№2

1) $f(z) = \dfrac{\sin z}{1-\tan z}$ $\qquad \tan z = 1 \Rightarrow z = \frac{\pi}{4} + \pi k,\ k \in \mathbb{Z}$

$$\lim_{z\to\frac{\pi}{4}+\varepsilon} f(z) = \lim_{z\to\frac{\pi}{4}+\varepsilon} \frac{\sin z}{1-\tan z} = \lim_{\varepsilon\to 0} \frac{\sin(\frac{\pi}{4}+\varepsilon)}{1-tg(\frac{\pi}{4}+\varepsilon)} = \infty$$

$$\sin\left(\frac{\pi}{4}+\varepsilon\right) = \frac{\sqrt{2}}{2}(\cos\varepsilon + \sin\varepsilon)$$

$$1-tg\left(\frac{\pi}{4}+\varepsilon\right) = 1 - \frac{1+tg\,\varepsilon}{1-tg\,\varepsilon} = \frac{-2\,tg\,\varepsilon}{1-tg\,\varepsilon}$$

$$f(\varepsilon) = \frac{\sqrt{2}}{2}\,\frac{\cos\varepsilon+\sin\varepsilon}{2tg\varepsilon}(tg\,\varepsilon - 1) \approx \frac{\sqrt{2}}{2}\,\frac{(1+\varepsilon)(\varepsilon-1)}{2\varepsilon} =$$

$tg\,\varepsilon \approx \varepsilon \quad \sin\varepsilon \approx \varepsilon$

$\cos\varepsilon \approx 1$

$$= \frac{\sqrt{2}}{4}\,\frac{\varepsilon^2-1}{\varepsilon} = \infty$$

$\Leftarrow$ полюс

$$\lim_{z\to\frac{5\pi}{4}} f(z) = \infty \ \text{— полюс}$$

$\uparrow$ ровно аналогично.

2) $f(z) = \dfrac{e^{\frac{c}{z-a}}}{e^{\frac{z}{a}}-1}$ $\quad \frac{z}{a}=0 \Rightarrow z=0.$

$z=a$ $z=2\pi i n a$

$$\lim_{z\to a+\varepsilon} f(z) = \lim_{z\to a+\varepsilon} \frac{e^{\frac{c}{z-a}}}{e^{\frac{z}{a}}-1} = \lim_{\varepsilon\to 0} \frac{e^{\frac{c}{\varepsilon}}}{e^{1+\frac{\varepsilon}{a}}-1} \ldots$$

рассм. с разных сторон.

$$\lim_{z\to a+} \frac{c}{z-a} = +\infty \Rightarrow e^{\frac{c}{z-a}} \to \infty.$$

$$\lim_{z\to a-} \frac{c}{z-a} = -\infty \quad e^{\frac{c}{z-a}} \to 0.$$

пределы не совпадают с разных сторон $\Rightarrow$ предела <u>нет</u>

т. изолированная

$$\lim_{z\to 2\pi i n a} f(z) = \lim_{z\to 2\pi i n a} \frac{e^{\frac{c}{z-a}}}{e^{\frac{z}{a}}-1} = \lim \frac{e^{\frac{c}{2\pi i n a-a}}}{\quad} \to \infty$$

некоторая "const"

полюс

$$e^{\frac{z}{a}}-1 \to e^{2\pi i n}-1$$

$$\cos(2\pi n) + i\sin(2\pi n) - 1 \to 0$$

№3

$z e^{\frac{1}{z}} e^{-\frac{1}{z^2}}$ $\qquad$ $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots$

$z = 0.$ $\qquad$ $e^{-\frac{1}{z^2}} = 1 - \frac{1}{z^2} + \frac{1}{2z^4} - \frac{1}{6z^3} + \dots$

$$\lim_{z\to 0+} z\, e^{\overset{\to +\infty}{\frac{1}{z}}}\, e^{\overset{\to -\infty}{-\frac{1}{z^2}}} \approx \lim_{z\to 0+}\left(z + 1 + \frac{1}{2z} + \frac{1}{6z^2} + \dots\right)\left(1 - \frac{1}{z^2} + \frac{1}{2z^4} - \frac{1}{6z^3} + \dots\right) =$$

$$= \infty.$$

$$\lim_{z\to 0-} z\, e^{\overset{\to -\infty}{\frac{1}{z}}}\, e^{\overset{\to -\infty}{-\frac{1}{z^2}}} = 0.$$

$\to$ не совпадают. Особая точка.

f - аналитическая функ. и изолирована.

w4

w1

1) $\int_C \frac{ze^z}{\tan z^2} dz$

особые точки

$\tan z^2 = 0$

$z^2 = 0 + \pi k$

$z = 0$ → центр круга, это вне нашей линии интегрирования

$z = \sqrt{\pi k}$

$$\int_C \frac{ze^z}{\tan z^2} dz = 2\pi i \operatorname{Res}_{z\to 0} f = 2\pi i \cdot 1$$

$$\lim_{z\to 0} \frac{ze^z}{\operatorname{tg} z^2} = \lim_{z\to 0} \frac{z(1+z)}{z^4 + \frac{z^6}{3}} = \lim_{z\to 0} \frac{1+z}{z} = \lim_{z\to 0} \frac{1}{z} + 1$$

$\Downarrow$

$\operatorname{Res}_{z\to 0} f = 1$

2)

$$\int_C e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right) dz = 2\pi i \operatorname{Res}_{z\to 0} f = 2\pi i \cdot 1$$

$\downarrow$

$$\left(1 - \frac{1}{z} + \dots\right)\left(\frac{1}{z} - \dots\right) \Rightarrow \frac{1}{z} \dots \Rightarrow \operatorname{Res}_{z\to 0} f = 1$$

3)

$$\int_C \frac{e^z}{z^n} dz = 2\pi i \operatorname{Res}_{z\to 0} f = \frac{2\pi i}{(n-1)!}$$

гособ. т.

$$\frac{e^z}{z^n} = \frac{1}{z^n}\left(1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!}\right) \Rightarrow$$

$$= \dots + \frac{1}{z}\frac{1}{(n-1)!} \dots \rightarrow \operatorname{Res} f = \frac{1}{(n-1)!}$$

№5

1) $\int\limits_{-\infty}^{\infty} \frac{x^4}{1+x^6}dx$ $= f(x)$

особые точки

$1+x^6=0$

$x^6=-1$

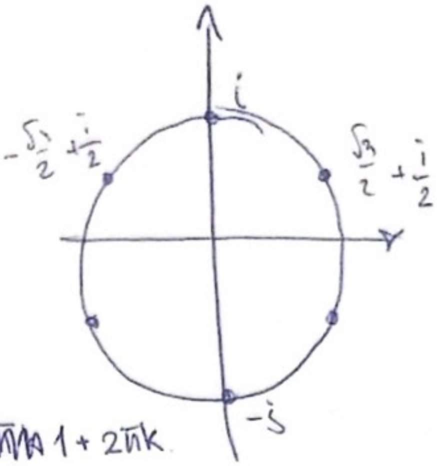
поворот $\frac{\pi}{3}$

или

$\sin 3x = +1$

$3x = 1 + 2\pi k$

$x = -\frac{\pi}{3} + \frac{2\pi k}{3}$



$\Rightarrow \int\limits_{-\infty}^{\infty} \frac{x^4}{1+x^6}dx =$

$= 2\pi i\left(\operatorname{Res}_{x\to\frac{\sqrt{3}}{2}+\frac{i}{2}} f(x) + \operatorname{Res}_{x\to-\frac{\sqrt{3}}{2}+\frac{i}{2}} f(x) + \operatorname{Res}_{x\to i} f(x)\right) = 2\pi i\left(\frac{1}{6\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)} + \frac{1}{6\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)} + \frac{1}{6i}\right) =$

$\operatorname{res}_{x=x_0} f(x) = \frac{f(x_0)}{g'(x_0)} = \frac{x_0^4}{6x_0^5} = \frac{1}{6x_0} \Rightarrow$ $= \frac{2\pi i}{6}\left(\frac{2}{\sqrt{3}+i} + \frac{2}{-\sqrt{3}+i} + \frac{1}{i}\right) =$

(делим на сопр. и знамен.) $= \frac{2\pi i}{6}\left(\frac{-2i\sqrt{3}-2+2i\sqrt{3}-2+(-1)-3}{(-1-3)i}\right) =$

$= \frac{\pi i}{3}\left(\frac{-8}{-4i}\right) = \frac{2\pi}{3}$ - ответ

2) $\int\limits_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta}d\theta$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$

$\cos 2\theta = \frac{e^{i2\theta}+e^{-i2\theta}}{2} = \frac{z^2+\frac{1}{z^2}}{2}$

$d\theta$ - ? $z = e^{i\theta}$

$dz = i\underbrace{e^{i\theta}}_{z}d\theta \Rightarrow d\theta = \frac{dz}{iz} \Rightarrow$

$= \int\limits_C \frac{\frac{z^2+\frac{1}{z^2}}{2}}{2+\frac{z+\frac{1}{z}}{2}} \cdot \frac{dz}{iz} = \int\limits_C \frac{z^2+\frac{1}{z^2}}{4+z+\frac{1}{z}} \cdot \frac{dz}{iz} = \frac{1}{i}\int\limits_C \frac{z^4+1}{4z^3+z^4+z^2}dz =$

$= -i\int\limits_C \frac{z^4+1}{z^2(4z+z^2+1)}dz$ → особые точки особ $z=0$ $\quad 4z+z^2+1=0.$

$D_1 = 4-1=3$

$z_{1,2} = -2 \pm \sqrt{3}$

$z_2 = -2+\sqrt{3}$ – внутри $\Rightarrow$

$z_1 = -2-\sqrt{3}$ – вне

$$\oint 2\pi i \left(\operatorname*{Res}_{z\to 0}^{①} f(x) + \operatorname*{Res}_{z\to -2+\sqrt{3}}^{②} f(x)\right)$$

① $\dfrac{d}{dz}\left(\dfrac{1+z^4}{z^2+4z+1}\right) =$

$$= \left(\frac{4z^3(z^2+4z+1)-(1+z^4)(2z+4)}{(z^2+4z+1)^2}\right)_{z\to 0} = -4$$

② $$\left.\frac{(1+z^4)(z-z_2)}{z^2(z-z_2)(z-z_1)}\right|_{z\to z_2} = \frac{1+(\sqrt{3}-2)^4}{(\sqrt{3}-2)^2(\underbrace{\sqrt{3}-2+2+\sqrt{3}}_{2\sqrt{3}})} = \frac{1}{(3-4\sqrt{3}+4)2\sqrt{3}} + \frac{3-4\sqrt{3}+4}{2\sqrt{3}} =$$

$$= \frac{1}{(7-4\sqrt{3})2\sqrt{3}} + \frac{7}{2\sqrt{3}} - 2 = \frac{1}{14\sqrt{3}-24} + \frac{7\sqrt{3}}{6} - 2 = \frac{6+7\cdot 14\cdot 3 - 24\cdot 7\sqrt{3}}{6\cdot 2(7\sqrt{3}-12)} - 2 =$$

$$= \frac{150-84\sqrt{3}}{6(7\sqrt{3}-12)} - 2 = \frac{25-14\sqrt{3}}{7\sqrt{3}-12} - 2 = \frac{6+7\sqrt{3}}{3} - 2 = \frac{7\sqrt{3}}{3} \Rightarrow$$

$$2\pi i \cdot (-i)\left(\frac{7\sqrt{3}}{3} - 4\right) = 2\pi\left(\frac{7\sqrt{3}}{3} - 4\right)$$ – Ответ

с i смр.

3) $$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = \int_{-\infty}^{+\infty} \frac{dx}{(x^2+|a|^2)(x^2+|b|^2)^2} = 2\pi i \left(\operatorname*{Res}_{z=i|a|} f(z) + \operatorname*{Res}_{z=i|b|} f(z)\right)$$

$x^2+|a|^2 = (x-|a|i)(x+|a|i)$

полюс

$x = \pm i|a|$

$x = \pm i|b|$

$$\operatorname*{Res}_{z=iA} f(z) = f(z)(z-i|a|)\Big|_{z\to i|a|} = \frac{1}{2i|a|(|b|^2-|a|^2)^2}$$

$$\operatorname*{Res}_{z=i|b|} f(z) = \frac{d}{dz} f(z)(z-i|b|)^2\Big|_{z\to i|b|} = \frac{d}{dz}\frac{1}{(z-i|a|)(z+i|a|)(z+i|b|)^2} =$$

$$= \frac{|a|^2-3|b|^2}{4i|b|^3(|a|^2-|b|^2)^2} \Rightarrow \frac{\pi(|a|+2|b|)}{2|a||b|^3(|a|+|b|)^2}$$

№6

$$\int_C \frac{z^5\,dz}{1+z^6} = 2\pi i \sum_{i=1}^{6} Res_i = -\underset{z\to\infty}{Res} f(z)$$

просуммировать Res.

полюс для обоих.

$\pm i$, $\frac{-\sqrt{3}}{2}+\frac{i}{2}$ $\frac{\sqrt{3}}{2}+\frac{i}{2}$

$-\frac{\sqrt{3}}{2}-\frac{i}{2}$ $\frac{\sqrt{3}}{2}-\frac{i}{2}$

$$\underset{z=\infty}{Res} f(z) = \underset{k\to 0}{Res}\left(-\frac{1}{k^2}\right) f\left(\frac{1}{k}\right) =$$

$$= \underset{k\to 0}{Res} -\frac{1}{k^2}\frac{\cancel{k^6}}{k^{5}(k^6+1)} =$$

$$= \underset{k\to 0}{Res} -\frac{1}{k(k^6+1)} = -\frac{1}{k^6+1}\Big|_{k\to 0} = -1$$

ответ: $-2\pi i\cdot(-1)$

$2\pi i$.

№7

$$\int_{-\infty}^{\infty} \frac{\sin^2 x\,dx}{x^2(x^2+1)} = \frac{1}{2}\left(\int_{-\infty}^{+\infty} \frac{1}{x^2(1+x^2)}dx - \int_{-\infty}^{+\infty} \frac{\cos 2x}{x^2(1+x^2)}dx\right)$$

① ②

$$① \int_{-\infty}^{+\infty} \frac{1}{x^2(1+x^2)}dx = \lim_{L\to\infty}\int_{-L}^{L} \frac{d\cancel{x}}{\cancel{x}^2(1+x^2)} = 2\pi i\left(Res(x=0)+Res(\cancel{x}=i)\right)$$

$x=0$ $x=\pm i$

2 1

$$Res(x=0) = \frac{d}{dz} f(\cancel{x})\cancel{x}^2\Big|_{\cancel{x}\to 0} = \frac{d}{d\cancel{x}}\left(\frac{1}{1+\cancel{x}^2}\right)\Big|_{\cancel{x}\to 0} =$$

$$= \frac{-2z}{(1+\cancel{x}^2)^2}\Big|_{z\to 0} = 0.$$

$$Res(\cancel{x}=i) = f(\cancel{x})(\cancel{x}-i)\Big|_{z\to i} = \frac{1}{\cancel{x}^2(\cancel{x}+i)}\Big|_{z\to i} = +\frac{1}{2i} = -\frac{i}{2} \Rightarrow$$

$$\Rightarrow = 2\pi i\left(0+\frac{i}{2}\right) = +\pi$$

② $$\int_{-\infty}^{+\infty} \frac{\cos 2x}{x^2(1+x^2)}\,dx = \mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{2ix}}{\cancel{x^2}(1+x^2)}\,dx = 2\pi i\left(\cancel{\quad} + \mathrm{res}(x=i)\right)$$

Т мнк не

$$\mathrm{res}(x=i) = \frac{e^{2ix}(x-i)}{x^2(x-i)(x+i)}\bigg|_{x\to i} = \frac{e^{-2}}{-2i} = \frac{e^{-2}i}{2}$$

$$= 2\pi i\left(\cancel{\quad} + \frac{e^{-2}i}{2}\right) = \cancel{\quad} - \pi e^{-2}$$

Итого:

$$\frac{1}{2}\left(+\pi\ \cancel{\quad} + \pi e^{-2}\right) = \frac{\pi}{2}\left(1 + \frac{1}{e^{+2}}\right) = \boxed{\frac{\pi}{2}\left(1+\frac{1}{e^2}\right)}$$

Сорем

№8

$$\int_0^{\infty} \frac{x \sin ax}{x^2+k^2}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x \sin ax}{x^2+|k|^2}\,dx = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{+\infty} \frac{e^{iax}}{x^2+|k|^2}$$

сингулярности:

$x = \pm i|k|$

$$\int_{-\infty}^{+\infty} \frac{e^{iax}}{x^2+|k|^2} = 2\pi i \,\mathrm{Res}\,(f(i|k|)) = $$

числитель и знаменатель

$$= 2\pi i \left.\frac{\cancel{x}\, e^{iax}}{2\cancel{x}}\right|_{x=i|k|} = 2\pi i \,\frac{e^{-a|k|}}{2} =$$

$= \pi i e^{-a|k|}$. → если у нас $a>0$ если $a<0$

вывод:

$\frac{1}{2}\pi i e^{-|a||k|}$ $\frac{1}{2}\pi i e^{a|k|}$

берем Im $\frac{1}{2}\pi e^{-|a||k|}\,\mathrm{sign}(a)$

№9

$$\int_{-\infty}^{+\infty} \frac{\cos(x-\frac{1}{x})}{1+x^2}\,dx = \mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{i(x-\frac{1}{x})}}{1+x^2}\,dx \qquad x=\pm i$$

$$= 2\pi i \,\mathrm{Res}\, f(i) \underset{2\pi i}{=} f(i)(x-i)\Big|_{x=i} = 2\pi i\,\frac{e^{i(x-\frac{1}{x})}}{x+i} = \frac{e^{-2}}{2i}\cdot 2\pi i = \pi e^{-2}$$

Ответ

w10

1) $$\int_0^{\infty} \frac{x-\sin x}{x^3}dx = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{x-\sin x}{x^3}dx = \frac{1}{2}\Big(\underbrace{\int_{-\infty}^{+\infty}\frac{1}{x^2}dx}_{①} -$$

$$- \underbrace{\operatorname{Im}\int_{-\infty}^{+\infty}\frac{e^{ix}}{x^3}dx}_{②}\Big)$$ $\quad \underline{x=0}$

① $2\pi i \operatorname{Res}(f(0)) = \frac{d}{dx}\frac{1}{x^2}\cdot x^2 \overset{\cdot 2\pi i}{\vee} = 0.$

② $2\pi i \operatorname{Res}(f(0)) = \overset{\cdot 2\pi i}{\vee}\frac{1}{(3-1)!}\cdot\frac{d^2}{dx^2}\left(\frac{e^{ix}x^3}{x^3}\right)\Big|_{x\to 0}=$

з - го входит.

$= \frac{2\pi i}{2} i^2 e^{ix}\Big|_{x\to 0} = -\frac{\pi i}{2}. \quad \operatorname{Im}\left(-\frac{\pi i}{2}\right) = -\frac{\pi}{2} \Rightarrow$

итог: $\frac{1}{2}\left(0+\frac{\pi}{2}\right) = \frac{\pi}{4}$ — ответ

2) $$\int_{-\infty}^{+\infty}\frac{e^{-iz}dz}{z^2+9} = -2\pi i \operatorname{Res}(f(-3i)) = -2\pi i \frac{e^{-iz}}{2z}\Big|_{z=-3i} =$$

$z=\pm 3i \quad = -2\pi i\cdot\frac{e^{-3}}{-2\cdot 3i} = \frac{\pi e^{-3}}{3}$ — ответ

№11

$$f(z) = z^3 \cos\frac{1}{z-2} \qquad \operatorname*{Res}_{z\to 0} f(z) = \operatorname*{Res}_{k\to 0}\left(-\frac{1}{k^2}\right) f\left(\frac{1}{k}\right) =$$

$z\to\infty$

$$= \operatorname*{Res}_{k\to 0} -\frac{1}{k^2}\cdot\frac{1}{k^3}\cos\frac{1}{\frac{1}{k}-2} = \operatorname*{Res}_{k\to 0} -\frac{1}{k^5}\cos\left(\frac{k}{1-2k}\right)$$

$$\cos\left(\frac{k}{1-2k}\right) \approx 1 - \frac{(k+2k^2+4k^3+\dots)^2}{2} + \frac{(k+2k^2+4k^3+\dots)^4}{4!} + \dots$$

домнож. на $\frac{1}{k^5}$ $\Rightarrow$ то, что нам нужно:

$$\Rightarrow -\frac{4}{2} - \frac{4}{2} + \frac{1}{4!} = -4 + \frac{1}{24} = -\frac{95}{24}$$

$$-2 - \frac{4}{2} - \frac{4}{2} + \frac{1}{4!} = -6 + \frac{1}{24} = -\frac{143}{24} \ *$$

$$\Rightarrow \frac{143}{24}$$

Ответ

№12

$$f(z) = \frac{1}{z^3 - z^5} \qquad \begin{matrix} z=-1 \\ z=0 \quad z=\infty \\ z=1 \end{matrix} \qquad \frac{1}{z^3(1-z^2)} = \frac{1}{z^3(1-z)(1+z)}$$

$$\operatorname{Res} f(0) = \frac{1}{(3-1)!}\frac{d^2}{dz^2}\left(\frac{z^3}{z^3(1-z)(1+z)}\right)\Bigg|_{z=0} = \frac{1}{2}\,\frac{2+6z^2}{(1-z^2)^3}\Bigg|_{z=0} = 1$$

$$\operatorname{Res} f(-1) = f(z)(z+1)\Big|_{z=-1} = \frac{1}{(-1)^3(1+1)} = -\frac{1}{2}$$

$$\operatorname{Res} f(1) = -\frac{1}{z^3(z+1)}\Bigg|_{z=1} = -\frac{1}{2}$$

$$\operatorname{Res} f(\infty) + \operatorname{Res} f(0) + \operatorname{Res} f(-1) + \operatorname{Res} f(1) = 0$$

z

$$\operatorname{Res} f(\infty) - 1 + 1 = 0 \Rightarrow \operatorname{Res} f(\infty) = 0$$

w13

1) $f(z) = \frac{\sin\frac{1}{z}}{1-z} = \left(\frac{1}{z} - \frac{1}{6z^3} + \frac{1}{5!z^5} - \dots\right)\left(1+z+z^2+z^3+z^4+\dots\right) =$

$z=0.$

$= \frac{1}{z} - \frac{1}{6z} + \frac{1}{5!z} - \dots = \frac{1}{z}\cdot\boxed{\sin(1)}$

2) $g(z) = e^{-e^{\frac{1}{z}}}$

$\operatorname*{Res}_{z\to\infty} g(z) = \operatorname*{Res}_{k\to 0}\left(-\frac{\exp(-\exp(k))}{k^2}\right) = \frac{d}{dk} - \frac{\exp(-\exp(k))}{k^2}\cdot k^2 \Big|_{k\to}$

$= +e^k e^{-e^k}\Big|_{k\to 0} = e^0 e^{-1} = \frac{1}{e}$.

$\operatorname*{Res}_{z\to 0} g(z) + \operatorname*{Res}_{z\to\infty} g(z) = 0$

$\operatorname*{Res}_{z\to 0} g(z) = -\frac{1}{e}$ - Ollu