Sample Test Team Round

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1 Instructions

- Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. **No calculators, smartwatches, phones, or computing devices are allowed.** No problems on the exam require the use of a calculator.
- The time allotted is **50 minutes**.
- Please write *simplified* answers for the problems.
- The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.
- SCORING: Each question is worth 5 points unless specified otherwise. There is no penalty for guessing.

For your answer to be *simplified*, you should:

(1) carry out any reasonable calculations. For instance, you should evaluate any expressions which will take negligible time to evaluate (such as $\frac{1}{13}+\frac{1}{17}$), as well as all products less than 10^4. Unreasonable calculations include large powers (e.g. 3^{20}), large factorials, products which are greater than 10^4, and trigonometric functions which cannot be expressed in terms of radicals.

(2) write rational numbers in lowest terms. Decimals are also acceptable, provided they are exact. $\frac{1}{7}$ may be written as $0.\overline{142857}$, but not, for example, as 0.142857. Unless specified otherwise, irrational numbers may never be approximated as rational (e.g. 3.14 and $\frac{22}{7}$ are not acceptable in place of π).

(3) move all square factors outside radicals (for instance, write $3\sqrt{7}$ instead of $\sqrt{63}$). However, denominators need not be rationalized. Both $\frac{2}{\sqrt{5}}$ and $\frac{2\sqrt{5}}{5}$ are acceptable. Radicands are also allowed to contain nested radicals. For example, there will be no penalty for not simplifying $\sqrt{3+2\sqrt{2}}$ as $1+\sqrt{2}$ but there will be for writing $\sqrt{\sqrt{75}}$ instead of $\sqrt{5\sqrt{3}}$.

2 Problems

Problem 1. A nonzero decimal number x is given. The decimal point of x is moved one digit to the left to obtain y and the decimal point of x is moved two digits to the right to obtain z. Suppose $|y - z| = 7.77$, find all possible values of x.

Problem 2. A garden of area A takes A minutes for a robot to irrigate. Suppose there are four circular gardens G_3, G_4, G_5, G_6 with radii $3,4,5,6$, and two robots. Once a robot starts irrigating a garden, it will continue to do so until the entire garden is irrigated. Once the work in one garden is completed, the robot then teleports to the next garden assigned to it if there is one. No two robots can work in the same garden. Find the minimum real number μ for which there exists an assignment of the two robots to gardens, such that all gardens can be irrigated in μ minutes.

Problem 3. Find the number of triples of integers (a,b,c) such that $\operatorname{lcm}(a,b) = 9, \operatorname{lcm}(b,c) = 35, \operatorname{lcm}(c,a) = 147$.

Problem 4. Two circles $\mathscr{C}_1, \mathscr{C}_2$ of radii $6m, 22m$ are externally tangent at a point T. Bug B_1 craws along $\mathscr{C}_1$ and bug B_2 along $\mathscr{C}_2$. Suppose they each crawl at $1m/s$ and both start crawling at T at time $0s$. Find the first time (in seconds) at which the distance between B_1 and B_2 reaches maximum.

Problem 5. A,B,C play a game, the outcome of each round is random and independent of previous rounds, A,B,C win with probabilities $\frac{3}{7}, \frac{2}{7}, \frac{1}{7}$ respectively (and a $\frac{1}{7}$ chance of a three-way tie). The three keep playing the game until someone wins. Find

1. **(+5)** the probability that B wins.
2. **(+10)** the expected number of rounds it takes A to win, given A wins.

Problem 6. **(+10)** Find the sum of all n such that the sum of positive divisors of n is equal to 124.

Problem 7. **(+10)** Find

$$\sum_{\substack{53 \mid m^2+n^2 \\ 0 \le m,n \le 52 \\ m,n \text{ integers}}} (-1)^{m^2+7n^2}$$

Problem 8. ABC, DEF are equilateral triangles with $AB = 4$ such that $D \in BC, E \in CA, F \in AB$ and $DE = \sqrt{7}$.

1. **(+5)** Find d, the distance from A to DE.
2. **(+10)** 6 tangents from D,E,F to the incircle of $\triangle ABC$ are constructed. Find the area of the convex hexagon bounded by the tangents (such that no part of any tangents lie within the hexagon).

Problem 9. Kelvin the Frog starts jumping from the point $(-273,-273)$. In each move it jumps from its current position (x, y) to one of $(x-1, y+4), (x+3, y), (x+1, y-4), (x-3, y)$. However, because temperature cannot be lower than $0K$, Kelvin cannot let any of its coordinates become ≤ -274.

1. **(+10)** Find the smallest number m for which there exists a sequence of m jumps Kelvin can use to make both of his coordinates positive.
2. **(+10)**Let N be the number of points in set $\{(x, y) : -273 \leq x, y < 0\}$ that can be reached by Kelvin. Find the remainder when N is divided by 1000.

Sample Test Team Round Solutions

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August 11, 2019

Problem 1. A nonzero decimal number x is given. The decimal point of x is moved one digit to the left to obtain y and the decimal point of x is moved two digits to the right to obtain z. Suppose $|y-z| = 7.77$, find all possible values of x.

Solution. Since $y = \frac{x}{10}, z = 100x$ it follows that $|x|(\frac{999}{10}) = \frac{777}{100}$, so $x = \boxed{\frac{7}{90}, -\frac{7}{90}}$. □

Problem 2. A garden of area A takes A minutes for a robot to irrigate. Suppose there are four circular gardens G_3, G_4, G_5, G_6 with radii $3, 4, 5, 6$, and two robots. Once a robot starts irrigating a garden, it will continue to do so until the entire garden is irrigated. Once the work in one garden is completed, the robot then teleports to the next garden assigned to it if there is one. No two robots can work in the same garden. Find the minimum real number μ for which there exists an assignment of the two robots to gardens, such that all gardens can be irrigated in μ minutes.

Solution. Since $3^2 + 4^2 + 5^2 = 50 > 3^2 + 6^2 > 4^2 + 5^2$ and a robot must take care of G_6, the optimal assignment is having one robot take care of G_3, G_6 and the other robot taking care of G_4, G_5, giving the answer $(3^2 + 6^2)\pi = \boxed{45\pi}$. □

Problem 3. Find the number of triples of integers (a, b, c) such that $\operatorname{lcm}(a, b) = 9, \operatorname{lcm}(b, c) = 35, \operatorname{lcm}(c, a) = 147$.

Solution. Note that 9 divides one of a, b from $\operatorname{lcm}(a, b) = 9$. However $\operatorname{lcm}(b, c), \operatorname{lcm}(c, a)$ are both not divisible by 9. Thus there are no a, b, c satisfying the above constraints and the answer is $\boxed{0}$. □

Problem 4. Two circles $\mathscr{C}_1, \mathscr{C}_2$ of radii $6m, 22m$ are externally tangent at a point T. Bug B_1 craws along $\mathscr{C}_1$ and bug B_2 along $\mathscr{C}_2$. Suppose they each crawl at $1m/s$ and both start crawling at T at time $0s$. Find the first time (in seconds) at which the distance between B_1 and B_2 reaches maximum.

Solution. $\boxed{66\pi}$ seconds after they most started crawling, the bugs are located on the antipodes of T on $\mathscr{C}_1, \mathscr{C}_2$, which is the maximum possible distance that can be achieved. □

Problem 5. A, B, C play a game, the outcome of each round is random and independent of previous rounds, A, B, C win with probabilities $\frac{3}{7}, \frac{2}{7}, \frac{1}{7}$ respectively (and a $\frac{1}{7}$ chance of a three-way tie). The three keep playing the game until someone wins. Find

1. **(+5)** the probability that B wins.
2. **(+10)** the expected number of rounds it takes A to win, given A wins.

Solution. 1. Let p_B denote the probability B wins. Then $p_B = \frac{2}{7} + \frac{1}{7} p_B$. Solving the equation gives $\boxed{p_B = \dfrac{1}{3}}$.

2. Given A wins, it is equivalent to consider the process with B, C removed such that A finishes the game with probability $\frac{3}{4}$ and waits for another round with probability $\frac{1}{4}$. In this case the expected number E_A of rounds A takes to win satisfies $E_A = \frac{3}{4} + \frac{1}{4}(1 + E_A)$. Solving gives $\boxed{E_A = \dfrac{4}{3}}$. □

Problem 6. **(+10)** Find the sum of all n such that the sum of positive divisors of n is equal to 124.

Solution. Let $n = \prod_{i=1}^{k} p_i^{a_i}$ where $p_1, \ldots, p_k$ are primes and a_i are positive integers for all i. Then the sum of positive divisors is

$$124 = \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1}$$

and each of the k terms $t_i = \frac{p_i^{a_i+1}-1}{p_i-1} \geq p_i + 1 \geq 3$. Since $124 = 31 \cdot 2^2$, $k \leq 2$. Now we split into two cases.

Case 1: $k = 1$. We have $\sum_{j=0}^{a_1} p_1^j = 124$. There are a few subcases:

- $a_1 = 1, p_1 = 123$ is not prime, false.
- $a_1 \geq 2$. Since $p_1 | 123$, $p_1 > 1$, $p_1 < 123/p_1$ and $123 = 3 \cdot 41$, $p_1 = 3$. Thus $3^{a_1+1} - 1 = 2(124) = 248$ which is impossible.

Thus there are no solutions when $k = 1$.

Case 2: $k = 2$. Up to permutation of the p_i, since $124 = 2^2 \cdot 31$, $(t_1, t_2) = (4, 31)$ is the only possible way to partition the factors of 124 such that each group has product ≥ 3. This means that $p_1 = 3, a_1 = 1$. Moreover, $31 = 2^5 - 1$, so $p_2 = 2, a_2 = 4$ is a solution. Suppose $p_2 \neq 2$, then $p_2 \geq 5$ is odd and for parity reasons, $2 | a_2$. Now note that $t_2 \geq 1 + p_2 + p_2^2 = 1 + 5 + 5^2 = 31$ with equality if and only if $p_2 = 5$. Therefore $p_2 = 5, a_2 = 2$ is also a solution.

In summary, $n = 3 \cdot 2^4, 3 \cdot 5^2 = 48, 75$ in Case 2, and the answer is $48 + 75 = \boxed{123}$. □

Problem 7. (+10) Find

$$\sum_{\substack{53|m^2+n^2\\0\le m,n\le 52\\m,n\text{ integers}}} (-1)^{m^2+7n^2}$$

Solution. Let $p=53$. The sum is the same as

$$1+\sum_{p|m^2+n^2,1\le m,n\le p-1} (-1)^{m^2+n^2}.$$

Since $m^2+(p-n)^2 = m^2+n^2+p(p-2n)$, it follows that $\frac{m^2+n^2}{p} = \frac{m^2+(p-n)^2}{p} - p + 2n \not\equiv \frac{m^2+n^2}{p}$ (mod 2) for $p|m^2+n^2$. Moreover, $(m,p-n)\in\mathbb{N}_{p-1}^2 \iff (m,n)\in\mathbb{N}_{p-1}^2$, here $\mathbb{N}_{p-1}$ denotes the set of all positive integers $\le p-1$. Therefore

$$\begin{aligned}\sum_{53|m^2+n^2,1\le m,n\le 52} (-1)^{m^2+n^2} &= \sum_{53|m^2+n^2,1\le m,n\le 52} \frac{(-1)^{m^2+n^2}+(-1)^{m^2+(p-n)^2}}{2}\\ &= \sum_{53|m^2+n^2,1\le m,n\le 52} 0 = 0\end{aligned}$$

and the answer is $\boxed{1}$. □

Problem 8. ABC, DEF are equilateral triangles with $AB=4$ such that $D\in BC, E\in CA, F\in AB$ and $DE=\sqrt{7}$.

1. **(+5)** Find d, the distance from A to DE.
2. **(+10)** 6 tangents from D,E,F to the incircle of $\triangle ABC$ are constructed. Find the area of the convex hexagon bounded by the tangents (such that no part of any tangents lie within the hexagon).

Solution. 1. Set $x=AF=BD=CE$, then $4-x=AE=CD=BF$. By Law of Cosines

$$x^2+(4-x)^2-2x(4-x)\cos 60^\circ = EF^2 = 7,$$

solving for x gives $x=1,3$. WLOG $x=3$. Since $d\cdot EF = 2[AEF] = x(4-x)\cdot\sin 60^\circ = \frac{x(4-x)\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$, it follows that $d=\boxed{\dfrac{3\sqrt{3}}{2\sqrt{7}}}$.

2. Let ℓ be the tangent from D to the incircle other than BC, and let ℓ intersect AC at D_1, and define E_1, F_1 similarly. Then the hexagon has area

$$[DEF]+3[DD_1E] = \frac{7\sqrt{3}}{4}+3[DD_1E].$$

Let $y=D_1C$, M_A, M_B the midpoints of BC, CA respectively. Let T be the point of contact of the incircle and DD_1. Then $DD_1 = DT+TD_1 = DM_A + D_1M_B = 1+(2-y) = 3-y$. Now Law of

Cosines on $\triangle D_1CD, \angle D_1CD$ gives $y^2+1-2y\cos 60^\circ = (3-y)^2$, solving gives $y=\frac{8}{5}$, thus D_1E and D_1D (which are both equal to $3-y$) are equal to $\frac{7}{5}$. Moreover, Law of Cosines in $\triangle DD_1C$ gives $\cos DD_1C = \frac{11}{14}$, thus $\sin DD_1E = \frac{5\sqrt{3}}{14}$. This gives

$$[DD_1E] = D_1E \cdot D_1D \cdot \frac{5\sqrt{3}}{14} \cdot \frac{1}{2} = \frac{7\sqrt{3}}{20}.$$

So the answer is

$$\frac{21\sqrt{3}}{20} + \frac{7\sqrt{3}}{4} = \boxed{\frac{14\sqrt{3}}{5}}.$$

$\square$

Problem 9. Kelvin the Frog starts jumping from the point $(-273,-273)$. In each move it jumps from its current position (x,y) to one of $(x-1,y+4),(x+3,y),(x+1,y-4),(x-3,y)$. However, because temperature cannot be lower than $0K$, Kelvin cannot let any of its coordinates become ≤ -274.

1. **(+10)** Find the smallest number m for which there exists a sequence of m jumps Kelvin can use to make both of his coordinates positive.

2. **(+10)**Let N be the number of points in set $\{(x,y): -273 \leq x, y < 0\}$ that can be reached by Kelvin. Find the remainder when N is divided by 1000.

Solution. 1. It is easy to see that any position (x,y) Kelvin can reach (with both coordinates ≥ -273) can be reached without going to any position having a coordinate ≤ -274.

Suppose we use A moves of $(-1,4)$ and B moves of $(3,0)$ (and A,B need not be positive). Kelvin ends at position $(-273,-273)+A(-1,4)+B(3,0) = (-273-A+3B,-273+4A)$. Thus we require $A \geq \frac{273}{4}$ and $B \geq \frac{273+A}{3}$. Therefore

$$A+B \geq \left\lceil \frac{273+\left\lceil\frac{273}{4}\right\rceil}{3} \right\rceil = \boxed{183}.$$

Equality is reached when $A=69, B=114$.

2. The set of points that can be reached by Kelvin without coordinate restrictions is $(-273-A+3B,-273+4A)$ $(A,B \in \mathbb{Z}) \equiv (-A+3B, 4A-1)$ $(A,B \in \mathbb{Z})$. For $y=4A-1 (0 \geq A \geq -68)$ the number of x such that (x,y) can be reached by Kelvin (with coordinate restrictions) is then the number of B with $3B-A \in [-273,-1]$. Thus the answer is $(91+91+90)\cdot\frac{69}{3} = \boxed{6356}$. $\square$