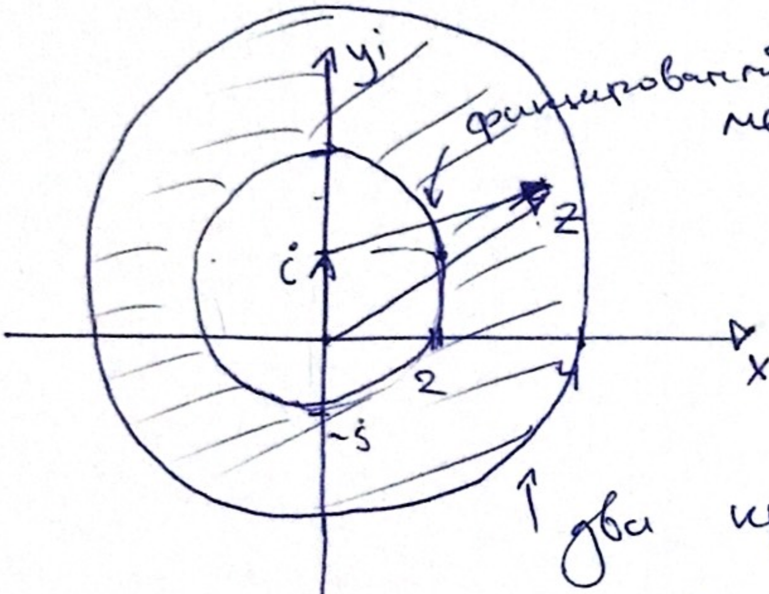
№1

$$2 \leq |z - i| \leq 4$$



фиксированный вектор, колеблющийся между значениями 2 и 4 =>

два круга с радиусами 2 и 4 и с центром в точке i.

Площадь: $S = 16\pi - 4\pi = 12\pi$.

№3

$\operatorname{Im} \frac{1}{z} = 1 \qquad z = x + iy$

$\operatorname{Im} \frac{\bar{z}}{z\bar{z}} = 1$

$\operatorname{Im} \frac{x - iy}{x^2 + y^2} = 1$

$-\frac{y}{x^2 + y^2} = 1$

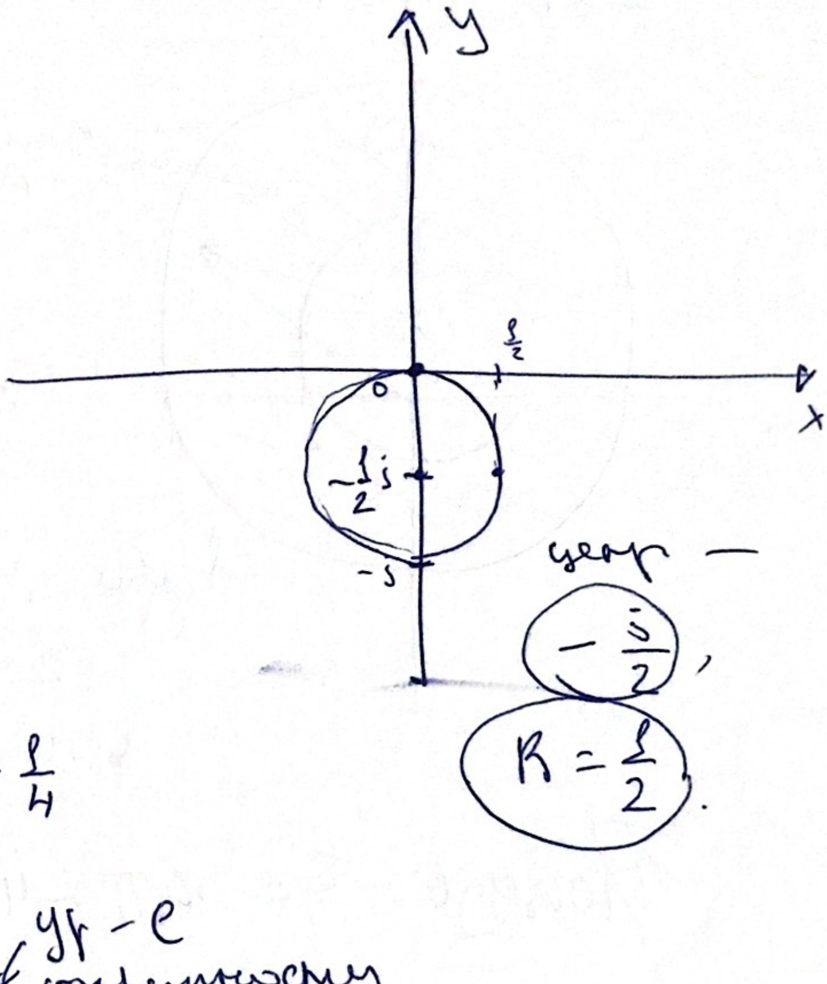
$-y = x^2 + y^2$

$-x^2 = y^2 + y$

$-x^2 = y^2 + y + \frac{1}{4} - \frac{1}{4}$

$-x^2 = \left(y + \frac{1}{2}\right)^2 - \frac{1}{4}$

$\left(y + \frac{1}{2}\right)^2 + x^2 = \frac{1}{4}$ ур-е окружности



центр — $-\frac{i}{2}$, $R = \frac{1}{2}$.

№1

$$1+2\varepsilon+3\varepsilon^2+\ldots+n\varepsilon^{n-1} = \frac{n}{\varepsilon-1}$$

если мы продифференцируем: $\varepsilon+\varepsilon^2+\varepsilon^3+\ldots+\varepsilon^n$, то получим

$$\varepsilon+\varepsilon^2+\ldots+\varepsilon^n = \varepsilon(1+\varepsilon+\varepsilon^2+\ldots+\varepsilon^{n-1}) = \varepsilon\frac{\varepsilon^n-1}{\varepsilon-1}$$

дифференцируем обе части

$$1+2\varepsilon+3\varepsilon^2+\ldots+n\varepsilon^{n-1} = \frac{\varepsilon^n-1}{\varepsilon-1} + \varepsilon\cdot\frac{n\varepsilon^{n-1}(\varepsilon-1)-(\varepsilon^n-1)}{(\varepsilon-1)^2} =$$

$$= \frac{\varepsilon^n-1}{\varepsilon-1} + \varepsilon\,\frac{\varepsilon^n(n-1)-n\varepsilon^{n-1}+1}{(\varepsilon-1)^2} = \frac{(\varepsilon^n-1)(\varepsilon-1)+\varepsilon^{n+1}(n-1)-n\varepsilon+\varepsilon}{(\varepsilon-1)^2} =$$

$$= \frac{n\varepsilon^{n+1}-\varepsilon^n(n+1)+1}{(\varepsilon-1)^2} = \frac{\varepsilon^n(n\varepsilon-n-1)+1}{(\varepsilon-1)^2} = \boxed{\frac{(n(\varepsilon-1)-1)\varepsilon^n+1}{(\varepsilon-1)^2}}$$

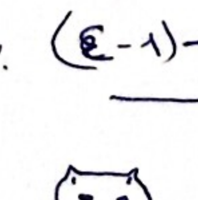
Ответ

Вопрос?

На сколько близки $\frac{(n(\varepsilon-1)-1)\varepsilon^n+1}{(\varepsilon-1)^2}$ и $\frac{n}{\varepsilon-1}$?

$$\frac{(n(\varepsilon-1)-1)\varepsilon^n+1}{(\varepsilon-1)^2} - \frac{n}{\varepsilon-1} = \frac{\varepsilon^n(n(\varepsilon-1)-1)+1-n\varepsilon+n}{(\varepsilon-1)^2}$$

$\Rightarrow$ при $n\to\infty$: $\dfrac{(\varepsilon-1)-\frac{1}{n}+\frac{1}{\varepsilon^n n}-\frac{\varepsilon}{\varepsilon^n}+\frac{1}{\varepsilon^n}}{\frac{(\varepsilon-1)^2}{\varepsilon^n n}} \to \infty$

ну такое 

№3.

$w:$ $\operatorname{Im} z = 1$

$z = x + i$

$$w(z) = z^3 + 3z - i = (x+i)^3 + 3(x+i) - i =$$

$$= x^3 + 2xi - \cancel{x} + ix^2 - 2x - i + \cancel{3x} + 3i - i =$$

$$= x^3 + 3x^2 i + i$$

$$\operatorname{Im} w(z) = 1 + 3x^2$$

совмещим $\Rightarrow$

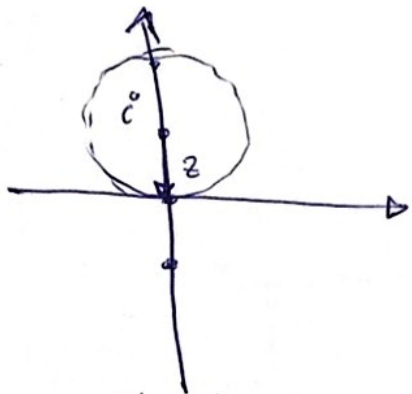
$\operatorname{Re} w(z) = x^3$

$$\boxed{\operatorname{Im} w = 1 + 3|\operatorname{Re} w|^{\frac{2}{3}}}$$

wii

$|z-i| = 1$

$w(z) = \frac{1}{z-2i}$



$|x+iy-i| = 1$

$(y-1)^2 = 1-x^2$

$y = \sqrt{1-x^2}+1$

$\sqrt{x^2+(y-1)^2} = 1$

$x^2+(y-1)^2 = 1$

Re(w)

$x^2 = 1-(y-1)^2$

$w(z) = \frac{1}{x+i(y-2)} = \frac{x-i(y-2)}{x^2-(y-2)^2}$

$\frac{\sqrt{y(2-y)}}{2(2-y)} = \frac{1}{2}\sqrt{\frac{y}{(2-y)}}$

$\operatorname{Re} w = \frac{x}{x^2-(y-2)^2} = \frac{\sqrt{1-(y-1)^2}}{\cancel{x^2}-\underbrace{2y^2+6y-5}_{-2y+4}} = \sqrt{\frac{y}{2-y}}$

$\frac{2-y}{\cancel{-2y^2+6y-5}\ \cancel{-2y}+4}$ $= \operatorname{Im} w$

$2-y$

$\operatorname{Im} w = \frac{-(y-2)}{x^2+(y-2)^2} = \frac{-(y-2)}{1-y^2+2y-1+y^2-4y+4} = \frac{-(y-2)}{-2y+4}$ = Im w

$\sqrt{1-y^2+2y-1} = \sqrt{2y-y^2} = \sqrt{y(2-y)}$

x^2-

$= \frac{2-y}{2(2-y)} = \frac{1}{2}$

w4

$$\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} = 0.$$

1) $2x + i2y = 0$

$x + iy = 0.$ → nem.

2) $2x + 2iy - 2iy - 2x = 0$

↓ ga

3) $-\frac{1}{(x+iy)^2} + \frac{1}{(x+iy)^2} = 0$

↓ ga

W5

1) $|f| = e^{r^2 \cos 2\varphi}$ $\quad z = re^{i\varphi}$ $\quad \varphi = \operatorname{arctg}\left(\frac{y}{x}\right)$

$z = x + iy$ $\quad \varphi = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$

$\varphi = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)$

$r = \sqrt{x^2+y^2}$

$$|f| = e^{(x^2+y^2)\left(2 \cdot \frac{x^2}{x^2+y^2} - 1\right)} = e^{x^2-y^2}$$

$$f(z) = R(x,y)e^{i\varphi(x,y)} \quad R(x,y) = e^{x^2-y^2}$$

$$\frac{\partial R}{\partial x} = R\frac{\partial \varphi}{\partial y} \qquad \frac{\partial R}{\partial y} = -R\frac{\partial \varphi}{\partial x}$$

$$2x\,\cancel{e^{(x^2-y^2)}} = \cancel{e^{(x^2-y^2)}}\frac{\partial \varphi}{\partial y} \qquad \frac{\partial \varphi}{\partial y} = 2x$$

$$+2y\,\cancel{e^{x^2-y^2}} = +\cancel{e^{x^2-y^2}}\frac{\partial \varphi}{\partial x} \qquad \frac{\partial \varphi}{\partial x} = 2y$$

$$\varphi = 2xy + \varphi(x)$$

$$\Rightarrow$$

$$\cancel{2}y + \frac{\partial \varphi}{\partial x} = \cancel{2}y$$

$$\frac{\partial \varphi}{\partial x} = 0.$$

$$f(z) = e^{x^2-y^2}e^{2ixy} = e^{x^2+2ixy-y^2}$$

$$x^2 + 2ixy - y^2 = x\underbrace{(x+iy)}_{z} + y(xi - y) = xz + y\frac{(-x-yi)}{i} =$$

$$= xz - \frac{yz}{i} = z\left(\frac{ix-y}{i}\right) = z^2 \Rightarrow f(z) = e^{z^2}$$

2) $\operatorname{Arg} f = xy \Rightarrow \varphi(x, y) = xy.$

$$\frac{\partial R}{\partial x} = Rx \qquad \frac{\partial R}{\partial y} = -Ry.$$

$$\ln R = \frac{x^2}{2} + \varphi(y). \qquad \cancel{e^{\frac{x^2}{2}+\varphi(y)}} \cdot \frac{d\varphi(y)}{dy} = -\cancel{e^{\frac{x^2}{2}+\varphi(y)}}\, y$$

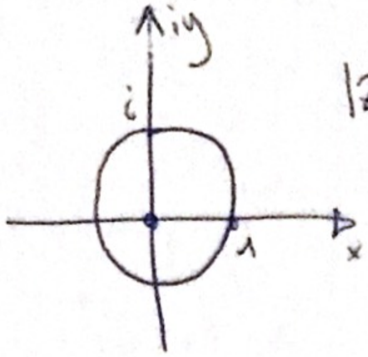
$$R = e^{\frac{x^2}{2} + \varphi(y)} \qquad d\varphi(y) = -y\,dy.$$

$$\varphi(y) = -\frac{y^2}{2}.$$

$$f(x, y) = e^{\frac{x^2}{2} - \frac{y^2}{2}} e^{ixy} = e^{\frac{x^2 + ixy - y^2}{2}} = e^{\frac{z^2}{2}}$$

w 7

$\int_C z\,dz$



$|z|=1$. $z=e^{i\varphi}$ $\quad \varphi\in[-\pi,\pi)$

$dz=ie^{i\varphi}d\varphi$

$\Downarrow$

$$\int_{-\pi}^{\pi} e^{i\varphi}\cdot ie^{i\varphi}d\varphi = i\int_{-\pi}^{\pi} e^{2i\varphi}d\varphi = \frac{1}{2}e^{i\varphi}\Big|_{-\pi}^{\pi} = \frac{1}{2}\big(\cos\pi + i\sin\pi - \cos-\pi +$$

$$+ i\sin-\pi\big) = \frac{1}{2}\big(-1+i\cdot 0-(-1)+i\cdot 0\big) = 0$$

$\int_C z^* dz \qquad z^* = e^{-i\varphi} \quad dz = ie^{i\varphi}d\varphi$

$$\int_{-\pi}^{\pi} e^{-i\varphi}\cdot ie^{i\varphi}d\varphi = i\int_{-\pi}^{\pi}d\varphi = i(\pi+\pi) = 2i\pi$$

w 8

$\int_C \frac{y\,dx - x\,dy}{x^2+y^2} =$ 1) $x=0$, $y=0$ $\quad x=\cos\varphi \quad dx = -\sin\varphi\, d\varphi$; $\quad y=\sin\varphi \quad dy=\cos\varphi\, d\varphi$

$$= \int_{-\pi}^{\pi}\frac{-\sin^2\varphi-\cos^2\varphi}{\cos^2\varphi+\sin^2\varphi}d\varphi = -\int_{-\pi}^{\pi}d\varphi = -2\pi$$

2) $x=2$, $y=0$ $\quad \int_C \frac{y\,dx-x\,dy}{x^2+y^2} = \cancel{\int_{-\pi}^{\pi}\frac{-\sin^2\varphi\, d\varphi-(\cos\varphi+2)\cos\varphi\, d\varphi}{x^2+y^2=1}} =$

$x=\cos\varphi+2$

$y=\sin\varphi$ $\quad \cancel{= -\int_{-\pi}^{\pi} 1+2\cos\varphi\, d\varphi = -2\pi - 2\sin\varphi\Big|_{-\pi}^{\pi} = -2\pi}$

$$= \int_{-\pi}^{\pi}\frac{-\sin^2\varphi\, d\varphi-(\cos\varphi+2)\cos\varphi\, d\varphi}{(\cos\varphi+2)^2+\sin^2\varphi} = -\int_{-\pi}^{\pi}\frac{1+2\cos\varphi}{4\cos\varphi+5}d\varphi = 0$$

№6

$f(z) = u(x,y) + i\vartheta(x,y)$ — гармоническая $\Rightarrow$
есть переход $\Rightarrow$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

1) $u = \varphi(x^2 - y^2) \Rightarrow \frac{\partial}{\partial x}\left(2x \frac{\partial \varphi}{\partial (x^2-y^2)}\right) + \frac{\partial}{\partial y}\left(2y \frac{\partial \varphi}{\partial (x^2-y^2)}\right) = 0$

$$\cancel{2 \frac{\partial \varphi}{\partial (x^2-y^2)}} + 4x^2 \frac{\partial^2 \varphi}{\partial (x^2-y^2)^2} - \cancel{2 \frac{\partial \varphi}{\partial (x^2-y^2)}} + 4y^2 \frac{\partial^2 \varphi}{\partial (x^2-y^2)^2} = 0$$

$$\cancel{\frac{\partial \varphi}{\partial (x^2-y^2)}} \frac{\partial^2 \varphi}{\partial (x^2-y^2)^2}(x^2+y^2) = 0$$

$$\Downarrow$$

$$\frac{\partial^2 \varphi}{\partial (x^2-y^2)^2} = 0 \Rightarrow \frac{\partial \varphi}{\partial (x^2-y^2)} = a \Rightarrow$$

$$\boxed{\varphi = a(x^2-y^2) + b}$$

$\frac{\partial u}{\partial x} = \frac{\partial \vartheta}{\partial y} \Rightarrow \frac{\partial \vartheta}{\partial y} = 2ax$ $\quad \frac{\partial \vartheta}{\partial x} = 2ay + \frac{\partial k}{\partial x} = +2ay$

$\frac{\partial u}{\partial y} = -\frac{\partial \vartheta}{\partial x}$ $\quad \Downarrow$

$\vartheta = 2ayx + k(x)$ $\quad \frac{\partial k}{\partial x} = 0 \Rightarrow$

$\vartheta = 2ayx$

$$f(z) = a(x^2-y^2) + b + 2iayx = a(\underbrace{x^2 + 2ixy - y^2}_{z^2}) + b =$$

$= az^2 + b$

2) $u = \varphi\left(\frac{y}{x}\right) \Rightarrow \frac{\partial}{\partial x}\left(-\frac{y}{x^2} \frac{\partial \varphi}{\partial (\frac{y}{x})}\right) + \frac{\partial}{\partial y}\left(\frac{1}{x} \frac{\partial \varphi}{\partial (\frac{y}{x})}\right) = 0$

$$\frac{2y}{x^3} \frac{\partial \varphi}{\partial (\frac{y}{x})} + \frac{y^2}{x^4} \frac{\partial^2 \varphi}{\partial (\frac{y}{x})^2} + \frac{1}{x^2} \frac{\partial^2 \varphi}{\partial (\frac{y}{x})^2} = 0$$

$$\frac{2y}{x^3}\frac{\partial \varphi}{\partial(\frac{y}{x})^2} + \frac{y^2}{x^4}\frac{\partial^2 \varphi}{\partial(\frac{y}{x})^2} + \frac{1}{x^2}\frac{\partial^2 \varphi}{\partial(\frac{y}{x})^2} = 0 \Rightarrow$$

$$\frac{2y}{x}\varphi\left(\frac{y}{x}\right) = -\frac{\partial \varphi}{\partial(\frac{y}{x})}\left(\frac{y^2}{x^2}+1\right) \Rightarrow \frac{2\frac{y}{x}}{(\frac{y}{x})^2+1}\partial\left(\frac{y}{x}\right) = -\frac{\partial\varphi(\frac{y}{x})}{\varphi(\frac{y}{x})} \Rightarrow$$

$$-\ln \underbrace{\varphi\left(\frac{y}{x}\right)}_{\partial(\frac{y}{x})^2} = \ln\left(\left(\frac{y}{x}\right)^2+1\right) + C_0$$

$$\frac{1}{\underbrace{\varphi\left(\frac{y}{x}\right)}_{\partial(\frac{y}{x})}} = C\left(\left(\frac{y}{x}\right)^2+1\right) \Rightarrow \cancel{\varphi\left(\frac{y}{x}\right)} \underbrace{\varphi\left(\frac{y}{x}\right)}_{\partial(\frac{y}{x})} = a\frac{1}{(\frac{y}{x})^2+1}$$

$$\cancel{\varphi\left(\frac{y}{x}\right) = a \operatorname{arctg}\left(\frac{y}{x}\right)+b} \quad \boxed{\varphi\left(\frac{y}{x}\right) = a\operatorname{arctg}\left(\frac{y}{x}\right)+b}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = a\frac{-1}{(\frac{y}{x})^2+1}\cdot\frac{y}{x^2} \Rightarrow v = \frac{a}{2}\left(-\ln(x^2+y^2)\right)+k(x)$$

$$\frac{y}{y^2+x^2}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{a}{x}\frac{1}{\frac{y^2}{x^2}+1} = -\frac{ax}{y^2+x^2} = -\frac{a}{2}\frac{2x}{x^2+y^2}+\frac{\partial k}{\partial x} \Rightarrow \frac{\partial k}{\partial x} = 0$$

$$f(z) = a\operatorname{arctg}\left(\frac{y}{x}\right)+b - \frac{ia}{2}\ln(x^2+y^2) = \cancel{\frac{ai}{2}\left(\ln\left(1+\frac{y^2}{x^2}\right)\right)}$$

$$\frac{ai}{2}\left(\ln\left(1-i\frac{y}{x}\right)-\ln\left(1+i\frac{y}{x}\right)\right) \quad \cancel{(x^2+y^2)^{-1}} + b =$$

$$= \cancel{\frac{ai}{2}\left(\ln\frac{x^2+y^2}{b^2(x^2+y^2)}\right)+b = \frac{ai}{2}\ln\left(\frac{1}{x^2}\right)+b}$$

$$= \frac{ai}{2}\left(\ln\left(\frac{1-i\frac{y}{x}}{\underbrace{(x^2+y^2)\left(1+i\frac{y}{x}\right)}_{\sim\frac{1}{z^2}}}\right)\right)+b = \frac{ai}{2}\ln\frac{1}{z^2}+b =$$

a – комплексное

$$a\ln\frac{1}{z^2}+b$$

wg

$$p(n) = \frac{1}{2\pi i}\int_C dz\, z^{-1-n}\prod_{k=1}^{\infty}\frac{1}{1-z^k}$$

$$\prod_{k=1}^{\infty}\frac{1}{1-z^k} \qquad \frac{1}{1-z} = 1+z+z^2+z^3\ldots$$

$$\frac{1}{1-z^2} = 1+z^2+z^4+\ldots$$

$$\frac{1}{1-z^3} = 1+z^3+z^6+\ldots \qquad \Rightarrow$$

$$\prod_{k=1}^{\infty}\frac{1}{1-z^k} = 1+z+2z^2+3z^3+5z^4+\ldots \Rightarrow$$

$$p(n) = \frac{1}{2\pi i}\int_C \left(z^{-1-n}+z^{-n}+2z^{1-n}+3z^{2-n}+5z^{3-n}\ldots\right)dz$$

$$= \operatorname*{Res}_{z=0}\left(z^{-1-n}+z^{-n}+2z^{1-n}+3z^{2-n}+5z^{3-n}+\ldots\right)$$

$$\underline{p(1) = 1} \qquad \underline{p(4) = 5}$$

w11

$$\frac{1+2z^2}{z^3+z^5} = \frac{A}{z^3}+\frac{B}{z^2}+\frac{C}{z}$$

$$\frac{1+2z^2}{\underbrace{z^3+z^5}_{1+z^2}} = \frac{A+zB+z^2C}{z^3}$$

$$\Downarrow$$

$$1+2z^2 = \left(A+zB+z^2C\right)\left(1+z^2\right)$$

не нужно

$$1+2z^2 = A+z^2A+zB+z^3B+z^2C+z^4C$$

$$\Downarrow$$

$$\cancel{A=1} \quad \cancel{B=0} \quad \cancel{C=0}$$

$$\begin{cases} A=1 \\ A+C=2 \\ zB+z^3B+z^4C=0 \end{cases}$$

$A=1$ $\quad B=0.$

$C=1$

$$zB+z^3B+z^4=0.$$

$$\cancel{B+z^2B+z^3=0}$$

$$\cancel{B(1+z^2)=-z^3}$$

$$\cancel{B=\frac{-z^3}{1+z^2}}$$

w:0

$y(1) = 0$

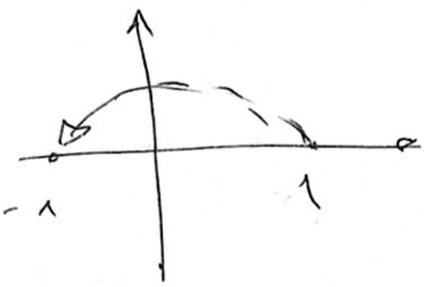
$y'(z) = \frac{1}{2z}$

$z = re^{i\varphi}$

$dz = r i e^{i\varphi} d\varphi$

$r = 1$

a)

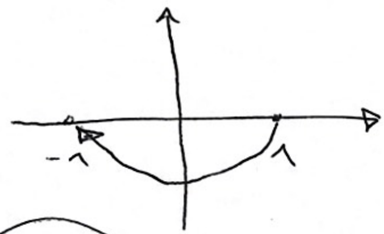


$y(-1) = \int_1^{-1} \frac{1}{2z} dz =$

$= \frac{1}{2} \int_0^{\pi} \frac{i e^{i\varphi} d\varphi}{e^{i\varphi}} =$

$= \frac{1}{2} i (\pi - 0) = \frac{\pi i}{2}$

b)



$y(-1) = \int_1^{-1} \frac{1}{2z} dz =$

$= \frac{1}{2} \int_0^{-\pi} i \, d\varphi = -\frac{\pi i}{2}$

№13

$$f = \frac{1}{z(z-1)} \qquad а)\ |z| \in (0;1) \qquad f = -\frac{1}{z} - \frac{1}{1-z} = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n$$

$$z_0 = 0 \qquad б)\ |z| \in (1; +\infty) \qquad f = \frac{1}{z^2\left(1-\frac{1}{z}\right)} = \frac{1}{z^2} \cdot$$

$$\cdot \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=2}^{\infty} \left(\frac{1}{z}\right)^n$$

№14

$$f = \frac{z}{z^2+1}$$ у функции f в т. z_0 — особенность =)

$$z_0 = i \qquad z_0 = i + \varepsilon$$

$$f = \frac{1}{2}\,\frac{1}{z-i} + \frac{1}{2}\,\frac{1}{z+i} = \frac{1}{2\varepsilon} + \frac{1}{2(\varepsilon+2i)} = \frac{1}{2\varepsilon} +$$

$$+ \frac{1}{4i\left(1+\frac{\varepsilon}{2i}\right)} = \frac{1}{2\varepsilon} + \frac{1}{4i}\sum_{n=0}^{\infty}(-1)^n \cdot \left(\frac{\varepsilon}{2i}\right)^n = \frac{1}{2(z-i)} +$$

$$+ \frac{1}{4i}\sum_{n=0}^{\infty}(-1)^n \left(\frac{z-i}{2i}\right)^n = \boxed{\frac{1}{2(z-i)} - \frac{i}{4}\sum_{n=0}^{\infty}(z-i)^n \left(\frac{i}{2}\right)^n}$$