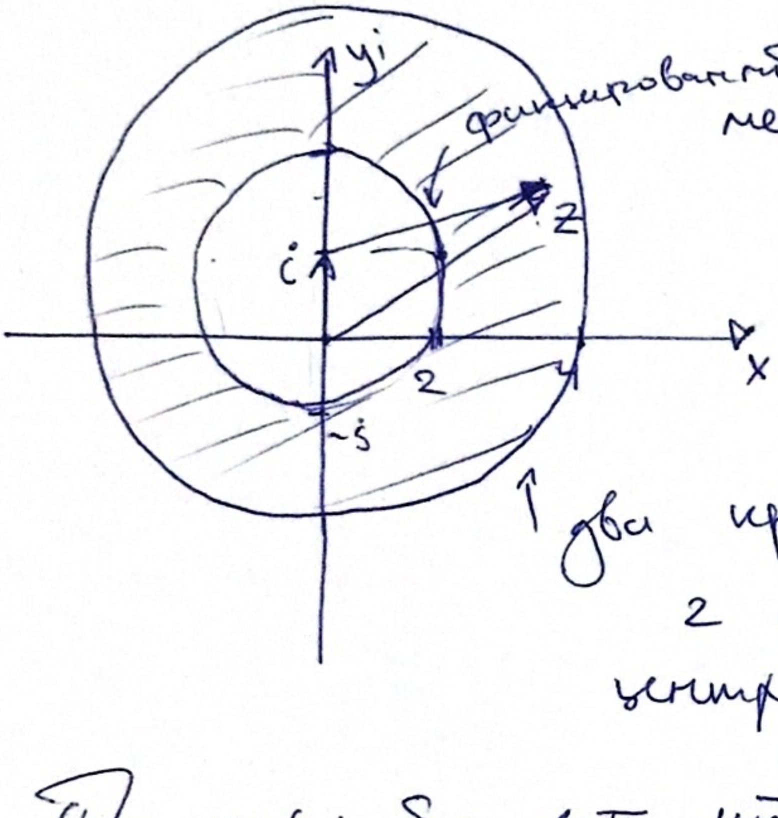
№1

$2 \le |z - i| \le 4$



фиксированный вектор, колеблющийся между значениями 2 и 4 =>

два круга с радиусами 2 и 4 и с центром в точке (i).

Площадь: $S = 16\pi - 4\pi = 12\pi$.

№3

$\mathrm{Im}\,\frac{1}{z} = 1 \qquad z = x + iy$

$$\mathrm{Im}\,\frac{\bar{z}}{z\bar{z}} = 1$$

$$\mathrm{Im}\,\frac{x - iy}{x^2 + y^2} = 1$$

$$-\frac{y}{x^2 + y^2} = 1$$

$$-y = x^2 + y^2$$

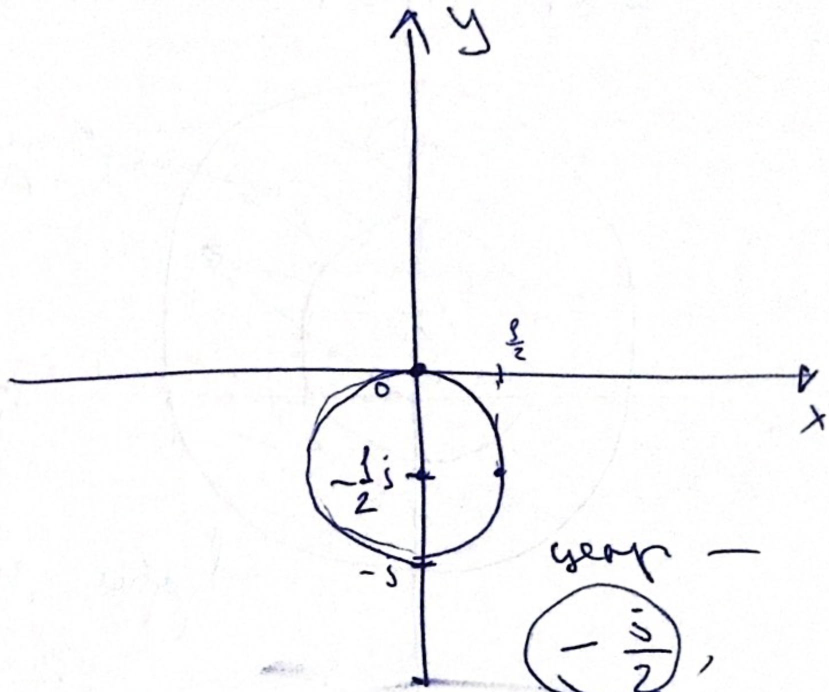
$$-x^2 = y^2 + y$$

$$-x^2 = y^2 + y + \frac{1}{4} - \frac{1}{4}$$

$$-x^2 = \left(y + \frac{1}{2}\right)^2 - \frac{1}{4}$$

$$\left(y + \frac{1}{2}\right)^2 + x^2 = \frac{1}{4}$$

ур-е окружности



центр — $-\frac{i}{2}$, $R = \frac{1}{2}$.

№ 1

$$1+2\varepsilon+3\varepsilon^2+\ldots+n\varepsilon^{n-1} = \frac{n}{\varepsilon-1}$$

если мы продифференцируем: $\varepsilon+\varepsilon^2+\varepsilon^3+\ldots+\varepsilon^n$, то получим

$$\varepsilon+\varepsilon^2+\ldots+\varepsilon^n = \varepsilon(1+\varepsilon+\varepsilon^2+\ldots+\varepsilon^{n-1}) = \varepsilon\frac{\varepsilon^n-1}{\varepsilon-1}$$

дифференцируем обе части

$$1+2\varepsilon+3\varepsilon^2+\ldots+n\varepsilon^{n-1} = \frac{\varepsilon^n-1}{\varepsilon-1}+\varepsilon\cdot\frac{n\varepsilon^{n-1}(\varepsilon-1)-(\varepsilon^n-1)}{(\varepsilon-1)^2} =$$

$$= \frac{\varepsilon^n-1}{\varepsilon-1}+\varepsilon\,\frac{\varepsilon^n(n-1)-n\varepsilon^{n-1}+1}{(\varepsilon-1)^2} = \frac{(\varepsilon^n-1)(\varepsilon-1)+\varepsilon^{n+1}(n-1)-n\varepsilon+\varepsilon}{(\varepsilon-1)^2} =$$

$$= \frac{n\varepsilon^{n+1}-\varepsilon^n(n+1)+1}{(\varepsilon-1)^2} = \frac{\varepsilon^n(n\varepsilon-n-1)+1}{(\varepsilon-1)^2} = \boxed{\frac{(n(\varepsilon-1)-1)\varepsilon^n+1}{(\varepsilon-1)^2}}$$

Ответ

Вопрос?

На сколько близки $\frac{(n(\varepsilon-1)-1)\varepsilon^n+1}{(\varepsilon-1)^2}$ и $\frac{n}{\varepsilon-1}$?

$$\frac{(n(\varepsilon-1)-1)\varepsilon^n+1}{(\varepsilon-1)^2}-\frac{n}{\varepsilon-1} = \frac{\varepsilon^n(n(\varepsilon-1)-1)+1-n\varepsilon+n}{(\varepsilon-1)^2}$$

при $n\to\infty$: $\dfrac{(\varepsilon-1)-\frac{1}{n}+\frac{1}{\varepsilon^n n}-\frac{\varepsilon}{\varepsilon^n}+\frac{1}{\varepsilon^n}}{\frac{(\varepsilon-1)^2}{\varepsilon^n n}} \to \infty$

ну такое 

№3.

w: $\operatorname{Im} z = 1$

$z = x + i$

$$w(z) = z^3 + 3z - i = (x+i)^3 + 3(x+i) - i =$$

$$= x^3 + 2x^2 i \cancel{-x} + ix^2 - 2x - i + \cancel{3x} + 3i - i =$$

$$= x^3 + 3x^2 i + i$$

$$\operatorname{Im} w(z) = 1 + 3x^2$$

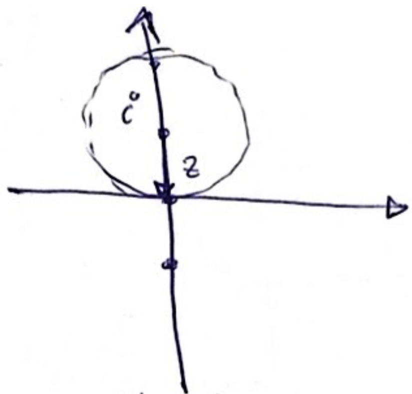
$$\operatorname{Re} w(z) = x^3$$

совмещаем $\Rightarrow$

$$\boxed{\operatorname{Im} w = 1 + 3|\operatorname{Re} w|^{\frac{2}{3}}}$$

wii

$|z - i| = 1$



$|x + iy - i| = 1$ $\quad (y-1)^2 = 1 - x^2$

$y = \sqrt{1-x^2} + 1$

$\sqrt{x^2 + (y-1)^2} = 1$

$w(z) = \dfrac{1}{z - 2i}$

$x^2 + (y-1)^2 = 1$ $\quad$ Re(w)

$x^2 = 1 - (y-1)^2$

$w(z) = \dfrac{1}{x + i(y-2)} = \dfrac{x - i(y-2)}{x^2 - (y-2)^2}$

$\dfrac{\sqrt{y(2-y)}}{2(2-y)} = \dfrac{1}{2}\sqrt{\dfrac{y}{(2-y)}}$

$\text{Re } w = \dfrac{x}{x^2 - (y-2)^2} = \dfrac{\sqrt{1-(y-1)^2}}{\cancel{x^2} - 2y^2 + 6y - 5} = \sqrt{\dfrac{y}{2-y}}$ $\quad \dfrac{2-y}{\cancel{[illegible]} \; \cancel{[illegible]}} = \text{Im } w$

$-2y + 4$ $\quad$ $2-y$

$\text{Im } w = \dfrac{-(y-2)}{x^2 + (y-2)^2} = \dfrac{-(y-2)}{1 - y^2 + 2y - \cancel{1} + y^2 - 4y + 4} = \dfrac{-(y-2)}{\cancel{[illegible]}}$ $= \text{Im } w$

$-2y + 4$

$\sqrt{1 - y^2 + 2y - 1} = \sqrt{2y - y^2} = \sqrt{y(2-y)}$ $\quad x^2 -$ $\quad = \dfrac{2-y}{2(2-y)} = \dfrac{1}{2}$.

W4

$$\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y} = 0.$$

1) $2x + i2y = 0$

$x + iy = 0. \to$ nem.

2) $2x + 2iy - 2iy - 2x = 0$

$\downarrow$ <u>ga</u>

3) $-\frac{1}{(x+iy)^2} + \frac{1}{(x+iy)^2} = 0$

$\downarrow$ <u>ga</u>

W5

1) $|f| = e^{r^2 \cos 2\varphi}$ $z = re^{i\varphi}$ $\varphi = \operatorname{arctg}\left(\frac{y}{x}\right)$

$z = x + iy$. $\varphi = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$

$\varphi = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)$

$r = \sqrt{x^2+y^2}$

$$|f| = e^{(x^2+y^2)\left(2 \cdot \frac{x^2}{x^2+y^2} - 1\right)} = e^{x^2-y^2}$$

$$f(z) = R(x,y)e^{i\varphi(x,y)} \qquad R(x,y) = e^{x^2-y^2}$$

$$\frac{\partial R}{\partial x} = R\frac{\partial \varphi}{\partial y} \qquad \frac{\partial R}{\partial y} = -R\frac{\partial \varphi}{\partial x}$$

$$2x\,\cancel{e^{(x^2-y^2)}} = \cancel{e^{(x^2-y^2)}}\frac{\partial \varphi}{\partial y} \qquad \frac{\partial \varphi}{\partial y} = 2x$$

$$+2y\,\cancel{e^{x^2-y^2}} = +\cancel{e^{x^2-y^2}}\frac{\partial \varphi}{\partial x} \qquad \frac{\partial \varphi}{\partial x} = 2y$$

$$\varphi = 2xy + \varphi(x)$$

$$\cancel{2}y + \frac{\partial \varphi}{\partial x} = \cancel{2}y \quad \Rightarrow$$

$$\frac{\partial \varphi}{\partial x} = 0.$$

$$f(z) = e^{x^2-y^2}e^{2ixy} = e^{x^2+2ixy-y^2}$$

$$x^2+2ixy-y^2 = x\underbrace{(x+iy)}_{z} + y(xi-y) = xz + y\frac{(-x-yi)}{i} =$$

$$= xz - \frac{yz}{i} = z\left(\frac{ix-y}{i}\right) = z^2 \quad \Rightarrow \quad f(z) = e^{z^2}$$

2) $\operatorname{Arg} f = xy \Rightarrow \varphi(x,y) = xy.$

$$\frac{\partial R}{\partial x} = Rx \qquad \frac{\partial R}{\partial y} = -Ry.$$

$$\ln R = \frac{x^2}{2} + \varphi(y). \qquad \cancel{e^{\frac{x^2}{2}+\varphi(y)}} \cdot \frac{d\varphi(y)}{dy} = -\cancel{e^{\frac{x^2}{2}+\varphi(y)}}\, y$$

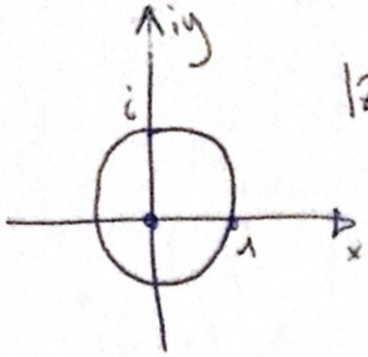
$$R = e^{\frac{x^2}{2}+\varphi(y)} \qquad d\varphi(y) = -y\,dy.$$

$$\varphi(y) = -\frac{y^2}{2}.$$

$$f(x,y) = e^{\frac{x^2}{2}-\frac{y^2}{2}} e^{ixy} = e^{\frac{x^2+ixy-y^2}{2}} = e^{\frac{z^2}{2}}$$

w 7

$\int_C z\,dz$



$|z| = 1 \quad z = e^{i\varphi}$

$dz = i e^{i\varphi} d\varphi \quad \varphi \in [-\pi, \pi)$

$$\int_{-\pi}^{\pi} e^{i\varphi}\cdot i e^{i\varphi} d\varphi = i\int_{-\pi}^{\pi} e^{2i\varphi} d\varphi = \frac{1}{2} e^{i\varphi}\Big|_{-\pi}^{\pi} = \frac{1}{2}\big(\cos\pi + i\sin\pi - \cos-\pi +$$

$$+ i\sin-\pi\big) = \frac{1}{2}\big(-1 + i\cdot 0 - (-1) + i\cdot 0\big) = 0$$

$\int_C z^* dz \qquad z^* = e^{-i\varphi} \quad dz = i e^{i\varphi} d\varphi$

$$\int_{-\pi}^{\pi} e^{-i\varphi}\cdot i e^{i\varphi} d\varphi = i\int_{-\pi}^{\pi} d\varphi = i(\pi + \pi) = 2i\pi$$

w8

$\int_C \frac{y\,dx - x\,dy}{x^2+y^2} =$ 1) $x = 0$, $y = 0$ $\quad x = \cos\varphi \quad dx = -\sin\varphi\, d\varphi$; $\quad y = \sin\varphi \quad dy = \cos\varphi\, d\varphi$

$$= \int_{-\pi}^{\pi} \frac{-\sin^2\varphi - \cos^2\varphi}{\cos^2\varphi + \sin^2\varphi} d\varphi = -\int_{-\pi}^{\pi} d\varphi = -2\pi$$

2) $x = 2$, $y = 0$ $\quad \int_C \frac{y\,dx - x\,dy}{x^2+y^2} = \cancel{\int_{-\pi}^{\pi}\frac{-\sin^2\varphi\, d\varphi - (\cos\varphi + 2)\cos\varphi\, d\varphi}{x^2+y^2 = 1}} =$

$x = \cos\varphi + 2$

$y = \sin\varphi$

$$\cancel{= -\int_{-\pi}^{\pi} 1 + 2\cos\varphi\, d\varphi = -2\pi - 2\sin\varphi\Big|_{-\pi}^{\pi} = -2\pi}$$

$$= \int_{-\pi}^{\pi} \frac{-\sin^2\varphi\, d\varphi - (\cos\varphi + 2)\cos\varphi\, d\varphi}{(\cos\varphi + 2)^2 + \sin^2\varphi} = -\int_{-\pi}^{\pi} \frac{1 + 2\cos\varphi}{4\cos\varphi + 5} d\varphi = 0$$

№6

$f(z) = u(x,y) + i\vartheta(x,y)$ — гармоническая $\Rightarrow$
есть первую $\Rightarrow$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

1) $u = \varphi(x^2 - y^2) \Rightarrow \frac{\partial}{\partial x}\left(2x \frac{\partial \varphi}{\partial (x^2-y^2)}\right) - \frac{\partial}{\partial y}\left(2y \frac{\partial \varphi}{\partial (x^2-y^2)}\right) = 0$

$$\cancel{2\frac{\partial \varphi}{\partial (x^2-y^2)}} + 4x^2 \frac{\partial^2 \varphi}{\partial (x^2-y^2)^2} - \cancel{2\frac{\partial \varphi}{\partial (x^2-y^2)}} + 4y^2 \frac{\partial^2 \varphi}{\partial (x^2-y^2)^2} = 0$$

$$\cancel{\frac{\partial \varphi}{\partial (x^2-y^2)}} \; \frac{\partial^2 \varphi}{\partial (x^2-y^2)^2}(x^2+y^2) = 0$$

$$\frac{\partial^2 \varphi}{\partial (x^2-y^2)^2} = 0 \Rightarrow \frac{\partial \varphi}{\partial (x^2-y^2)} = a \Rightarrow$$

$$\boxed{\varphi = a(x^2-y^2) + b}$$

$\frac{\partial u}{\partial x} = \frac{\partial \vartheta}{\partial y} \Rightarrow \frac{\partial \vartheta}{\partial y} = 2ax$ $\quad \frac{\partial \vartheta}{\partial x} = 2ay + \frac{\partial k}{\partial x} = +2ay$

$\frac{\partial u}{\partial y} = -\frac{\partial \vartheta}{\partial x}$ $\quad \vartheta = 2ayx + k(x)$ $\quad \frac{\partial k}{\partial x} = 0 \Rightarrow$

$\vartheta = 2ayx$

$$f(z) = a(x^2-y^2) + b + 2iayx = a(\underbrace{x^2 + 2ixy - y^2}_{z^2}) + b =$$

$$= a z^2 + b$$

2) $u = \varphi\left(\frac{y}{x}\right) \Rightarrow \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\frac{\partial \varphi}{\partial\left(\frac{y}{x}\right)}\right) + \frac{\partial}{\partial y}\left(\frac{1}{x}\frac{\partial \varphi}{\partial\left(\frac{y}{x}\right)}\right) = 0$

$$\frac{2y}{x^3}\frac{\partial \varphi}{\partial\left(\frac{y}{x}\right)} + \frac{y^2}{x^4}\frac{\partial^2 \varphi}{\partial\left(\frac{y}{x}\right)^2} + \frac{1}{x^2}\frac{\partial^2 \varphi}{\partial\left(\frac{y}{x}\right)^2} = 0$$

$$\frac{2y}{x^3}\frac{\partial\varphi}{\partial(\frac{y}{x})^2} + \frac{y^2}{x^4}\frac{\partial^2\varphi}{\partial(\frac{y}{x})^2} + \frac{1}{x^2}\frac{\partial^2\varphi}{\partial(\frac{y}{x})^2} = 0 \Rightarrow$$

$$\frac{2y}{x}\varphi\left(\frac{y}{x}\right) = -\frac{\partial\varphi}{\partial(\frac{y}{x})}\left(\frac{y^2}{x^2}+1\right) \Rightarrow \frac{2\frac{y}{x}}{(\frac{y}{x})^2+1}\,\partial\left(\frac{y}{x}\right) = -\frac{\partial\varphi(\frac{y}{x})}{\varphi(\frac{y}{x})} \Rightarrow$$

$$-\ln\frac{\varphi(\frac{y}{x})}{\partial(\frac{y}{x})^2} = \ln\left(\left(\frac{y}{x}\right)^2+1\right)+C_0$$

$$\frac{1}{\frac{\varphi(\frac{y}{x})}{\partial(\frac{y}{x})}} = C\left(\left(\frac{y}{x}\right)^2+1\right) \Rightarrow \frac{\varphi(\frac{y}{x})}{\partial(\frac{y}{x})} = a\frac{1}{(\frac{y}{x})^2+1}$$

$$\Downarrow$$

$$\boxed{\varphi\left(\frac{y}{x}\right) = a\,\mathrm{arctg}\left(\frac{y}{x}\right)+b}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = a\frac{-1}{(\frac{y}{x})^2+1}\cdot\frac{y}{x^2} \Rightarrow v = \frac{a}{2}\left(-\ln(x^2+y^2)\right)+k(x)$$

$$\Updownarrow$$

$$\frac{y}{y^2+x^2}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{a}{x}\frac{1}{\frac{y^2}{x^2}+1} = -\frac{ax}{y^2+x^2} = -\frac{a}{2}\frac{2x}{x^2+y^2}+\frac{\partial k}{\partial x} \Rightarrow \frac{\partial k}{\partial x}=0$$

$$\Downarrow$$

$$f(z) = \underbrace{a\,\mathrm{arctg}\left(\frac{y}{x}\right)}_{=}+b - \frac{ia}{2}\ln(x^2+y^2) = \cancel{\frac{ai}{2}\left(\ln\left(1+\frac{y^2}{x^2}\right)\right)}$$

$$\frac{ai}{2}\left(\ln\left(1-i\frac{y}{x}\right)-\ln\left(1+i\frac{y}{x}\right)\right) \quad \cancel{(x^2+y^2)^{-1})+b} =$$

$$= \cancel{\frac{ai}{2}\left(\ln\frac{x^2+y^2}{b^2(x^2+y^2)}\right)+b = \frac{ai}{2}\ln\left(\frac{1}{x^2}\right)+b}$$

$$= \frac{ai}{2}\left(\ln\left(\frac{1-i\frac{y}{x}}{(x^2+y^2)(1+i\frac{y}{x})}\right)\right)+b = \frac{ai}{2}\ln\frac{1}{z^2}+b =$$

$$\sim\frac{1}{z^2}$$

$$a\ln\frac{1}{z^2}+b$$

a – комплексное

wg

$$p(n) = \frac{1}{2\pi i} \int_C dz\, z^{-1-n} \prod_{k=1}^{\infty} \frac{1}{1-z^k}$$

$$\prod_{k=1}^{\infty} \frac{1}{1-z^k} \qquad \frac{1}{1-z} = 1 + z + z^2 + z^3 \dots$$

$$\frac{1}{1-z^2} = 1 + z^2 + z^4 + \dots$$

$$\frac{1}{1-z^3} = 1 + z^3 + z^6 + \dots \quad \Rightarrow$$

$$\prod_{k=1}^{\infty} \frac{1}{1-z^k} = 1 + z + 2z^2 + 3z^3 + 5z^4 + \dots \Rightarrow$$

$$p(n) = \frac{1}{2\pi i} \int_C \left(z^{-1-n} + z^{-n} + 2z^{1-n} + 3z^{2-n} + 5z^{3-n} \dots \right) dz$$

$$= \operatorname*{Res}_{z=0} \left(z^{-1-n} + z^{-n} + 2z^{1-n} + 3z^{2-n} + 5z^{3-n} + \dots \right)$$

4

$$\underline{p(1) = 1} \qquad \underline{p(4) = 5}$$

w11

$$\frac{1+2z^2}{z^3+z^5} = \frac{A}{z^3} + \frac{B}{z^2} + \frac{C}{z}$$

$$\frac{1+2z^2}{\underbrace{z^3+z^5}_{1+z^2}} = \frac{A+zB+z^2C}{z^3}$$

$\Downarrow$

$$1+2z^2 = (A+zB+z^2C)(1+z^2)$$

не нужно

$$1+2z^2 = A + z^2A + zB + z^3B + z^2C + z^4C$$

$\Downarrow$

$$\begin{cases} A = 1 \\ A + C = 2 \\ zB + z^3B + z^4C = 0 \end{cases}$$

$A = 1$ $\quad B = 0.$

$C = 1$

$$\cancel{zB + z^3B + z^4 = 0}$$

$$\cancel{B + z^2B + z^3 = 0}$$

$$\cancel{B(1+z^2) = -z^3}$$

$$\cancel{B = -\frac{z^3}{1+z^2}}$$

wω

$y(1) = 0$

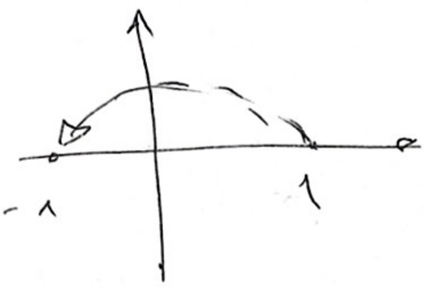
$y'(z) = \frac{1}{2z}$

$z = re^{i\varphi}$

$dz = rie^{i\varphi}d\varphi$

$r = 1$

a)

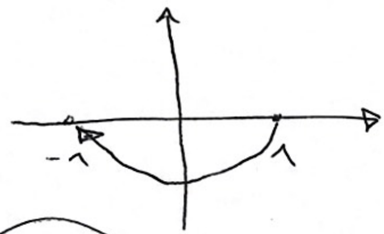


$y(\cancel{-1}) = \int^{-1} \frac{1}{2z}\,dz =$

$= \frac{1}{2}\int_0^{\pi} \frac{ie^{i\varphi}\,d\varphi}{e^{i\varphi}} =$

$= \frac{1}{2} i(\pi - 0) = \frac{\pi i}{2}$

b)



$y(\cancel{-1}) = \int_1^{-1} \frac{1}{2z}\,dz =$

$= \frac{1}{2}\int_0^{-\pi} i\,d\varphi = -\frac{\pi i}{2}$

№13

$f = \frac{1}{z(z-1)}$ а) $|z| \in (0;1)$ $f = -\frac{1}{z} - \frac{1}{1-z} = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n$

$z_0 = 0$ б) $|z| \in (1; +\infty)$ $f = \frac{1}{z^2(1-\frac{1}{z})} = \frac{1}{z^2} \cdot$

$\cdot \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=2}^{\infty} \left(\frac{1}{z}\right)^n$

№14

$f = \frac{z}{z^2+1}$ у функции f в т. z_0 — особенность. =)

$z_0 = i$ $z_0 = i + \varepsilon$

$$f = \frac{1}{2}\frac{1}{z-i} + \frac{1}{2}\frac{1}{z+i} = \frac{1}{2\varepsilon} + \frac{1}{2(\varepsilon+2i)} = \frac{1}{2\varepsilon} +$$

$$+ \frac{1}{4i\left(1+\frac{\varepsilon}{2i}\right)} = \frac{1}{2\varepsilon} + \frac{1}{4i}\sum_{n=0}^{\infty}(-1)^n \cdot \left(\frac{\varepsilon}{2i}\right)^n = \frac{1}{2(z-i)} +$$

$$+ \frac{1}{4i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n = \boxed{\frac{1}{2(z-i)} - \frac{i}{4}\sum_{n=0}^{\infty}(z-i)^n\left(\frac{i}{2}\right)^n}$$

№12

$$f(z) = \frac{1}{z(e^z - 1)} = \frac{1}{z\left(1 + z + \underbrace{\frac{z^2}{2} + \ldots}_{\text{не интересно}} - 1\right)} \approx \frac{1}{z \cdot z} = \frac{1}{z^2}$$

$\boxed{\text{2-ой порядок полюса.}}$

$$f(z) = \frac{1}{z\left(1 + z + \frac{z^2}{2} + \ldots - 1\right)} \approx \frac{1}{z\left(z + \frac{z^2}{2}\right)} = \frac{1}{z^2\left(1 + \frac{z}{2}\right)} \approx \frac{1}{z^2}\left(1 - \frac{z}{2} + \frac{z^2}{4} + \ldots\right)$$

$\boxed{\text{Ответ: } -\frac{1}{2}}$