 \bullet 42,$$

$$b_2(t) = \int_0^2 mm_2(t) = 5 \bullet 42, \text{ and}$$

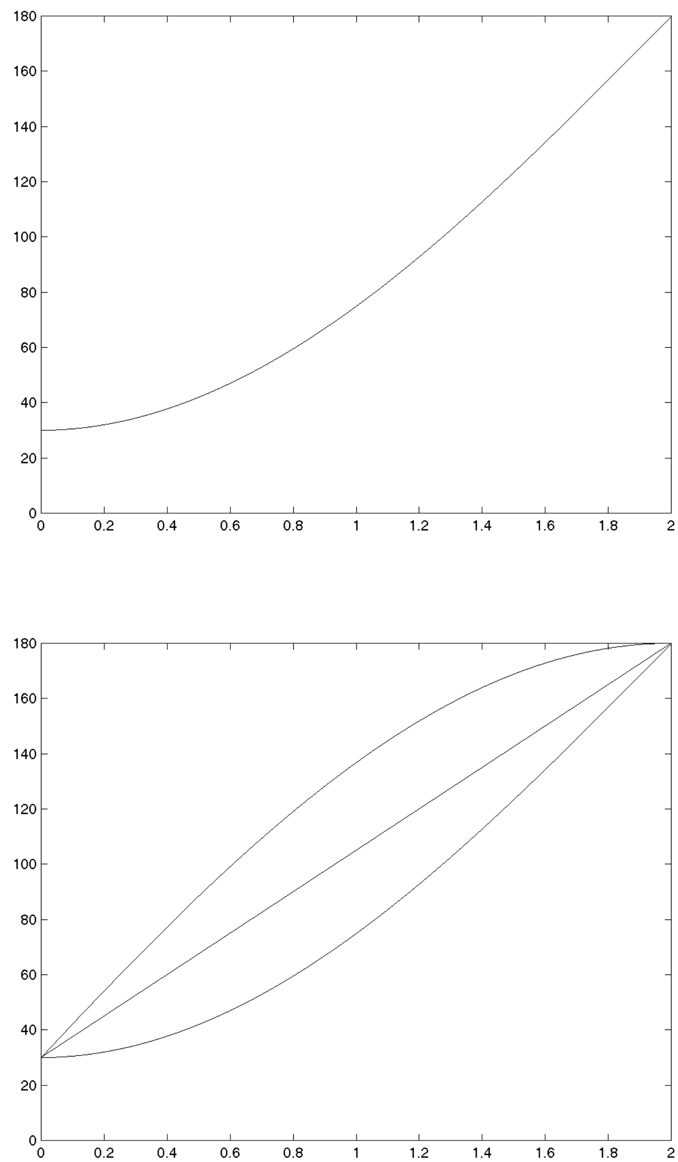
$$b_3(t) = \int_0^2 mm_3(t) = 4 \bullet 42. \quad (12)$$

Each repetition of the theme occurs a perfect fifth higher than last, and is presented in parallel sonorities of increasing density (progressing from a single pitch to a perfect fifth dyad to a triad of stacked perfect fourths, etc.).

As there are no closed solutions for the integral of trigonometric functions with a non-integer exponent, $b_1(t)$ and $b_3(t)$ must be approximated with the aid of a computer.[12] Once the tempo functions were established, it was necessary to find the precise temporal location of each successive sixteenth-note, setting $b(t) = \tau$ for each voice, with τ ranging from 0 to the total number of beats for each voice in increments of $\frac{1}{4}$, providing a non-linear grid on which to begin composing the canon theme. Due to

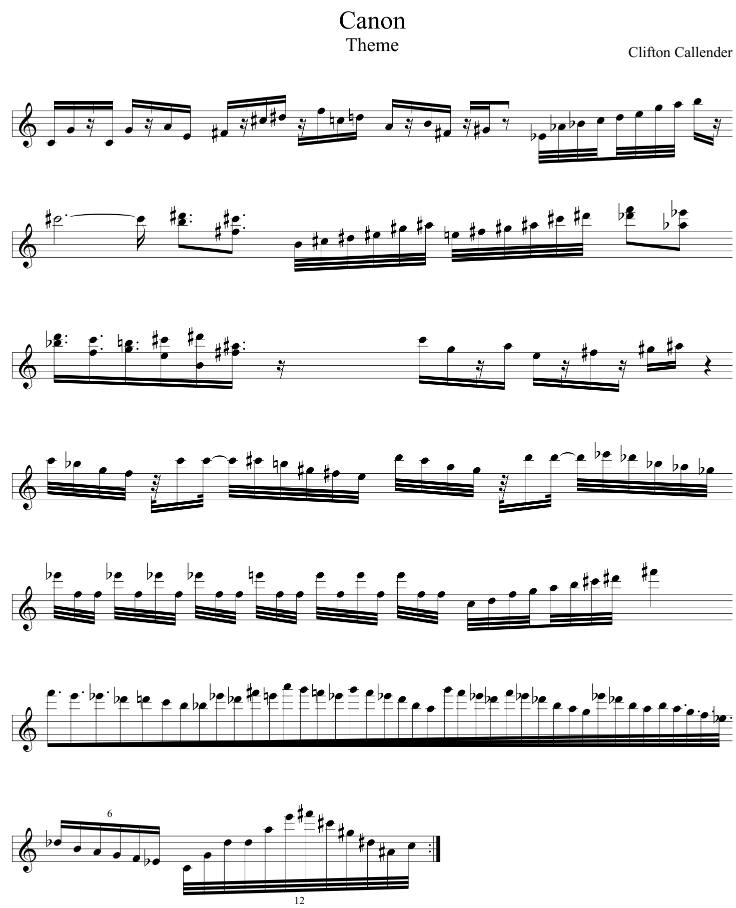


EXAMPLE 4: GRAPHS OF TEMPO FUNCTIONS FOR EACH VOICE IN *CANON*.



EXAMPLE 4 (CONT.)

the extreme ranges of tempi, the theme was constructed to work well melodically and harmonically at slower tempi, and within a kind of textural polyphony as the tempi become increasingly faster. As an example, consider the first twenty thirty-second notes of the fifth system. At slower tempi the minor- and major-seventh descents are heard primarily as



EXAMPLE 5: *CANON* THEME.

melodic gestures and the inherent 3+2+2+3 rhythmic structure is readily perceived. However, at faster tempi the upper and lower notes separate into two distinct auditory streams (Bregman 1990), the melodic and rhythmic details become increasingly less perceptible, and the effect is that of a rhythmic trill. The theme is also designed with numerous salient markers (such as the just-mentioned minor-/major-seventh example, the chords of systems two and three, and the notated accelerando within the large-scale accelerando in system six) to aid the listener in perceiving the location of each voice in its respective repetition of the canon theme. These markers help to bring out the gradual increase and subsequent decrease in the echo distance between voices. The upper and middle voices diverge and then reconverge over the course of the entire piece, as do the middle and lower voices. Since the upper voice states the theme twice more than the lower voice, these voices have an additional convergence in the middle of the piece. This internal convergence occurs at approximately 60" on the fourth sixteenth-note of beat three, so that the area immediately around this convergence features two voices stating the opening phrase of the theme simultaneously, the only time this occurs other than the very beginning of the piece.

RELATIVITY

relativity is the last movement of a work for string quartet, *Luminous Signals*, which is intended for live performance with each player listening to an independent click track via headphones. The tempo design is an elaboration of the design for *Canon* with two distinct accelerandi, one interpolated within the other. The primary accelerando is similar to that used in *Canon*, while the secondary accelerando is inserted into the primary accelerando near the end of the work:

$$\text{vln. I} = mm_o(t) = 118.386454\left(\frac{t}{0.598242}\right)^{2.203901} + 60\,,$$

$$\text{vln. II} = mm_1(t) = 104.795468\left(\frac{t}{0.598242}\right)^{1.836087} + 60\,,$$

$$\text{vla.} = mm_2(t) = 90.079141\left(\frac{t}{0.598242}\right)^{1.437818} + 60\,, \text{ and}$$

$$\text{vc.} = mm_3(t) = 73.901362\left(\frac{t}{0.598242}\right)^{1} + 60 \quad \text{(Example 6).} \qquad (13)$$

In the secondary accelerando each voice is reset to the original tempo of quarter-note equals 60 and accelerates quickly, over the course of 58 beats (and a duration of precisely 0.598242'), to the tempo at which the voice was progressing before being the insertion of a second accelerando. At this point, each voice is governed by the primary accelerando for the remainder of the movement (Example 7).

The secondary accelerando can be generalized to any situation in which a given passage consists of known beginning and ending tempi, the length over which the accelerando takes place, and the desired number of beats to occur within the passage:

$$mm(t) = \lambda\left(\frac{t}{\xi}\right)^{a} + \upsilon\ , \tag{14}$$

where υ is the initial tempo, λ is the difference between ending and beginning tempi, and ξ is the desired length of the section.[13] (In *relativity*, ξ must be the same in each voice to preserve the symmetry of the primary accelerando.) a has the effect of warping the function to obtain the desired number of beats; for $\lambda > 0$, as a increases $b(t)$ decreases, and vice versa; for $\lambda < 0$ as a increases $b(t)$ increases, and vice versa. The appropriate value of a is obtained by evaluating $b(t)$ with $t = \xi$, and setting $b(\xi) = \tau$, the desired number of beats:

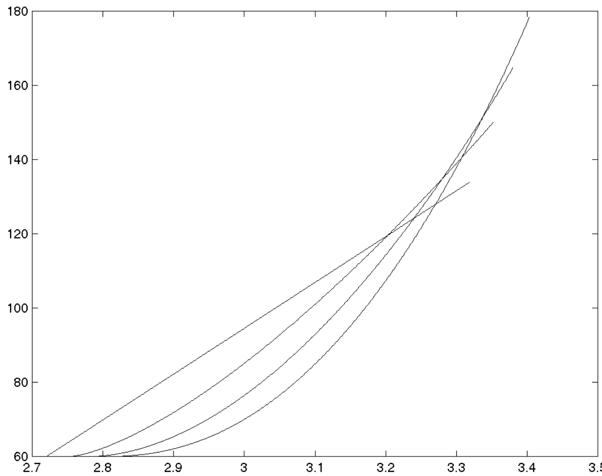
$$b(t) = \upsilon t + \frac{\lambda t^{a+1}}{(a+1)\xi^{a}}\ ,$$

$$b(\xi) = \upsilon\xi + \frac{\lambda\xi}{a+1} = \tau\ ,$$

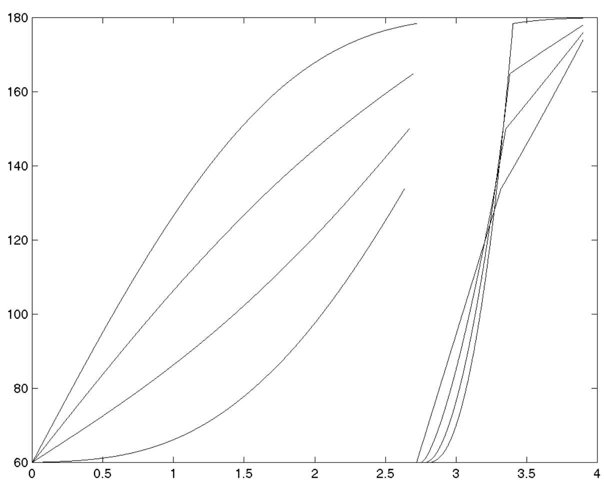
$$a = \frac{\lambda\xi}{\tau - \nu\xi} - 1\ . \tag{15}$$

Exponential Acceleration

As a final example, we consider a tempo function in which acceleration is a constant with respect to a logarithmic scale of tempo. For example, if a voice accelerates from M.M. 30 to M.M. 60 over a given duration, the voice should accelerate from M.M. 60 to M.M. 120 over the same dura-



EXAMPLE 6: GRAPH OF SECONDARY ACCELERANDO IN *RELATIVITY*



EXAMPLE 7: GRAPH OF PRIMARY AND SECONDARY ACCELERANDI IN *RELATIVITY*

tion. If the acceleration, $\omega(t)$, raises the exponential growth in tempo by a factor of 1 every ξ minutes, then

$$\omega(t) = \frac{1}{\xi},$$

and .

$$\int\omega(t) = \frac{t}{\xi}.$$

Converting $\int\omega(t)$, a function of instantaneous tempo in a logarithmic scale, to a linear scale yields

$$mm(t) = \upsilon\beta^{\int\omega(t)} = \upsilon\beta^{\frac{t}{\xi}} \text{, and} \tag{16}$$

$$b(t) = \frac{\xi\upsilon\beta^{\frac{t}{\xi}}}{\ln\beta}, \tag{17}$$

where β is the ratio between tempi separated by the temporal distance ξ, and υ is the initial tempo. If this tempo structure were treated as a canon, a composer may choose to bring in the second voice either after τ beats of the first voice have elapsed, or when the first voice reaches a tempo of κ. For the former case, setting $b(t) = \tau$:

$$t = \frac{\xi\, ln\left(\frac{\tau\, ln\beta}{\xi\upsilon}\right)}{ln\beta}, \tag{18}$$

representing the time at which the second voice should enter to yield an imitative distance of τ beats. For the latter case, setting $mm(t) = \kappa$:

$$t = \xi\frac{\ln\left(\frac{\kappa}{\upsilon}\right)}{\ln\beta}, \tag{19}$$

representing the time at which the first voice reaches a tempo of κ.

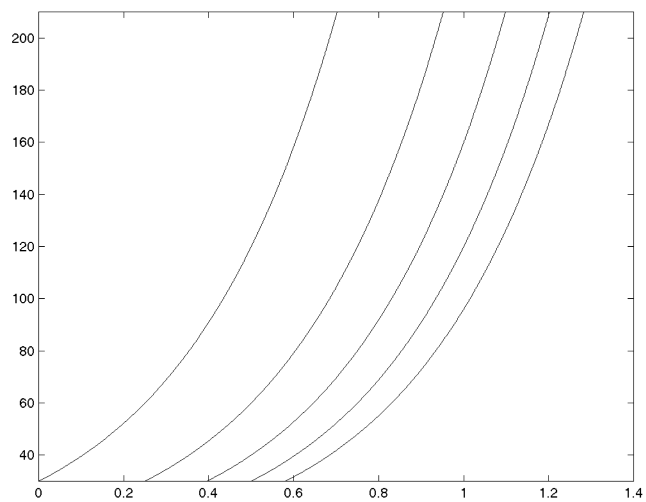
Example 8 shows an example of a canon built on this tempo structure, in which each voice begins at the same tempo, and the ratio between successive voices proceeds 2:1, 3:2, 4:3, 5:4, and so forth. After the 1st entry, every i^{th} entry enters when the $(i - 1)^{th}$ entry reaches the tempo of

$$\kappa = \upsilon\frac{i}{i-1}.$$

The time at which every i^{th} entry, $i \geq 1$, enters is thus,

$$t = \sum_{n=1}^{i} \xi \frac{ln\left(\frac{n}{n-1}\right)}{ln\beta}. \qquad (20)$$

Because acceleration has been defined as a constant with respect to a logarithmic scale, the tempo ratio between any two voices will always remain the same.



EXAMPLE 8: GRAPH OF TEMPO FUNCTIONS FOR EXPONENTIAL ACCELERATION.

Conclusion

The preceding examples barely scratch the surface of the compositional potentials of formalized accelerando. However, this surface is greatly expanded through a more general approach to acceleration, which focuses on variations in tempi over time rather than recursive operations on durations. With the aid of computers, composers can take advantage of the rich relations between beats, tempo, and acceleration. Using computer-aided mathematics programs, one need not be fully trained in the techniques of calculus in order to explore these ideas (although some proficiency is certainly helpful) as long as the underlying concepts are clearly understood. As we have seen, these ideas are not strictly limited to mechanical or electrical instruments; acoustic works with live musicians may also incorporate these tempo schemes by having performers listen to independent click tracks.

As a final thought, it would be very useful to have functions for differentiation and integration incorporated within a computer-assisted composition program, such as Patchwork or Max/MSP. This would have the advantage of applying the techniques of calculus within a flexible environment for manipulating frequency, rhythm, and other musical parameters. Another possibility would be the development of a program which would translate between existing computer-aided mathematics programs, such as Mathematica, and programs such as Patchwork or Max/MSP, providing a substantial increase in the methods for manipulating musical data.

References

MIDI and mp3 files of *Canon* are available at the author's website:
<http://www.niu.edu/music/callender/excerpts/canon.mid>
<http://www.niu.edu/music/callender/excerpts/canon.mp3>

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Notes

1.For simplicity, throughout this paper I will refer to both acceleration and deceleration as an accelerando, with ritardando being implied by a negative accelerando.

2. Large-scale continuous accelerando is a factor in some non-canonic studies, such as No. 23, and some canonic studies—including Nos. 34 and 48—are based on large-scale graduated accelerandi, in which the changes in tempi occur as a series of discrete steps. The final section of String Quartet No. 3 is the only other example of a canon based on continuous accelerando. (Gann, 163)

3. The first movement of his String Quartet No. 3 is a particularly good example of this technique. See Gann's discussion on pp. 279–80. (Note that the convergence point for any pair of voices does not align with the convergence point of any other pairs of voices in this movement. It may be more appropriate to speak of a "convergence area" for this example.)

4. The initial beat of the accelerando is the 0th beat.

5. Indeed, there is a formal limit on the length of this type of acceleration, since (again for positive acceleration) over time the durations become infinitely small. The limit of this length is given by

$$\lim_{1 \to \infty} \sum_{n=0}^{i} dr^n = \frac{d}{1-r}, \text{ for r<1.}$$

6. In this bizarre physics there would be no such thing as a "free-fall;" G-forces would continue to mount throughout one's fall!

7. David Epstein (1995) derives tempo and acceleration curves by taking the first and second order derivatives of a function obtained by plotting beats with respect to time of performances of accelerandi and ritardandi.

8. For a primer on the basic concepts and techniques of calculus I recommend Berlinski 1995.

9. This is certainly no mathematical breakthrough; had Newton somehow been introduced to Nancarrow's music, he would have immediately noted the connection between beats, tempo, and accelerando, and the motion of physical bodies under gravity. However, as musi-

cians tend to focus on duration, rather than time, with regard to rhythm, we often overlook the potentials inherent in these relations.

10. The tempo and acceleration functions for Nancarrow's geometric accelerando are

$$g(t) = \frac{r-1}{(d+t(r-1))\ln r} \text{ and}$$

$$h(t) = -\frac{(r-1)^2}{(d+t(r-1))^2 \ln r} ,$$

respectively.

11. $mm_1(t)$ refers to the upper voice, $mm_2(t)$ to the middle voice, and $mm_3(t)$ to the lower voice.

12. These integrals were approximated using the NIntegrate function in Mathematica.

13. Alternatively, the situation may be modeled by setting λ to be the ratio between beginning and ending tempi, rather than their difference:

$$g(t) = (\lambda - 1)\left(\frac{t}{\xi}\right)^a \upsilon + \upsilon .$$