Sample Test Individual Round

Grant Yu

August 11, 2019

1 Instructions

- Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. **No calculators, smartwatches, phones, or computing devices are allowed.** No problems on the exam require the use of a calculator.
- The time allotted is **75 minutes**.
- The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.
- SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.

2 Problems

Problem 1. In a drawer of 72 socks, the number of red, yellow, green, blue, purple socks are $22, 21, 20, 7, 2$ respectively. How many socks must one take out to guarantee finding two of the same color?

(A) 3 **(B)** 6 **(C)** 7 **(D)** 21 **(E)** 67

Problem 2. Egbert and Beatrice are running around a circular track clockwise, starting at a point P. Suppose it takes Egbert 12 minutes and Beatrice 9 minutes to run a lap, how many laps will Beatrice have run when they first meet at P again?

(A) 4 **(B)** 6 **(C)** 24 **(D)** 36 **(E)** 108

Problem 3. Winston is given a four-digit integer n and he adds a decimal point to the decimal representation of n to obtain the number m. Suppose $m + n = 2182.4$, what is n?

(A) 1884 **(B)** 1904 **(C)** 1914 **(D)** 1944 **(E)** 1984

Problem 4. N students are being divided into k classrooms. The number of people in classroom i is c_i $(1 \le i \le k)$ with $c_1 \le c_2 \le ... \le c_k$. If the number of classrooms increases by 1 then it can happen that every classroom contains exactly 6 people. If the number of classrooms decreases by 1 then a division exists with every classroom containing exactly 9 people. Suppose we divide N students into the k classrooms such that $c_k - c_1$ is minimized, what is c_k?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Problem 5. How many prime numbers use all of the decimal digits and each exactly once?

(A) 0 **(B)** 2 **(C)** $10! - 9! - 2 \cdot 8! + 7!$ **(D)** $10! - 9! + 8!$ **(E)** $10! - 9! + 2 \cdot 8!$

Problem 6. A driver drives from station A to station B at a rate of 50 mph. On the way back his rate is 30 mph. Suppose the total time he spent on the road is 6 hours, How many minutes does the driver spend on the way from A to B?

(A) 100 **(B)** 120 **(C)** 135 **(D)** 144 **(E)** 150

Problem 7. $P_1P_2...P_7$ is a regular heptagon. Find the *radian* measure of the acute angle formed between P_1P_4 and P_3P_7.

$$\textbf{(A)}\ \frac{\pi}{14} \qquad \textbf{(B)}\ \frac{\pi}{7} \qquad \textbf{(C)}\ \frac{1\pi}{3} \qquad \textbf{(D)}\ \frac{2\pi}{7} \qquad \textbf{(E)}\ \frac{3\pi}{7}$$

Problem 8. Let $S = \{\gamma, \phi, e, \pi\}$ where $\gamma \approx 0.577$ is the Euler-Mascheroni Constant, $\phi \approx 1.618$ the golden ratio, $e \approx 2.718, \pi \approx 3.142$ are defined as usual. Three *distinct* numbers a, b, c are taken out of S to make a trapezoid of height a parallel sides of lengths b, c. Find the maximum value of the area of a trapezoid that can be made this way.

$$\textbf{(A)}\ \gamma\pi \qquad \textbf{(B)}\ \frac{\phi(\gamma+e)}{2} \qquad \textbf{(C)}\ \frac{\pi(\phi+e)}{2} \qquad \textbf{(D)}\ \frac{\gamma(5\pi+3e)}{2} \qquad \textbf{(E)}\ \frac{\pi^2+e^2}{2}$$

Problem 9. Suppose we have an infinite supply of \$5, \$7, \$13 bills. The smallest number of bills it takes to pay a fee of \$188 is n, and the number of ways to pay a fee of \$188 with the smallest number of bills is m. Find $m + n$.

$$\textbf{(A)}\ 17 \qquad \textbf{(B)}\ 18 \qquad \textbf{(C)}\ 20 \qquad \textbf{(D)}\ 21 \qquad \textbf{(E)}\ 22$$

Problem 10. Find the remainder when

$$\sin\pi + 2\cos 2\pi + 3\sin 3\pi + 4\cos 4\pi + \ldots + 2019\sin 2019\pi$$

is divided by 1000.

$$\textbf{(A)}\ 90 \qquad \textbf{(B)}\ 108 \qquad \textbf{(C)}\ 180 \qquad \textbf{(D)}\ 590 \qquad \textbf{(E)}\ 608$$

Problem 11. A sequence obeys the relation $s_n = s_{n-1} + s_{n-2} + s_{n-3}$ and $s_1 = 1, s_2 = 10, s_3 = 100$. Find the remainder when s_{2019} is divided by 8.

$$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 7$$

Problem 12. A *non-rectangular* parallelogram $ABCD$ has a perimeter of p. Suppose the distance from A to BC is 14, the distance form A to CD is 18 and all sidelengths of $ABCD$ are integers. Find the smallest possible value of p for which such a parallelogram exists.

$$\textbf{(A)}\ 36 \qquad \textbf{(B)}\ 54 \qquad \textbf{(C)}\ 64 \qquad \textbf{(D)}\ 88 \qquad \textbf{(E)}\ 96$$

Problem 13. Find the area of the quadrilateral formed by the negative coordinate axes and the lines $y = x + 4, 3y = 2x - 10$.

$$\textbf{(A)}\ \frac{218}{3} \qquad \textbf{(B)}\ 80 \qquad \textbf{(C)}\ \frac{261}{3} \qquad \textbf{(D)}\ 90 \qquad \textbf{(E)}\ 108$$

Problem 14. A square R of sidelength 2 is constructed within a square S of sidelength 3, such that R, S have parallel sides and their centers coincide at O. Now we rotate R by 45 degrees around O to form another square T. The closest distance from a vertex of S to a vertex of T can be written as $\frac{\sqrt{m-n\sqrt{k}}}{2}$ where k is not divisible by the square of any prime and m, n, k are positive integers. Find $m+n+k$.

(A) 24 **(B)** 30 **(C)** 36 **(D)** 40 **(E)** 48

Problem 15. Points A_1, B_1 are chosen on the sides AB, BC of a rectangle $ABCD$ (endpoints included), with $\gamma = \frac{AB}{BC}$. Find the maximum possible value of $\frac{[A_1B_1D]}{BC^2}$.

(A) $\dfrac{\gamma}{\sqrt{7}}$ **(B)** $\dfrac{\gamma}{2}$ **(C)** $\dfrac{\gamma^2}{2}$ **(D)** $\dfrac{1}{\sqrt{7}}$ **(E)** $\dfrac{1}{2}$

Problem 16. Let D be the reflection of the orthocenter H of $\triangle ABC$ over the midpoint of BC. If $AB = 3, BC = 6, CA = 7$, the length of AD can be written as $\frac{m\sqrt{n}}{k}$ where m, k are relatively prime positive integers and n is not divisible by the square of any prime. Find $m+n+k$.

(A) 44 **(B)** 60 **(C)** 74 **(D)** 88 **(E)** 118

Problem 17. A triangle ABC is given. Points D, E are chosen on segments AC, BC. Suppose $AD = \frac{1}{6}DC$ and the area of $\triangle CDE$ is half that of $\triangle ABC$. Find $BE : EC$.

(A) $\dfrac{2}{3}$ **(B)** $\dfrac{5}{12}$ **(C)** $\dfrac{5}{7}$ **(D)** $\dfrac{6}{7}$ **(E)** $\dfrac{11}{12}$

Problem 18. Find the maximum value of the real-valued function

$$f(x) = (201 + 28\sqrt{x-1} - 9\sqrt{13-4x})^{\log_{45-4x}(8)}.$$

(A) $174^{\log_{41} 8}$ **(B)** 20 **(C)** 27 **(D)** $+\infty$ **(E)** Does not exist

Problem 19. What is the maximum value of $\sin x + 4\cos x$ ($x \in \mathbb{R}$)?

(A) $\dfrac{5}{\sqrt{2}}$ **(B)** $\dfrac{1+4\sqrt{3}}{2}$ **(C)** 4 **(D)** $\sqrt{17}$ **(E)** 5

Problem 20. In $\triangle ABC$, D, E, F are the midpoints of AB, BC, CA, respectively. Suppose $AB = 10, CD = 9$ and $CD \perp AE$. Find BF.

(A) $\sqrt{57}$ **(B)** $2\sqrt{15}$ **(C)** $2\sqrt{29}$ **(D)** $3\sqrt{13}$ **(E)** $5\sqrt{5}$

Problem 21. How many integers n are there such that $|4x-n|+|x-4n| \geq \frac{n^4}{4^4}$ holds identically over all $x \in \mathbb{R}$?

(A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

Problem 22. 10 people are sitting around a circular table. Find the number of ways to choose some people, such that there are two chosen people sitting next to each other.

(A) 337 **(B)** 511 **(C)** 684 **(D)** 728 **(E)** 901

Problem 23. $ABCD$ is a tetrahedron with $AB = CD = AC = BD = 3, AD = 4, BC = 2$. The radius of the sphere circumscribing $ABCD$ can be written as $\frac{\sqrt{p}}{q}$ where no square of any prime divisor of q also divides p. Find $p+q$.

(A) 48 **(B)** 69 **(C)** 90 **(D)** 111 **(E)** 132

Problem 24. Given regular 11-gon $P := P_1...P_{11}$, we assign a number to each vertex of P randomly with each number in $1,...,11$ used exactly once. Find the expected number of arithmetic progressions of length 3 appearing on P.

For example, $P_1...P_{11} \to 4,7,10,2,3,11,8,5,6,9,1$ has only three such progressions, namely $(1,4,7)$, $(4,7,10)$ and $(11,8,5)$.

(A) $\frac{1}{3}$ **(B)** $\frac{4}{9}$ **(C)** $\frac{7}{11}$ **(D)** $\frac{5}{9}$ **(E)** $\frac{13}{22}$

Problem 25. How many base-nine numbers using each of the digits $5,6,7,8$ exactly once are divisible by 11?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 12

Sample Test Individual Round Solutions

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Problem 1. In a drawer of 72 socks, the number of red, yellow, green, blue, purple socks are $22, 21, 20, 7, 2$ respectively. How many socks must one take out to guarantee finding two of the same color?

(A) 3 **(B)** 6 **(C)** 7 **(D)** 21 **(E)** 67

Solution. The answer is $\boxed{B}$. 5 socks is not enough since we can take one of each color. Since there are only 5 colors, when we take 6 socks there must be one color with ≥ 2 socks. □

Problem 2. Egbert and Beatrice are running around a circular track clockwise, starting at a point P. Suppose it takes Egbert 12 minutes and Beatrice 9 minutes to run a lap, how many laps will Beatrice have run when they first meet at P again?

(A) 4 **(B)** 6 **(C)** 24 **(D)** 36 **(E)** 108

Solution. $\frac{\text{lcm}(12,9)}{9} = 4 \implies \boxed{A}$. □

Problem 3. Winston is given a four-digit integer n and he adds a decimal point to the decimal representation of n to obtain the number m. Suppose $m + n = 2182.4$, what is n?

(A) 1884 **(B)** 1904 **(C)** 1914 **(D)** 1944 **(E)** 1984

Solution. The decimal point must have been added between the original tenth and units digit of n, so $m + n = 1.1n$. On the other hand, $m + n = 2182.4$, thus $n = \frac{2182.4}{1.1} = 1984 \implies \boxed{E}$. □

Problem 4. N students are being divided into k classrooms. The number of people in classroom i is c_i $(1 \leq i \leq k)$ with $c_1 \leq c_2 \leq \ldots \leq c_k$. If the number of classrooms increases by 1 then it can happen that every classroom contains exactly 6 people. If the number of classrooms decreases by 1 then a division exists with every classroom containing exactly 9 people. Suppose we divide N students into the k classrooms such that $c_k - c_1$ is minimized, what is c_k?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Solution. Let k be the number of classrooms. From the givens,

$$(k+1)\cdot 6 = N = (k-1)\cdot 9$$

so that $k=5$, hence $N=36$. Intuitively, the smallest max − min difference is reached when students are evenly distributed, which occurs when the distribution is $7,7,7,7,8$. Thus the largest number of people contained is $8 \implies \boxed{E}$.

Comment: Here we present a proof of the intuitive fact. We have already shown that $k=5$. Note that $c_5 - c_1 > 0$, else $5c_1 = 36$, contradiction. Thus $c_5 - c_1 \geq 1$. When $c_5 = 8, c_1 = ... = c_4 = 7$ the equality holds. Now we show that this is the only equality case. Let a be the number of $c_i = m$, b the number of $c_i = m+1$, then $a+b=5$ and $am + b(m+1) = 36$, hence $5m+b=36$ which means $5|36-b$, so $5|1-b$. However $0 \leq b \leq 5$, so $b=1$. This forces $m=7$ which gives the aforementioned equality case. □

Problem 5. How many prime numbers use all of the decimal digits and each exactly once?

(A) 0 **(B)** 2 **(C)** $10!-9!-2\cdot 8!+7!$ **(D)** $10!-9!+8!$ **(E)** $10!-9!+2\cdot 8!$

Solution. There are 0($\implies \boxed{A}$). The sum of digits is divisible by 3 which makes all such numbers divisible by 3. □

Problem 6. A driver drives from station A to station B at a rate of 50 mph. On the way back his rate is 30 mph. Suppose the total time he spent on the road is 6 hours, How many minutes does the driver spend on the way from A to B?

(A) 100 **(B)** 120 **(C)** 135 **(D)** 144 **(E)** 150

Solution. Suppose the distance between A, B is d. Since it took $\frac{d}{50}$ hours to get from A to B and $\frac{d}{30}$ on the way back, the total time spent is $\frac{d}{50}+\frac{d}{30} = 6$ hours. The answer is then $\frac{d}{50} = \frac{\frac{d}{50}}{\frac{d}{50}+\frac{d}{30}}\cdot 6\cdot 60 = 135. \implies \boxed{C}$ □

Problem 7. $P_1P_2...P_7$ is a regular heptagon. Find the *radian* measure of the acute angle formed between P_1P_4 and P_3P_7.

(A) $\frac{\pi}{14}$ **(B)** $\frac{\pi}{7}$ **(C)** $\frac{1\pi}{3}$ **(D)** $\frac{2\pi}{7}$ **(E)** $\frac{3\pi}{7}$

Solution. Let ω be the circumcircle of the regular heptagon. Take arc measures with respect to ω we get $\angle(P_1P_4, P_3P_7) = \frac{\widehat{P_7P_1}+\widehat{P_3P_4}}{2} = \frac{\frac{2\pi}{7}+\frac{2\pi}{7}}{2} = \frac{2\pi}{7}$. $\Longrightarrow$ $\boxed{D}$. □

Problem 8. Let $S = \{\gamma, \phi, e, \pi\}$ where $\gamma \approx 0.577$ is the Euler-Mascheroni Constant, $\phi \approx 1.618$ the golden ratio, $e \approx 2.718, \pi \approx 3.142$ are defined as usual. Three *distinct* numbers a, b, c are taken out of S to make a trapezoid of height a parallel sides of lengths b, c. Find the maximum value of the area of a trapezoid that can be made this way.

(A) $\gamma\pi$ **(B)** $\frac{\phi(\gamma+e)}{2}$ **(C)** $\frac{\pi(\phi+e)}{2}$ **(D)** $\frac{\gamma(5\pi+3e)}{2}$ **(E)** $\frac{\pi^2+e^2}{2}$

Solution. Note that $\gamma < \phi < e < \pi$ and the area of the trapezoid is $a \cdot \frac{b+c}{2}$, which is maximized only if $ab + ac$ is maximized. Note that if any of $a, b, c = \min S$ we can always replace it with an unused number in S that is larger to increase $ab + ac$. Thus $\{a, b, c\} = \{\phi, e, \pi\}$. Among all such choices of a, b, c, since $ab + bc + ca$ is constant, it follows that $ab + ac$ is maximized iff bc is minimized, which occurs when b, c are the smallest two among ϕ, e, π, giving $a = \pi$. Therefore the maximum value of the area of a trapezoid that can be made this way is $\frac{\pi(\phi+e)}{2}$ $\boxed{\Longrightarrow C}$. □

Problem 9. Suppose we have an infinite supply of \$5, \$7, \$13 bills. The smallest number of bills it takes to pay a fee of \$188 is n, and the number of ways to pay a fee of \$188 with the smallest number of bills is m. Find $m + n$.

(A) 17 **(B)** 18 **(C)** 20 **(D)** 21 **(E)** 22

Solution. Suppose we use k \$13 bills. It can be easily verified that $k \neq 14$. When $k = 13$, there is only one combination $13 \cdot 13 + 2 \cdot 7 + 5 = 188$. This uses 16 bills. To see that we cannot form 188 with ≤ 16 bills in any other way, note that when $k \leq 12$, the number of bills worth 7 or less is $\geq \left\lceil \frac{188-13k}{7} \right\rceil$, so the total number of bills used is $\geq \frac{188-13k}{7} + k = \frac{188-6k}{7} \geq \frac{188-6\cdot12}{7} > 16$, so ≥ 17 bills are needed.

Therefore $m = 1, n = 16$ and the answer is 17 $\Longrightarrow$ $\boxed{A}$.

□

Problem 10. Find the remainder when

$$\sin\pi + 2\cos 2\pi + 3\sin 3\pi + 4\cos 4\pi + \ldots + 2019\sin 2019\pi$$

is divided by 1000.

(A) 90 **(B)** 108 **(C)** 180 **(D)** 590 **(E)** 608

Proposed by Ethan Yu

Solution. Note that $\sin(2k-1)\pi = 0, \cos 2k\pi = 1$ for all k integers, hence the sum is just $2+4+6+\ldots+2018 \equiv \boxed{90} \pmod{1000}$, hence $\boxed{A}$. $\square$

Problem 11. A sequence obeys the relation $s_n = s_{n-1} + s_{n-2} + s_{n-3}$ and $s_1 = 1, s_2 = 10, s_3 = 100$. Find the remainder when s_{2019} is divided by 8.

(A) 0 **(B)** 2 **(C)** 4 **(D)** 5 **(E)** 7

Let t_n be the remainder when s_n is divided by 8. Then

$$(t_n)_{n\geq 1} = (1,2,4,7,5,0,4,1,5,2,0,7,1,0,0,1,||\,1,2,4,7,5,\ldots)$$

is periodic with period 16, thus $t_{2019} = t_3 = 4 \implies \boxed{D}$.

Problem 12. A *non-rectangular* parallelogram $ABCD$ has a perimeter of p. Suppose the distance from A to BC is 14, the distance form A to CD is 18 and all sidelengths of $ABCD$ are integers. Find the smallest possible value of p for which such a parallelogram exists.

(A) 36 **(B)** 54 **(C)** 64 **(D)** 88 **(E)** 96

Solution. Note that $14BC = [ABCD] = 18CD$ and $p = 2(BC+CD)$. The former guarantees the existence of a positive integer t such that $BC = 9t, CD = 7t$. Moreover $CD > 14, BC > 18$(for example, by dropping altitude from C to AD at H and consider right triangle HCD) so $\min t = 3$. Therefore $\min p = 96 \implies \boxed{E}$. $\square$

Problem 13. Find the area of the quadrilateral formed by the negative coordinate axes and the lines $y = x+4, 3y = 2x-10$.

(A) $\frac{218}{3}$ **(B)** 80 **(C)** $\frac{261}{3}$ **(D)** 90 **(E)** 108

Solution. The lines $\ell_1 : y = x+4$, $\ell_2 : 3y = 2x-10$ intersect at $A := (-22,-18)$. ℓ_1 has x-intercept at $B := (-4,0)$, ℓ_2 has y-intercept at $C := (0, \frac{-10}{3})$. Let O be the origin. Then the quadrilateral in consideration is $ABOC$. Then

$$[ABOC] = [ABO] + [AOC] = \frac{1}{2}\left(4\cdot 18 + \frac{10}{3}\cdot 22\right) = \frac{218}{3} \implies \boxed{A}$$

$\square$

Problem 14. A square R of sidelength 2 is constructed within a square S of sidelength 3, such that R, S have parallel sides and their centers coincide at O. Now we rotate R by 45 degrees around O to form another square T. The closest distance from a vertex of S to a vertex of T can be written as $\frac{\sqrt{m-n\sqrt{k}}}{2}$ where k is not divisible by the square of any prime and m, n, k are positive integers. Find $m+n+k$.

(A) 24 (B) 30 (C) 36 (D) 40 (E) 48

Let $S = ABCD$ and $T = EFGH$, with E closest to line AB among the vertices of T. let M be the midpoint of AB. Then the closest distance is $EA = \sqrt{EM^2 + MA^2} = \frac{\sqrt{26-12\sqrt{2}}}{2}$, thus the answer is $40 \implies \boxed{D}$.

Problem 15. Points A_1, B_1 are chosen on the sides AB, BC of a rectangle $ABCD$ (endpoints included), with $\gamma = \frac{AB}{BC}$. Find the maximum possible value of $\frac{[A_1B_1D]}{BC^2}$.

(A) $\dfrac{\gamma}{\sqrt{7}}$ (B) $\dfrac{\gamma}{2}$ (C) $\dfrac{\gamma^2}{2}$ (D) $\dfrac{1}{\sqrt{7}}$ (E) $\dfrac{1}{2}$

Solution. Let P be the point such that PA_1BB_1 is a rectangle. Then $AP \le AD \implies [APD] \le [ADA_1] \implies [A_1DP] \le \frac{[ADPA_1]}{2}$, equality iff $P \in CD$. Likewise $[B_1DP] \le \frac{[CDPB_1]}{2}$. Combining with $[A_1PB_1] = \frac{[A_1PB_1B]}{2}$ it follows that $[A_1DB_1] \le \frac{[ABCD]}{2}$, equality if and only if $A_1 = A, B_1 = B$. Since $[ABCD] = BC^2\gamma$ the answer is $\frac{\gamma}{2} \implies \boxed{B}$.

□

Problem 16. Let D be the reflection of the orthocenter H of $\triangle ABC$ over the midpoint of BC. If $AB = 3, BC = 6, CA = 7$, the length of AD can be written as $\frac{m\sqrt{n}}{k}$ where m, k are relatively prime positive integers and n is not divisible by the square of any prime. Find $m + n + k$.

(A) 44 (B) 60 (C) 74 (D) 88 (E) 118

Solution. Since $\angle BHC = 180° - A$, we deduce that D lies on the circumcircle of $\triangle ABC$. Thus, $\angle BAO = 90° - C = \angle CBH = \angle BCD = \angle BAD$, so $D \in AO$. Thus $AD = 2AO = \frac{AB \cdot BC \cdot CA}{2[ABC]}$. By Heron's Formula $[ABC] = \sqrt{80}$. Thus the answer is $AD = \frac{63\sqrt{5}}{20}$ giving the answer $63 + 5 + 20 = 88 \implies \boxed{D}$. □

Problem 17. A triangle ABC is given. Points D, E are chosen on segments AC, BC. Suppose $AD = \frac{1}{6}DC$ and the area of $\triangle CDE$ is half that of $\triangle ABC$. Find $BE : EC$.

(A) $\dfrac{2}{3}$ (B) $\dfrac{5}{12}$ (C) $\dfrac{5}{7}$ (D) $\dfrac{6}{7}$ (E) $\dfrac{11}{12}$

Solution.

$$2 = \frac{[ABC]}{[DCE]} = \frac{AC \cdot BC \sin C}{DC \cdot EC \sin C} = \frac{AC \cdot BC}{DC \cdot EC} = \frac{7}{6} \cdot \frac{BC}{EC}$$
$$\implies \frac{BE}{EC} = \frac{BC}{EC} - 1 = \frac{5}{7} \implies \boxed{C}.$$

□

Problem 18. Find the maximum value of the real-valued function

$$f(x) = (201 + 28\sqrt{x-1} - 9\sqrt{13-4x})^{\log_{45-4x}(8)}.$$

(A) $174^{\log_{41} 8}$ **(B)** 20 **(C)** 27 **(D)** $+\infty$ **(E)** Does not exist

Solution. The domain of f is $[1, \frac{13}{4}]$.

Note that the function is strictly increasing because $28\sqrt{x-1}, -9\sqrt{13-4x}, \log_{45-4x}(8)$ are strictly increasing, and $201+28\sqrt{x-1}-9\sqrt{13-4x} \geq 201+28\sqrt{1-1}-9\sqrt{13-4\cdot 1} > 1$. Thus $f(x) \leq f(\frac{13}{4}) = \boxed{27}$. □

Problem 19. What is the maximum value of $\sin x + 4\cos x$ $(x \in \mathbb{R})$?

(A) $\dfrac{5}{\sqrt{2}}$ **(B)** $\dfrac{1+4\sqrt{3}}{2}$ **(C)** 4 **(D)** $\sqrt{17}$ **(E)** 5

Solution. The answer is $\boxed{D}$.

Method 1:

$$\sin x + 4\cos x \overset{\text{Cauchy-Schwartz}}{\leq} \sqrt{(1^2+4^2)(\sin^2 x + \cos^2 x)} = \sqrt{17}.$$

Equality when $\sin x = \cos x/4$, i.e., $x = \tan^{-1}\frac{1}{4}$.

Method 2:

$$\sin x + 4\cos x = \sqrt{17}\left(\frac{1}{\sqrt{17}}\sin x + \frac{4}{\sqrt{17}}\cos x\right) = \sqrt{17}\sin\left(x + \cos^{-1}\frac{1}{\sqrt{17}}\right)$$

which attains a maximum value of $\sqrt{17}$.

Method 3: Let $f(x) = \sin x + 4\cos x$. Then $f'(x) = \cos x - 4\sin x$ changes from positive to negative at $x^* = \cot^{-1}(4) + 2\pi k$ $(k \in \mathbb{Z})$. Because f is periodic and differentiable everywhere, $\max f = f(x^*) = \sqrt{17}$. □

Problem 20. In $\triangle ABC$, D, E, F are the midpoints of AB, BC, CA, respectively. Suppose $AB = 10, CD = 9$ and $CD \perp AE$. Find BF.

(A) $\sqrt{57}$ **(B)** $2\sqrt{15}$ **(C)** $2\sqrt{29}$ **(D)** $3\sqrt{13}$ **(E)** $5\sqrt{5}$

Solution. Let G be the centroid of $\triangle ABC$. We have

$$GD = \frac{CD}{3} = 3 \implies AG = 4 \implies GE = 2.$$

Now by median length formula on AE and $\triangle ABC$, we find $AC^2 = 52$, hence $BF = \sqrt{\frac{2a^2+2c^2-b^2}{4}} = 3\sqrt{13} \implies \boxed{D}$. □

Problem 21. How many integers n are there such that $|4x-n|+|x-4n| \geq \frac{n^4}{4^4}$ holds identically over all $x \in \mathbb{R}$?

(A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

Solution. The answer is $\boxed{C}$; the numbers are $-9,\ldots,9$. It is easy to see that $n = 0$ satisfies the condition. Now suppose without loss of generality that $n \geq 1$, since n satisfies the condition if and only if $-n$ does as well. For $n \geq 1$, $f(x) := |4x-n|+|x-4n|$ changes slope at only $x = \frac{n}{4}, 4n$ so it suffices to check the value of $|4x-n|+|x-4n|$ for these two values of x. Note that $f(\frac{n}{4}) = 4n - \frac{n}{4}$, $f(4n) = 15n$, so $n \geq 1$ satisfies the condition if and only if

$$4n - \frac{n}{4} \geq n^4/4^4$$

which is equivalent to $n \leq 4\sqrt[3]{15}$. Now verify that $n = 9$ satisfies the above inequality and $n = 10$ does not, the answer is $9 \cdot 2 + 1 = 19$. □

Problem 22. 10 people are sitting around a circular table. Find the number of ways to choose some people, such that there are two chosen people sitting next to each other.

(A) 337 **(B)** 511 **(C)** 684 **(D)** 728 **(E)** 901

Solution. Let M be the number of ways to select some people with no two sitting next to each other. We calculate M first. Label the people $1,2,\ldots,10$ counterclockwise. When 1 is chosen, $2,10$ cannot be. Delete positions $10,1,2$ on the circle and "unwrap" the circle from here to form a line consisting of 7 individuals $3,4,\ldots,9$, it is then equivalent to select some people from 7 standing in a line such that no two are are adjacent. Replace 7 with n and call this number l_n. There are l_7 ways in the case where 1 is chosen. Now suppose 1 is not chosen. Then we delete 1 and unwrap the circle to form a line consisting of 9 individuals, giving l_9 ways to choose in this case. Thus $m = l_7 + l_9$.

For n people standing in a line, we number them $1,2,\ldots,n$ from left to right. If n is chosen then $n-1$ cannot be, so there are l_{n-2} ways in this case. Otherwise, there are l_{n-1} ways. Thus $l_n = l_{n-1} + l_{n-2} \ \ \forall \ \ n \geq 3$ and $l_2 = 3, l_1 = 2$. This gives $l_7 = 34, l_9 = 89$ so that $M = 123$. Thus the answer is $2^{10} - M = 901 \implies \boxed{E}$ □

Problem 23. $ABCD$ is a tetrahedron with $AB = CD = AC = BD = 3, AD = 4, BC = 2$. The radius of the sphere circumscribing $ABCD$ can be written as $\frac{\sqrt{p}}{q}$ where no square of any prime divisor of q also divides p. Find $p+q$.

(A) 48 **(B)** 69 **(C)** 90 **(D)** 111 **(E)** 132

Solution. Let O_A, O_D denote the circumcenters of $\triangle BAC, \triangle BDC$ respectively. Let O be situated in space such that $OO_A \perp ABC, OO_D \perp BCD$. Then O is the cirucmcenter of tetrahedron $ABCD$, since $OA = OB = OC$ and $OB = OC = OD$. Let $\theta = \angle O_A M O_D$ where M is the midpoint of BC. We calculate $O_D D = \frac{9\sqrt{2}}{8}$ (by setting $O_D D = x, MO_D = 2\sqrt{2} - x, O_D C = O_D D, O_D C^2 = O_D M^2 + MC^2$ and solve for x, say) and $O_D O = \frac{7\sqrt{2}}{8} \tan\frac{\theta}{2}$. Consider isosceles $\triangle AMD$. Since $AM = MD = 2\sqrt{2}$ it follows that in fact $\theta = 90°$, hence $OO_D = \frac{7\sqrt{2}}{8}$. The circumradius is then

$$OD = \sqrt{O_D D^2 + OO_D^2} = \frac{\sqrt{2}}{8}\sqrt{7^2 + 9^2} = \frac{\sqrt{65}}{4} \implies \boxed{B}.$$

□

Problem 24. Given regular 11-gon $P := P_1 ... P_{11}$, we assign a number to each vertex of P randomly with each number in $1, ..., 11$ used exactly once. Find the expected number of arithmetic progressions of length 3 appearing on P.

For example, $P_1 ... P_{11} \to 4, 7, 10, 2, 3, 11, 8, 5, 6, 9, 1$ has only three such progressions, namely $(1, 4, 7)$, $(4, 7, 10)$ and $(11, 8, 5)$.

(A) $\frac{1}{3}$ **(B)** $\frac{4}{9}$ **(C)** $\frac{7}{11}$ **(D)** $\frac{5}{9}$ **(E)** $\frac{13}{22}$

Solution. Suppose there are N arithmetic progressions, then the probability of $P_i P_{i+1} P_{i+2}$ forming an arithmetic progression (indices taken modulo 11) is $\frac{N}{11 \cdot 10 \cdot 9}$. By linearity of expectation, the expected number of good indices is simply $\frac{N}{10 \cdot 9}$. Moreover, an arithmetic progression of length 3 with elements in $1, ..., 11$ is uniquely determined by its first and last terms, and exists if and only if the first and last terms have the same parity and are different. Thus, $N = 5 \cdot 4 + 6 \cdot 5 = 50$, and the answer is $\frac{5}{9} \implies \boxed{D}$. □

Problem 25. How many base-nine numbers using each of the digits $5, 6, 7, 8$ exactly once are divisible by 11?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 12

Solution. The answer is 3 ($\implies \boxed{B}$).

First we find the so-called divisibility rule for 11 in base 9 for four digit numbers. Note that $\overline{abcd}_9 = 9^3a + 9^2b + 9c + d \equiv 3a + 4b - 2c + d \pmod{11}$.

By the "divisibility rule for 11 in base 9", $11|\overline{abcd}_9 \iff 11|3a + 4b - 2c + d$. Let $(w, x, y, z) = (a-5, b-5, c-5, d-5)$ so that $\{w, x, y, z\} = \{0, 1, 2, 3\}$, the "divisibility rule" in base 9 becomes $11|3w + 4x - 2y + z + 30$, which is

$$11|3w + 4x - 2y + z - 3. \qquad (*)$$

By Rearragement Inequality, since $-2 < 1 < 3 < 4$ we have

$$-2\cdot 0 + 1 + 3\cdot 2 + 4\cdot 3 \geq -2y + z + 3w + 4x \geq -2\cdot 3 + 2 + 3\cdot 1 + 4\cdot 0$$

So that $-2y + z + 3w + 4x \in \{3, 14\}$.

Case 1: $-2y + z + 3w + 4x = 3$, so $z + 3w + 4x \leq 9$. This means that $x \leq 2$.

1. If $x = 0$ then $-2y + z + 3w = 3$, so $3|y + z$, so $w = 3$, meaning $-2y + z = -6$ which is impossible.
2. If $x = 1$, then $-2y + z + 3w = -1$, giving $3|y + z + 1$. Since $y, z, w = 0, 2, 3$ in some order it follows that $3|w$, so in fact $9|-2y + z + 1$ $(= -3w)$. If $y = 2$ then $z + 1 = 4$, so $z = 3$. This yields a solution $(w, x, y, z) = (0, 1, 2, 3)$. Otherwise $z = 2$ and we have $9|-2y + 3$ and $-2y + 3 \in [-3, 3]$ which means $-2y + 3 = 0$, impossible.
3. If $x = 2$, then $-2y + z + 3w = -5$, so $-2y < -5$ giving $y = 3$. This means that $z + 3w = 1$, so $3|z - 1 \implies z = 1$. This also gives $w = 0$. Hence a solution $(0, 2, 3, 1)$

Thus there is only 1 solution in **Case 1**.

Case 2: $-2y + z + 3w + 4x = 14$.

1. $y = 3$: $z + 3w + 4x = 20$ but $z + 3w + 4x \leq 0 + 3\cdot 1 + 4\cdot 2 = 11$, contradiction.
2. $y = 2$: $z + 3w + 4x = 18$ but $z + 3w + 4x \leq 0 + 3\cdot 1 + 4\cdot 3 = 15$, contradiction.
3. $y = 1$: $z + 3w + 4x = 16$. In this case $4|z - w$, impossible.
4. $y = 0$: $z + 3w + 4x = 14$. Since $4|z - w + 2$, z, w have the same parity, hence $x = 2$, giving $z + 3w = 6$, so $w = 1, z = 3$, hence a solution $(1, 2, 0, 3)$.

$\square$