Problem 6

Solution

Proof. Let $f(n)$ denote a function that for any n gives the sum of 1^2 up to n^2. Assume the function is of the form $f(n) = an^3 + bn^2 + cn + d$ where a, b, c, d are constants. Then,

$$\begin{aligned} d &= 0 \\ a + b + c + d &= 1 \\ 8a + 4b + 2c + d &= 5 \\ 27a + 9b + 3c + d &= 14 \end{aligned}$$

Solving this system of equations, $a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}$, and $d = 0$. Therefore, $f(n) = \frac{n}{6}(2n+1)(n+1)$ after simplification.

Assuming $f(n)$ is correct, it can be shown that $f(n+1)$ is also correct.

$$\begin{aligned} f(n+1) &= f(n) + (n+1)^2 \\ \frac{n^3}{3} + \frac{3n^2}{2} + \frac{13n}{6} + 1 &= \frac{n^3}{3} + \frac{n^2}{2} + n^2 + \frac{n}{6} + 2n + 1 \\ \frac{n^3}{3} + \frac{3n^2}{2} + \frac{13n}{6} + 1 &= \frac{n^3}{3} + \frac{3n^2}{2} + \frac{13n}{6} + 1 \end{aligned}$$

Therefore, by induction, $f(n)$ is the correct function. $\square$