Sample Test Final Round

GRANT YU

August 11, 2019

1 Instructions

- Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. **No calculators, smartwatches, phones, or computing devices are allowed.** No problems on the exam require the use of a calculator.
- The time allotted is **50 minutes**.
- Please write *simplified* answers for the problems.
- The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.
- SCORING: Each question is worth 20 points. There is no penalty for guessing. Individuals are ranked based on the sum of the final round score and the individual round score.

For your answer to be *simplified*, you should:

(1) carry out any reasonable calculations. For instance, you should evaluate any expressions which will take negligible time to evaluate (such as $\frac{1}{13}+\frac{1}{17}$), as well as all products less than 10^4. Unreasonable calculations include large powers (e.g. 3^{20}), large factorials, products which are greater than 10^4, and trigonometric functions which cannot be expressed in terms of radicals.

(2) write rational numbers in lowest terms. Decimals are also acceptable, provided they are exact. $\frac{1}{7}$ may be written as $0.\overline{142857}$, but not, for example, as 0.142857. Unless specified otherwise, irrational numbers may never be approximated as rational (e.g. 3.14 and $\frac{22}{7}$ are not acceptable in place of π).

(3) move all square factors outside radicals (for instance, write $3\sqrt{7}$ instead of $\sqrt{63}$). However, denominators need not be rationalized. Both $\frac{2}{\sqrt{5}}$ and $\frac{2\sqrt{5}}{5}$ are acceptable. Radicands are also allowed to contain nested radicals. For example, there will be no penalty for not simplifying $\sqrt{3+2\sqrt{2}}$ as $1+\sqrt{2}$ but there will be for writing $\sqrt{\sqrt{75}}$ instead of $\sqrt{5\sqrt{3}}$.

2 Problems

Problem 1. The sequence $(a_n)_{n=1}^{\infty}$ satisfies $a_1 = 2, a_n = na_{n-1} - 1 - \frac{1}{n(n-1)}$ for all $n \geq 2$. Find

$$\sum_{i=1}^{2019} ia_i.$$

Problem 2. Given $\triangle ABC$ and $D \in BC, E \in AB$ such that $AD = 2, BC = 3, AC = 4$ and $\angle DAB = \angle ACB = \angle BDE < 90°$, the length of CE can be written as $\frac{\sqrt{p}}{q}$ where no square of any prime divisor of q also divides p. Find $p + q$.

Problem 3. The sequence $t_1, t_2, \ldots$ is defined by $t_1 = 2, t_n = 2^{t_{n-1}}$ for all $n \geq 2$. For each $n \geq 1$, let r_n be the remainder of t_n upon division by 641. Find the number of distinct values in the sequence $(r_n)_{n=1}^{\infty}$.

Problem 4. Let N be the number of ways to stack $2^{2^{2^{2^2}}} = 2^{65536}$ 2-shaped blocks with weights $1, 2, 3, 4, 5, \ldots, 2^{2^{2^{2^2}}}$, such that the weight of the block immediately below each block (except the bottom one) has weight at least three less than the block above. That is, N is the number of permutations $\sigma = (\sigma_i)_{i=1}^{2^{2^{2^{2^2}}}}$ of $\{1, 2, 3, 4, 5, \ldots, 2^{2^{2^{2^2}}}\}$ such that $\sigma_{i+1} \geq \sigma_i - 3 \ \ \forall \ \ 1 \leq i < 2^{2^{2^{2^2}}}$. Find the remainder of N upon division by $2^{2^{2^{2^2}}}$.

Problem 5. Let S be the sum of all positive integers N such that $\frac{1}{N}$ has *minimal* repeating block of length 6 and every digit to the right of the decimal point in its decimal representation is part of some repeating block. Find the remainder when S is divided by 1000.

Problem 6. A bored mathematician draws the full curve

$$x^8 + 4x^6y^2 + 6x^4y^4 + 2x^4y + 4x^2y^6 + 4x^2y^3 - x^2 + y^8 + 2y^5 = 0, \qquad (x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

then adds 11 equally-spaced rays pointing away from the origin O, intersecting the curve at points $P_1, \ldots, P_{11}$ with P_{11} lying on the positive x-axis. Find

$$\prod_{i=1}^{11} |OP_i|.$$

Sample Test Final Round Solutions

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Problem 1. The sequence $(a_n)_{n=1}^{\infty}$ satisfies $a_1 = 2, a_n = na_{n-1} - 1 - \frac{1}{n(n-1)}$ for all $n \geq 2$. Find

$$\sum_{i=1}^{2019} ia_i.$$

Solution. A direct induction gives $a_n = n! + \frac{1}{n}$. Thus

$$\sum_{i=1}^{2019} ia_i = \sum_{i=1}^{2019} (i \cdot i! + 1) = \sum_{i=1}^{2019} ((i+1)! - i!) + 2019 = \boxed{2020! + 2018}$$

$\square$

Problem 2. Given $\triangle ABC$ and $D \in BC, E \in AB$ such that $AD = 2, BC = 3, AC = 4$ and $\angle DAB = \angle ACB = \angle BDE < 90°$, the length of CE can be written as $\frac{\sqrt{p}}{q}$ where no square of any prime divisor of q also divides p. Find $p + q$.

Solution. Let $c = AB, d = BE, x = BD$, then $3 - x = DC$. From $\triangle BED \sim \triangle ADB$ we get $x^2 = cd$. Moreover $DE||AC$, hence $\frac{x}{d} = \frac{3}{c}$. Cancelling x gives $\underline{9d = c^3}$; cancelling d gives $\underline{3x = c^2}$. Applying Stewart's Theorem on $\triangle ABC$ and AD we obtain

$$c^2(3-x) + 16x = (4 + x(3-x)) \cdot 3.$$

Substituting $3x$ for c^2 and solve for x, we obtain $x = \frac{3}{4}$. This gives $c = \frac{3}{2}$. Now calculate $\cos B = \frac{-19}{36}$, so $EC = \frac{\sqrt{661}}{8}$. $\square$

Problem 3. The sequence $t_1, t_2, \ldots$ is defined by $t_1 = 2, t_n = 2^{t_{n-1}}$ for all $n \geq 2$. For each $n \geq 1$, let r_n be the remainder of t_n upon division by 641. Find the number of distinct values in the sequence $(r_n)_{n=1}^{\infty}$.

Solution. Note that the smallest m for which $2^m \equiv 1 \pmod{641}$ is $m = 64$, hence $2^a \equiv 2^b \pmod{641} \iff a \equiv b \pmod{64}$. Next we note that $t_1 = 2, t_2 = 2^2, t_3 = 2^4, t_4 = 2^{16}, t_5 = 2^{2^{16}}$ and so on, with $64 = 2^6 | \log_2(t_i)$ for all $i \geq 5$. Therefore $t_i \equiv 1 \pmod{641}$ for all $i \geq 5$. Therefore the only remainders are congruent to t_1, t_2, t_3, t_4, t_5 respectively, and the answer is $\boxed{5}$. $\square$

Problem 4. Let N be the number of ways to stack $2^{2^{2^{2^2}}} = 2^{65536}$ 2-shaped blocks with weights $1,2,3,4,5,\ldots,2^{2^{2^{2^2}}}$, such that the weight of the block immediately below each block (except the bottom one) has weight at least three less than the block above. That is, N is the number of permutations $\sigma = (\sigma_i)_{i=1}^{2^{2^{2^{2^2}}}}$ of $\{1,2,3,4,5,\ldots,2^{2^{2^{2^2}}}\}$ such that $\sigma_{i+1} \geq \sigma_i - 3 \;\; \forall \;\; 1 \leq i < 2^{2^{2^{2^2}}}$. Find the remainder of N upon division by $2^{2^{2^{2^2}}}$.

Solution. Place blocks in order of increasing weights. We say that a stack is *valid* if the weights of blocks in the stack (when read from top to bottom) $\sigma_1,\ldots,\sigma_m$ form a permutation of $\{1,\ldots,m\}$ and satisfies $\sigma_{i+1} \geq \sigma_i - 3 \;\; \forall \;\; 1 \leq i < m$. Let a_n be the number of valid stacks with $m = n$. Check that when $n \geq 4$,

1. when n is taken out of a valid stack, the resulting stack is still valid with $m = n-1$;
2. given any valid stack with $n-1$ blocks, the set of all positions that we can insert the nth block and still have a valid stack are exactly the locations immediately above $n-3, n-2, n-1$ in the stack of $n-1$ blocks.

The above two observations imply that $a_n = 3a_{n-1}$ for all $n \geq 4$. Moreover any arrangement of 3 blocks is valid, hence $a_3 = 3! = 6$. This means that $a_n = 3^{n-3}a_3 = 3^{n-2}\cdot 2$ for all $n \geq 3$. Thus $N = 2\cdot 3^{2^{2^{2^{2^2}}}-2}$, which leaves a remainder of $\boxed{\dfrac{2+2^{2^{2^{2^2}}}}{9}}$ upon division by $2^{2^{2^{2^2}}}$. $\square$

Problem 5. Let S be the sum of all positive integers N such that $\frac{1}{N}$ has *minimal* repeating block of length 6 and every digit to the right of the decimal point in its decimal representation is part of some repeating block. Find the remainder when S is divided by 1000.

Solution. We say that a number $\frac{p}{q}$ $(p,q \in \mathbb{N}, (p,q) = 1)$ is *fully repetitive* if every digit to the right of the decimal point in its decimal representation is part of some repeating block.

Lemma: $\frac{p}{q}$ is fully repetitive if and only if $(q,10) = 1$.

Proof: Given a fully repetitive number $r = m + 0.\overline{a_1 a_2 \ldots a_s}$ $(m \in \mathbb{N}, a_1,\ldots,a_s \in [0,9]\cap\mathbb{Z})$, by geometric series formula it follows that

$$r = m + \frac{0.\overline{a_1 a_2 \ldots a_s}}{1-10^{-s}} = m + \frac{\overline{a_1 a_2 \ldots a_s}}{10^s - 1},$$

therefore when written as a reduced fraction, the denominator of r divides $10^s - 1$ which is relatively prime to 10. Now suppose a positive rational number $u = m + 0.v_1 v_2 \ldots v_k \overline{u_1 u_2 \ldots u_s}$ is not fully repetitive, $k \geq 1$ and $v_k \neq u_s$ (else v_k can be included as part of some repeating block). Then $10^k u = v + 0.\overline{u_1 \ldots u_s}$ where $v \in \mathbb{N}$. Writing $v + 0.\overline{u_1 \ldots . u_s}$ as a reduced fraction $\frac{p}{q}$ it follows that $(q,10) = 1$ and $u = \frac{p}{10^k q}$. If u can still be written as a fraction with denominator relatively prime to 10, then $10^k|p$. However $p|u_1...u_s+(10^s-1)v_1v_2...v_k$. Thus $10^k|u_1u_2...u_s+10^sv_1...v_k-v_1...v_k$. In any case, as $s\ge 1$ it follows that $10|u_s-v_k$, hence $u_s=v_k$, contradiction. ■

Back to the main problem. It is equivalent to consider the N that are relatively prime to 10. As observed in the proof of the lemma, any repetitive number r has denominator dividing 10^s-1 if and only if its repeating block has period dividing s. Thus we look for the number of positive integers N relatively prime to 10 such that N divides 10^6-1 but none of $10^2-1, 10^3-1$. By Principle of Inclusion-Exclusion

$$S=\sum_{d|10^6-1} d-\sum_{d|10^2-1} d-\sum_{d|10^3-1} d+\sum_{d|10-1} d$$

as $10^6-1=3^3\cdot 37\cdot 7\cdot 11\cdot 13$, $10^3-1=3^3\cdot 37$, $10^2-1=3^2\cdot 11$,

$$S=40\cdot 38\cdot 8\cdot 12\cdot 14-40\cdot 38-13\cdot 12+13\equiv \boxed{217} \pmod{1000}.$$

□

Problem 6. A bored mathematician draws the full curve

$$x^8+4x^6y^2+6x^4y^4+2x^4y+4x^2y^6+4x^2y^3-x^2+y^8+2y^5=0, \qquad (x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$$

then adds 11 equally-spaced rays pointing away from the origin O, intersecting the curve at points $P_1,\dots,P_{11}$ with P_{11} lying on the positive x-axis. Find

$$\prod_{i=1}^{11}|OP_i|.$$

Solution. By completing the square, the given equation is equivalent to

$$(\heartsuit):(x^2+y^2)^4+2(x^2+y^2)^2y-x^2=0.$$

Letting $r=\sqrt{x^2+y^2}>0$ and $\theta\in\mathbb{R}$ such that $x=r\cos\theta, y=r\sin\theta$, $(\heartsuit)$ becomes $r^8+2r^5\sin\theta-r^2\cos^2\theta=0$. As $(x,y)\neq 0$ it follows that $r\neq 0$, so

$$(\heartsuit)\iff r^6+2r^3\sin\theta-\cos^2\theta=0\iff r=\sqrt[3]{-\sin\theta\pm 1}.$$

However since $r>0$ it suffices to consider $r=\sqrt[3]{-\sin\theta+1}=\sqrt[3]{1+\cos(\frac{\pi}{2}+\theta)}=\sqrt[3]{2\cos^2(\frac{\pi}{4}+\frac{\theta}{2})}$, $0\le\theta<2\pi, \theta\neq\frac{\pi}{2}$. Thus,

$$\prod_{i=1}^{11}|OP_i|^3=2^{11}\left(\prod_{k=0}^{10}\cos\left(\frac{\pi}{4}+\frac{\pi k}{11}\right)\right)^2=-2^{-11}\prod_{k=0}^{21}2\cos\left(\frac{\pi}{4}+\frac{\pi k}{11}\right).$$

Let $\omega=e^{\frac{i\pi}{11}}, \tau=e^{\frac{i\pi}{4}}$. Note that

$$\prod_{k=0}^{21}2\cos\left(\frac{\pi}{4}+\frac{\pi k}{11}\right)=\prod_{k=0}^{21}\left(\omega^k\tau+\frac{1}{\omega^k\tau}\right)=\frac{1}{\tau^{22}\omega^{\sum_{k=1}^{21}k}}\prod_{k=0}^{21}(\omega^{2k}\tau^2+1)$$

and $\tau^{22} = \tau^2 = i, \omega^{\sum_{k=1}^{21} k} = (\omega^{11})^{21} = -1$, so

$$\prod_{i=1}^{11} |OP_i|^3 = -2^{-11} i \left(\prod_{k=0}^{10} (\omega^{2k} i + 1) \right)^2 = -2^{-11} i \left(\prod_{k=0}^{10} (\omega^{2k} i + 1) \right)^2 = -2^{-11} i \left(\underbrace{\prod_{k=0}^{10} (i - \omega^{2k})}_{\spadesuit} \right)^2 .$$

Since $x^{11} - 1 = \prod_{k=0}^{10} (x - \omega^{2k})$ for all $x \in \mathbb{C}$, the product ($\spadesuit$) equals to $i^{11} - 1 = -i - 1 = \sqrt{2} e^{\frac{5\pi i}{4}}$, hence

$$\prod_{i=1}^{11} |OP_i| = \sqrt[3]{2^{-10}} = \boxed{\frac{1}{8\sqrt[3]{2}}},$$

$\square$