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Seiberg Witten Theory and the Differential Topology of 4-manifolds

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0 Introduction

Dimension four is profoundly special for all three fields: Topology, Geometry and Physics

- From a classical topological perspective, dimension four lies between the well-understood realms of low- and high-dimensional topology. A striking example is the failure of the smooth h-cobordism theorem in dimension four, which holds in all dimensions $n \geq 5$. One direct consequence is the existence of exotic smooth structures on $\mathbb{R}^4$.
- Geometrically, the rotation Lie Group $SO(4)$ is not a simple group, leading to the split of its Lie Algebra $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ which gives rise to the direct sum split of the bundle of two forms $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. This is profound for multiple reasons, one of them being that curvature is a two form allowing for the study of (anti-)self-dual equations. Furthermore this split can be linked to the spin geometry and space of conformal structures on a 4-manifold providing a deep connection to topology.
- From a physical point of view, four is the (at least macroscopic) dimension of space-time. The tight relationship to geometry was first established with Einsteins theory of relativity, which models gravity as the curvature of space-time. At least as profound is the not yet fully mature development of quantum field theory modeling quantum fields with bundles over space time. This has brought nothing short of a revolution to the mathematical study of 4-manifolds.

The aim of this thesis is to give an overview of the different perspectives that these three fields offer on the differential topology of 4-manifolds and to give an introduction into Seiberg-Witten Theory.

The **first chapter** begins by summarizing key results from the classical topology of four-manifolds. This includes the h-cobordism theorem and its smooth failure in dimension four, leading to exotic $\mathbb{R}^4$'s, as well as Freedman's topological classification of simply connected four-manifolds via intersection forms. This provides essential context for understanding why dimension four is so special from a topological perspective and why gauge-theoretic results in this setting are so remarkable.
We also introduce Čech cohomology, which provides a unified framework for discussing bundles and cohomology. This leads to an elegant definition of some important characteristic classes, which not only play a fundamental role in the classical topology of 4-manifolds but are also crucial in understanding $U(1)$ bundles and $spin^c$-structures. Later, in section 3.3 we will build on these results to prove that $spin^c$ structures always exist on 4-manifolds allowing for the rich algebraic structure that underpins the seiberg-witten equations.

Following Freedman's classification [Fre82], the open question was which intersection forms could be realized by smooth 4-manifolds. Donaldson's 1983 breakthrough [DON83], inspired by Yang–Mills equations from physics, showed that the only positive definite intersection form realizable by a smooth 4-manifold is the identity matrix, revealing severe constraints on smooth structures.
Even more remarkably, in 1994 Edward Witten [Wit94] used purely physical reasoning to construct a "dual theory" to Donaldson theory — now known as Seiberg-Witten theory. It provided a much simpler framework for studying four-manifodls while yielding even stronger results than donaldsons theory.

In the **second chapter**, we explore the origins of these physics-inspired approaches and the profound interplay between physics and topology. In the first section, we show how concepts such as fiber bundles and the covariant exterior derivative naturally arouse in quantum mechanics and electromagnetism and how magnetic monopoles can carry topological charge. These ideas reveal that connections—or gauge fields—are fundamental physical objects. Given that spin structures play an important role in Seiberg–Witten theory, we also discuss how the concept of spin emerged as a deeper symmetry of quantum mechanics.
In the second section, we reverse the perspective and show how physical theories have contributed to topology. We examine how Yang–Mills theory, conceived as a generalization of Maxwell's theory [YM54], led to the development of the anti-self-dual equations—a breakthrough that ultimately enabled Donaldson to revolutionize the study of 4-manifolds. Finally, we offer an introductory glimpse into topological quantum field theory, a framework that lies at the heart of Witten's dualization of Donaldson theory to obtain Seiberg–Witten theory [WIT88].

In the **third chapter** we start the business of doing rigorous mathematics. The purpose of this chapter is to introduce the preliminaries for understanding the Seiberg-Witten equations. As the study of these equations intersects geometry, analysis, and algebra, we will develop the essential tools from each of these fields.
We begin by setting up the geometric framework. In the first section, we introduce the basic language of fiber bundles, connections, and curvature. Next, we shift our focus to an analytic perspective. The second section covers elliptic complexes and and the Hodge–Dirac operator. These tools will be key in studying the fundamental complex appearing in Seiberg-Witten Theory. Finally, in the third—and most detailed—section, we turn to the algebraic

structures underlying the theory. Here, we explore Clifford algebras, spin groups, and spin structures. This will lead us to the (complex) Spin–Dirac operator, which appears explicitly in one of the Seiberg–Witten equations. We also show that all four-manifold admit $spin^c$-structure guaranteeing that Seiberg–Witten theory can be applied to all smooth four-manifolds

After laying down all the necessary preliminaries, we are ready to delve into the gauge theoretic approach as employed by Seiberg–Witten and Donaldson theory. The *general strategy* behind this approach is quite different from that encountered in Chapter 1, where we saw that many classical results about the topology of four-manifolds were obtained by directly studying the manifold via its handle decomposition, its homotopy type, or its homology. In contrast, the gauge theoretic approach is more indirect and much more in the spirit of physics.
In this approach, we consider specific bundles over the manifold that exhibit a high degree of symmetry or possess a rich algebraic structure. On these bundles, we study geometric structures—typically by means of partial differential equations (PDEs) involving sections, connections, and curvature. In the language of physics, these objects correspond to physical entities such as matter fields or gauge fields that obey physical laws.
The key insight is that by carefully choosing the right bundles and PDEs, the solution spaces of these equations—when considered up to gauge equivalence—encode important topological information about the underlying manifold. We call these solution spaces the moduli spaces, denoted by $\mathfrak{M}$. Under certain circumstances, particularly when we can avoid so-called "reducible solutions," these moduli spaces are smooth, finite-dimensional, and orientable manifolds that are either compact or can be reasonably compactified.

In the **final and most detailed chapter**, we will study the Seiberg–Witten equations and pursue this general strategy with great success. We included a detailed outline of this chapter at the beginning.

Our main references are [Sco05; Don97; Mor96; Nab97; Nab00; Nic00].

All spaces are assumed paracompact.

1 Classical Topology of 4-Manifolds

Before delving into the geometric and gauge-theoretic approaches to smooth 4-manifolds, we first set the stage by summarizing key concepts and results from the classical topology of 4-manifolds. These results provide essential context for understanding why dimension four is so special from a topological perspective and why the gauge theoretic results are so spectacular. For more detail we refer the reader to [Sco05].

1.1 Higher dimensions and the h-cobordism theorem

The h-cobordism theorem is one of the corner stones in the theory of high dimensional manifolds. It shows that if you can construct a h-cobordism between two smooth simply connected manifolds, then they are already diffeomorphic [1]. This is a very powerful result. For example, it almost immediately implies the poincare conjecture in dimension 5 or higher. The smooth failure in dimension 4 on the other hand always produces exotic $\mathbb{R}^4$'s. Let us be just a bit more precise:

Definition 1.1.1. Cobordism

A cobordism between two oriented n-manifolds M and N is an oriented (n+1)-manifold W, such that:

$$\partial W = \bar{M} \cup N =: -M + N$$

A trivial cobordism between M and N is $W = M \times [0, 1]$.
If W deformation retracts to M (or N) , we call W a **h-cobordism**. If M and N and W are simply connected this is equivalent to:

$$H_*(W, M; \mathbb{Z}) = 0$$

The remarkable fact about h-cobordism is that in dimension $n \geq 5$ is in fact equivalent to diffeomporhism. Even though this is not in the scope of this thesis, I want to briefly sketch the proof to illustrate how entirely different the techniques of those "classical" results are from those employed in gauge theory which we will encounter later in greater detail.

Theorem 1.1.1. Smooth h-cobordism theorem

Let M and N be oriented simply connected n-dimensional manifolds with $n \geq 5$. If they are h-cobordant through a simply connected $(n + 1)$-manifold W, then there is a diffeomorphism:

$$M \cong N$$

Proof. A **Morse function** $f : W \to [0, 1]$ on W gives rise to a handle decomposition. The critical points of the function correspond to the addition of handles, and the topology of W changes when passing through a critical point. The decomposition can be manipulated through **handle sliding**, **cancellation**, and **creation**. One can show that if certain handle intersections cancel each other geometrically, one can entirely cancel all handles obtaining a trivial cobordism proving the result.
The condition $H_*(W, M; \mathbb{Z}) = 0$ can be shown to imply that these handel intersections cancel algebraically. In dimension 5 or higher one can always smoothly embed (whitney) disks, transforming this algebraic triviality into geometric triviality. □

In dimension 4, the proof of the h-cobordism fails because there is not enough space to smoothly embded (whitney) disks. Instead every disk might have self intersections. One can remove the self-intersecions by embedding even more whitney disks, but in the general those will also have self-intersections.
The idea of a casson was to repeat this procedure indefinitely. By a result from Michael Freedman, the resulting casson disk turns out to be homeomorphic to a disk without self-intersection!

[1] A refinement of the h-cobordism theorem for non-simply connected manifolds is known as the s-cobordism theorem.

Corollary 1.1.1. Homeomorphic h-cobordism theorem for smooth 4-manifolds

If M and N are smooth oriented 4-manifolds that are h-cobordant, then they are homeomorhic, but not necessarily diffeomorphic. Such a phenomenom is called **exotic**.

In fact, any non-trivial h-cobordism between simply connected smooth 4-manifolds exhibits (small) **exotic** $\mathbb{R}^4$**'s**! The following startling theorem gives some insight in how the diffeomorphic h-cobordism fails and how it leads to exotic $\mathbb{R}^4$'s:

Theorem 1.1.2. Akbulut cork

Let M and N be smooth simply connected 4-manifolds that are h-cobordant by W that is not diffeomorphic to $M \times [0,1]$ (but by the above theorem is homeomorphic). Then there is an open sub-h-cobordism U that is homeomorphic to $\mathbb{R}^4 \times [0,1]$ and contains a compact contractible sub-h-cobordism K such that W and U are trivial h-cobordisms outside K, meaning:

$$W \setminus int(K) \cong \big(M \setminus K\big) \quad U \setminus K \cong \big(U \cap M \setminus K\big) \times [0,1]$$

and $U \cap M$ and $U \cap N$ are homeommorphic to $\mathbb{R}^4$ (but not diffeomorphic), meaning they are **exotic** $\mathbb{R}^4$**'s**. Further, K is a non-trivial sub-h-cobordism between two diffeomorphic manifolds that when restricted to the boundary is an involution. Those diffeomorphic manifolds that are connect by K are are called **Akbulut cork**s.

Remark 1.1. *With freedmans result on casson handles one can now also prove the h-cobordism theorem for topological manifolds: After proving that they also admit a handle decomposition (note that one can not define differentiable morse functions any more), one can proceed analogously to the prove for smooth manifolds and use casosn handles to remove self-interections. The result is that simply-connected topological 4-manifolds that are h-cobordant are homeomorphic.*

1.2 Intersection Form and 4-Manifolds

In this section we will introduce the key invariant of 4-manifolds: The intersection form. It encodes how embedded surfaces intersect and plays the key role in classifying simply connected 4-manifolds.

Let's start with some fundamental facts about 4-manifolds required to understand the intersection form:

Lemma 1.1. *Every class in $H^2(M,\mathbb{Z})$ corresponds to (the isomorphism class of) a complex line bundle.*

Proof. Recall that every cohomology class $\alpha \in H^m(M,\mathbb{Z})$ can be represented by the homotopy class of maps from M into a Eilenberg-MacLane space $[M, K(m,\mathbb{Z})]$ and that isomorphism classes of G-bundles can be represented by the homotopy class of maps from M into a classifying space $[M, \mathcal{B}G]$. It turns out that

$$K(2,\mathbb{Z}) = \mathbb{C}P^\infty = \mathcal{B}U(1)$$

implying that cohomology 2-classes correspond bijectively to isomorphism classes of complex line bundles. This bijection is concretely realized by the first chern class

$$H^2(M,\mathbb{C}) \ni \alpha \mapsto L_\alpha \text{ s.t. } c_1(L_\alpha) = \alpha$$

□

An important consequence of this is:

Proposition 1.2.1. Homology by embded surfaces

Every class in $a \in H_2(M,\mathbb{Z})$ can be represented by an embedded surface S_a.

Proof. In the simply connected case there is a direct construction, but in the general case we can apply the previous lemma: Let $\alpha \in H^2(M, \mathbb{Z})$ be poincare dual to $x \in H_2(M, \mathbb{Z})$ which corresponds to a line bundle L_α with $c_1(L_\alpha) = \alpha$. Picking a generic section of $s \in \Gamma(L_\alpha)$. It's zero set $s^{-1}(0)$ will be an embedded surface poincare dual to α and thus represent x. □

Let M be a closed oriented 4-manifold. Note that any two generic surfaces are transverse and thus only intersect at finitely many points each picking up a sign depending on whether the orientation of the direct sum of the tangent spaces at the intersection point agrees with the orientation of TM at this point. The **algebraic intersection number** is then given by summing these local contributions over all intersection points.

Definition 1.2.1. Intersection Form

Let S_a and S_b be surfaces representing $a, b \in H^2(M, \mathbb{Z}$. The **intersection form** is the bilinear map:

$$Q_M : H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \to \mathbb{Z}, \quad (a, b) \mapsto S_a \cdot S_b$$

Note that by linearity, Q_M will vanish on torsion elements.

To prove that the intersection form is even well-defined, one can show that it is equivalent to the pairing of the cup product:

Lemma 1.2. *Let $\alpha, \beta \in H^2(M, \mathbb{Z})$ be poincare dual to $a, b \in H_2(M, \mathbb{Z})$. Then:*

$$Q_M(a, b) = \big(\alpha \cup \beta\big)([M]) =: Q_M(\alpha, \beta)$$

where $\cup : H^p(M, \mathbb{Z}) \times H^q(M, \mathbb{Z}) \to H^{p+q}(M, \mathbb{Z})$, $[M] \in H_2(M, \mathbb{Z})$ is the fundamental class of M and $\langle , \rangle$ is the evaluation pairing of cohomology and homology. Because Q_M vanishes on torsion elements, we can injectively include the free part of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$ and thus work in deRham cohomology and write:

$$Q_M(\alpha, \beta) = \int_M \alpha \wedge \beta$$

Proof. Poincare Duality and local expression. □

Some important properties of the intersection form are:

$$rank(Q_M) = b_2(M) \quad sign(Q_M) = sign(M) = b_2^+(M) - b_2^-(M) \quad Q_{M\sharp N} = Q_M \oplus Q_N$$

where $\sharp$ denotes connected sum.
Bit by bit it was proved that the intersection form classifies simply-connected 4-manifolds. First it was proved that the intersection form offers a classification up to homotopy (Whiteheads Theorem) and then up to h-cobordism (Walls Theorem).
Together with Freedmans homeomorphic h-cobordism theorem, Walls Theorem implies:

Corollary 1.2.1. Classification of 4-manifolds

Freedman (1982): If M and N are smooth simply-connected 4-manifolds with isomorphic intersection form, then they are homeomorphic.

Freedman went even further by constructing a topological manifold for every possible intersection form that satisfies the necessary criteria. Since these forms are algebraically classified, this led to a complete classification of topological simply connected 4-manifolds.

The key remaining question is: **which intersection forms can also be realized by smooth 4-manifolds?** In 1983, Donaldson ([DON83]) provided a breakthrough by using gauge theory inspired by the Yang-Mills equations from physics. He proved that the only positive definite intersection form that can be realized by a smooth 4-manifold is the identity matrix. This result stunned mathematicians, as it imposed unexpected and severe constraints on the smooth category.

Even more remarkably, in 1994 Edward Witten ([Wit94]) applied physical reasoning to construct a dual theory—Seiberg-Witten theory—which provided a much simpler framework while yielding even stronger results about smooth 4-manifolds. In the next chapter, we will give a rough idea of how physics intertwines with topology, followed by a detailed introduction to Seiberg-Witten theory in the following chapter.

Before that, we want to talk in some more detail about characteristic classes as they are important classical invariants of 4-manifolds that can also be defined in the gauge theoretic framework (via the chern weil method) and will be essential in understanding $spin^c$-structures.

1.3 Characteristic classes

There are many quite different approaches to characteristic classes, each highlighting different aspects of the theory. Since our main focus is on the second Stiefel-Whitney class and the first Chern class, which admit elegant descriptions in terms of Čech cohomology, we will adopt this approach. One of its key advantages is that we can talk about bundles and cohomology in a unified way. This perspective will also facilitate our discussion of bundles and $spin^c$ structures, which play a central role in Seiberg-Witten theory.

With this in mind, let us begin with a brief digression on Čech cohomology.

1.3.1 Čech Cohomology

We will not develop the theory in full detail and generality but just describe the parts that are important for our purpose.
Čech cohomology directly uses the open covering defining the manifold to extract topological information in an algebraic form. The simplest version of čech cohomology turns out to be equivalent to ordinary (singular) cohomology but the generalizations it allows to describe bundles is where the real power lies.
Throughout this section let $\{U_\alpha\}_{\alpha \in I}$ be an open covering and, for now, G be an abelian group. We define an n-cochain $\varphi \in \check{C}^n(\{U_\alpha\})$ to be a set of locally-constant function on the intersection of (n+1) open sets:

$$\varphi = \{\varphi_{\alpha_0 \dots \alpha_n} : U_{\alpha_0} \cap \dots \cap U_{\alpha_n} \to G\} \in \check{C}^n(M, G)$$

We define the following in the usual way. The coboundary

$$(\delta\varphi)_{\alpha_0 \dots \alpha_{n+1}} = \sum_{k=0}^{n+1} (-1)^k \varphi_{\alpha_0 \dots \hat{\varphi}_k \dots \varphi_{n+1}}$$

A cocylce: $\varphi \in \check{C}^n(M, G)$ is a $\check{C}$eck cocycle iff $\delta\varphi = 0$ and a $\check{C}$eck coboundary iff $\varphi = \delta\alpha$ for $\alpha \in \check{C}^{n-1}(M, G)$. The cohomology groups:

$$\check{H}^n(\{U_\alpha\}, G) = \{\varphi \in \check{C}^n(\{U_\alpha\}, G) \big| \delta\varphi = 0\} / \{\delta\alpha \in \check{C}^n(\{U_\alpha\}, G) \big| \alpha \in \check{C}^{n-1}(\{U_\alpha\}, G)\}$$

These groups might depend on the chosen cover, but refining the open cover removes that dependence. We hence define the **Čech cohomology groups** by taking the direct limit over refinements of the open covering:

$$\check{H}^k(M, G) = \varinjlim \check{H}(\{U_\alpha\}, G)$$

We want to remark that it is not always necessary to take the direct limit but if suffices to find **good cover** ie a covering for which all open sets and all intersections are contractible. We can now state:

Theorem 1.3.1. Čech cohomology and singular cohomology

Čech cohomology and singular cohomology are isomorphic:

$$\check{H}^k(M, G) \cong H^k(M, G)$$

We extend the theory in two important ways that will allow us to describe bundles with čech cohomology:
1. We do not require G to be abelian anymore - this will result in $\check{H}^1(M, G)$ not being a group any more and $\check{H}^k(M, G) = \emptyset$ for $k \geq 2$.
2. We remove the requirement that cochains be locally constant functions but instead allow them to be smooth G-valued functions $C^\infty(G)$.

Having done these replacements, we can define cochains and cocycles analogous as before, but since the group is not abelian anymore the order matters. We prefer to use multiplicative notation and define the coboundary for $f \in \check{H}^0(M, C^\infty(G))$ and $\varphi \in \check{C}^1(M, C^\infty(G))$ by:

$$(\delta f)_{\alpha\beta} = f_\alpha \cdot f_\beta^{-1} \quad (\delta\varphi)_{\alpha\beta\gamma} = \varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha}$$

The action of $\delta f \in im(\delta_0)$ on $\varphi \in ker(\delta_1)$ is:

$$(\varphi \cdot \delta f)_{\alpha\beta} = f_\alpha \cdot \varphi_{\alpha\beta} \cdot f_\beta^{-1}$$

Consequently:

$$[\varphi] = [\varphi'] \in \check{H}^1(M, C^\infty(G)) \Leftrightarrow \exists f \in \check{C}^0(M, C^\infty(G)) \text{ s.t. } \varphi_{\alpha\beta} = (\varphi' \cdot \delta f)_{\alpha\beta} = f_\alpha \cdot \varphi'_{\alpha\beta} \cdot f_\beta^{-1}$$

Note that $\check{H}^1(M, C^\infty(G))$ is not a group any more but instead just a pointed set with distinguished element $\varphi_{\alpha\beta} = id$.

Theorem 1.3.2. Classification of principal bundles

The set of all principal G-bundles (up to isomorphism) over M $\mathcal{P}_G(M) = [M, \mathcal{B}G]$ is given by čech-cohomology with coefficients in smooth G-valued functions:

$$\mathcal{P}_G(M) = \check{H}^1(M, C^\infty(G))$$

The following construction will provide an elegant way to define characteristic classes: Any short exact sequence of groups

$$0 \to G_0 \to G_1 \to G_2 \to 0$$

leads to a long exact sequence of pointed sets (ie the image of one map coincides with the preimage of the distinguished point):

$$\cdots \to \check{H}^k(M, C^\infty(G_0)) \to \check{H}^k(M, C^\infty(G_1)) \to \check{H}^k(M, C^\infty(G_2)) \to \check{H}^{k+1}(M, C^\infty(G_0)) \to \dots$$

Since continuous functions into discrete groups are locally constant: $\check{H}^k(M, C^\infty(G)) = H^k(M, G)$ for discrete groups G.

1.3.2 Euler class

Let $E \to M$ be real oriented vector bundle of rank $r \leq n$ over a closed oriented n-manifold. The euler class $e(E) \in H^{n-r}(M; \mathbb{Z})$ measures the obstruction to the existence of nowhere-vanishing sections: Consider a generic section $s \in \Gamma(E)$. It's zero locus $s^{-1}(0)$ will be a submanifold of M of d which represents a homology class in $[s^{-1}(0)] \in H_{n-r}(M, \mathbb{Z})$. We define the euler class to be it's poincare dual:

$$e(E) = [s^{-1}(0)]^* \in H^r(M, \mathbb{Z})$$

Remark 1.2. *For a rank n vector bundle the zero locus will just be set of signed points (where the sign of p is given by $det(ds_p)$).*
For the tangent bundle the sum of these signed points it the euler characteristic $\chi(M)$.

In the chern-weil approach the euler class is given by:

$$e(E) = Pf(F_\nabla) \in H^r_{dR}(M, \mathbb{Z})$$

where Pf is the pfaffian, F_∇ is any curvature form on E and $H^r_{dR}(M, \mathbb{Z})$ is the image of the map $H^r(M, \mathbb{Z}) \to H^r_{dR}(M, \mathbb{R})$.

1.3.3 Stiefel-Whitney classes

In general: The k-th Stiefel-Whitney class $w_k(E) \in H^k(M^4, \mathbb{Z}_2)$ measures obstructions to finding 4-k+1 independent sections on the k-skeleton of M. One can construct the classes pretty directly according to this general scheme. For space reason we prefer a more distiled approach.

1. Stiefel-Whitney Class

The short exact sequence of groups:

$$0 \to SO(k) \to O(k) \overset{det}{\to} \mathbb{Z}_2 \to 0$$

leads to a long exact sequence in čech cohomology:

$$0 \to H^0(M, \mathbb{Z}_2) \to \check{H}^1(M, C^\infty(SO(k))) \to \check{H}^1(M, C^\infty(O(k))) \overset{w_1}{\to} H^1(M, \mathbb{Z}_2) \to 0$$

where $det : O(k) \to \mathbb{Z}_2$ induces a map $w_1 : \check{H}^1(M, C^\infty(O(k))) \to H^1(M, \mathbb{Z}_2)$ assigning a princicpal $O(k)$-bundle to it's first stiefel-whtiney class $w_1(E) \in H^1(M, \mathbb{Z}_2)$. If $w_1(E) = 0$ then, by exactness, it lies in the image of the prevous map meaning that it comes from a principal SO(k) bundle: it is orientable. The concrete čech cocycle representing $w_1(E)$ can be constructed from the $O(k)$ cocycle $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(k)\}$ representing the bundle E:

$$w_1(E) = \{det(g_{\alpha\beta}) : U_\alpha \cap U_\beta \to \mathbb{Z}_2\} \in \check{H}^1(M, \mathbb{Z}_2) = H^1(M, \mathbb{Z}_2)$$

2. Stiefel-Whitney Class

The short exact sequence of groups [2]

$$0 \to \mathbb{Z}_2 \to Spin(k) \to SO(k) \to 0$$

leads to the long exact sequence of pointed sets:

$$\cdots \to H^1(M, \mathbb{Z}_2) \to \check{H}^1(M, C^\infty(Spin(k))) \to \check{H}^1(M, C^\infty(SO(k))) \overset{w_2}{\to} H^2(M, \mathbb{Z}_2)$$

where w_2 is taken as the definition of the second stiefel-whitney class. By exactness, if $w_2(E) = 0$ then the $SO(k)$ bundle must come from a $Spin(k)$ bundle and these are classified by $H^1(M, \mathbb{Z}_2)$. The concrete čech cocycle representing $w_2(E)$ can be constructed by lifting the set of $SO(k)$-valued maps $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(k)\}$ representing E to any set of $Spin(k)$-valued maps $\{\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(k)\}$ and then defining:

$$w_2(E) = \{w_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z}_2\} = \{\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z}_2\}$$

1.3.4 First chern class

Let $E \to M$ be complex vector bundle of complex rank r and real rank $n = 2r$. Recall that the determinant bundle $det(E) = \Lambda^n(E)$ is a complex line bundle. We define the first chern class of a bundle E via this determinant bundle:

$$c_1(E) = c_1(det(E)) = e_2(det(E)) \in H^2(M, \mathbb{Z})$$

The description of the first chern class in čech cohomology is quite eluminating:
Equipping E with a hermitian metric, reduces the structure group from $GL(r, \mathbb{C})$ to $U(r)$. The bundle is defined by the cocycle:

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(r)\} \in \check{H}^1\big(M, C^\infty(U(r))\big)$$

The cocycle defining the line bundle $det(E)$ is:

$$\{det(g_{\alpha\beta}) : U_\alpha \cap U_\beta \to U(1)\} \in \check{H}^1\big(M, C^\infty(U(1))\big)$$

We can always lift the collection of maps defining this cocycle to another collection of maps

$$\{\theta_{\alpha\beta} = \frac{1}{2\pi i} log(det(g_{\alpha\beta})) : U_\alpha \cap U_\beta \to \mathbb{R}\} \text{ s.t. } \{det(g_{\alpha\beta}) = exp(2\pi i\theta_{\alpha\beta}) : U_\alpha \cap U_\beta \to U(1)\}$$

Because $\{g_{\alpha\beta}\}$ and consequently $\{det(g_{\alpha\beta})\}$ satisfy the cocycle condition, we have that:

$$c_{\alpha\beta\gamma} := \theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha} \in \mathbb{Z}$$

Note that these are (locally) constant maps (because $c_{\alpha\beta\gamma}$ is smooth and $\mathbb{Z}$ valued) and one can further show that they satisfy the cocycle conidtion. Thus

$$\{c_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z}\} \in \check{H}^2(M, \mathbb{Z}) \cong H^2(M, \mathbb{Z})$$

and we can take this as definition of the first chern class which makes rigorous the intuition that the first chern class measures the rotations/twist of the determinant $U(1)$-bundle.

[2] We will introduce the spin group later in detail - for now it suffices to know that it is the universal double cover of SO(n). See proposition ??

1.3.5 Integral lifts of $w_2(E)$

The second stiefel whitney class plays an important role in understanding spin and *spin*^{c} structures. Great insight comes from a special case of Wu's formula (see [Mil10] §11 for more generality):

Theorem 1.3.3. Wu's Formula

$\exists$ an integral class $\underline{w} \in H_2(M, \mathbb{Z})$ which is a characteristic element of the intersection form:

$$\underline{w} \cdot x = x \cdot x \pmod 2 \qquad \forall x \in H^2(M, \mathbb{Z})$$

if and only if $\exists$ $\underline{w} \in H_2(M, \mathbb{Z})$ satisfying

$$H_2(M, \mathbb{Z}_2) \ni w_2(TM)^* = \underline{w} \pmod 2$$

Remark 1.3. *Characteristic elements are not unique. If $\underline{w}$ is a characteristic element then so is $\underline{w} + 2x$ for any $x \in H_2(M, \mathbb{Z})$.*

In order to prove that inegral lifts of $w_2(TM)$ always exists, it is enough to show that the intersection form always has a characteristic element. This follows from purely algebraic reasons. Hence

Theorem 1.3.4. Integral lift of $w_2(TM)$ on 4-manifolds

For a smooth oriented 4-manifold, there exists an integral lift of $w_2(TM)^* \in H_2(M, \mathbb{Z}_2)$.

2 Topology and Physics

In the previous chapter, we explored the "classical" topology of four-manifolds, which may seem far removed from physics. However, ideas from physics unexpectedly revolutionized the field when Donaldson, in 1983 [DON83], used the anti-self-dual Yang-Mills equations to obtain groundbreaking results in differential topology. This was not the end of the story. In 1994, Edward Witten [Wit94] astonished the mathematical community by introducing a much simpler set of equations—now known as the Seiberg-Witten equations—which, according to his physical resoning, should be equivalent to the anti-self-duality equations. These equations will be the focus of the next chapter.

Before diving into them, we take a brief excursion into physics to understand how topology first entered physics and, in turn, how physics profoundly influenced topology. Excelent references for the interplay of topology, geometry and physics are [Nab97; Nab00].

2.1 Topology Enters Physics

2.1.1 Quantum Mechanics and the exterior covariant derivative on $U(1)$ Bundles

Experiments in the early 20th century suggested that particles exhibit wave-like behavior. A first approach was to describe a particle by a complex-valued wave function $\phi : \mathbb{R}^3 \to \mathbb{C}$. One of the fundamental equations governing its evolution is the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t}\phi = H\phi,$$

where H, the Hamiltonian, determines the system's dynamics. To understand its form, recall that in quantum mechanics, observables correspond to hermitian operators whose eigenvalues represent possible measurement outcomes. The probability of measuring λ in state ψ is given by

$$\mathbb{P}(\lambda) = |\langle v_\lambda, \psi\rangle|^2.$$

Since probabilities remain unchanged under local phase shifts $\psi \to e^{i\theta}\psi$, this transformation leaves all observables unaffected—an instance of **gauge freedom**.

By Noether's theorem, every conserved quantity corresponds to a symmetry. In quantum mechanics, symmetries are represented by one-parameter subgroups of a Lie group acting unitarily, generated by anti-hermitian operators. If A is an observable, the associated symmetry transformation is generated by iA. Applying this to time translation, the Schrödinger equation states that $-iH$ generates time evolution, identifying H as the energy operator, analogous to the classical Hamiltonian.

Remark 2.1. *The eigenvalues of H represent the possible energy measurements, and the corresponding eigenvalue equation is known as the* ***time-independent Schrödinger equation****.*

For a free particle, the only energy contribution is kinetic, given classically by $\frac{p^2}{2m}$. In quantum mechanics, the Hamiltonian takes the form

$$H = \frac{\hbar^2}{2m}(-i\nabla)^2,$$

leading to the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi = \frac{\hbar^2}{2m}(-i\nabla)^2\psi.$$

Since gauge freedom allows replacing $\psi \to e^{in\theta}\psi$, substituting this into the equation yields:

$$\hbar\big(i\frac{\partial}{\partial t} + n\frac{\partial\theta}{\partial t}\big)\psi' = \frac{\hbar^2}{2m}\big(-i\nabla - n\nabla\theta\big)^2\psi'.$$

This deviates from the original equation, suggesting that derivatives transform as:

$$i\vec{\partial} = (i\frac{\partial}{\partial t}, -i\nabla) \to i\vec{\partial}' = i\vec{\partial} + n(\frac{\partial\theta}{\partial t}, -\nabla\theta).$$

This is a first instance of a **gauge transformation** in physics.

To make sense of this mathemaically, consider a principal $U(1)$-bundle $L \to M$ with connection 1-form A, exterior covariant derivative d_A, and an equivariant function $\tilde{\psi}$. A local gauge $s \in \Gamma(M|_U)$ pulls these back to:

$$s^* d_A = d + \mathcal{A}, \quad s^*\tilde{\psi} = \psi.$$

With a Minkowski metric, index raising gives:

$$d\psi = \frac{\partial\psi}{\partial t}dt + \nabla\psi \cdot (dx, dy, dz) \quad \mapsto \quad \vec{\partial}\psi = \left(\frac{\partial\psi}{\partial t}, -\nabla\psi\right).$$

Changing gauges via $s' = se^{-in\theta}$ transforms:

$$\psi' = e^{in\theta}\psi, \quad \vec{\partial}' = \vec{\partial} - ni(\frac{\partial\theta}{\partial t}, -\nabla\theta).$$

This matches exactly with the transformation law physicists initially introduced heuristically showing that one should appropriately model the wave function as an equivariant function on a $U(1)$ bundle which is acted on by the exterior covariant derivative.

2.1.2 Electromagnetism and harmoinc 2-forms

One of the great triumphs of theoretical physics is Maxwell's equations, which unified electricity and magnetism into a single framework. They not only synthesized experimental laws into an elegant theory but also led to a surprising prediction: oscillating electric and magnetic fields propagate as waves at a speed matching the measured velocity of light. Moreover, this speed was found to be independent of the observer, contradicting Galilean relativity and inspiring Einstein's theory of relativity. Relativity revealed that electric and magnetic fields are not separate entities but aspects of a unified electromagnetic field that transform under changes in reference frames.

We now introduce Maxwell's equations in the relativistic setting using differential forms. Assuming flat Minkowski spacetime (appropriate when gravitational effects are negligible), we choose coordinates x^0, x^1, x^2, x^3, where x^0 represents time, and x^i $(i = 1, 2, 3)$ represent spatial dimensions. Define the 1-form electric field ϵ and 2-form magnetic field B as:

$$\epsilon = E^1 dx^1 + E^2 dx^2 + E^3 dx^3,$$

$$B = B^1 dx^2 \wedge dx^3 + B^2 dx^3 \wedge dx^1 + B^3 dx^1 \wedge dx^2.$$

The electromagnetic field tensor is the 2-form:

$$F = \epsilon \wedge dt + B,$$

which in components is:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}.$$

The classical motion of a charged particle with 4-momentum π is governed by:

$$\frac{d\pi}{d\tau} = qF(., u),$$

where τ is proper time is an observer-independent equation. Maxwell's equations in vacuum arise as the Euler-Lagrange equations of the action:

$$S = -\frac{1}{4}\int_M ||F||^2 = \frac{1}{2}\int(|B|^2 - |E|^2).$$

These equations state that F is a harmonic 2-form:

$$dF = 0, \quad d^*F = 0,$$

where $*$ is the Hodge star. The space of solutions depends on the topology of spacetime. In Minkowski space, $H^2_{dR}(\mathbb{R}^{1,3}) = \mathbb{R}$, meaning a one-parameter family of possible electromagnetic fields.

For example, the classical Coulomb field of a charge n at the origin, with only radial electric components, is given by:

$$F = \frac{n}{r^3}(x^1 dx^1 + x^2 dx^2 + x^3 dx^3) \wedge dx^0.$$

This field admits a global potential A, satisfying $F = dA$, with:

$$A = -\frac{n}{r}dx^0.$$

However, A is not unique, as adding any exact form $d\theta$ preserves F, demonstrating another instance of **gauge freedom**. We will later see how this connects to the gauge freedom encountered in quantum mechanics.

2.1.3 Magnetic Monopoles and Topological Charge of a $U(1)$ Bundle

The field tensor for a magnetic monopole with charge g is given by:

$$F = \frac{g}{r^3}(x^1 dx^2 \wedge dx^3 - x^2 dx^1 \wedge dx^3 + x^3 dx^1 \wedge dx^2).$$

Since F has no time component, it simplifies in spherical coordinates (r, θ, φ) for a time slice $x^0 = 0$:

$$F = g \sin\varphi \, d\varphi \wedge d\theta.$$

The cohomology class of F is nontrivial, as seen by integrating over the two-sphere $S^2 \subset \mathbb{R}^3 \subset \mathbb{R}^{1,3}$:

$$\frac{1}{2\pi}\int_{S^2} F = \frac{g}{2\pi}\int \sin\varphi \, d\varphi \wedge d\theta = 2g.$$

If magnetic charge were quantized, similar to electric charge, the integral would take only discrete values. This suggests imposing the quantization condition:

$$\frac{1}{2\pi}[F] \in H^2_{dR}(M, \mathbb{Z}).$$

Thus, F can be interpreted as the first Chern class of a $U(1)$-bundle:

$$\frac{1}{2\pi}[F] = c_1(L),$$

where $iF \in \Omega^2(M, \mathfrak{u}(1))$ is the curvature 2-form of L. The magnetic charge corresponds to half the Chern number, a topological invariant measuring how often the bundle twists.

Remark 2.2. *Since F is independent of time and radius, one can restrict the $U(1)$-bundle to S^2. Setting $g = \frac{n}{2}$, we obtain:*

- $n = 0$: *trivial bundle,*
- $n = 1$: *Hopf bundle $S^3 \to S^2$,*
- $n = 2$: $\mathbb{R}P^3 = SO(3) \to S^2$,
- *more generally,* $S^3/\mathbb{Z}_n \to S^2$.

Despite the elegance of this model, magnetic monopoles remain experimentally unobserved. However, these observations strongly suggest that electro-magnetism should also formulated in terms of a $U(1)$ bundle.

2.1.4 The Significance of Connections

The curvature iF of a $U(1)$-bundle can be globally regarded as a $\mathfrak{u}(1)$-valued 2-form on M because $U(1)$ is abelian.[1] More generally, it should be seen as an ad-invariant form on L, denoted $i\tilde{F} \in \Omega^2_{\text{ad}}(L, \mathfrak{u}(1))$. If $i\tilde{A}$ is a connection 1-form on L, then:

$$\tilde{F} = d\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] = d\tilde{A}.$$

A local section $s \in \Gamma(L|_{U_\alpha})$ allows pulling this back to M:

$$F = s^*\tilde{F} = d(s^*\tilde{A}) =: d\mathcal{A}.$$

For an electric monopole, we saw that $[F] = 0$ and $F = dA$ globally, meaning L is trivial and $\mathcal{A}$ can be globally defined. For a magnetic monopole, however, $[F] \neq 0$, meaning L is nontrivial, and $\mathcal{A}$ can only be defined locally.

[1] Since $U(1)$ is abelian, the adjoint bundle is trivial: $iF \in \Omega^2_{\text{ad}}(L, \mathfrak{u}(1)) = \Omega^2(M, \mathfrak{u}(1))$.

Initially, connections were seen as mere computational tools for obtaining electromagnetic fields. However, the Aharonov-Bohm experiment demonstrated that the potential itself has physical significance [AB59]. This perspective aligns with the interpretation of $\mathcal{A}$ as a connection on the $U(1)$-bundle, influencing the Schrödinger equation of a charged particle in an external field. Writing

$$\mathcal{A}^\sharp = in(V, \vec{A}),$$

the Schrödinger equation reads:

$$\left(i\frac{\partial}{\partial t} - nV\right)\psi = \frac{1}{2m}\big(- i\vec{\nabla} - n\vec{A}\big)^2\psi.$$

Under a gauge transformation $s' = se^{-in\theta}$, the connection transforms as $\mathcal{A}' = \mathcal{A} - ind\theta$, leading to:

$$\mathcal{A}'^\sharp = in(V, \vec{A}) + in\left(- \frac{\partial\theta}{\partial t}, \vec{\nabla}\theta\right),$$

and the transformed Schrödinger equation becomes:

$$\left(i\frac{\partial}{\partial t} - n(V - \frac{\partial\theta}{\partial t})\right)\psi = \frac{1}{2m}\big(- i\vec{\nabla} - n(\vec{A} + \vec{\nabla}\theta)\big)^2\psi.$$

This highlights that the fundamental quantity governing electromagnetism is the connection of the $U(1)$-bundle which looks different in different in gauges. In a sense, a photon can be interpreted as a derivative acting on electrons.

2.1.5 Spin - A deeper symmetry of quantum mechanics

A fundamental assumption in physics is that physical systems exhibit rotational symmetry, meaning they should transform under an $SO(3)$ representation. However, the discovery of spin challenged this assumption.

The Stern-Gerlach experiment showed that electrons exhibit an intrinsic angular momentum—spin—that takes only two discrete values, "up" and "down," rather than a continuous range. This suggests that the wave function must include an additional spin component:

$$\psi(x, y, z, t) = \begin{pmatrix} \psi_1(x, y, z, t) \\ \psi_2(x, y, z, t) \end{pmatrix} \in \mathbb{C}^2.$$

However, there is no nontrivial representation of $SO(3)$ on $\mathbb{C}^2$, seemingly contradicting rotational symmetry. This is resolved by the quantum mechanical fact that $\pm\psi$ represent the same physical state, allowing for a two-valued representation instead.

Definition 2.1.1. n-valued representation of G

Let G_1 be a connected group and $\rho : G \to GL(V)$ be a representation of that group. Further, let G be a group isomorphic to G_1/N where N is a discrete normal subgroup of G_1 which is not contained in the kernel of ρ. Identifying elements of G with G_1/N or more specific defining $\rho(g) := \{\rho(h) \in GL(V) | h \in gN\}$ with $n = |\rho(g)| = |\rho(N)|$ we call ρ *a n-valued representation of G*

To resolve the apparent conflict introduced by the discovery of spin, we can thus look for 2-valued representations of $SO(3)$ which are exactly the representations of $G_1 = Spin(3) = SU(2)$.

2.2 Physics Enters Topology

In this section, we explore how physical theories, particularly gauge theories, have influenced topology.

2.2.1 Yang-Mills Theory

We previously saw how quantum particles interact with an electromagnetic field through a $U(1)$ connection. This naturally leads to the question: can other structure groups describe different physical phenomena? In 1954, Yang and Mills [YM54] extended Maxwell's theory by introducing an $SU(2)$ gauge symmetry, inspired by isospin descirption of nucelei. Remarkably, they discovered many formulas from bundle theory purely from physical intuition, without knowledge of fiber bundles.

Their theory considers an $SU(2)$ bundle $E \to M$ with curvature F_A, and the Yang-Mills action:

$$YM(A) = -\frac{1}{4}\int_M ||F_A||^2 dvol = -\frac{1}{4}\int_M \mathrm{tr}(F_A \wedge *F_A).$$

The associated Euler-Lagrange equations are:

$$d_A * F_A = 0,$$

analogous to Maxwell's equations. Since F_A is a curvature 2-form, it also satisfies the Bianchi identity $d_A F_A = 0$.

Of particular interest are absolute minima of the Yang-Mills action which correspond to solutions of the anti-self-duality equation:

$$*F_A = -F_A.$$

Solutions to these equations, known as instantons, first appeared as anti-self-dual connections on the $SU(2)$ bundle $S^3 \hookrightarrow S^7 \to S^4$ in the physics literature [Bel+75]. Their study led to Donaldson theory, revolutionizing the differential topology of 4-manifolds.

2.2.2 Topological Quantum Field Theory

The influence of physics on the topology of four-manifolds extends beyond suggesting the right equations and solving them in simple cases. Edward Witten provided a deeper interpretation of Donaldson invariants in terms of a supersymmetric quantum field theory, which ultimately led him to the Seiberg-Witten invariants. While I do not yet fully understand the details, I will attempt to convey some fundamental ideas.

We have previously seen the operator formulation of quantum mechanics, but an alternative approach—introduced by Paul Dirac and Richard Feynman—known as the *path integral formulation*, plays a central role in modern physics. This formulation, though notoriously difficult to make rigorous, is conceptually appealing and remarkably effective.

Starting with a field theory as discussed above, we denote by $\mathcal{F}$ the space of field configurations (e.g., sections or connections on bundles) that make the action stationary. The theory possesses symmetries described by a Lie group G, under which physically indistinguishable configurations lie in the same orbit. The moduli space of physical configurations is then given by

$$\mathfrak{M} = \mathcal{F}/G.$$

Observables in the theory can be understood as real-valued functions

$$O : \mathfrak{M} \to \mathbb{R},$$

which assign a measurable quantity to a physical configuration. In the path integral formulation, "quantization" assigns expectation values to observables:

$$\langle O \rangle = \int_{\mathfrak{M}} e^{-\frac{S(\phi)}{e^2}} O([\phi])[D\phi],$$

where $[D\phi]$ is a (generally non-existent) measure on $\mathfrak{M}$, and e is the coupling constant. If these expectation values are independent of the underlying metric of M, the theory is called a **topological quantum field theory (TQFT)**.

Remark 2.3. *When Michael Freedman[2] learned that topological quantum field theories yield observables measuring topological properties of the underlying space, he saw an extraordinary application: many topological properties are computationally intractable (NP-hard or even undecidable). However, if a physical system could realize a TQFT, one could bypass these computational difficulties by simply measuring! This insight led Freedman to propose the idea of a* ***topological quantum computer****.*

Such devices are now shown equivalent to standard quantum computers, but they have a major advantage: due to their topological nature, they are inherently resistant to noise. Freedman now leads a research team at Microsoft Station Q exploring this possibility. The only known naturally occurring TQFTs are two-dimensional anyons, which play a crucial role in the fractional quantum Hall effect. The primary challenge in building a topological quantum computer remains engineering feasibility.

[2] According to his own account, see [Mic24]

The first TQFT was constructed by Edward Witten in 1988 [Wit94] where he interpreted Donaldson invariants as expectation values of certain observables. He arrived at this theory by quantizing a supersymmetric gauge theory with field content

$$A \in \mathcal{A}(P), \quad \underbrace{\phi \in \Omega^0(M, ad(P)), \lambda \in \Omega^0(ad(P))}_{\text{bosonic}},$$

$$\underbrace{\eta \in \Omega^0(M, ad(P)), \chi \in \Omega^1(M, ad(P)), \psi \in \Omega^2_+(M, ad(P))}_{\text{fermionic}}.$$

Writing $\xi = (A, \phi, \lambda, \eta, \chi, \psi)$, the Donaldson-Witten action takes the form

$$S_{DW} = \int_M \operatorname{tr}\Big(-\frac{1}{4} F_A \wedge *F_A - \frac{1}{4} F_A \wedge F_A + \frac{1}{2}[\psi, \psi]\phi + i d_A \chi \wedge \psi$$

$$-2i[\chi, *\chi]\lambda + i * (\phi \Delta_0 \lambda) + \chi \wedge *d_A \eta \Big).$$

Quantizing the theory via the path integral leads to the partition function:

$$Z = \int_{\mathfrak{M}} e^{-\frac{S_{DW}(\xi)}{e^2}} [D\xi].$$

The constant e is the coupling constant, which significantly affects the quantized theory. In the *weak coupling limit* ($e \ll 1$), perturbation theory can often be used, while the *strong coupling limit* remains challenging.

In the weak coupling limit, Witten identified the partition function with the zeroth Donaldson invariant. Moreover, he argued that, due to the system's inherent symmetries, the partition function and expectation values should be independent of the coupling constant. This suggested that insights into Donaldson theory could be gained by analyzing the strong coupling limit—a challenge that remained open in 1988.

Six years later, Seiberg and Witten [SW94a] [SW94b] discovered how to perform calculations in the strong coupling regime, leading Witten to Donaldson's dual theory, now known as **Seiberg-Witten theory**.

3 Preliminaries

The purpose of this chapter is to introduce the preliminaries for understanding the Seiberg-Witten equations. The study of these equations intersects geometry, analysis, and algebra, so we will develop the essential tools from each of these fields. For more detailed expositions and proofs, we refer to [Law94; Nab97; Nab00; Wen22; Nat10; Mit11]

The first section establishes the geometric framework by introducing bundles, connections, and curvature. The second section takes an analytic perspective, discussing elliptic complexes and and the Hodge–Dirac operator. As a central tool in Hodge theory, this operator plays a crucial role in Seiberg–Witten theory, allowing us to show that the moduli space of (irreducible) solutions is a finite-dimensional manifold and to compute its dimension. Finally, the third and most detailed section explores the algebraic structures underlying the Seiberg-Witten equations, namely Clifford algebras, spin groups, and spin structures. Here, we introduce the Spin–Dirac operator, which appears explicitly in one of the Seiberg–Witten equations. We also show that all four-manifold admit $spin^c$-structure ensuring that Seiberg–Witten theory can be applied to all smooth four-manifolds

3.1 Bundles, Connections, and Curvature

This section will not develop the theory of bundles from scratch but rather recall the most important definitions and result that will be important in understanding seiberg-witten theory.

3.1.1 Bundles

Definition 3.1.1. Group of Gauge Transformations

Let $E_1 \to X$ and $E_2 \to X$ be two fiber bundles with standard fiber F. A **G-bundle isomorphism** is a diffeomorphism $\Phi : E_1 \to E_2$ respecting the fiber structure, meaning:

$$1) \text{ Fiber-preserving: } \pi_2 \circ \Phi = \pi_1 \quad 2) \text{ G-equivariance: } \Phi(p \cdot g) = \Phi(p) \cdot g$$

We call the group (under composition) of G-bundle Isomorphisms of a principal bundle P, the (global) **Group of Gauge Transformations** $\mathcal{G}(P)$. Since G acts freely and transitively on the fibers of P, for every $p \in P$ there is a unique $g \in G$ s.t. $p \cdot g = \Phi(p)$. It turns out to be very useful to think of Φ as:

$$\Phi(p) = p \cdot \psi(p) \quad \text{where} \quad \psi : P \to G$$

Locally, a gauge transformation acts as a change of local section/gauge

$$\Phi(s_\alpha(x)) = s_\alpha(x) \cdot \psi(s_\alpha(x)) =: s_\alpha(x) \cdot g_\alpha(x) =: s_\alpha^g(x)$$

Since local sections correspond to local trivializations, we can think of gauge transformation as change of local trivializations.

Theorem 3.1.1. Classifying space

Suppose $EG \to BG$ is a principal bundle and P is weakly contractible, then all (isomorphism classes of) principal G-bundles over X are classified by homotopy classes of a maps from X into BG denoted $[X, BG]$. The bijection is given by:

$$\phi : [X, BG] \to \mathcal{P}_G X, \quad f \mapsto f^* EG$$

We call BG the classifying space and EG the universal bundle.

Example 3.1. *The classifying space for vector bundles of fiber dimension k is the Grassmanian of k-dimensional subspaces in* $\mathbb{R}^\infty$ *and the universal bundle is the tautological bundle over* $\mathbb{R}P^\infty$

One should regard principal bundles as a prototype for all G-bundles with the same (equivalence class) of transition functions. In fact, we can construct all other G-bundles with the associated bundle construction

Theorem 3.1.2. Assoicated bundle construction

Let P be the unique (up to isomorphism) principal G-bundle from a given set (equivalence class) of transition functions. We can then construct the G-bundle E with standard fiber F acted on by G via ρ by using the balanced product:

$$P^{\rho} = P \times_{\rho} F$$

$$[x, p] = [y, q] \quad \Leftrightarrow \quad (y, q) = (x \cdot g, \rho(g^{-1}) \cdot p)$$

We will call this bundle the **associated bundle** with fiber F to the principal bundle P.

This construction is very powerful, as it will allow us to transfer geometric structures and results from a principal bundle to any associated bundle.

Definition 3.1.2. Spaces of sections

The (infinite dimensional) vector space of smooth local sections $s_\alpha : U_\alpha \to E|_{U_\alpha}$ of a bundle E is denoted by $\Gamma(E|_{U_\alpha})$. The vector space of smooth global sections of the bundle of differential k-forms is denoted $\Omega^k(M) = \Gamma(\Lambda^k(T^*M))$ and the space of smooth global sections of V-valued or E-valued k-forms is denoted $\Omega^k(M, V) = \Gamma(\Lambda^k(T^*M) \otimes V)$ or $\Omega^k(M, E) = \Gamma(\Lambda^k(T^*M) \otimes E)$ respectively.

Transition functions of a bundle encode the relative "twist and stretch" of fibers between overlapping trivializations. This perspective treats the fibers over every patch as trivial, and accounts for the globally non-trivial structure by prescribing the relative "twist and stretch" on overlapping regions.
Connections provide an alternative perspective. Instead of focusing solely on the discrete changes at overlaps, connections describe how the bundle "twists and stretches" continuously as you go from fiber to fiber. If our bundle is smooth, this continuous description allows us to track the behavior of fibers infinitesimally, enabling a deeper geometric understanding of the bundle's structure and its relation to the base manifold.

3.1.2 Connections

Definition 3.1.3. Connections on smooth fiber bundles

There are multiple ways one can describe a **connection** on a smooth fiber bundle $E \to M$:

1. Parallel transport
2. Horizontal subbundle
3. Connection 1-Form This is the projection along this horizontal subbundle, projecting into the vertical subbundle:
$$\Omega^1(E, VE) \ni K : TE \to VE$$
4. Covariant Derivative

One can naturally extend the covariant derivative to an exterior covariant derivative

Definition 3.1.4. Exterior covariant derivative

Regarding sections $s \in \Gamma(E) = \Omega^0(M, E)$ as E valued 0-forms on M, one can regard the covariant derivative of a section as an exterior derivative of E-valued 0-forms, called the **exterior covariant derivative**:

$$d_\nabla s(X) := \nabla_X s$$

It extends to all of $\Omega^*(M, E)$. One can describe this extension (just like the covariant derivative itself) axiomatically using the graded leibniz rule or in local coordinates by a formula analogous to the one for the ordinary exterior derivative but using ∇ for differentiation.

Remark 3.1. *The perspective of a connection as a derivative of E-valued forms, reveals that calculus with bundle valued objects requires one to take into account an "intrinics change" due to the "twist and stretch" of the fibers. This allows one to model external influence on bundle valued objects for example the influence of the electro magnetic field on charged particles as we have seen in the last chapter.*

To simplify the connection on principal bundles we need

Definition 3.1.5. Fundamental vector field

The action of G on P can be differentiated at the identity to identify $\mathfrak{g} = T_{id}G$ with V_pE. For $X \in \mathfrak{g}, p \in P, \sigma_p(g) = p \cdot g$ we define the **fundamental vector field** to be:

$$X^F(p) := (T\sigma_p)_{id}(X) = \frac{d}{dt} p \cdot exp(tX)\big|_{t=0} \in V_pP$$

Where $exp(X) : \mathbb{R} \to G, t \mapsto exp(tX)$ is the flow of the left invariant Vector field of X starting at $exp(0) = id \in G$ which happens to be the same as the flow of the right invariant vector field.

Lemma 3.1. *The push forward of the fundamental vector field of $X \in \mathfrak{g}$ by $g \in G$ equals the fundamental vector field of $ad_{g^{-1}}(X)$:*

$$(R_g)_* X^F = (ad_{g^{-1}}(X))^F$$

Proof.

$$(R_g)_* X^F(pg) = (TR_g)_p \circ (T\sigma_p)_{id}(X) = T(R_g \circ \sigma_p)_{id}(X)$$

$$= T(h \mapsto phg = \sigma_{pg} \circ Ad_{g^{-1}}(h))(X) = (T\sigma_{pg})_{id} \circ (TAd_{g^{-1}})_{id}(X) = (ad_{g^{-1}}(X))^F(pg)$$

□

Definition 3.1.6. Connection on principal bundle

The extra structure on a principal bundle allows us to convert connections into $\mathfrak{g}$-valued 1-forms which will make this important object much easier to deal with. The details of this construction can be found in most text books or lecture notes. The upshot is that the connection 1-form $K \in \Omega^1(P, VE)$ is equivalent to $A \in \Omega^1(P, \mathfrak{g})$ satisfying:

1. $A(X^F(p)) = X \quad \forall p \in P, X \in \mathfrak{g}$
2. $R_g^* A = ad_{g^{-1}} \circ A$
   We will denote the space of connections on P by $\mathcal{A}(P)$.

To connect forms on manifolds with forms on bundles, we have to restrict the forms on the bundles by symmetry

Definition 3.1.7. Tensorial forms

We say that a V-valued k-form $\hat{\omega} \in \Omega^k(P, V)$ is **tensorial of type** ρ if it satisfies

$$R_g^* \hat{\omega} = \rho(g^{-1}) \cdot \hat{\omega} \quad \text{and} \quad \hat{\omega} = \hat{\omega} \circ H$$

where R_g is right multiplication, ρ is a representation of G on V, $p \in P_x, x \in M, X \in TP_p$, and H is the horizontal projection (of all k input vectors). We will denote such forms by: $\Omega^k_\rho(P, V) \subset \Omega^k(P, V)$

The following theorem provides the crucial link between bundle valued forms on M and equivariant V-valued forms on principal bundles.

Theorem 3.1.3. Tensorial forms descend

The space of V-valued k-forms of tensorial type ρ is isomorphic to k-forms on M with values in the associated bundle:

$$\Omega^k(M, P \times_\rho V) \cong \Omega^k_\rho(P, V)$$

Proof. One easily checks that a local isomorphism is given by pullback via a local section and that the isomorphism defined in that way is actually independent of the local section. □

Remark 3.2. *For V-valued 0-forms $\phi \in \Omega^0_\rho(P, V)$, being tensorial of type ρ just means that:*

$$\phi(p \cdot g) = \rho(g^{-1}) \cdot \phi(p)$$

*In physics, such objects occur (locally after choosing a section and coordinate chart) as the wave functions of particles. For $V = \mathbb{C}^k$ they are called **matter fields**. The representation ρ is characteristic of the particle. The former theorem 3.1.3 assures that matter fields are actually globally defined, $\mathbb{C}^k$-valued objects on the base manifold (space-time).*

Remark 3.3. *The space of connections $\mathcal{A}(P)$ on a principal bundle P is not tensorial but any difference is. This shows that $\mathcal{A}(P)$ is an affine space based on $\Omega^1_{ad}(P, \mathfrak{g}) \cong \Omega^1(M, ad(P))$*

We can use the correspondence between bundle valued form on M and tensorial forms on the the bundle to get an easier description of the exterior covariant derivative.

Theorem 3.1.4. Covariant exterior derivative on bundle

For $\hat{\omega} \in \Omega^k_\rho(P, V)$ evaluating the exterior derivative of $\hat{\omega}$ on the horizontal part of the input vectors, preserves the ρ-equivariance and is equal to the exterior derivative evaluated on the input vectors plus the wedge product of the connection with the form as correction term

$$d_A\hat{\omega}(X_0, X_1, ..., X_k) := d\hat{\omega}(H(X_0), ..., H(X_k)) = d\hat{\omega}(X_0, ..., X_k) + A \wedge \hat{\omega}(X_0, ..., X_k)$$

This is also called **covariant exterior derivative**. It corresponds via theorem 3.1.3 to the exterior covariant derivative $d_\nabla : \Omega^k(M, P^\rho) \to \Omega^{k+1}(M, P^\rho)$ from definition 3.1.4:

$$d_\nabla \omega = [p, d_A\hat{\omega}] \in \Omega^{k+1}(M, P^\rho)$$

Note, that graded Leibniz rules satisfied by d, translate to graded Leibniz rules for d_∇. Those can be used for axiomatic definition and to prove the explicit formula for d_∇ from definition 3.1.4.

Remark 3.4. *We have seen in the last chapter (see 2.1.1), how the covariant exterior derivative of matter fields (see 3.2) arouse on its own accords in physics to keep the Schrödinger equation invariant under local phase changes. Let $\varphi : U_\alpha \to \mathbb{R}^n$ be a coordinate chart of M, G be a Matrix Lie Group with Lie Algebra $\mathfrak{g}$, $s \in \Omega^0(U_\alpha, P)$ be a local section of a principal G-bundle $P \to M$, $A \in \mathcal{A}(P)$ be a connection on P, and $\phi \in \Omega^0_{ad}(P, \mathbb{C}^k)$.*
Locally the exterior covariant derivative is given by

$$(s \circ \varphi^{-1})^*(d_A\phi) = d(\phi \circ (s \circ \varphi^{-1})) + ((s \circ \varphi^{-1})^* A) \cdot (\phi \circ (s \circ \varphi^{-1})) =: d\phi_\alpha + A_\alpha \cdot \phi_\alpha$$

where $\phi_\alpha \in \Omega^0(\varphi_* U_\alpha, \mathbb{C}^k) = \Omega^0(\mathbb{R}^n, \mathbb{C}^k)$ *and* $A_\alpha \in \Omega^1(s_* U_\alpha, \mathfrak{g}) = \Omega^1(\mathbb{R}^n, \mathfrak{g})$*. Now choosing a basis* $\{T_i\}$ *of* $\mathfrak{g}$ *(consisting of matrices), we can write for* $X \in \mathbb{R}^n$

$$A_\alpha(X) = (A_\alpha)_i dx^i(X) = (A_\alpha)^k_i T_k dx^i(X) = (A_\alpha)^k_i T_k X^i \in \mathfrak{g} \subset M^{k\times k}$$

and thus arrive at the formula physicists use for computation:

$$d\phi_\alpha + A_\alpha \cdot \phi_\alpha = \left(\frac{\partial \phi_\alpha}{\partial x^i} + \big((A_\alpha)^k_i T_k\big)\phi_\alpha\right) dx^i$$

Before moving on to curvature, we want to state the general formula for how a connections transform under gauge transformation. Recall that it was this formula that physicists discovered by requiring the schrödinger equation to be gauge invariant.

Lemma 3.2. *The action of* $\Phi \in \mathcal{G}(P)$ *on* $A \in \mathcal{A}(P)$ *is given by:*

$$\Phi \cdot A_{\Phi(p)} = \Phi^* A_{\Phi(p)} = ad_{(\psi(p))^{-1}} \circ A_p + (TL_{(\psi(p))^{-1}})_{\psi(p)} \circ (T\psi)_p$$

where $\Phi(p) = p \cdot \psi(p)$ *with* $\psi \in \Omega^0_\rho(P, G)$

It is now time to take quick look at curvature.

3.1.3 Curvature

Definition 3.1.8. Curvature 2-form

The curvature 2-Form $F_A \in \Omega^2(P, \mathfrak{g})$ of a principal bundle P is defined by:

$$F_A(\xi, \eta) := H^* dA(\xi, \eta) = -A([H\xi, H\eta])$$

where $H : TP \to HP$ is the horizontal projection and $\xi, \eta \in TP$.

Proof. Using a well-knownn formula for 2-forms, we get:

$$H^* dA(\xi, \eta) = \mathcal{L}_{H\xi} \underbrace{A(H\eta)}_{=0} - \mathcal{L}_{H\eta} \underbrace{A(H\xi)}_{=0} - A([H\xi, H\eta]) = -A([H\xi, H\eta])$$

□

Theorem 3.1.5. Second structural equation

$$F_A = dA + \frac{1}{2}[A, A]$$

Lemma 3.3. *The curvature 2-Form* F_A *is ad-equivariant, meaning:*

$$R_g^* F_A = ad_{g^{-1}} \circ F_A$$

and horizontal, meaning that it vanishes on vertical vectors. Thus it is tensorial of type ad

$$F_A \in \Omega^2_{ad}(P, \mathfrak{g}) \subset \Omega^2(P, \mathfrak{g})$$

and according to theorem 3.1.3 corresponds to a

$$F_\nabla \in \Omega^2(M, ad(P))$$

Theorem 3.1.6. Curvature tensor and curvature form

For $\omega \in \Omega^k_\rho(P, V)$ the following equation holds:

$$d_A^2\omega = F_A \wedge \omega \in \Omega^{k+2}_\rho(P, V)$$

which under theorem 3.1.3 translates into

$$d_\nabla^2\hat{\omega} = F_\nabla \wedge \hat{\omega} \in \Omega^{k+2}(M, P^\rho)$$

for $\hat{\omega} \in \Omega^k(M, E^\rho)$ and the bundle valued curvature form $\Omega_A \in \Omega^2(M, ad(P))$. Note that $\wedge : \Omega^2(M, ad(P)) \times \Omega^k(M, P^\rho) \to \Omega^{k+2}(M, P^\rho)$
For $\eta \in \Omega^0(M, P^\rho) = \Gamma(P^\rho)$ this bundle valued form is better known as the **Riemannian curvature tensor** $R : TM \oplus TM \oplus P^\rho \to P^\rho$:

$$R(.,.)\eta = F_\nabla \wedge \eta \in \Omega^2(M, P^\rho)$$

Using the explicit definition definition 3.1.4 of d_∇ we obtain an explicit formula for the curvature tensor:

$$R(X, Y)\eta = F_\nabla \wedge \eta = d_\nabla(\nabla\eta)(X, Y) = \nabla_X\nabla_Y\eta - \nabla_Y\nabla_X\eta - \nabla_{[X,Y]}\eta$$

3.2 Elliptic Complexes and Hodge Theory

Elliptic complexes provide a natural generalization of the de Rham complex, offering a powerful analytical framework for studying geometric and topological structures via the Hodge–Dirac operator. In the context of Seiberg–Witten theory, these tools are essential for proving that the moduli space of irreducible solutions is a finite-dimensional manifold and for determining its dimension.

We will not go into too much analytical detail, such as Sobolev spaces and regularity theory, but instead focus on the geometric and topological implications of these concepts.

Definition 3.2.1. Differential Operator

Let E, F be vector bundles over M of rank k and l respectively. A linear map $D : \Gamma(E) \to \Gamma(F)$ is called a **differential operator of order m** if, by identifying $\Gamma(E|_U)$ with $C^\infty(U, \mathbb{R}^k)$ and $\Gamma(F|_V)$ with $C^\infty(V, \mathbb{R}^l)$ via local trivializations, the operator is written as:

$$C^\infty(U, \mathbb{R}^k) \to C^\infty(V, \mathbb{R}^l), f \mapsto \sum_{|\alpha| \leq m} c_\alpha \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \sum_i \alpha_i$, and $c_\alpha : U \to Hom(\mathbb{R}^k, \mathbb{R}^l)$.

We remark that every differential operator has a formal adjoint. The behaviour of a differential operator is fundamentally determined by

Definition 3.2.2. Principal Symbol

Let $\pi : T'M \to M$ be the bundle of non-zero cotangent vectors. The **principal symbol** $\sigma(D)$ of a differential operator $D : \Gamma(E) \to \Gamma(F)$ is the unique smooth fiber-preserving map

$$\sigma(D) : T'M \to Hom(E, F)$$

which for every $\xi_p \in T'_pM$ satisfies:

$$\sigma(D)(\xi_p) = \left(\eta(p) \mapsto \frac{1}{m!} D(f^m \eta)(p) \right) \in Hom(E_p, F_p)$$

where $f \in C^\infty$ with $f(p) = 0, d_p f = \xi_p$ and $\eta \in \Gamma(E)$ are arbitrary (ie $\sigma(D)$ does not depend on them). One can alternatively think of $\sigma(D)$ as a bundle homomorphism:

$$\sigma(D) : \pi^* E \to \pi^* F$$

where a point $(\xi_p, q) \in \{(\xi_p, q) \in T'M \times \pi^* E \mid q \in \pi^*(i\Lambda^0(M))_{\xi_p}\} \cong \pi^* E \subset T'M \times \pi^* E$ [a] is mapped to

$$\sigma(D)(\xi_p, \eta_p) := (\xi_p, \sigma(D)(\xi)\eta(p)) \in \pi^* F \subset T'M \times \pi^* F$$

[a] In general a bundle $E \to M$ can naturally be embedded in $M \times E$ by $q \mapsto (\pi(q), q) \in M \times E$

In many applications, one is confronted with

Definition 3.2.3. Differential Complex

Let $E_0, ..., E_n$ be differentiable vector bundles over a compact differentiable manifold M and $D_0, ..., D_{n-1}$ be differential operators of order k. A **differential complex** is a sequence

$$\Gamma(E_0) \overset{D_0}{\to} \Gamma(E_1) \overset{D_1}{\to} \dots \overset{D_{n-1}}{\to} \Gamma(E_n)$$

s.t. $D_{i+1} \circ D_i = 0$ and $im(D_i) \subseteq ker(D_{i+1})$. If the corresponding sequence of symbols

$$0 \to \pi^* E_0 \overset{\sigma(D_0)}{\to} \pi^* E_1 \overset{\sigma(D_1)}{\to} \dots \overset{\sigma(D_{n-1})}{\to} \pi^* E_n \to 0$$

is exact (ie $im(\sigma(D_i)) = ker(\sigma(D_{i+1}))$), we call the complex an **elliptic complex** and write

$$\mathcal{E} = \bigoplus_i E_i, \quad \mathcal{E}^{\text{even}} = \bigoplus_i E_{2i}, \quad \mathcal{E}^{\text{odd}} = \bigoplus_i E_{2i+1}$$

Further, we define cohomology groups (vector spaces) by (with the obvious trivial extensions at both ends of the complex):

$$H^k(\mathcal{E}) = ker(D_k)/im(D_{k-1})$$

As $ker(D_k)$ and $im(D_k)$ are infinite dimensional vector spaces, it is a priori not clear that these quotient spaces are finite dimensional. It will turn out to be the case for elliptic complexes.

A single differential operator $D : E \to F$ defines a differential complex with symbol sequence:

$$0 \to \pi^* E \overset{\sigma(D)}{\to} \pi^* F \to 0$$

It is elliptic if $ker(\sigma(D)) = 0$ and $\pi^* F = im(\sigma(D))$ or equivalently if $\sigma(D) : \pi^* E \to \pi^* F$ is invertible. In this case, if L defines an elliptic complex, we call the D an **elliptic operator**. These have nice properties

Theorem 3.2.1. Elliptic operatos have finite dimensional kernels

Let $D : \Gamma(E) \to \Gamma(F)$ be an elliptic operator, then the vector spaces

$$ker(D) \quad \text{and} \quad ker(D^*)$$

are finite dimensional.

Proof. Here is a very rough outline of what has to be shown in order to prove this:

1. A Hilbert Space is finite dimensional if its closed and bounded subsets are compact
2. We can extend differential operators $D : \Gamma(E) \to \Gamma(F)$ to operators between Hilbert Spaces (Sobelev Spaces)
3. (the extension of) Elliptic differential operators satisfy a global estimate from which it follows (together with 1) that ker(D) is finite dimensional and im(D) is closed
4. Elliptic regularity implies that the kernel consists of smooth sections

□

Another remarkable property of elliptic operators is:

Theorem 3.2.2. Direct sum splitting

Let $D : \Gamma(E) \to \Gamma(F)$ be an elliptic operator, then we have the following direct sum splitting:

$$\Gamma(F) = im(D) \perp ker(D^*) \quad \Gamma(E) = im(D^*) \perp ker(D)$$

Proof. Note that we get the second equality by exchanging D with D^*. It is therefore sufficient to show the first.
1. We prove that $ker(D^*) = im(D)^\perp$:
$\xi \in ker(D^*) \subset \Gamma(F) \Leftrightarrow \forall \eta \in \Gamma(E) : 0 = \langle D^*\xi, \eta\rangle = \langle \xi, D\eta\rangle \Leftrightarrow \xi \in im(D)^\perp$.
2. We now show that $im(D)$ and $ker(D^*)$ span all of $\Gamma(F)$:
If $\Gamma(F)$ was a Hilbert space, this would follow immediately. Since $L^2(F) = H^0(F)$ is a Hilbert space, we have $H^0(F) = im(\tilde{D}) \oplus im(\tilde{D})^\perp = im(\tilde{D}) \oplus ker(\tilde{D}^*)$ for the unique extensions of D and D^*. Now, for $\xi \in \Gamma(F) \subset H^0(F)$, we have $\xi = \tilde{D}\eta + \xi'$ with $\xi' \in ker(\tilde{D}^*) = ker(D^*)$ (which we have already shown is smooth). Thus $\tilde{D}\eta = \xi - \xi'$ is also smooth and elliptic regularity implies that also η is smooth and thus $\xi \in im(D) \oplus ker(D^*)$.

□

Corollary 3.2.1. Cokernel of elliptic operator

For an elliptic operator D:
$$coker(D) = \Gamma(F)/im(D) = ker(D^*)$$

We can now define a Dirac and Laplace operator for any differential complex

Definition 3.2.4. Hodge Dirac and Hodge Laplace Operator

For an elliptic complex $\mathcal{E}$ we can define the following operators:
$$D = D_0 \oplus \cdots \oplus D_{n-1}$$
with adjoint operator D^*. The **Hodge Dirac Operator** $\not{D}_\mathcal{E}$ of the complex $\mathcal{E}$ is defined by:
$$\not{D}_\mathcal{E} := D + D^* :$$
and squares to the **Hodge Laplace Operator** $\Delta_\mathcal{E}$:
$$\Delta_\mathcal{E} = \not{D}_\mathcal{E}^2 = DD^* + D^*D$$
Both $\Delta_\mathcal{E}$ and $\not{D}_\mathcal{E}$ are self-adjoint.

These operators will be elliptic iff the differential complex is elliptic:

Theorem 3.2.3. Ellipticity of differential complex

For an differential complex $\mathcal{E}$, the following are equivalent

1. $\mathcal{E}$ is elliptic
2. $\not{D}_\mathcal{E}$ is elliptic
3. $\Delta_\mathcal{E}$ is elliptic

It is this elliptic operator from which we can extract most information. To prove the 2 main theorems, we need 2 lemmas:

Lemma 3.4.
$$ker(\Delta_\mathcal{E}) = ker(D) \cap ker(D^*) = ker(\not{D}_\mathcal{E})$$
is finite dimensional.

Proof. First equality:
We have $ker(\Delta) \subseteq ker(D) \cap ker(D^*)$:
$$x \in ker(\Delta) \Rightarrow 0 = \langle \Delta x, x\rangle = \langle (DD^* + D^*D)x, x\rangle = \langle D^*x, D^*x\rangle + \langle Dx, Dx\rangle$$
$$\Rightarrow x \in ker(D) \cap ker(D^*)$$

and also $ker(D) \cap ker(D^*) \subseteq ker(DD^* + D^*D) = ker(\Delta)$.
Second equality:
We have $ker(D) \cap ker(D^*) \subseteq ker(D + D^*) = ker(\not{D})$ and also $ker(\not{D}) \subseteq ker(\not{D}^2) = ker(\Delta)$
The finite dimensionality follows from that $\Delta_{\mathcal{E}}$ is an elliptic operator. □

Lemma 3.5.

$$im(\Delta_{\mathcal{E}}) = im(D_{\mathcal{E}}) \perp im(D^*_{\mathcal{E}})$$

Proof. $im(D_{\mathcal{E}})$ and $im(D^*_{\mathcal{E}})$ are orthogonal, because: $\langle D_{\mathcal{E}}\xi, D^*_{\mathcal{E}}\eta\rangle = \langle D^2_{\mathcal{E}}\xi, \eta\rangle = 0$ for any $\xi, \eta \in \Gamma(\mathcal{E})$
$\subseteq$:

$$im(\Delta_{\mathcal{E}}) = im(D^*_{\mathcal{E}}D_{\mathcal{E}} + D_{\mathcal{E}}D^*_{\mathcal{E}}) \subseteq im(D^*_{\mathcal{E}}D_{\mathcal{E}}) \oplus im(D_{\mathcal{E}}D^*_{\mathcal{E}}) \subseteq im(D_{\mathcal{E}}) \oplus im(D^*_{\mathcal{E}})$$

$\supseteq$: By theorem theorem 3.2.2 and because $\Delta_{\mathcal{E}}$ is self-adjoint, we have $\Gamma(\mathcal{E}) = im(\Delta_{\mathcal{E}}) \perp ker(\Delta_{\mathcal{E}})$. To show that $im(D_{\mathcal{E}}) \oplus im(D^*_{\mathcal{E}}) \subseteq im(\Delta_{\mathcal{E}})$ we can thus show that it is in the orthogonal complement of $ker(\Delta_{\mathcal{E}})$. So let $\omega \in ker(\Delta_{\mathcal{E}})$ and $\xi = D_{\mathcal{E}}\eta + D^*_{\mathcal{E}}\xi' \in im(D_{\mathcal{E}}) \oplus im(D^*_{\mathcal{E}})$, then:

$$\langle \xi, \omega\rangle = \langle \xi, D^*_{\mathcal{E}}\omega\rangle + \langle \xi, D_{\mathcal{E}}\omega\rangle = 0$$

because by lemma 3.4 $ker(\Delta_{\mathcal{E}}) = ker(D_{\mathcal{E}}) \cap ker(D^*_{\mathcal{E}})$.

□

We can finally state the two powerful results we were seeking to establish:

Theorem 3.2.4. Generalized Hodge Decomposition

$$\Gamma(\mathcal{E}) = ker(\Delta_{\mathcal{E}}) \perp im(D^*_{\mathcal{E}}) \perp im(D_{\mathcal{E}}) = ker(\not{D}_{\mathcal{E}}) \perp im(D^*_{\mathcal{E}}) \perp im(D_{\mathcal{E}})$$

Proof. Since $\Delta_{\mathcal{E}}$ is elliptic, by theorem 3.2.2 we have a splitting:

$$\Gamma(\mathcal{E}) = ker(\Delta_{\mathcal{E}}) \perp im(\Delta^*_{\mathcal{E}}) = ker(\Delta_{\mathcal{E}}) \perp im(\Delta_{\mathcal{E}})$$

By lemma 3.5 we further have:

$$\Gamma(\mathcal{E}) = ker(\Delta_{\mathcal{E}}) \perp im(D_{\mathcal{E}}) \perp im(D^*_{\mathcal{E}})$$

□

Theorem 3.2.5. Generalized Hodge Theorem

Writing $\Delta_k = \Delta_{\mathcal{E}}|_{E_k}$ and $\not{D}_k = \not{D}_{\mathcal{E}}|_{E_k}$, the generalized **Hodge Theorem** asserts that we have an isomorphism:

$$ker(\Delta_k) = ker(\not{D}_k) \cong H^k(\mathcal{E}) = \frac{ker(D_k)}{im(D_{k-1})}$$

Proof. The isomorphism is given by the quotient projection:

$$ker(\Delta_k) \ni \omega \mapsto [\omega] = \omega + im(D_{k-1}) \in H^k(\mathcal{E})$$

To show injectivity, it suffices to show that the kernel of this quotient projection is trivial: If $[\omega] = 0$, then $\omega \in im(D_{k-1}) \subset im(D_{\mathcal{E}})$ which by theorem 3.2.4 is orthogonal to $ker(\Delta_{\mathcal{E}}) \supset ker(\Delta_k)$.
For surjectivity, take $\omega \in ker(D_k)$ and apply the hodge decomposition theorem 3.2.4:

$$\omega = \omega_0 + D_{k-1}\xi + D^*_k\eta$$

for $\omega_0 \in ker(\Delta_{\mathcal{E}}), \xi \in \Gamma(E_{k-1}), \eta \in \Gamma(E_k)$. We wish to show that $D^*_k\eta = 0$ because then $\omega_0 \mapsto [\omega_0] = [\omega]$. For this, apply, D_k to ω:

$$D_k\omega = 0 = D_k\omega_0 + D_kD_{k-1}\xi + D_kD^*_k\eta = 0 + 0 + D_kD^*_k\eta$$

as by lemma 3.4 $ker(\Delta) = ker(D_{\mathcal{E}}) \cap ker(D^*_{\mathcal{E}}) \subset ker(D_k)$ and $D_kD_{k-1} = 0$. Now:

$$0 = \langle D_kD^*_k\eta, \eta\rangle = \langle D^*_k\eta, D^*_k\eta\rangle$$

and thus $D^*_k\eta = 0$ □

We can extract even more information from an elliptic complex.

Definition 3.2.5. Index and Euler Characteristic

The index of an elliptic differential operator D is the (finite) number:

$$ind(D) = dim(ker(D)) - dim(coker(D)) = dim(ker(D)) - dim(ker(D^*))$$

For an elliptic complex the hodge-dirac operator $\not{D}$ is self-adjoint and if we restrict $\not{D}^+ := \not{D}|_{\mathcal{E}^{\text{even}}} : \mathcal{E}^{\text{even}} \to \mathcal{E}^{\text{odd}}$ its formal adjoint is $\not{D}^- := \not{D}|_{\mathcal{E}^{\text{odd}}}$. The index becomes an interesting object called the **Euler Characteristic** $\chi(\mathcal{E})$:

$$ind(\not{D}^+) = dim(ker(\not{D}^+)) - dim(ker(\not{D}^-)) = \sum_{\text{k even}} dim(ker(\not{D}_k)) - \sum_{\text{k odd}} dim(ker(\not{D}_k))$$

$$= \sum_{i=0}^{n} (-1)^i dim(H^i(\mathcal{E})) =: \chi(\mathcal{E})$$

Arguably, one of the greatest achievement in 20th century mathematics is the following theorem which exhibits the index of an elliptic as a topological property obtained by integrating certain characteristic classes over M:

Theorem 3.2.6. Atiyah-Singer Index Theorem

The index of an elliptic operator can be calculated as an integral over M of certain characteristic classes of M.

Example 3.2. *For the deRham complex*

$$0 \to \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \to 0$$

deRhams theorem asserts that the cohomology groups are isomorphic to singular cohohomology, implying in particular that the dimension of the deRham cohomology groups equals the betti numbers. The operator $\not{D} = d + d^*$ is called the Hodge-Dirac operator and the index of $\not{D}^+ = \chi(\mathcal{E})$ becomes the ordinary Euler characeristic

$$\chi(\mathcal{E}) = \chi(M) = \sum_{i=0}^{n} (-1)^i b_i(M)$$

By the Atiyah-Singer Index Theorem, this can be expressed in terms of topological data, leading to ***Chern-Gauss-Bonnet theorem:***

$$\chi(M) = \langle e(TM), [M] \rangle$$

where e(TM) denotes the Euler class and TODO more?
On an even dimensional manifold, one can use the chern weil method to express the cap product as:

$$\chi(M) = \int_M e(\Omega_\nabla)$$

for the curvature 2-form Ω_∇ of TM.

3.3 Spin Geometry

In this section, we explore the algebraic structures underlying the Seiberg–Witten equations. We begin by introducing Clifford algebras and their globalization to Clifford algebra bundles. This leads naturally to the study of Clifford modules and the challenge of globalizing them, which motivates the concepts of spin- and $spin^c$-structures. These structures induce associated spinor bundles that serve as irreducible modules for the Clifford algebra bundle and are part of the seiberg-witten configuration space.

While there are generally topological obstructions to the existence of such structures, we'll prove that $spin^c$ structures exists on all 4-manifolds making this powerful algebraic structures applicable to the study of four-manfolds. Finally, we introduce the Spin–Dirac operator, providing another generalization of the dirac operator arising in classical hodge theory on the deRham complex. This operator plays a crucial role in the Seiberg–Witten equations, appearing explicitly in one equation.

3.3.1 Clifford Algebras

Let's start by defining clifford multiplication in a familiar context. For $\alpha \in \Lambda^1(V), \beta \in \Lambda^k(V)$ or $\alpha \in \Lambda^k(V), \beta \in \Lambda^1(V)$ we define **Clifford Multiplication** by:

$$\alpha \cdot \beta = \alpha \wedge \beta - \alpha \wedge^* \beta$$

Note that for $\alpha, \beta \in \Lambda^1(V)$, the interior product is just the inner product and we get the **Clifford Relations:**

$$\alpha \cdot \beta + \beta \cdot \alpha = -2\langle \alpha, \beta \rangle$$

More generally, we define

Definition 3.3.1. Clifford Algebra

Given a finite dimensional vector space V over $\mathbb{R}$ with nondegenerate symmetric bilinear form $\langle , \rangle$, we call the real unit algebra over V satisfying the Clifford Relations the **Clifford Algebra** Cl(V). More concretely, Cl(V) is the real unit algebra generated by V satisfying:

$$v_1 \cdot v_2 + v_2 \cdot v_1 = -2\langle v_1, v_2 \rangle 1_{Cl(V)} \quad \forall v_1, v_2 \in V$$

or equivalently

$$v^2 := v \cdot v = -\langle v, v \rangle 1_{Cl(V)}$$

which yields the above relation by linearity.
One can complexify the Clifford Algebra to get:

$$\mathbb{C}l(V) := Cl(V) \otimes \mathbb{C}$$

For $V = \mathbb{R}^n$ with the standard inner product, we will also write $Cl_n := Cl(n) := Cl(\mathbb{R}^n)$.

Remark 3.5. *The exterior algebra $\Lambda(V)$ can be characterized by the universal property [1] that for an algebra $\mathcal{A}$ every linear map $f : V \to \mathcal{A}$ satisfying:*

$$f(v)f(w) + f(w)f(v) = 0_{\mathcal{A}} \; \forall v, w \in V$$

extends uniquely to an algebra homomorphism:

$$\tilde{f} : \Lambda(V) \to \mathcal{A}$$

Similarly, the Clifford Algebra can be characterized by the universal property that every linear map f satisfying:

$$f(v)^2 = -\langle v, v \rangle 1_{\mathcal{A}}$$

extends uniquely to an algebra homomorphism:

$$\tilde{f} : Cl(V) \to \mathcal{A}$$

[1] Meaning that all other algebras $\tilde{\Lambda}$ satisfying this property, must be isomorphic to $\Lambda(V)$

The Clifford Algebra naturally inherits some structure from the underlying vector space.
Inner product: There is a natural inner product on $Cl(V)$ defined analagously to the one on $\Lambda(V)$ by declaring monomials $e_{i_1} \dots e_{i_k}, 0 \leq k \leq n$ to be an orthonormal basis and extending by linearity.
Group representations: A group representation $\rho : G \to GL(V)$ extends to a representation $\tilde{\rho} : G \to Gl(Cl(V))$ defined by:

$$\tilde{\rho}(g)(v_1 \cdot v_2 \cdot \dots \cdot v_k) = \rho(g)v_1 \cdot \rho(g)v_2 \cdot \dots \cdot \rho(g)v_k$$

As these structures are defined completely analogous for the exterior algebra, it is not hard to see:

Proposition 3.3.1. Isomorphism exterior algebra and clifford algebra

As a vector space $Cl(V)$ is naturally isomorphic to ΛV (but not as an algebra):

$$Cl(V) \cong \Lambda V, \quad e_{i_1} \cdot e_{i_2} \cdot \dots \cdot e_{i_k} \mapsto e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}, 0 \leq k \leq n$$

In fact, we have already seen that one can think of $Cl(V) \cong Cl(\Lambda^1(V))$ as $\Lambda^* V$ equipped with Clifford multiplication [2].
In particular, Cl(V) has a natural $\mathbb{Z}$-grading:

$$Cl(V) = Cl^{(0)}(V) \oplus Cl^{(1)}(V) \oplus \cdots \oplus Cl^{(n)}(V)$$

corresponding to $Cl^{(k)}(V) \cong \Lambda^k(V)$ and $\mathbb{Z}_2$-grading:

$$Cl(V) = Cl_0(V) \oplus Cl_1(V) \cong \Lambda^{\text{even}} V \oplus \Lambda^{\text{odd}} V$$

which respects clifford multiplication:

$$Cl_i(V) \cdot Cl_j(V) \subseteq Cl_{i+j \bmod 2}(V)$$

making Cl(V) into **$\mathbb{Z}_2$-graded Algebra**.

Example 3.3. *Clifford Algebras generalize many known algebras. For example real, complex and quaterionic numbers:*

$$Cl(\mathbb{R}^0) = \mathbb{R} \quad Cl(\mathbb{R}) = \mathbb{C} \quad Cl(\mathbb{R}^2) = \mathbb{H}$$

Classification of Clifford Algebras

Clifford Algebras are very well understood and fully classified. In the even dimensional case (which will mainly concern us) one can readily prove that the complexified clifford algebra is isomorphic to a endomorphism algebra: For n = 2m, we can define the standard orthogonal complex structure J on V. Extending this to $V \otimes \mathbb{C}$, we get a decomposition into $\pm i$ Eigenspaces:

$$V \otimes \mathbb{C} = W \oplus \bar{W} = span\{\frac{e_{2j-1} + ie_{2j}}{\sqrt{2}} | 1 \leq j \leq m\} \oplus span\{\frac{e_{2j-1} - ie_{2j}}{\sqrt{2}} | 1 \leq j \leq m\}$$

A real inner product $\langle , \rangle_{\mathbb{R}}$ on V can bilinearly be extended to $V \otimes \mathbb{C}$, making W and $\bar{W}$ maximally isotropic subspaces because (without loss of generality, assume the inner product is not negative, otherwise change the direction of the inequalities):

$$0 \leq \langle w_1, w_2 \rangle = \langle Jw_1, Jw_2 \rangle = i^2 \langle w_1, w_2 \rangle \leq 0 \text{ for } w_1, w_2 \in W$$

and analogously for $\bar{W}$. It thus defines a bilinear form pairing W and $\bar{W}$. We can now define an linear map f from $V \otimes \mathbb{C}$ to $End(\Lambda^*(W))$ defined by:

$$f(w + \bar{w}) := \left(\psi \mapsto (w + \bar{w}) \cdot \psi := \sqrt{2}(w \wedge \psi - \bar{w} \wedge^* \psi)\right)$$

where $\wedge^*$ uses the bilinear form pairing W and $\bar{W}$. f satisfies the clifford relations:

$$f(v)f(u) + f(u)f(v) = -2\langle v, u \rangle 1_{End(\Lambda^*(V))}$$

[2]The exact relationship is that Cl(V) is the quantization of the exterior algebra $\Lambda^* V$.

and thus, by the universality property (see remark 3.5) f extends to a unique homomorphism:

$$\tilde{f} : \mathbb{C}l(V) \to End(\Lambda^* V)$$

ie a representation of $\mathbb{C}l(V)$. For dimensionality reasons, we get:

Theorem 3.3.1. Classification of complex clifford algebras in even dimension

If V is even dimensional, then $V \otimes \mathbb{C} = W \oplus \bar{W}$ splits in $\pm i$ Eigenspaces and

$$\mathbb{C}l_n \cong End(\Lambda^* W)$$

Proof. We have to show that the clifford relations are satisfied for f. Let $u = w + \bar{w} \in V \otimes \mathbb{C} = W \oplus \bar{W}$.

$$f(u)^2\psi = 2\big(w \wedge (w \wedge \psi - \bar{w} \wedge^* \psi) - \bar{w} \wedge^* (w \wedge \psi - \bar{w} \wedge^* \psi)\big)$$

$$= 2\big(\underbrace{w \wedge (w \wedge \psi)}_{=0} - w \wedge (\bar{w} \wedge^* \psi) - \bar{w} \wedge^* (w \wedge \psi) + \bar{w} \wedge^* (\bar{w} \wedge^* \psi)\big)$$

$$= 2\big(w \wedge (\bar{w} \wedge^* \psi) - \big((\bar{w} \wedge^* w) \wedge \psi - w \wedge (\bar{w} \wedge^* \psi)\big) + 0\big) = -2(\bar{w} \wedge^* w) \wedge \psi$$

$$= -2\langle \bar{w}, w\rangle\psi = -(\langle w, \bar{w}\rangle + \langle \bar{w}, w\rangle)\psi = -\langle w + \bar{w}, w + \bar{w}\rangle\psi = -\langle u, u\rangle\psi$$

□

Before stating the full classifiaction result, note that every finite dimensional vector space is isomorphic to $\mathbb{R}^n$ and hence will the corresponding Clifford Algebras and that nondegenerate symmetric bilinear forms are equivalent to nondegenerate quadratic forms. We thus have one clifford algebra in each dimension for each isomorphism class of nondegenerate quadratic form which can always be diagonalized and hence labeled by the signature (p,q).
In the real case, every non-degenerate quadratic form is equivalent to: $Q(u) = \sum_{i=0}^{p} u_i^2 - \sum_{j=p+1}^{n} u_j^2$.
In the complex case $-u_k^2 = (iu_k)^2$ and thus every quadratic form is equivalent to $Q(u) = \sum u_i^2$ making the classification significantly easier: In more general the classification result is

Theorem 3.3.2. Classification of Clifford Algebras

For complex clifford algebras:

$$\text{If n is even: } \mathbb{C}l(\mathbb{R}^n) \cong End(\mathbb{C}^N) \cong M_N(\mathbb{C}) \text{ for } N = 2^{\frac{n}{2}}$$

$$\text{If n is odd: } \mathbb{C}l(\mathbb{R}^n) \cong M_N(\mathbb{C}) \oplus M_N(\mathbb{C}) \text{ for } N = 2^{\frac{n-1}{2}}$$

The real case is a bit more complicated. Denoting by $\mathbb{R}^{p+q}$ the vector space $\mathbb{R}^n$ with quadratic form of signature (p, q), the Clifford Algebra $Cl(\mathbb{R}^{p+q})$ depends on $p - q$ mod 8. The following tables summarizes the classification (left for even dimension, right for odd):

$p - q$ mod 8	$Cl(\mathbb{R}^{p+q})$ ($N = 2^{\frac{n}{2}}$)
0	$M_N(\mathbb{R})$
2	$M_N(\mathbb{R})$
4	$M_{N/2}(\mathbb{H})$
6	$M_{N/2}(\mathbb{H})$

$p - q$ mod 8	$Cl(\mathbb{R}^{p+q})$ ($N = 2^{\frac{n-1}{2}}$)
1	$M_N(\mathbb{R}) \oplus M_N(\mathbb{R})$
3	$M_N(\mathbb{C})$
5	$M_{N/2}(\mathbb{H}) \oplus M_{N/2}(\mathbb{H})$
7	$M_N(\mathbb{C})$

Global Algebra structure

Denoting by $\tilde{\mu} : SO(n) \to GL(Cl(\mathbb{R}^n))$ the extension of the standard representation of $SO(n)$ on $\mathbb{R}^n$ to $Cl(\mathbb{R}^n)$ and P_M^{SO} the frame bundle of TM, we can easily globalize the clifford algebra:

Definition 3.3.2. Clifford Algebra Bundle

The real and complex **Clifford Algebra Bundle** Cl(M) and $\mathbb{C}l(M)$ are defined as the associated bundles

$$Cl(M) := P_M^{SO} \times_{\tilde{\mu}} Cl(\mathbb{R}^n) \quad \mathbb{C}l(M) := P_M^{SO} \times_{\tilde{\mu}} \mathbb{C}l(\mathbb{R}^n)$$

The induced connections are both denoted ∇^c.

By defining clifford multiplication fiberwise the algebra structures carry over to the section spaces $\Gamma(\mathbb{C}l(M)) \supset \Gamma(Cl(M))$. More concretely, using the associated bundle construction, we define for $x \in M$, $p' = p \cdot g^{-1} \in P_{M,x}^{SO}$ and $a, a' \in \mathbb{C}l(\mathbb{R}^4)$:

$$[p, a] \cdot [p', a'] = [p, a] \cdot [p, \mu(g^{-1})a'] = [p, a \cdot \mu(g^{-1})a'] \in P_M^{SO} \times_{\tilde{\mu}} \mathbb{C}l(\mathbb{R}^4) = \mathbb{C}l(M)$$

A straight forward calculation shows that clifford multiplication does not depend on p and p' showing that these vector spaces have a well-defined **algebra structure**.
Furthermore, we should remark that the induced connections ∇^c define derivations on those algebras, meaning:

$$\nabla^c(a \cdot a') = (\nabla^c a) \cdot a' + a \cdot \nabla^c a'$$

for $a, a' \in \Gamma(\mathbb{C}l(M))$ which can be proves using the associated bundle construction and the leibniz rule of the exterior derivative.
Now, the local vector space isomorphisms, naturally carry over to vector bundle isomorphisms

$$\Lambda^n(M) = P_M^{SO} \times_\mu \Lambda^*(\mathbb{R}^n) \cong P_M^{SO} \times_\mu Cl(\mathbb{R}^n) = Cl(M)$$

suggesting a new perspective on the dirac operator of the deRham complex.

Proposition 3.3.2. Dirac Operator of deRham complex with clifford multiplication

Let ∇^c be an induced Levi-Cevita connection on $Cl(M) \cong \Lambda^n(M)$ and $\{e^i\}$ be an orthonormal basis of $\Omega^1(U_\alpha)$ (local frame) with dual basis $\{e_i\}$ of $\Gamma(TM|_{U_\alpha})$. We can then locally express the **Hodge-Dirac Operator**:

$$\not{D} = d + d^* = \sum_i e^i \wedge \nabla^c_{e_i} - e^i \wedge^* \nabla^c_{e_i} = \sum_i e^i \cdot \nabla^c_{e_i}$$

which is independent of the chosen basis.

Writing the dirac operator of the deRham complex like this suggests that one could generalize this important operator to clifford module bundles (if they exist). It turns out that this is a powerful generalization and it will be this operator that appears in the seiberg-witten equations. To pursue this, we first have to talk about clifford modules and representations.

3.3.2 Clifford Representations

Definition 3.3.3. Algebra Representation

A **representation of an Algebra** $\mathcal{A}$ is an algebra homomorphism: $\zeta : \mathcal{A} \to End(V)$ where V is a finite dimensional vector space also called a **$\mathcal{A}$-module**. If $W \subset V$ is ζ-invariant ie $\zeta(a)w \in W$ for all $a \in \mathcal{A}, w \in W$, it defines a subrepresentation $\xi' : \mathcal{A} \to End(W)$. A representation ζ is said to be **irreducible** if it has only trivial subrepresentations.

Representations of the clifford algebra turn out to be fully reducible. In order to prove this, we need to briefly digress to group representations.

Lemma 3.6. *Let G be a compact Lie group. Any representation $\rho : G \to Aut(V)$ can be made unitary by equipping V with the inner product:*

$$\langle\langle v, w\rangle\rangle := \int_G \langle \rho(g)v, \rho(g)w\rangle \mu(dg)$$

where $\langle,\rangle$ is any inner product on V, and $d\mu$ is the Haar measure [3]*. For any such representation, we will always assume that such a inner product has been chosen and is thus unitary. For finite groups the haar measure is just $\mu(g) = \frac{1}{|G|}$ and the integral reduces to a sum:*

$$\langle\langle v, w\rangle\rangle = \int_G \langle \rho(g)v, \rho(g)w\rangle \mu(dg) = \sum_{g\in G} \langle \rho(g)v, \rho(g)w\rangle \frac{1}{|G|}$$

Proof.

$$\langle\langle \rho(h)v, \rho(h)w\rangle\rangle = \int_G \langle \rho(g)\rho(h)v, \rho(g)\rho(h)w\rangle \mu(dg) = \int_G \langle \rho(gh)v, \rho(gh)w\rangle \mu(dg)$$
$$= \int_G \langle \rho(g)v, \rho(g)w\rangle \mu(dg) = \langle\langle v, w\rangle\rangle$$

Because G acts freely and transitively on itself. □

Corollary 3.3.1. Full reducibility of compact lie group representations

A representation of a compact lie group G is fully reducible, meaning that it splits as a direct sum of irreducible representations.

Proof. V is a reducible G-module iff there is a non-trivial subspace $V_0 \subset V$ that is ξ-invariant under. Because ξ is unitary, $V_0^\perp$ must also be ξ-invariant because for any $v_0 \in V_0, v_1 \in V_0^\perp, g \in G$:

$$\langle\langle v_0, \xi(g)v_1\rangle\rangle = \langle\langle \underbrace{\xi(g^{-1})v_0}_{\in V_0}, v_1\rangle\rangle = 0 \Rightarrow \xi(g)v_1 \in V_0^\perp$$

We can thus iteratively split ξ into a direct sum of submodules until they do not have any subrepresentations, meaning they are irreducible □

The Clifford Group Cliff_n is defined to be the group (under clifford multiplication) of elements $\pm e_{i_1} \cdot \ldots \cdot e_{i_k}, 0 \leq k \leq n$. Note that $\xi : \mathrm{Cliff}_n \to Aut(V)$ with $\xi(-1_{\mathrm{Cliff}_n}) = -id_{Aut(V)}$ are in bijective correspondent with $\bar{\xi} : \mathbb{C}l_n \to End(V)$ by:

$$\bar{\xi}(\sum_{k=0}^{n} u^{i_1\ldots i_k} e_{i_1} \cdot \ldots \cdot e_{i_k}) = \sum_{k=0}^{n} u^{i_1\ldots i_k} \xi(e_{i_1} \cdot \ldots \cdot e_{i_k})$$

Where we used the Einstein-Summation convention to avoid a second sum symbol. [4]

Proposition 3.3.3. Full reducibility of clifford algebra representations

Every representation of $\mathbb{C}l(n)$ is completely reducible, meaning it splits as a direct sum of irreducible representations.

Proof. Since representations of the clifford algebra are in bijective correspondence with group representations of Cliff_n, we can show that those are completely reducible. This follows immediately from corollary 3.3.1 as Cliff_n is a finite group. □

Since Clifford Algebras are made of matrix algebras, and every Matrix Algebras $M_n(\mathbb{F})$ has unique irreducible representations, given by the natural action on $\mathbb{F}^n$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ we get:

[3] For any $\omega \in \Lambda^n(T_{id}G)$ there is unique left-invariant $\omega^L = (L_G)_*\omega \in \Omega^n(G)$ with $\omega^L(id) = \omega$ which can be used to define a measure via integration: $\mu_\omega(U) := \int_U \omega^L$. Since $\Lambda^n(T_{id}G)$ is 1-dimensional, all of them are related by scalar multiplication. We define the Haar measure to be the choice that normalizes G ie for which $\mu(G) = 1$

[4] We required $\xi(-id_{\mathrm{Cliff}_n}) = -id_{Aut(V)}$ so that $\xi(-e_{i_1} \cdot \ldots \cdot e_{i_k}) = \xi(-1u) = \xi(-1)\xi(u) = -\xi(e_{i_1} \cdot \ldots \cdot e_{i_k})$ and thus is compatible with the $\mathbb{F}$-linearity an algebra homomorphism $\bar{\xi}$ satisfies (for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$).

Theorem 3.3.3. Classification of irreducible clifford representations

The classification of clifford algebras in theorem 3.3.2 provides a classification of all irreducible representations of Clifford Algebras. We denote an irreducible module by Ψ. For n even Ψ is unique and

$$\mathbb{C}l_n = End(\Psi) = \Psi^* \otimes \Psi \cong \Psi \otimes \Psi$$

where the last isomorphism is given my the metric.

Example 3.4. ***The matrix model of*** $\mathbb{C}l(4)$ *Because* $\mathbb{C}l(\mathbb{R}^4)$ *will play an important role in Seiberg-Witten Theory we will explicitly describe the matrix algebra it is isomorphic to (which will naturally include a description of the representation). To achieve this, we first describe the isomorphism* $Cl(\mathbb{R}^4) \cong M_2(\mathbb{H})$*. This requires identifying* $\mathbb{R}^4$ *with quaternions* $\mathbb{H}$ *and embedding it into the Algebra* $M_2(\mathbb{H})$ *of 2 x 2 quaternionic matrices:*

$$\mathbb{R}^4 = \mathbb{H} \subset M_2(\mathbb{H}), \quad q \mapsto x = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$$

For the basis of $\mathbb{R}^4$*:*

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} e_3 \mapsto \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} e_4 \mapsto \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

Note that the inner product on $\mathbb{R}^4$ *can be written as:*

$$\langle q_1, q_2 \rangle_{\mathbb{R}} = \frac{1}{2}(\bar{q}_1 q_2 + \bar{q}_1 q_2) = \frac{1}{4}\big((\bar{q}_1 + \bar{q}_2)(q_1 + q_2) - (\bar{q}_1 - \bar{q}_2)(q_1 - q_2)\big)$$

$$= \frac{1}{4}\big(det(x+y) + det(x-y)\big)$$

(in particular $|q|^2 = det(x)$*) which extends to an inner product on* Cl_4*:* $\langle x, y \rangle := \frac{1}{4}(det(x+y) - det(x-y))$*. One can now easily verify that*

$$Cl(4) \cong M_2(\mathbb{H})$$

The natural $\mathbb{Z}^2$ *grading:*

$$Cl(4) = Cl_0(4) \oplus Cl_1(4)$$

$$Cl_i(4)Cl_j(4) \subseteq Cl_{i+j \text{ (mod 2)}}(4)$$

corresponds to

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix} + \begin{pmatrix} 0 & q_{12} \\ q_{21} & 0 \end{pmatrix}$$

To describe the complexification $\mathbb{C}l(4) = Cl(4) \otimes \mathbb{C}$*, we can identify* $\mathbb{H}$ *with* $\mathbb{C}^2$*:*

$$\mathbb{H} \cong \mathbb{C}^2, \quad \alpha + \beta j \mapsto \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

Which for the basis of $\mathbb{H}$ *reads*

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} := \mathbb{I} \quad i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} := I \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} := J \quad k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} := J$$

and translates the generators of the Clifford Algebra $\mathbb{R}^4 \subseteq Cl(\mathbb{R}^4) \otimes \mathbb{C}$ *to:*

$$e_1 \mapsto \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \quad e_2 \mapsto \begin{pmatrix} 0 & I \\ I & o \end{pmatrix} \quad e_3 \mapsto \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \quad e_4 \mapsto \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

One easily verifies:

$$\mathbb{C}l(4) \cong M_4(\mathbb{C})$$

It is now natural to ask whether it is possible to globalize clifford modules to clifford module bundles. Because every clifford representation is completly reducible, we are most interested in globalizing irreducible clifford modules as all others can be constructed from these.

Definition 3.3.4. Clifford Bundle

A **Clifford Bundle** is a bundle whose standard fiber is an irreducible Clifford Module.

It turns out that there a topological obstruction to defining clifford bundles: We will see that the clifford algebra naturally contains the spin group (and $spin^c$ group in the complex case). Irreducible representations of clifford algebras can be restricted to these groups, giving the famous spin $\frac{1}{2}$ representations. A manifold whose structure group can be lifted to spin (or $spin^c$) will thus naturally carry Clifford Bundles by the associated bundle construction.
Before discussing spin groups lets remark a special case guaranteeing the existence of $spin^c$ structures:

Remark 3.6. *Let M be an even dimensional manifold that admits an almost-complex structure J on TM. We can then proceed analogously as we did for theorem 3.3.1 but globally to obtain a clifford bundle directly: There is a decomposition $TM^c = TM \otimes \mathbb{C}$ into TM^c_+ and TM^c_-. The clifford algebra bundle can be expressed as*

$$\mathbb{C}l(M) = End(\Lambda^*(TM^c_+))$$

where $\Lambda^*(TM^c_+)$ is a (complex) clifford bundle.

3.3.3 Spin groups

For an algebra A, let $A^\times$ denote the group of invertible elements. The group $Pin(n)$ is the subgroup of $Cl^\times(n) \subset Cl(n)$ generated by all $x \in \mathbb{R}^n$ of unit length. Analogously $Pin^{\mathbb{C}}(n)$ is the subgroup of $\mathbb{C}l^\times(n) \subset \mathbb{C}l(n)$ generated by elements $x \in \mathbb{R}^n$ of unit length with an additional U(1) factor.

The Spin group can now be defined as:

$$Spin(n) = Pin(n) \cap Cl_0(n)$$

and analogously:

$$Spin^c(n) = Pin^{\mathbb{C}}(n) \cap \mathbb{C}l_0(n)$$

It is easy to see that:

$$Spin^c(n) = Spin(n) \times_{\mathbb{Z}_2} U(1)$$

Example 3.5. ***Spin in the matrix model of***
$\mathbb{C}l(4)$ Using the matrix model introduced in example 3.4, we can identify the Pin group to be all elements generated by: $\begin{pmatrix} 0 & q \\ -\overline{q} & 0 \end{pmatrix}$ for unit quaternions $q \in Sp(1)$. Analogously the $Pin^{\mathbb{C}}$ group is generated by $\lambda \begin{pmatrix} 0 & Q \\ -\overline{Q} & 0 \end{pmatrix}$ for $Q \in SU(2)$ and $\lambda \in U(1)$. The Spin groups become:

$$Spin \cong \left\{ \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \;\middle|\; q_1, q_2 \in Sp(1) \right\} \cong Sp(1) \times Sp(1) \cong SU(2) \times SU(2)$$

and the complexified $spin^c$ group is:

$$spin^c \cong \left\{ \lambda \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \;\middle|\; u_1, u_2 \in SU(2), \lambda \in U(1) \right\}$$

$$= \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \;\middle|\; u_1, u_2 \in U(2), det(u_1) = det(u_2) \right\}$$

The significance of the group $Spin(n)$ is that it is the universal double cover of $SO(n)$. To prove this we need the following lemma:

Lemma 3.7. *The center $\mathcal{Z}(Cl(V))$ is equal to $Cl^{(0)}(V)$ if n=dim(V) is even and equal to $Cl^{(0)}(V) \oplus Cl^{(n)}(V)$ if n is odd. In any case:*

$$\mathcal{Z}(Cl(V)) \cap Cl_0(V) = Cl^{(1)}(V)$$

Proof. An element in the center of Cl(V) is also in the center of $V \subset Cl(V)$ and an element in the center of $V \subset Cl(V)$ is also in the center of Cl(V) because basis elements are products of vectors in V.
$Cl^{(0)} = \mathbb{R}id$ is obviously in the kernel. For a basis element $e_I := e_{i_1} \cdot e_{i_2} \cdot ... \cdot e_{i_k}$ of $Cl^{(k)}, k \geq 1$ and a vector $e_j \in V$:

$$e_j e_I = (-1)^k e_I e_j \text{ for } j \notin I$$

$$e_j e_I = (-1)^{k-1} e_I e_j \text{ for } j \in I$$

We see that e_I can not commute will all of V if both cases can occur. For $k = n$ only the second case can occur, and thus $Cl^{(n)}$ is in also in the center if n is odd. □

Proposition 3.3.4. Spin double covers SO

Spin(n) is the universal double cover of SO(n). The double cover is given by the group homomorphism:

$$p_{Spin} : Spin(n) \to SO(n), u \mapsto ad_u$$

Because SO(n) is a Lie Group, so is Spin(n).

Proof. Defining the adjoint action of invertible Elements $Cl^\times(n)$ on $Cl(n)$ for $u \in Cl^\times(n)$ by:

$$ad_u : Cl(n) \to Cl(n), \quad p \mapsto upu^{-1}$$

which for $x \in \mathbb{R}^n \subset Cl(n)$ with $|x| = 1$ and any $v \in \mathbb{R}^n$ becomes:

$$ad_x(v) = -xvx$$

we have

$$x(vx) + x(xv) = xvx - v = -2x\langle v, x\rangle$$
$$\Leftrightarrow xvx = v - 2\langle v, x\rangle x$$

Since $\langle v, x\rangle x$ is the projection of v to x, subtracting it once from v, annihilates the x component, and subtracting it again, negates the original x component. xvx is thus the reflection of v through the hyperplane $x^\perp$. Since Spin(n) is generated by an even number of vectors of unit length, $Spin(n) \subset Cl^\times(n)$ acts on $\mathbb{R}^n \subset Cl(n)$ via the adjoint representation by an even number of ref

lections. Because any rotation can be represented by an even number of reflections by the cartan-dieudonné theorem, we have a surjective map

$$p_{Spin} : Spin(n) \twoheadrightarrow SO(n), \quad u \mapsto ad_u$$

The kernel obviously contains $\mathbb{Z}_2 = \pm 1$, and one can see that this is the entire kernel, by $uxu^{-1} = x \ \forall x \in Cl(n) \Leftrightarrow u \in Z(Cl(n)) = span\{e_0\}$ Since u is of unit length (as $u \in Spin(4)$), we get $u = \pm e_o$. □

The analogous statement for $spin^c$ is:

Proposition 3.3.5. $Spin^c$ **double covers** $SO \times U(1)$

There is a double cover

$$p_{spin^c} : Spin^c(n) = Spin(n) \times U(1)/\mathbb{Z}_2 \to SO(n) \times U(1)$$

that looks like p_{Spin} over $Spin(n)$ and like the squaring map $U(1) \ni z \mapsto z^2 \in U(1)$ over U(1). We will write:

$$p_{spin^c}(\xi) = (\lambda_{SO}(\xi), \lambda_{U(1)}(\xi))$$

Because $SO(n) \times U(1)$ is a Lie group, so is $spin^c(n)$.

To understand and work with the spin groups in a smooth framework we need to understand the infinitesimal generators aka the lie algebras.

The Lie Algebras

Proposition 3.3.6. Lie algebras of spin groups

A basis for the Lie Algebra $\mathfrak{spin}(n)$ of $Spin(n)$ is given by: $\{\frac{1}{2}e_ie_j\}_{i<j}$. The exponential function for Spin(n) is given by:

$$exp(e_ie_jt) = cos(t) + e_ie_jsin(t)$$

and the bracket by the commutator $[u, v] = uv - vu$
Similarly, a basis for the Lie algebra $\mathfrak{spin}^{\mathbb{C}}(n)$ of the group $Spin^c(n)$ is given by $\mathfrak{spin}(n) \oplus \mathfrak{u}(1)$. The exponential map is given by:

$$exp((e_ie_j + i\theta)t) = (cos(t) + e_ie_jsin(t))expi\theta t$$

Proof. We only prove the second part. The first is analogous. Consider the path:

$$\gamma(t) = (e_isin(t) + e_jcos(t))expi\theta t(-e_isin(t) + e_jcos(t))expi\theta t$$
$$= (cos(2t) + e_ie_jsin(2t))exp2i\theta t \in spin^c(n)$$

satisfying $\gamma(0) = id$ and $\gamma'(0) = 2e_ie_j + 2i\theta$, proving $\mathfrak{spin}(n) \oplus i\mathbb{R} \subseteq \mathfrak{spin}^{\mathbb{C}}(n)$. For dimensionality reason, they are in fact equal (because $dim(\mathfrak{spin}^{\mathbb{C}}) = dim(spin^c) = dim(SO(n) \times U(1))$. Also by the defining properties of the exponential map, we can write $\gamma(t) = exp(2e_ie_jt + 2i\theta t)$.

□

The appearance of the factor 2 in the previous proof corresponds to the fact that we are dealing with double covers. This factor also manifests itself in the lie algebra isomorphism:

Proposition 3.3.7. Lie algebra isomorphism $\mathfrak{spin}$ and $\mathfrak{so}$

Differentiating the double cover $p_{Spin} : Spin(n) \to SO(n)$ at the identity gives the Lie Algebra isomorphism:

$$(p_{Spin})_* : \mathfrak{spin}(n) \cong \mathfrak{so}(n), \quad \frac{1}{2}e_ie_j \mapsto e_i \wedge e_j$$

A rotation in the i-j-plane by angle θ is given by:

$$exp(e_ie_j\frac{\theta}{2}) = cos(\frac{\theta}{2}) + e_ie_jsin(\frac{\theta}{2}) \in Spin(n)$$

Proof. For the path $\gamma(t) = exp(e_ie_jt)$ with inverse $\gamma^{-1}(t) = epx(-e_ie_jt)$ and $v \in \mathbb{R}^n$:

$$p_{Spin}(\gamma(t))v = ad_{\gamma(t)}v = \big(cos(t) + e_ie_jsin(t)\big)v\big(cos(t) - e_ie_jsin(t)\big)$$
$$= v(cos^2(t) + sin^2(t)) + cos(t)sin(t)\big(e_ie_jv - ve_ie_j\big)$$

taking the derivative at the identity gives:

$$(p_{Spin})_*(e_ie_j)v = \frac{d}{dt}p_{Spin}(exp(e_ie_jt))v|_{t=0} = (e_ie_jv - ve_ie_j) = \sum_{k=1}^{n}(e_ie_jv_ke_k - v_ke_ke_ie_j)$$

$$= (-e_iv_j + e_jv_i - (-v_ie_j + v_je_i)) + \sum_{k \neq i,j}(e_ie_jv_ke_k - e_ie_jv_ke_k) = 2(e_j\langle e_i, v\rangle - e_i\langle e_j, v\rangle) = 2e_i \wedge e_j(v)$$

□

Proposition 3.3.8. Lie algebra isomrorphism $\mathfrak{spin}$ and $\mathfrak{so} \oplus \mathfrak{u}(1)$

Differentiating the double cover $p_{spin^c} : spin^c(n) \to SO(n) \times U(1)$ at the identity gives the Lie Algebra isomorphism:

$$(p_{Spin^c})_* : \mathfrak{spin}^{\mathfrak{c}}(n) \cong \mathfrak{so}(n) \oplus \mathfrak{u}(1), \quad e_ie_j + i\theta \mapsto 2(e_i \wedge e_j + i\theta)$$

Proof. For the path $\gamma(t) = exp((e_ie_j + i\theta)t)$ with inverse $\gamma^{-1}(t) = epx(-(e_ie_j + i\theta)t)$ and $\xi = (v, u) \in \mathbb{R}^n \times \mathbb{C}$

$$p_{spin^c}(\gamma(t))\xi = \big(\lambda_{SO}(\gamma(t)), \lambda_{U(1)}(\gamma(t))\big)\xi = (p_{Spin}(exp(e_ie_jt))v, exp(2i\theta t)u)$$

Taking the derivative at the identity gives:

$$(p_{spin^c})_*(e_ie_j + i\theta)v = ((p_{Spin})_*(e_ie_j)v, \frac{d}{dt}exp2i\theta tu|_{t=0}) = 2(e_i \wedge e_j(v), i\theta u)$$
$$= 2(e_i \wedge e_j + i\theta)(\xi)$$

□

Having discussed the lie group and algebra structure, it is now time to look at the spin representations.

Spin Representations

Recall that our motivation for studying spin groups, was to globalize irreducible clifford modules. We hence define:

Definition 3.3.5. Spin $\frac{1}{2}$ representation

The spin $\frac{1}{2}$ representation $\tau : Spin(n) \to Aut(\Psi)$ is defined by restricting an irreducible representation of $\mathbb{C}l(n) \to End(\psi)$ to $Spin(n) \subset \mathbb{C}l(n)$. The representation obtained by restricting to $Spin^c(n) \subset \mathbb{C}l(n)$ is also denoted τ or $\tau_{\mathbb{C}}$ if we want to emphasize the complex nature.

Analyzing this representation gives:

Proposition 3.3.9. Spin $\frac{1}{2}$ representation

Let G be $Spin(n)$ or $Spin^c(n)$. If n is odd, the spin $\frac{1}{2}$ representations $\tau : G \to Aut(\Psi)$ of dimension $\frac{n-1}{2}$ (real or complex respectively) are independent of the irreducible representation of $\mathbb{C}l(n)$ it is restricted from. For n even, the spin representation τ splits as a direct sum of two inequivalent irreducible representations $\tau^{\pm}$ of dimension $\frac{n-2}{2}$ on

$$\Psi = \Psi^+ \oplus \Psi^-$$

and the exchange relations hold

$$e_i \cdot \Psi^{\pm} \subset \Psi^{\mp}$$

None of the representations descend to a representation of SO(n).

If it is possible to globalize spin groups we can use the spin $\frac{1}{2}$ representation to construct clifford bundles. We will now discuss when this is possible.

3.3.4 The Global Picture

Spin Structures

We have seen that $Spin(n)$ is a double cover of $SO(n)$. Given a $SO(n)$ bundle one can hence try to construct a principal $Spin(n)$ bundle by lifting the corresponding $SO(n)$ cocycle to a $Spin(n)$ cocycle. Recall that choosing a metric on M (which can always be done), reduces the structure group to $SO(n)$ and the frame bundle P_M^{SO} is thus described by a $SO(n)$-valued chech cocycle (see Section 1.3.1):

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(n)\} \in \check{H}^1(M, C^\infty(SO(n)))$$

We have already seen that lifting the defining maps against the double cover $p_{Spin} : Spin(n) \to SO(n)$ to get maps

$$\{\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(n)\}$$

leads to the second stiefel-whitney class $w_2(TM) \in \check{H}^2(M, \mathbb{Z}_2) = H^2(M, \mathbb{Z}_2)$ (see subsection 1.3.3) whose maps satisfy

$$w_{\alpha\beta\gamma}(x) := \tilde{g}_{\alpha\beta}(x)\tilde{g}_{\beta\gamma}(x)\tilde{g}_{\gamma\alpha}(x) = \pm id \in \mathbb{Z}_2$$

If $w_2(TM) = 0$, then the maps $\{\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(n)\}$ satisfy the cocycle condition meaning they define a cocycle in cech cohomology and hence a Spin(n) bundle:

Definition 3.3.6. Spin structures

If $w_2(TM) = 0$ one can lift the frame bundle $\pi_{SO} : P_M^{SO} \to M$ to a prinicpal $Spin(n)$-bundle $\pi_{Spin} : P_M^{Spin} \to M$ that double covers the frame bundle by a map $\Pi_{Spin} : P_M^{Spin} \to P_M^{SO}$ satisfying $\Pi_{Spin}(p \cdot u) = \Pi_{Spin}(p) \cdot p_{Spin}(u)$ fitting in the commutative diagram:

$$\begin{tikzcd} P_M^{Spin} \arrow[rr, "\Pi_{Spin}"] \arrow[dr, "\pi_{Spin}"'] & & P_M^{SO} \arrow[dl, "\pi_{SO}"] \\ & M & \end{tikzcd}$$

We call a lift from P_M^{SO} to P_M^{Spin} a **spin structure** and if M admits a spin structure (if $w_2(TM) = 0$) we call M a **spin manifold**. As we have seen in 1.3.3 spin structures are classified by $H^1(M, \mathbb{Z}_2)$ if they exists.

Analogously, one can lift the $SO(n)$-valued cocycle $\{g_{\alpha\beta}\}$ of the frame bundle to $spin^c$-valued maps:

$$\{\tilde{g}^{\mathbb{C}}_{\alpha\beta} : U_\alpha \cap U_\beta \to spin^c(n)\}$$

If they define a cocycle we get:

Definition 3.3.7. ***Spin^c*** **structures**

A $spin^c$ **structure** $\mathfrak{s} := \{\tilde{g}^{\mathbb{C}}_{\alpha\beta}\}$ is a lift of the frame bundle P_M^{SO} to a principal $Spin^c(n)$-bundle $\pi_{spin^c} : P_M^{Spin^c} \to M$ that double covers a principal bundle $P_M^{SO} \times_M \mathcal{L}^2$ [a] where $\mathcal{L}^2$ is a principal $U(1)$-bundle. Recall that $\lambda_{U(1)} : Spin^c(n) = Spin(n) \times U(1)/\mathbb{Z}_2 \to U(1)$ looks like the squaring map over $U(1)$. The cocycle of $\mathcal{L}^2(\mathfrak{s})$ is given by:

$$\{\lambda_{U(1)} \circ \tilde{g}^{\mathbb{C}}_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)\}$$

The double cove $\Pi_{spin^c} : P_M^{Spin^c} \to P_M^{SO} \times_M \mathcal{L}^2$ satisfies $\Pi_{spin^c}(p \cdot \xi) = \Pi_{spin^c}(p) \cdot p_{spin^c}(\xi) = \Pi_{spin^c}(p) \cdot (\lambda_{SO}(\xi), \lambda_{U(1)}(\xi))$ for $p \in P_M^{Spin^c}, \xi \in Spin^c$ and fits in the commutative diagram:

$$\begin{tikzcd} P_M^{spin^c} \arrow[rr, "\Pi_{spin^c}"] \arrow[dr, "\pi_{spin^c}"'] & & P_M^{SO} \times_M \mathcal{L}^2 \arrow[dl, "\pi_{SO,det}"] \\ & M & \end{tikzcd}$$

The components of this projection are denoted analogous to the components of the group projections they come from: $\Pi_{spin^c}(p) = (\Lambda_{SO}(p), \Lambda_{U(1)}(p))$.

[a] Here $E \times_M F$ is the fibered product of two bundles $E, F \to M$ which is bundle over M whose fiber over $p \in M$ is $E_p \times F_p$ defined by taking the product $E \times F \to M \times M$ and identifying the diagonal $M \times M$ with M.

We can understand $spin^c$ structures better by understanding it's connection to the second stiefel-whitney class:

Theorem 3.3.4. Existence of ***spin^c*** **structures**

The second stiefel-whtiney class $w_2(TM) \in H^2(M, \mathbb{Z}_2)$ can be lifted to an integral class $\underline{w} \in H^2(M, \mathbb{Z}_2)$ if and only if there exists $spin^c$ structures $\mathfrak{s}$ double covering $P_M^{SO} \times \mathcal{L}^2$ where $\mathcal{L}^2$ is the line bundle corresponding to an integral lift $\underline{w}$ (see lemma 1.1). We we call this integral lift of $w_2(TM)$ the first chern class of $\mathfrak{s}$:

$$c_1(\mathfrak{s}) := c_1(\mathcal{L}^2) = \underline{w}$$

Proof. We will explicitly construct the $spin^c$ cocycle from the the cocycles of the frame bundle and the line bundle $\mathcal{L}^2$ corresponding to $\underline{w}$ after giving a short proof:
Consider the short exact sequence of groups:

$$0 \to \mathbb{Z}_2 \to Spin^c(4) \to SO(4) \times U(1) \to 0$$

leading to the long exact sequence of pointed sets (see subsection 1.3.1):

$$\cdots \to H^1(M, \mathbb{Z}_2) \to \check{H}^1(M, C^\infty(Spin^c(4))) \to \check{H}^1(M, C^\infty(SO(4))) \oplus \check{H}^1(M, C^\infty(U(1))) \to H^2(M, \mathbb{Z}_2)$$

The last map can be shown to map a principal $SO(4) \times U(1)$-bundle $P \times_M \mathcal{L}^2$ to $w_2(P) + c_1(\mathcal{L}^2)$ (mod 2) $\in H^2(M, \mathbb{Z}_2)$. By exactness, the bundle comes from a principal $Spin^c(4)$-bundle if this is zero which happens if and only if $w_2(P) = c_1(\mathcal{L}^2)$ (mod 2) or in other words if the second stiefel-whitney class of P lifts to an integral class (which corresponds to a line bundle).

Explicit construction:
The integral lift $\underline{w} \in H^2(M, \mathbb{Z})$ of $w_2(TM) \in H^2(M, \mathbb{Z}_2)$ corresponds to a line bundle $\mathcal{L}^2$ which corresponds to a U(1)-valued cocycle $\{l_{\alpha\beta}\}$. A square root bundle $\mathcal{L}$ ie a complex line bundle s.t.

$$\mathcal{L}^2 = \mathcal{L} \otimes \mathcal{L}$$

exists if and only if the collections of maps $\{\sqrt{l_{\alpha\beta}}\}$ satisfies the cocycle condition. The obstruction to building this bundle is measured by a $\check{C}ech$ 2-cochain:

$$c_{\alpha\beta\gamma} = \sqrt{l_{\alpha\beta}} \cdot \sqrt{l_{\beta\gamma}} \cdot \sqrt{l_{\gamma\alpha}} \in \{-1, 1\}$$

which can be seen to be a cocycle and thus defining a cohomology class (because locally constant functions with codomain $\{-1, 1\}$ are constant). Note that $c_{\alpha\beta\gamma} = 1$ if and only if the total rotation described by this cochain is even and recall from subsection 1.3.4 that the first chern class measures the total rotations. Thus, $\{c_{\alpha\beta\gamma}\}$ represent the mod 2 reduction of $c_1(\mathcal{L}^2)$ which was by definition equal to $\underline{w}$. Hence, the obstruction to building a square root bundle of $\mathcal{L}^2$ is $w_2(T_M)$.
We already know, that $w_2(T_M)$ is also the obstruction cocycle to lifting the original $SO(4)$-valued cocycle $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(4)\}$ to maps $\{\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(4)\}$ that satisfy the cocycle condition. It is explicitly given by:

$$w_{\alpha\beta\gamma} = \tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\gamma} \cdot \tilde{g}_{\gamma\alpha}$$

Since $c_{\alpha\beta\gamma}$ and $w_{\alpha\beta\gamma}$ are cohomologuous, they differ by a cech coboundary:

$$c_{\alpha\beta\gamma} = w_{\alpha\beta\gamma} \cdot (\delta\epsilon)_{\alpha\beta\gamma} = (\epsilon_{\alpha\beta}\tilde{g}_{\alpha\beta})(\epsilon_{\beta\gamma}\tilde{g}_{\beta\gamma})(\epsilon_{\gamma\alpha}\tilde{g}_{\gamma\alpha})$$

for $\epsilon_{\alpha\beta} : U_\alpha \cap U_\beta \to \{-1, 1\}$ meaning we can modify the lifts $\{\tilde{g}_{\alpha\beta}\}$ to $\{\tilde{g}'_{\alpha\beta}\} := \{\epsilon_{\alpha\beta}\tilde{g}_{\alpha\beta}\}$ such that

$$\mathfrak{s} := \{\sqrt{l_{\alpha\beta}} \cdot \tilde{g}'_{\alpha\beta} : U_\alpha \cap U_\beta \to spin^c(4)\}$$

satisfies the cocycle condition, giving us a $spin^c$ **structure** on M.

□

With that we can prove the important corollary:

Corollary 3.3.2. All 4-manifolds admit $spin^c$ structures

All 4-manifolds admit $spin^c$ structures.

Proof. We have seen in Section 1.3.5 that $w_2(TM)$ can be lifted to an integral class for any 4-manifold. We can thus apply the last theorem 3.3.4 proving that any 4-manifold admits $spin^c$ structures. □

Another fact now becomes crystal clear:

Corollary 3.3.3. Local description of $P_M^{Spin^c}$

We can locally think of the principal bundle $P_M^{Spin^c}$ as the tensor product $P_M^{Spin} \otimes \mathcal{L}$ where $\mathcal{L}$ is the square root bundle of $\mathcal{L}^2$. Both bundles only exists globally if $w_2(TM) = 0$.

Having understood the construction and existence of *spin*- and *spin*^{c}-structures, we can now define connections on the lifted principal bundles.

Definition 3.3.8. Connetions on *spin* and *spin*^{c} structures

If M admits a spin-structure, the double cover $\Pi_{Spin} : P_M^{Spin} \to P_M^{SO}$, can be used to lift the Levi-Cevita connection A_{LC} on P_M^{SO} to a connection

$$A_{Spin} := \Pi^*_{Spin} A_{LC} : TP_M^{Spin} \to \mathfrak{so}(n) \cong \mathfrak{spin}(n)$$

If M admits a $spin^c$ structure, there is no canonical connection on $P_M^{spin^c}$. Instead, the double cover $\Pi_{spin^c} = (\Lambda_{SO}, \Lambda_{U(1)}) : P_M^{spin^c} \to P_M^{SO} \times_M \mathcal{L}^2(\mathfrak{s})$ can be used to lift a connection defined on $P_M^{SO} \times_M \mathcal{L}^2$. The first component carries the canonical Leve-Cevita connection A_{LC} and together with a connection $A_{\mathcal{L}^2}$ on $\mathcal{L}^2(\mathfrak{s})$, we get a connection on $P_M^{spin^c}$:

$$A_{spin^c} := A_{Spin} + A_{\mathcal{L}^{1/2}} := \Pi^*_{spin^c}(A_{LC} + A_{\mathcal{L}^2})$$

$$= \Lambda^*_{SO} A_{LC} + \lambda^*_{U(1)} A_{\mathcal{L}^2} : TP_M^{spin^c} \to \mathfrak{so}(n) \oplus \mathfrak{u}(1) \cong \mathfrak{spin}^{\mathbb{C}}(n)$$

With the global notion of spin and $spin^c$ structures, we can now define the corresponding module bundles: spinor bundles:

Spinor bundles

Definition 3.3.9. Spinor bundles

Let M be a smooth manifold with frame bundle P_M^{SO}. If the structure group can be lifted to $Spin(n)$, ie if M is a **Spin manifold** then we can pick a spin structure and define the **spinor bundle** of M

$$S(M) := P_M^{Spin} \times_\tau \Psi$$

and for n even the **positive spinor bundle** and **negative spinor bundle**

$$S^{\pm}(M) := P_M^{Spin} \times_{\tau^{\pm}} \Psi^{\pm}$$

Analogously, if the structure group can be lifted to $Spin^c(n)$, then we define the **complex spinor bundles**:

$$S_{\mathbb{C}}(M) := P_M^{Spin^c} \times_\tau \Psi_{\mathbb{C}}$$

and for n even

$$S^{\pm}_{\mathbb{C}}(M) := P_M^{Spin^c} \times_{\tau^{\pm}} \Psi^{\pm}_{\mathbb{C}}$$

The connections induced from the principal spin bundles (see definition 3.3.8) via the usual associated bundle construction are all denoted ∇^s.

Recall that our motivation for constructing spinor bundles was to obtain clifford bundles (bundles whose fibers are irreducible clifford modules). The following proposition shows that we succeeded:

Proposition 3.3.10. The spinor bundle is a clifford bundle

The spinor bundles (complex and real) are clifford bundles. In particular, there is a natural action of the clifford algebra bundle (see definition 3.3.2) on the spinor bundle via pointwise clifford multiplication:

$$\mathbb{C}l(M) \times S_{\mathbb{C}}(M) \to S_{\mathbb{C}}(M)$$

$$[q, \xi] \cdot [q', \psi] = [q, \xi] \cdot [q, \tau(g)\psi] = [q, \xi \cdot \tau(g)\psi] := [q, \bar{\tau}(\xi)\tau(g)\psi]$$

making the vector space $\Gamma(S_{\mathbb{C}}(M))$ into a $\Gamma(\mathbb{C}l(M))$-module (see 3.3.1) where $\bar{\tau} : \mathbb{C}l_n \to End(\Psi)$ is the representation that the $spin^c$ representation τ is restricted from and $g \in Spin^c(n), q = q'g^{-1} \in P_M^{Spin^c}, \xi \in \mathbb{C}l(\mathbb{R}^n), \psi \in \Psi_{\mathbb{C}}$. In fact

$$\mathbb{C}l(M) \cong End(S_{\mathbb{C}}) \cong S_{\mathbb{C}}^* \otimes S_{\mathbb{C}} \cong S_{\mathbb{C}} \otimes S_{\mathbb{C}}$$

The real case is analogous.

Proof. Note that

$$\mathbb{C}l(M) = P_M^{SO} \times_\mu \mathbb{C}l(n) = P_M^{Spin} \times_{\mu \circ p_{Spin}} \mathbb{C}l(n) = P_M^{Spin} \times_{\mu \circ p_{Spin}} End(\Psi_{\mathbb{C}})$$

A calculation shows that the multiplication is independent of $q \in \mathbb{C}l(M)_x$. □

There is an important restriction of this action:

Remark 3.7. *If $dim(M)$ is even, then*

$$\mathbb{C}l(M) \supset TM \otimes \mathbb{C} \cong Hom(S_{\mathbb{C}}^{\pm}, S_{\mathbb{C}}^{\mp}) \cong (S_{\mathbb{C}}^{\pm})^* \otimes S_{\mathbb{C}}^{\mp}$$

Let's record 2 important compatibility relations

Lemma 3.8. *1. Clifford multiplication acts by isometries:*

$$\langle u \cdot \psi, u \cdot \psi' \rangle = \langle \psi, \psi' \rangle$$

for $u \in \Gamma(TM) \subset \mathbb{C}l(M)$ with $|u| = 1$ and $\psi, \psi' \in \Gamma(S(M))$. 2. The induce connections on $\mathbb{C}l(M)$ and $S_{\mathbb{C}}(M)$ are compatible will the clifford action of $\Gamma(\mathbb{C}l(M))$ on $S_{\mathbb{C}}(M)$:

$$\nabla^s(a \cdot \psi) = \nabla^c a \cdot \psi + a \cdot \nabla^s \psi$$

for $a \in \Gamma(\mathbb{C}l(M)), \psi \in \Gamma(S(M))$.

Proof. 1. Recall that a representation of a compact lie group is unitary (see lemma 3.6) and note that on every fiber $u_x \in Pin(T_x M)$.
2. Follows again from using the associated bundle construction and the leibniz rule of the exterior derivative. □

As suggested by proposition 3.3.2 we can now generalize the dirac operator acting on clifford module bundles and in particular on spinor bundles.

The Spin-Dirac operator

We start with the definition of the "plain" Spin-Dirac operator. It is this operator that is most commonly refered to when talking about dirac operators:

Definition 3.3.10. Spin-Dirac Operator

The **Spin-Dirac Operator** is locally defined by:

$$\partial\!\!\!/ := \sum_i e_i \cdot \nabla^s_{e_i} : \Gamma(S(M)) \to \Gamma(S(M))$$

for a local orthonormal basis $e_1 \dots e_n$ of $\Gamma(TM|_U)$. The **Complex Spin-Dirac Operator**

$$\partial\!\!\!/_{\mathbb{C}} : \Gamma(S_{\mathbb{C}}(M)) \to \Gamma(S_{\mathbb{C}}(M))$$

is defined completely analogous. We will often drop the $\mathbb{C}$ subscript if we want to talk about both operators in the same time of it is clear from context which operator is used. If we want to explicitly show the dependece on the connection $A \in \mathcal{A}(P_M^{Spin^c})$ we will denote the operator by $\partial\!\!\!/_A$.

This should be promptly appended by:

Lemma 3.9. *The Dirac operator $\partial\!\!\!/$ is independent of the choice of orthonormal basis.*

Let's state the most important properties of this important operator:

Proposition 3.3.11. Properties of the Spin-Dirac operator

The Spin-Dirac operator $\partial\!\!\!/$ is formally self-adjoint:

$$\langle \partial\!\!\!/\psi, \psi' \rangle = \langle \psi, \partial\!\!\!/\psi' \rangle$$

and is elliptic.
For n even, the dirac operator on spinor bundles maps positive spinor fields to negative spinor fields

$$\partial\!\!\!/^+ := \partial\!\!\!/|_{\Gamma(S^+(M)} : \Gamma(S^+(M)) \to \Gamma(S^-(M))$$

Proof. The proves of the first two statements can be found in most text books or lecture notes talking about dirac operators. The last statements follows from remark 3.7 and the local expression of the dirac operator. □

We can generalize this dirac operator to have coefficients in a bundle:

Definition 3.3.11. Dirac operator with coefficients in a bundle

Let E be any complex vector bundle with hermitian metric and unitary connection. There is an induced hermitian metric and unitary connection ∇ on the product bundle $S \otimes E$ where S is the spinor bundle. We can then define clifford multiplication on $S \otimes E$ by letting it act trivially on the second factor and define the **dirac operator with coefficients in E** locally the same way as before:

$$\partial\!\!\!/_E(\psi \otimes \sigma) = \sum e_i \cdot \nabla_{e_i}(\psi \otimes \sigma)$$

Example 3.6. *By recalling that $\Lambda^*(M) \cong Cl(M) \cong S(M) \otimes S(M)$ this definition allows us to express the hodge-dirac operator of the deRham complex $\mathcal{E}$ as the spin-dirac operator with coefficients in the spinor bundle*

$$d + d^* = D\!\!\!\!/_{\mathcal{E}} = \partial\!\!\!/_{S(M)}$$

We have finally discussed all preliminaries and are ready to delve into Seiberg-Witten Theory.

4 Seiberg-Witten-Theory

In this chapter we will finally study the Seiberg-Witten equations. As mentioned in the introduction, our goal is to understand the space of solutions to the Seiberg-Witten equations $Sol(SW(\mathfrak{s}))$ modulo gauge equivalence - the moduli space of solutions $\mathfrak{M}(\mathfrak{s}) = Sol(SW(\mathfrak{s}))/\mathcal{G}(\mathfrak{s})$ - and to extract topological information about the base manifold M from it. The main references for this chapter are [Mor96; Nic00; Moo01; Don97; Sco05]

One way to extract information from $\mathfrak{M}(\mathfrak{s})$ is to consider it as a submanifold of some larger (infinite dimensional) configuration space modulo equivalence, called the moduli space of configurations $\mathcal{B}(\mathfrak{s}) = \mathcal{C}/\mathcal{G}$. We will show that $\mathfrak{M}$ is a **finite dimensional, orientable, compact manifold** whenever $b_2^+(M) \geq 1$ and thus determines an integral homology class $[\mathfrak{M}] \in H_{dim(\mathfrak{M})}(\mathcal{B}, \mathbb{Z})$. One can then evaluate certain natural cohomology classes of $\mathcal{B}$ on $[\mathfrak{M}]$. Of course the result will a priori depend on the metric but it turns out that if $b_2^+(M) \geq 2$ the moduli space $\mathfrak{M}$ is unique up to cobordism so that $[\mathfrak{M}(\mathfrak{s})]$ does not depend on the metric and can be used in that way to define topological invariants of $(M, \mathfrak{s})$ where $\mathfrak{s}$ is the chosen $spin^c$ structure on M of which there only exists finitely many as we will prove (one can thus also define the invariant by averaging over them).

In order to understand the moduli space $\mathfrak{M}$ we have to understand two maps. Because we work with smooth manifolds we can obtain local results by studying the linearizations.

1. The action of the group of gauge transformation $\mathcal{G}$ on a configuration $c \in \mathcal{C}$

$$\sigma_c : \mathcal{G} \to \mathcal{C}$$

   If this map is injective, i.e. if the action is free, the moduli space of configurations $\mathcal{B} = \mathcal{C}/\mathcal{G}$ will be a manifold near c.

2. The solution space of the Seiberg-Witten equations can be expressed as the zero set of the Seiberg-Witten map:

$$sw : \mathcal{C} \to \mathcal{X}$$

   If the derivative of this map is a fredholm operator, then almost all $\eta = sw(c) \in \mathcal{X}$ will be regular values and we can apply the implicit function theorem to obtain a finite dimensional manifold structure for the moduli space of the perturbed Seiberg-Witten equations $\mathfrak{M}_\eta$.

For every $c \in Sol(SW_\eta)$ the two linearizations at c combine nicely to the **fundamental complex**:

$$0 \to T_{id}\mathcal{G} \to T_c\mathcal{C} \to T_\eta\mathcal{X} \to 0$$

which turns out to be elliptic. The condition for σ_c to be (locally) injective is encoded in this complex as the vanishing of the 0th cohomology group and the condition for $sw(c) = \eta$ to be a regular value is encoded as the vanishing of the 2nd cohomology group. The 1st cohomology group will turn out to be the tangent space $T_{[c]}\mathfrak{M}_\eta$ so that the euler characteristic will give us the dimension of the moduli space if we can ensure the vanishing if the 0th and 2nd cohomology groups.
The vanishing of the 2nd cohomology group is ensured because we perturb the equations and the vanishing of the 0th group will follow from a neat transversality argument whenever $b_2^+(M) \geq 1$.

Outline

Because this is by far the longest chapter, we provide a quick outline:
The setup: In the first section we will discuss the basic ingredients for the equations. We will discuss the space of configurations $\mathcal{C}$, the squaring map and the group of gauge transformations. We then introduce the equations and discuss the analytical setup needed to use powerful results from banach space theory.

The Moduli Space of Configurations: In this section, our goal is to show that the quotient space $\mathcal{B} = \mathcal{C}/\mathcal{G}$ possesses a smooth manifold structure near *irreducible configurations*. This is achieved via the *Slice Theorem*, which demonstrates that if the action of $\mathcal{G}$ is free (i.e., if the corresponding linearization $T\sigma_c$ is injective), then local slices in $\mathcal{C}$ can be identified with open neighborhoods in $\mathcal{B}$.

The Moduli Space of Seiberg–Witten Monopoles: After proving that the moduli space of Seiberg-Witten monopoles is even well-defined, we express the solution space of the Seiberg–Witten equations as the zero set of the Seiberg–Witten map $sw : \mathcal{B} \to \mathcal{X}$. We then discuss the need for perturbing the equation in order to guarantee regularity and introduce the linearization of the Seiberg-Witten map sw_{*c} which has to be a fredholm operator in order for the level set to be a finite dimensional manifold.

The Elliptic Complex: We discuss the fundamental complex already mentioned above and use it to show that the linearized Seiberg-Witten map sw_{*c} at irreducible solutions c is always fredholm proving that an open neighborhood of irrdeducible solutions is a finite dimensionality manifold. We end this section by computing its dimension.

Reducible Monopoles: In this section we use a neat transversality argument to prove that if $b_2^+(M) \geq 1$ for a generic metric there will be no reducible solutions meaning that in this case the analysis from before is applicable and the entire moduli space will be a finite dimensional manifold. Finally, we state the result that for $b_2^+(M) \geq 2$ the cobordism class of $\mathfrak{M}$ is independent of the metric.

Compactness, Finiteness, and Regularity: This section will have a very analytical flavour. We will prove bounds on solutions to SW_η and use gauge transformations to increase the regularity of the equations. With these results we can then prove three important results: 1. the moduli space is always compact, 2. the moduli space consists only of smooth configurations and 3. there are only finitely many $spin^c$-structures for which the moduli space is non-empty, making the computation of the invariants tractable.

Orientation: We will sketch how one can orient the moduli space which is key for $[\mathfrak{M}_\eta]$ to define an integer homology class instead of just a $\mathbb{Z}_2$ class.

The Invariants: Finally, we briefly discuss the construction of the Seiberg-Witten invariants.
Throughout this chapter M will be a smooth closed oriented riemannian 4-manifold.

4.1 The set up

In this section we introduce the Seiberg-Witten equations along with all the basic ingredients. We start with the configuration space.

4.1.1 The configuration space

Recall that hoosing a $spin^c$-structure $\mathfrak{s}$ on M (which always exists on 4-manifolds by corollary 3.3.2) is equivalent to defining a $Spin^c$-bundle $P_M^{Spin^c}$ that double covers $P_M^{SO} \times \mathcal{L}^2$ where $c_1(\mathcal{L}^2) = \underline{w}$ is an integral lift of the second stiefel-whitney class (see theorem 3.3.4).

The Connection
One part of the configuration space is a connection

$$A_{\mathcal{L}^2} \in \mathcal{A}(\mathcal{L}^2)$$

The negative spinor field
The other part is a negative complex spinor field (see definition 3.3.9)

$$\psi \in \Gamma(\mathcal{W}^+) := \Gamma(S_{\mathbb{C}}^+) \cong \Omega^0_{\tau^+}(P_M^{Spin^c}, \Psi^+)$$

We can describe this spinor field more explicitly in the our case of dimension 4. Recall that there is an exceptional lie algebra isomorphism $\mathfrak{spin}^c(4) = \mathfrak{spin}(4) \oplus \mathfrak{u}(1) = \mathfrak{spin}(3) \oplus \mathfrak{spin}(3) \oplus \mathfrak{u}(1) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ which integrates to $Spin^c(4) = SU(2) \times SU(2) \times U(1)/\mathbb{Z}_2$. This give us projections:

$$\mathcal{W}^- \circlearrowleft \quad U(2) \overset{\rho_-}{\leftarrow} Spin^c(4) = SU(2) \times SU(2) \times \lambda/\mathbb{Z}_2 \overset{\rho_+}{\rightarrow} U(2) \quad \circlearrowright \mathcal{W}^+$$

$$u_2 \cdot \lambda \leftarrow\!\shortmid [u_1, u_2, \lambda] \mapsto u_1 \cdot \lambda$$

where the loop arrows symbolize that $\mathcal{W}^\pm$ are induced from the respective group components.

The Configuration Space
With this we can now state that the configuration space $\mathcal{C}$ that is the domain if the Seiberg-Witten equations is:

$$\mathcal{C} = \Gamma(\mathcal{W}^+) \times \mathcal{A}(\mathcal{L}^2)$$

4.1.2 The squaring map

We have seen that the exceptional lie algebra isomorphism $\mathfrak{spin}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ splits a $Spin^c$-bundle in two $U(2)$ bundles. The same isomorphism can be written $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and integrates to a map $SO(4) \to SO(3) \times SO(3)$ giving projection maps:

$$\Lambda^2_- \circlearrowleft \quad SO(3) \overset{\lambda_-}{\leftarrow} SO(4) = Spin(3) \times Spin(3)/\mathbb{Z}_2 \overset{\lambda_+}{\rightarrow} SO(3) \quad \circlearrowright \Lambda^2_+$$

$$p_{Spin}(u_2) \hookleftarrow [u_1, u_2] \mapsto p_{Spin}(u_1)$$

This leads in particular to a split of the $SO(4)$-bundle $SO(4) \hookrightarrow \Lambda^2(M) = P^{SO}_M \times_\mu \Lambda(\mathbb{R}^4) \to M$ into two $SO(3)$-bundles $SO(3) \hookrightarrow \Lambda^2_+(M) \to M$ and $\Lambda^2_-(M)$ [1] . It's sections are self-dual and anti-self-dual forms respectively. One of the reasons why this is significant is because the curvature of a principal bundle P can be regarded as a section of the twisted bundle $\Lambda^2(M) \otimes ad(P)$ which also splits accordingly. Donaldson Theory is the study of (gauge equivalence classes of) connections on prinicpal SU(2) bundles whose curvature form F_A projects trivially to the self dual part ie $F_A^+ = 0$. In Seiberg-Witten theory the bundle of interest is a U(1)-bundle $\mathcal{L}^2$ whose curvature 2-form is a section of $\Lambda^2(M) \otimes (M \times i\mathbb{R}) = i\Lambda^2(M)$ and the object of study are not connections whose curvature projects trivially to the self-dual part, but whose curvature projects to negatuve spinor fields that are annihilated by the dirac operator. The map establishing this correspondence is called the **squaring map** σ fitting nicely into the commutative diagram:

$$\begin{tikzcd}
\mathcal{W}^- \circlearrowleft & U(2) \arrow[d] & \mathrm{Spin}^{\mathbb{C}}(4) \arrow[l, "\rho_-"'] \arrow[r, "\rho_+"] \arrow[d] & U(2) \arrow[d] & \circlearrowright \mathcal{W}^+ \arrow[d, "\sigma"] \\
i\Lambda^2_- \circlearrowleft & SO(3) & SO(4) \arrow[l, "\lambda_-"'] \arrow[r, "\lambda_+"] & SO(3) & \circlearrowright i\Lambda^2_+ \\
 & & \circlearrowright TM & &
\end{tikzcd}$$

Because of this peculiarity of dimension 4, the positive and negative spinnor bundles $\mathcal{W}^\pm$ are also called **self-dual** or **anti-self-dual** spinnor bundles respectively. We will now define the squaring map more explicitly.

Lemma 4.1. *Since $\Lambda^*(M) \otimes \mathbb{C} \overset{f}{\cong} \mathbb{C}l(M)$ as vector bundles, we can naturally define clifford multiplication of $\Lambda^*(M)$ on $\mathcal{W}^+$ by first sending it through the isomorphism f. Because $e_i \cdot \mathcal{W}^\pm \subset \mathcal{W}^\mp$ for $e_i \in T_pM$ clifford multiplication of $\Lambda^2(M)$ preserves $\mathcal{W}^\pm$ and we can regard clifford multiplication as a linear map*

$$\cdot \ : \Omega^2(M) \to \Gamma(End(\mathcal{W}^\pm)) \subset \Gamma(\mathbb{C}l_0(M))$$

$$e^i \wedge e^j \mapsto \left(\psi \mapsto (e^i \wedge e^j) \cdot \psi := e_i e_j \cdot \psi\right)$$

where the last expression is true for any local orthonormal basis $\{e^i\}$ of $\Omega^1(U_\alpha)$. Restricting this map to $\Omega^2_+(M)$ gives an isomorphism to trace-free endomorphisms of $\Gamma(\mathcal{W}^+)$:

$$\rho : \Omega^2_+(M) \cong \Gamma(End_0(\mathcal{W}^+))$$

Proof. One can for example use the explicit matrix model to prove that. □

[1]The $\mathbb{CP}^1$-bundle $Z = S_{\sqrt{2}(\Lambda^2_+) \to M}$ is called the twistor space of M. One can define $Spin^c(4)$-structures geometrically as complex line bundles over Z that have degree 1 over every fiber. The existence of them is equivalent to the fact that Z can always be expressed as a projectivization of a complex vector bundle $\mathcal{W}^+ \to M$. See [LeB21]

Definition 4.1.1. Squaring map

The **squaring map** $\sigma : \Gamma(\mathcal{W}^+) \to i\Omega^2_+(M)$ is given by:

$$\sigma(\psi) = \rho^{-1}((\psi \otimes \psi^*)_0)$$

where $(\psi \otimes \psi^*)_0 = \psi \otimes \psi * -\frac{1}{2}tr(\psi \otimes \psi^*)id = \psi \otimes \psi^* - \frac{1}{2}|\psi|^2 id$ is trace free. More explicitly, σ is given by:

$$\sigma(\psi) = -\frac{1}{4}i\Big((|\psi_1|^2 - |\psi_2|^2) \cdot b_1 - 2 \cdot Im(\psi_1\bar{\psi}_2) \cdot b_2 - 2 \cdot Re(\psi_1\bar{\psi}_2) \cdot b_3\Big)$$

$$= -\frac{1}{4}\Big((\psi^* I\psi) \cdot b_1 + (\psi^* J\psi) \cdot b_2 + (\psi^* K\psi) \cdot b_3\Big)$$

where $\{b_1, b_2, b_3\}$ is the standard basis of local self-dual forms given by a local orthonormal basis of T^*M.

The formula can easily be verified by applying ρ. Note that if we restrict the map on a fiber to the 3-sphere of radius 2 $S^3_2 \subset \mathcal{W}^+_p \cong \mathbb{C}^2$, the image is the 2-sphere of radius $\sqrt{2}$ $S^2_{\sqrt{2} \subset i(\Lambda^2_+(M))_p}$ (because $|b_i| = \sqrt{2}$) and $\sigma(e^{i\theta}\psi) = \sigma(\psi)$ and the map is none other than the **hopf fibration**. Because

$$\sigma(r\psi) = r^2\sigma(\psi)$$

one can regard σ as a "squared cone" on fiber-wise hopf maps.

4.1.3 The Group of Gauge Transformations

We have seen in definition 3.3.8 that the connection of $P^{Spin^c}_M$ is determined by choosing a connection on $\mathcal{L}^2$. The appropriate notion of gauge transformations of $P^{Spin^c}_M$ is hence the subgroup of bundle isomorphisms that act trivially on the $SO(n)$ part ie those $\sigma : P^{Spin^{\mathbb{C}}}_M \to P^{Spin^{\mathbb{C}}}_M$ for which the projection $\Lambda_{SO} : P^{Spin^c}_M \to P^{SO}_M$ is invariant:

$$\Lambda_{SO} \circ \sigma = \Lambda_{SO}$$

Proposition 4.1.1. Group of Gauge Transformations

The Group of Gauge Transformations $\mathcal{G}(\mathfrak{s})$ can be identified with:

$$\mathcal{G}(\mathfrak{s}) = \Big\{\sigma_g : P^{Spin^{\mathbb{C}}}_M \to P^{Spin^{\mathbb{C}}}_M, \sigma_g(p) = p \cdot g(\pi_{P^{Spin^{\mathbb{C}}}_M}(p)) \text{ for } g \in \Gamma(M \times U(1))\Big\} \cong \Gamma(M \times U(1))$$

where $U(1)$ acts on $P^{Spin^c}_M$ by the inclusion $U(1) \subset Spin^c(4)$.

Note that the σ_g's in the definition are equivariant (because U(1) is in the center of $Spin^c$), fiber preserving maps (and thus actually bundle isomorphisms see **??**.

The action of $\mathcal{G}(\mathfrak{s})$ on the solution space $Sol(SW)$ is given by pullback. We will now write explicit formulas for the various pullbacks.

Action on Spinors The action of $g \in \mathcal{G}(\mathfrak{s})$ on self-dual spinors, regarded as equiavariant maps $\tilde{\psi} \in \Omega^0_{\tau^+}(P^{Spin^{\mathbb{C}}}_M, \Psi^+)$ is given by:

$$(\tilde{\psi} \cdot g)(p) := \sigma^*_g\tilde{\psi}(p) = \tilde{\psi}(\sigma_g(p)) = \tilde{\psi}(p \cdot g(x)) = g^{-1}(x)\tilde{\psi}(p)$$

where $x = \pi_{Spin^c}(p)$ Regarding self-dual spinors as sections $\psi \in \Gamma(\mathcal{W}^+)$ this translates to:

$$\psi \cdot g = g^{-1}\psi$$

Action on connections

To describe the action of $g \in \mathcal{G}(\mathfrak{s})$ on $A_{\mathcal{L}^2} \in \mathcal{A}(\mathcal{L}^2)$, we note that σ_g induces an isomorphism σ'_g of $\mathcal{L}^2(\mathfrak{s})$ defined so that the following diagram commutes:

$$\begin{array}{ccc} P_M^{Spin^{\mathbb{C}}} & \xrightarrow{\sigma_g} & P_M^{Spin^{\mathbb{C}}} \\ \lambda_{U(1)}\big\downarrow & & \big\downarrow\lambda_{U(1)} \\ \mathcal{L}^2(\mathfrak{s}) & \xrightarrow{\sigma'_g} & \mathcal{L}^2(\mathfrak{s}) \end{array}$$

A simple computation yields the explicit formula:

$$\sigma'_g(\lambda_{U(1)}(p)) = \lambda_{U(1)}(\sigma_g(p)) = \lambda_{U(1)}(p \cdot g(\pi_{Spin^c}(p))) = \lambda_{U(1)}(p) \cdot det(g(\pi_{Spin^c}(p))) = \lambda_{U(1)}(p) \cdot (g(\pi_{Spin^c}(p))^2$$

$$\Rightarrow \sigma'_g(u) = u \cdot (g \circ \pi_{det})^2 =: u \cdot g'^2$$

for $u \in \mathcal{L}^2(\mathfrak{s})$. Note again that because $U(1)$ is abelian, the σ'_g's are equivariant and thus bundle isomorphism. The action of $g \in \mathcal{G}(\mathfrak{s})$ on $A_{\mathcal{L}^2} \in \mathcal{A}(\mathcal{L}^2)$ is defined by pullback with σ'_g:

$$A_{\mathcal{L}^2} \cdot g := (\sigma'_g)^* A_{\mathcal{L}^2} = g'^{-2} A_{\mathcal{L}^2} g'^2 + 2g'^{-1} dg' = A_{\mathcal{L}^2} + 2g'^{-1} dg'$$

where we used the standard formula for a gauge transformation of a connection (see lemma 3.2).
We want to remark that the connections $A_{spin^c} \in \mathcal{A}(P_M^{Spin^c})$ transforms without a factor of 2.
Action on configurations
Thus the action of the configuration space is:

$$(\psi, A) \cdot g = \left((g \circ \pi_{Spin^c})^{-1}\psi, A_{\mathcal{L}^2} + 2(g \circ \pi_{det})^{-1} d(g \circ \pi_{det})'\right)$$

4.1.4 The Equations

We are now ready to state the Seiberg-Witten equations:

$$\begin{cases} \partial\!\!\!/_{A_{spin^c}} \psi = 0, \\ F^+_{A_{\mathcal{L}^2}} = \sigma(\psi) \end{cases} \qquad (SW)$$

where $\partial\!\!\!/_{A_{spin^c}} =: \partial\!\!\!/^+_{A_{spin^c}} : \Gamma(\mathcal{W}^+) \to \Gamma(\mathcal{W}^-)$ is the complex spin-dirac operator acting on complex spinor fields. Solutions (φ, A) to those equations are referred to as **Seiberg-Witten monopoles**. The space of solutions, denoted by $Sol(SW)$ forms a subspace inside the infinte dimensional configuration space $\mathcal{C} = \Gamma(\mathcal{W}^+) \times Conn(\mathcal{L})$. Note that $Sol(SW)$ depends on the chosen metric and the chosen $Spin^{\mathbb{C}}$ structure $\mathfrak{s}$ which we will sometimes denote explicitly to remind ourselves of that fact.

4.1.5 Analytical setup

Morally, the above definitions are the right ones. Technically, though, spaces of smooth sections are not sufficiently well behaved to use powerful results like the sard-smale theorem or the inverse and implicit function theorem. The issue is that smooth function spaces are not complete. Using local sections and coordinates, one can easily construct a sequence of smooth local sections that converges to non-smooth section. What we need are banach or even better hilbert spaces. One way to produce a hilbert space, is to enlarge the space of smooth sections to L^p-sections. Unfortunately, elements in L^p spaces might not be differentiable (not even continuous) so we cannot do calculus with these spaces. The standard approach to overcome this issue is to enlarge the spaces to Sobolev spaces $W^{k,p}$ that are k-times weakly differentiable and whose weak derivatives are also in L^p. Just like L^p these spaces are all banach spaces and in the case of $p = 2$ it is a hilbert space. Skipping a lot of details on how to define these spaces, we enlarge the spaces of interest. We will use upper indices to indicate the sobolev extension we are working with. Fixing a smooth connection $A_0 \in \mathcal{A}(\mathcal{L}^2)$ (and remembering that connections forma an affine space) we define:

$$\mathcal{A}^2 = \mathcal{A}^{2,2}(\mathcal{L}^2) = \{A_0 + ia \big| a \in \Gamma^{2,2}(\Lambda^1(M))\}$$

and extend the configuration space to

$$\mathcal{C}^2 = \mathcal{C}^{2,2} = \Gamma^{2,2}(\mathcal{W}^+) \times \mathcal{A}^{2,2}(\mathcal{L}^2)$$

The solution space to SW in $\mathcal{C}^2$ Whenever the sobolev property plays no essential role, we will drop the indices for notational clarity.

4.2 The moduli space of configurations

To prove the moduli space of configurations $\mathcal{B} = \mathcal{C}/\mathcal{G}$ has a manifold structure near irreducible configurations, we first have to linearize the action of $\mathcal{G}$ on $c \in \mathcal{C}$. After this, we will define reducible configurations and prove the slice theorem.
Let's first understand the involved tangent spaces:

Lemma 4.2. *The group of gauge transformations $\mathcal{G}(\mathfrak{s})$ can be extended to a hilbert lie group $\mathcal{G}^{3,2}$ modeled by $i\Gamma^{3,2}(M)$ whose lie algebra is*

$$\mathfrak{g}(\mathfrak{s}) = i\Gamma^{3,2}(M)$$

Lemma 4.3. *The tangent space of $\mathcal{C}^{2,2} = \Gamma^{2,2}(\mathcal{W}^+) \times \mathcal{A}^{2,2}(\mathcal{L}^2)$ at every $(\psi, A_{\mathcal{L}^2})$ is:*

$$T_{(\psi,A)}\mathcal{C}^{2,2} = \Gamma^{2,2}(\mathcal{W}^+) \times i\Gamma^{2,2}\big(\Lambda^1(M)\big)$$

Proposition 4.2.1. Linearized action of $\mathcal{G}$ on $c \in \mathcal{C}$

The action of $g \in \mathcal{G}(\mathfrak{s})$ on $c = (\psi, A_{\mathcal{L}^2}) \in \Gamma(\mathcal{W}^+) \times \mathcal{A}(\mathcal{L}^2)$ can be understood as a map

$$f_{(\psi,A_{\mathcal{L}^2})} : \mathcal{G}^{3,2}(\mathfrak{s}) \to \Gamma^{2,2}(\mathcal{W}^+) \times \mathcal{A}^{2,2}(\mathcal{L}^2), \quad g \mapsto (\psi, A_{\mathcal{L}^2}) \cdot g = (g^{-1}\psi, A_{\mathcal{L}^2} + 2g^{-1}dg)$$

that can be differentiated at the identity:

$$D_{c\cdot\mathcal{G}} := (Tf_{(\psi,A_{\mathcal{L}^2})})_{id} : \mathfrak{g}(\mathfrak{s}) = i\Gamma^{2,2}(M) \to \Gamma^{2,2}(\mathcal{W}^+) \times i\Gamma^{2,2}\big(\Lambda^1(M)\big), \quad a \mapsto (-a\cdot\psi, 2da)$$

where $(a\cdot\psi)(p)$ is given by clifford multiplication of $a(p) \in i\mathbb{R} \subset \mathbb{C}l(T_pM)$ with $\psi(x) \in \mathcal{W}_p^+$. The kernel is the tangent space to the stabalizer. It's formal adjoint is given by:

$$D^*_{(\psi,A)\cdot\mathcal{G}}(\varphi, a) = 2\delta a + iIm(\langle\varphi,\psi\rangle)$$

Proof. Let $a : M \to i\mathbb{R}$ be in $i\Gamma(M) = \mathfrak{g}(\mathfrak{s})$. Then $g(t) = e^{ta}$ is a curve in $\mathcal{G}(\mathfrak{s})$ with $g(0) = id, a := g'(0) = a$ and compute the derivative. □

As hinted at in the outline, reducible configurations are those for which the stabilizer is not trivial. The following lemma characterizes **reducible configurations**.

Lemma 4.4. *For $(\psi, A_{\mathcal{L}^2}) \in \mathcal{C}$ the following are equivalent:*
1. $Stab(\psi, A_{\mathcal{L}^2})$ is not trivial
2. $\psi = 0$
3. $Stab(\psi, A_{\mathcal{L}^2}) \cong U(1)$
4. $D_{c\cdot\mathcal{G}}$ has non-trivial kernel
If any of those conditions is met, we call $c = (\psi, A) \in \mathcal{C}$ ***reducible****.*

Proof. An element $(\psi, A_{\mathcal{L}^2})$ is fixed by $\pm id \neq g \in \mathcal{G}(\mathfrak{s})$ iff

$$(\psi, A_{\mathcal{L}^2}) \cdot g = (g^{-1}\psi, A_{\mathcal{L}^2} + 2g^{-1}dg) = (\psi, A_{\mathcal{L}^2}) \Leftrightarrow \psi = 0 \text{ and } g = e^{i\theta} = const$$

showing the equivalence of all statements. □

One can show that the space of irreducible configuration $\mathcal{C}^*$ is open in $\mathcal{C}$ and so will be the moduli spaces $\mathcal{B}^*, \mathfrak{M}^*$ of irreducible equivalence classes.
To prove the slice theorem we will need the following technical lemma:

Lemma 4.5. *Let $(\psi_n, A_n), (\tilde{\psi}_n, \tilde{A}_n)$ be sequences in $\mathcal{C}^{2,2}(\mathfrak{s})$ converging to $(\psi, A), (\tilde{\psi}, \tilde{A}) \in \mathcal{C}^{2,2}(\mathfrak{s})$ respectively and $g_n \in \mathcal{G}^{3,2}(\mathfrak{s})$ s.t.*

$$(\psi_n, A_n) \cdot g_n = (\tilde{\psi}_n, \tilde{A}_n)$$

Then, there exists a subsequence $\tilde{g}_n$ converging to $g \in \mathcal{G}^{3,2}(\mathfrak{s})$ and

$$(\psi, A) \cdot g = (\tilde{\psi}, \tilde{A})$$

With that we can prove that there are local slices giving the moduli space $\mathcal{B}^*$ a local manifold structure.

Theorem 4.2.1. Slice Theorem

If $(\psi, A_{\mathcal{L}^2}) \in \mathcal{C}$ is irreducible, then there is a neighborhood

$$U_{(\psi,A),\epsilon} := \left\{ (\psi, A) + (\varphi, a) \middle| (\varphi, a) \in ker(D^*_{c\cdot\mathcal{G}}), ||(\varphi, a)|| < \epsilon \right\} \subset \mathcal{C}$$

that projects injectively into the moduli space $\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}$ giving it a local smooth hilbert manifold structure.

Proof. We prove that

$$\sigma : U_{(\psi,A),\epsilon} \times \mathcal{G}(\mathfrak{s}) \to \mathcal{C}(\mathfrak{s}), \quad \big((\varphi, a), g\big) \mapsto \big((\psi, A) + (\varphi, a)\big) \cdot g$$

is a local diffeomeorphism. The derivative at $\big((\psi, A), id\big)$ can be calculated using a generalized product/leibniz rule to be:

$$T\sigma_{\big((\psi,A),id\big)} = \big(id_{ker(D^*_{c\cdot\mathcal{G}})}, D_{c\cdot\mathcal{G}}\big) : ker(D^*_{c\cdot\mathcal{G}}) \oplus i\Gamma(M) \to ker(D^*_{c\cdot\mathcal{G}}) \oplus im(D_{c\cdot\mathcal{G}})$$

The map is obviously surjective and it is also injective because (ψ, A) is irreducible by assumption and thus a linear isomorphism. One can also show from general sobolev properties that it is bounded and thus the inverse function theorem for banach spaces, there is a a neighborhood $V_{id,\delta} \subset \mathcal{G}(\mathfrak{s})$ of the identity and a diffeomorphism:

$$\tilde{\sigma} : U_{(\psi,A),\epsilon} \times V_{id,\delta} \to \mathcal{C}(\mathfrak{s})$$

Next, we want to show that this diffeomorphism extends (for a perhaps smaller ϵ) to $\sigma' : U_{(\psi,A),\tilde{\epsilon}} \times \mathcal{G}(\mathfrak{s}) \to \mathcal{C}(\mathfrak{s})$. Because it is a local diffeomorphism we just have to show that it is bijective. We proceed the proof by contraposition. Assume σ' is not bijective for any $\tilde{\epsilon}$. We want to show that then, there cannot be a local diffeomorphism $\tilde{\sigma}$. If σ' is not bijective, there exists sequences a_n, b_n in $U_{(\psi,A),\tilde{\epsilon}}$ converging to $0 \in U_{(\psi,A),\tilde{\epsilon}}$ and a sequence g_n in $\mathcal{G}(\mathfrak{s})$ s.t.

$$\big((\psi, A) + a_n\big) \cdot g_n = \sigma'(a_n, g_n) = \sigma'(b_n, id) = (\psi, A) + b_n \text{ but } (a_n, g_n) \neq (b_n, id)$$

By lemma 4.5 there is a subsequence of g_n converging to $g \in \mathcal{G}(\mathfrak{s})$ which must satisfy $\sigma'(0, g) = \sigma'(0, id)$ implying that $g = id$ as we assumed that (ψ, A) is irreducible. Thus, for sufficiently large n, $\sigma'(, g_n) = \tilde{\sigma}(, g_n)$ showing that $\tilde{\sigma}$ is not bijective. □

4.3 The moduli space of SW monopoles

Before we discuss the Seiberg-Witten moduli space, let's first ensure that it is well-defined.

Lemma 4.6. *The solution space of the Seiberg-Witten equations $Sol^2(SW) \subset \Gamma^2(\mathcal{W}^+) \times \mathcal{A}^2(\mathcal{L}^2)$ is invariant under the action of $\mathcal{G}^{3,2}(\mathfrak{s})$ on $\Gamma^2(\mathcal{W}^+) \times \mathcal{A}^2(\mathcal{L}^2)$.*

Proof. The invariance of the curvature equation follows from a straightforward calculation and the fact that F_A is invariant under gauge transformation because $U(1)$ is abelian. The invariance of the dirac equation follows from the fact that $\nabla^{A,g}(\psi \cdot g) = \big(\nabla^A \psi\big) \cdot g$ which can be verified using the formulas given above (note that the factor of 2 from the gauge transformation and $\frac{1}{2}$ from the action of $\mathfrak{spin}^c$ cancel). □

Assuming $(\psi, A) \in \mathcal{C}$ is irreducible, we can identify a neighborhood of moduli space of SW monopoles with:

$$\mathfrak{M}^2(\mathfrak{s}) \cap U_{[\psi,A_{\mathcal{L}^2}]} = Sol^2(SW) \cap U_{(\psi,A),\epsilon} \subset \mathcal{B}^2(\mathfrak{s})$$

which we can also write (also dropping the sobolev indices) as:

$$Sol(SW) \cap U_{(\psi,A),\epsilon} = (sw|_{U_{(\psi,A),\epsilon}})^{-1}(0)$$

for the **Seiberg Witten map**

$$sw : \Gamma(\mathcal{W}^+) \times \mathcal{A}(\mathcal{L}^2) \to \Gamma(\mathcal{W}^-) \times i\Omega^2_+(M)$$
$$(\psi, A_{\mathcal{L}^2}) \mapsto \big(\partial\!\!\!/_{A_{Spin^c}}\psi, F^+_{A_{\mathcal{L}^2}} - \sigma(\psi)\big)$$

We would like to apply the implicit function theorem to sw. Let's first compute the linearization:

Lemma 4.7. *The derivative of the Seiberg Witten map $c = (\psi, A_{\mathcal{L}^2}) \overset{sw}{\mapsto} \left(\partial\!\!\!/_{A_{Spin^c}}\psi, F^+_{A_{\mathcal{L}^2}} - \sigma(\psi)\right)$ is given by:*

$$sw_{*c} : \Gamma(\mathcal{W}^+) \oplus i\Omega^1(M) \to \Gamma(\mathcal{W}^-) \oplus i\Omega^2(M)$$

$$(\varphi, a) \mapsto \left(\partial\!\!\!/\varphi + \frac{1}{2}a \cdot \psi, -D_\psi \varphi + d^+ a\right) = \begin{pmatrix} \partial\!\!\!/ & \cdot\frac{1}{2}\psi \\ -D_\psi & d^+ \end{pmatrix} \begin{pmatrix} \varphi \\ a \end{pmatrix}$$

where $D_\psi = \psi \otimes \varphi^* + \varphi \otimes \psi^* + \langle \psi, \varphi \rangle_{\mathbb{R}}$

Proof. □

We compute

$$sw_{*(\psi, A_{\mathcal{L}^2})}(\varphi, a) = \frac{d}{dt} sw(\psi + t\varphi, A_{\mathcal{L}^2} + ta)|_{t=0}$$

Using $\partial\!\!\!/_{A'} = \partial\!\!\!/_A + \frac{1}{2}a\cdot$ which one can see by using the associated bundle construction and the leibniz rule for d, the first component becomes:

$$\frac{d}{dt}\partial\!\!\!/_{A'}(\psi + t\varphi) = \frac{d}{dt}\underbrace{\partial\!\!\!/_A\psi}_{=0} + \frac{1}{2}ta \cdot \psi + t(\partial\!\!\!/_A\varphi + \frac{1}{2}ta \cdot \varphi)\Big|_{t=0}$$

$$= \partial\!\!\!/_A\varphi + \frac{1}{2}a \cdot \psi$$

The first addend of the second component becomes:

$$\frac{d}{dt}F^+_{A_{\mathcal{L}^2}+ta}|_{t=0} = \frac{d}{dt}\left(d(A_{\mathcal{L}^2} + ta)\right)^+ + \frac{1}{2}\left([A + ta, A + ta]\right)^+\Big|_{t=0} = d^+ a$$

To compute the second addent, note that ρ^{-1} is a linear isomorphism of vector spaces whose derivative is just ρ^{-1} itself. Thus:

$$\frac{d}{dt}\sigma(\psi + t\varphi)|_{t=0} = \frac{d}{dt}\rho^{-1}\Big(\left((\psi + t\varphi) \otimes (\psi + t\varphi)^*\right)_0\Big)|_{t=0} = \rho^{-1}\Big(\frac{d}{dt}\left((\psi + t\varphi) \otimes (\psi + t\varphi)\right)_0|_{t=0}\Big)$$

The argument becomes:

$$\frac{d}{dt}\left((\psi + t\varphi) \otimes (\psi + t\varphi)^*\right)|_{t=0} = \left(\psi \otimes \varphi^* + \varphi \otimes \psi^*\right)_0 = \psi \otimes \varphi^* + \varphi \otimes \psi^* + \frac{\overbrace{\langle \psi, \varphi \rangle_{\mathbb{C}} + \langle \varphi, \psi \rangle_{\mathbb{C}}}^{=\langle \psi, \varphi \rangle_{\mathbb{R}}}}{2}$$

One can prove that sw_{*c} is always onto the first component using the unique continuation property of the dirac operator (which we excluded for space reasons). It is harder to prove that the map is always onto the second factor for a generic metric [2] so we will instead perturb the second Seiberg-Witten equation to:

$$F^+_{A_{\mathcal{L}^2}} = \sigma(\psi) + \eta$$

for $\eta \in i\Omega^2_+(M)$. We will denote the perturbed Seiberg-Witten equation by SW_η. The perturbed solution space can then locally be described by:

$$\mathfrak{M}_\eta(\mathfrak{s}) = Sol(SW_\eta) \cap U_{(\psi, A), \epsilon} = (sw|_{U_{(\psi, A), \epsilon}})^{-1}(0, \eta)$$

If $(sw|_{U_{(\psi, A_{\mathcal{L}^2}), \epsilon}})_{*(\psi, A_{\mathcal{L}^2})} = sw_{*(\psi, A_{\mathcal{L}^2})}|_{ker(D^*_{c \cdot \mathcal{G}})}$ is a fredholm map for all $(\psi, A_{\mathcal{L}^2}) \in \mathcal{C}$, then the sard-smale theorem in combination with the implicit function theorem guarantees, that this will be smooth finite dimensional manifold for almost all perturbations:

Theorem 4.3.1. Sard-Smale Theorem + Implicit function theorem

Let $f : X \to Y$ be a smooth map between banach manifolds and assume that $Tf_x : TX \to TY$ is fredholm at all $x \in X$. The **Sard-Smale Theorem** says that almost all $y \in Y$ are regular values, meaning that for all $\tilde{x} \in f^{-1}(y)$ the kernel $ker(Tf_{\tilde{x}})$ is surjective. The **implicit function theorem** then implies that the preimage $f^{-1}(y)$ is either empty or a smooth manifold of dimension:

$$dim(f^{-1}(y)) = ind(Tf_{\tilde{x}}) = dim(ker(Tf_{\tilde{x}}))$$

So our goal is to prove that $sw_{*c}|_{ker(D^*_{c \cdot \mathcal{G}})}$ is fredholm for all irreducible $c \in Sol(SW_\eta(\mathfrak{s}))$. We will be able to do this by using the fundamental complex.

[2] For a proof see **[Fri00]**

4.4 The elliptic complex

We can summarize the analysis of the last two sections in the **fundamental complex** $\mathcal{E}(c, \mathfrak{s})$:

$$0 \to \underbrace{\mathfrak{g}(\mathfrak{s})}_{=i\Omega^0(M)} \xrightarrow{D_{c\cdot\mathcal{G}}} \underbrace{T_c\mathcal{C}}_{=\Gamma(\mathcal{W}^+)\oplus i\Omega^1(M)} \xrightarrow{sw_{*c}} \underbrace{T_{(0,\eta)}\mathcal{X}}_{=\Gamma(\mathcal{W}^-)\oplus i\Omega^2_+(M)} \to 0$$

where $D_{c\cdot\mathcal{G}} : a \mapsto (-a\cdot\psi, 2da)$ is the infinitesimal action. We have seen that the moduli space of configurations $\mathcal{B}(\mathfrak{s}) = \mathcal{C}(\mathfrak{s})/\mathcal{G}(\mathfrak{s})$ is a smooth manifold in the neighborhood of all $[c] = [\psi, A] \in \mathcal{B}(\mathfrak{s})$ for which $ker(D_{c\cdot\mathcal{G}}) = 0$ is injective. In this case c is an irreducible configuration. Further, we have seen that the moduli space of Seiberg-Witten monopoles $\mathfrak{M}_\eta$ is a finite dimensional manifold for almost all $\eta \in i\Omega^2_+(M)$ if c is irreducible and $sw_{*c}|_{D^*_{c\cdot\mathcal{G}}}$ is fredholm. After proving that the fundamental complex $\mathcal{E}(c, \mathfrak{s})$ is elliptic for $c \in Sol(SW_\eta)$ we can encode these two conditions into the cohomology of $\mathcal{E}(c, \mathfrak{s})$ and prove that $sw_{*c}|_{D^*_{c\cdot\mathcal{G}}}$ is fredholm if c is irreducible. Finally, we can compute the virtual dimension [3] of $\mathfrak{M}_\eta$ using the Atiyah-Singer index theorem.

Proposition 4.4.1. Ellipticity of the fundamental complex

The fundamental complex $\mathcal{E}(c, \mathfrak{s})$ is an elliptic complex for all $c \in Sol(SW_\eta)$ and all $spin^c$ structures $\mathfrak{s}$.

Proof. 1. We have to show that it is an elliptic complex ie that $sw_{*(\psi,A_{\mathcal{L}^2})} \circ (-\cdot\psi, 2d) = 0$. Let $f \in i\Omega^0(M)$

$$sw_{*(\psi,A_{\mathcal{L}^2})}\big((-f\cdot\psi, 2df)\big) = \big(-\partial\!\!\!/(f\cdot\psi) + df\cdot\psi, D_\psi(f\cdot\psi) + 2d^+(df)\big)$$

One can easily proof that the dirac operator satisfies a leibniz rule as it is defined by the covariant derivative. Thus, the first component becomes:

$$-\partial\!\!\!/(f\cdot\psi) + df\cdot\psi = -df\cdot\psi - f\cdot\underbrace{\partial\!\!\!/\psi}_{=0} + df\cdot\psi = 0$$

The first addend of the second component is:

$$D_\psi(f\cdot\psi) = \psi\otimes(f\cdot\psi)^* + (f\cdot\psi)\otimes\psi^* + \langle\psi, f\cdot\psi\rangle_\mathbb{R} = \psi\otimes\langle -, f\cdot\psi\rangle_\mathbb{C} + f\cdot\psi\otimes\langle -,\psi\rangle_\mathbb{C}$$
$$= \bar{f}\cdot\psi\otimes\psi^* + f\cdot\psi\otimes\psi^* = 0$$

where we used that f is imaginary valued and thus $f\cdot\psi \perp \psi$ and $\bar{f} = -f$.
The second added of the second component is also zero because:

$$d^+d = pr_+ \circ d \circ d = 0$$

2. We have to show that the complex is elliptic ie that the (principal) symbol sequence is exact. Because the symbol of a differential operator only depends on the highest order term, the symbol of the first operator is just twice the symbol of d or

$$\sigma\big((-\cdot\psi, 2d)\big) : \pi^*\big(i\Lambda^0(M)\big) \to \pi^*\big(\mathcal{W}^+ \times i\Lambda^1(M)\big)$$

$$\sigma\big((-\cdot\psi, 2d)\big)(\xi_p, \theta_p) = \sigma\big((0, 2d)\big)(\xi_p, \theta_p) = \Big(\xi_p, \big((0, 2(d_pf\cdot\theta_p + f(p)d_p\theta))\big)\Big) = \big(\xi_p, (0, 2\xi_p\cdot\theta_p)\big)$$

for $(\xi_p, \theta_p) \in \pi^*(i\Lambda^0(M)) \subset T'M \times \pi^*(i\Lambda^0(M)), \pi^*\big(i\Lambda^0(M)\big)$, any $f \in C^\infty(M)$ satisfying $f(p) = 0$ and $d_pf = \xi_p$ and any $\theta \in i\Omega^0(M)$ satisfying $\theta(p) = \theta_p$. The symbol of the second operator is:

$$\sigma(sw_{*(\psi,A_{\mathcal{L}^2})}) : \pi^*\big(\mathcal{W}^+ \times i\Lambda^1(M)\big) \to \pi^*\big(\mathcal{W}^- \times i\Lambda^2_+(M)\big)$$

$$\sigma(sw_{*(\psi,A_{\mathcal{L}^2})})\big(\xi_p, (\varphi_p, a_p)\big) = \sigma\big(\partial\!\!\!/ \oplus d^+\big)\big(\xi_p, (\varphi_p, a_p)\big) = \xi_p\cdot\varphi_p + pr_+ \circ \xi_p \wedge a_p$$

where the symbol of the dirac operator and of d^+ can easily be calculated in the same way as above using the respective leibniz rules.
Because $T^*M \cong = \Lambda^1(M)$ and $\theta(p) \in i\mathbb{R}$, the image of the first symbol is $\pi^*\big(\{0\} \times i\Lambda^1(M)\big) \subset \pi^*\big(\mathcal{W}^+ \times i\Lambda^1(M)\big)$ which is exactly the kernel of the second symbol because:

$$\xi_p\cdot\varphi_p + pr_+ \circ \xi_p \wedge a_p = 0 \Leftrightarrow \varphi_p = 0 \text{ and } a_p = \xi_p\cdot\theta_p \in i\Lambda^1_p(M), \theta_p \in i\mathbb{R}$$
$$\Leftrightarrow (\xi_p, (\varphi_p, a_p)) \in \pi^*\big(\{0\} \times i\Lambda^1(M)\big) \subset \pi^*\big(\mathcal{W}^+ \times i\Lambda^1(M)\big)$$

□

[3] We say "virtual" because the moduli space might also be empty. If it is not empty its dimension will be the virtual dimension.

Having shown that differential complex $\mathcal{E}(c,\mathfrak{s})$ is always elliptic we can apply hodge theory.

Corollary 4.4.1. Hodge Theory applied to the fundamental complex

The hodge decomposition theorem 3.2.4 gives:

$$i\Omega^0(M) = ker(D_{c\cdot\mathcal{G}}) \oplus_\perp im(D^*_{c\cdot\mathcal{G}})$$

$$\Gamma(\mathcal{W}^+) \oplus i\Omega^1(M) = ker(sw_{*(\psi,A_{\mathcal{L}^2})}) \cap ker(D^*_{c\cdot\mathcal{G}}) \oplus_\perp im((sw_{*(\psi,A_{\mathcal{L}^2})})^*) \oplus_\perp im(D_{c\cdot\mathcal{G}})$$

$$\Gamma(\mathcal{W}^-) \oplus i\Omega^2_+(M) = ker((sw_{*(\psi,A_{\mathcal{L}^2})})^*) \oplus_\perp im(sw_{*(\psi,A_{\mathcal{L}^2})})$$

and by the generalized hodge theorem 3.2.4 finite dimensional cohomology groups (vector spaces):

$$H^0(c,\mathfrak{s}) = ker(D_{c\cdot\mathcal{G}})$$

$$H^1(c,\mathfrak{s}) = \frac{ker(sw_{*c})}{im(D_{c\cdot\mathcal{G}})} = ker(sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})})$$

$$H^2(c,\mathfrak{s}) = \frac{\Gamma(\mathcal{W}^-) \oplus i\Omega^2_+(M)}{im(sw_{*c})} = ker((sw_{*c})^*)$$

where for $H^1(c,\mathfrak{s})$ we used that $ker(sw_{*c}) \perp im((sw_{*c})^*)$.

In particular note that $H^0(c,\mathfrak{s}) = ker(D_{c\cdot\mathcal{G}}) = 0$ if and only if c is irreducible and that $H^2(c,\mathfrak{s}) = 0$ if and only if sw_{*c} is surjective.

With that we can prove:

Proposition 4.4.2. Linearized Seiberg-Witten map is fredholm

$sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})}$ is always fredholm for any $c \in Sol(SW_\eta)$.

Proof. 1. That the dimensions of the kernel if finite dimensional can be seen directly:

$$dim(ker(sw_{*(\psi,A_{\mathcal{L}^2})}|_{ker(D^*_{c\cdot\mathcal{G}})})) = dim(H^1(c,\mathfrak{s})) < \infty$$

2. To see that the cokernel is finite dimensional we first note that:

$$coker(sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})}) = \underbrace{ker((sw_{*(\psi,A_{\mathcal{L}^2})})^*) \oplus_\perp im(sw_{*(\psi,A_{\mathcal{L}^2})})}_{=\Gamma(\mathcal{W}^-)\oplus i\Omega^2_+(M)} \Big/ \underbrace{im(sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})})}_{=im(sw_{*c})} = ker((sw_{*c})^*)$$

where we used the fact that the Seiberg-Witten equations are invariant under gauge transformations, which is equivalent to saying that sw takes the same value on any orbit meaning that the image of sw is the same if restricted to slices which translates on the level of tangent spaces to $im(sw_{*c}) = im(sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})})$. Thus:

$$dim(coker(sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})})) = dim(H^2(c,\mathfrak{s})) < \infty$$

3. Finally, we can see that the image of sw_{*c} is closed because complements in hilbert spaces are closed (and we extended our spaces to hilbert spaces):

$$im(sw_{*c})^\perp = ker((sw_{*c})^*) \Rightarrow im(sw_{*c}) = im(sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})}) \text{ is closed}$$

□

The conditions for the implicit function theorem are thus satisfied and we get:

Corollary 4.4.2. Virtual dimension of the moduli space

If $c = (\psi, A_{\mathcal{L}^2}) \in Sol(SW_\eta)$ is irreducible, then for almost all pertutbations η the map $sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})}$ is surjective, $H^0(c,\mathfrak{s}) = 0, H^2(c,\mathfrak{s}) = 0$ and the moduli space of SW monopoles in an open neighborhood of $[c]$ is either empty or has a local finite dimensional manifold structure of dimension:

$$dim(\mathfrak{M}_\eta(\mathfrak{s}) \cap U_{[c]}) = dim(H^1(c,\mathfrak{s}))$$

Proof. Because $sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})}$ is fredholm and onto the first factor, the preimage $(sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})})^{-1}(0,\eta)$ will be a smooth finite dimensional manifold of dimension $dim(ker(sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})}))$ for almost a η by theorem 4.3.1. By corollary 4.4.1 the dimension is equal to $H^1(c,\mathfrak{s})$.
Further, the fact that $(0,\eta)$ is a regular value means that $sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})}$ is surjective. As we have seen in the proof of proposition 4.4.2 2. we have $im(sw_{*c}|_{ker(D^*_{c\cdot\mathcal{G}})}) = im(sw_{*c})$ which by corollary 4.4.1 means that $H^2(c,\mathfrak{s}) = 0$. □

This is great news because in this case the euler characteristic of $\mathcal{E}(c,\mathfrak{s})$ reduces to $-dim(H^1(c,\mathfrak{s}))$ and can be calculated using the Atiyah-Singer Index theorem (see theorem 3.2.6).

Lemma 4.8. *The euler characteristic of $\mathcal{E}(c,\mathfrak{s})$ is:*

$$\chi(\mathcal{E}(c,\mathfrak{s})) = \frac{1}{4}\big(3\ sign(M) + 2\chi(M) - c_1(\mathfrak{s})^2\big)$$

If $H^0(c,\mathfrak{s}) = 0$ then for a generic perturbation $\eta \in i\Omega^2_+(M)$ the dimension of $\mathfrak{M}_\eta(\mathfrak{s})$ near $[c] = [\psi, A_{\mathcal{L}^2}] \in \mathfrak{M}_\eta$ is:

$$dim(\mathfrak{M}(\mathfrak{s}) \cap U_{[c]}) = -\chi(\mathcal{E}(c,\mathfrak{s})) = \frac{1}{4}\big(c_1(\mathfrak{s})^2 - 3\ sign(M) - 2\chi(M)\big)$$

Proof. One can deform the complex to get rid of all zeroth order terms without changing the index.One is then left with one complex involving d and d^+ and can directly read of the index in terms of (partial) betti numbers (by hodge theory) and the other only involving only $\not{\partial}_A$whose index can be calculated using the atiyah singer index theorem.
In corollary 4.4.2 we have already seen that $H^2(c,\mathfrak{s}) = 0$ for a generic perturbation and if $H^0(c,\mathfrak{s}) = 0$ □

To summarize, we have seen that $\mathfrak{M}_\eta(\mathfrak{s})$ is a smooth finite dimensional manifold if reducible solution can be avoided (iff $H^0(c,\mathfrak{s}) = 0$) and that we can compute its dimension. What is still open, is the question under what circumstances reducible solution can be avouided. We will address this issue now.

4.5 Reducible monopoles

According to lemma 4.4 $c = (\psi, A_{\mathcal{L}^2}) \in \mathcal{C}(\mathfrak{s})$ is reducible iff $\psi = 0$. In this case, the Seiberg-Witten equations reduce to:

$$F^+_{A_{\mathcal{L}^2}} = 0$$

which is just the equation for anti-self dual connection on the line bundle $\mathcal{L}^2$. If we can ensure that no anti-self-dual connections can exists on $\mathcal{L}^2$, then there will be no reducible solution to the Seiberg-Witten equations. This will turn out to always be the case if $b_2^+(M) \geq 1$. The prove in a neat transversality argument. Let's first recall:

Lemma 4.9. *The space of Curvature 2-forms $F_A \in \Omega^2(M, ad(\mathcal{L})) = i\Omega^2(M)$ on a line bundle $\mathcal{L}$ is the same as the space of closed 2-form $\Phi \in i\Omega^2(M)$ whose cohomology class is $[\Phi] = -2\pi i c_1(\mathcal{L})$.*

Hence, anti-self-dual curvature forms can only exist if closed anti-self-dual 2-forms exist. The following lemma crucially restricts the space of closed anti-self-dual 2-forms:

Lemma 4.10. *A (anti)-self-dual 2-forms $\omega \in \Omega^2_\pm(M)$ is closed if and only if it is coclosed if and only if it is harmonic.*

Proof.

$$\pm d\omega = \pm d(*\omega) = \pm d^*\omega$$

□

Giving us the condition:

Proposition 4.5.1. Condition for existence of anti-self-dual connections

A line bundle $\mathcal{L} \to M$ admits anti-self-dual connections if and only if $c_1(\mathcal{L})$ can be represented by an anti-self-dual harmonic form ie if and only if

$$c_1(\mathcal{L}) \in \mathcal{H}^2_-(M,g)$$

4.5.1 The transversality argument

As indicated in the last proposition, $\mathcal{H}^2_-$ depends on the metric (actually on the conformal class). We will consider the space of all $\mathcal{H}^2_-$ and show that the subspace of $\mathcal{H}^2_-$'s containing $c_1(\mathcal{L})$ is a $b_2^+(M)$-dimensional subspace. Finally, if $b_2^+(M) \geq 1$, the map assigning $\mathcal{H}^2_-$ to a metric will be **transverse** to this subspace. Hence, $\mathcal{H}^2_-(M,g)$ will not contain $c_1(\mathcal{L})$ for a generic metric g and hence admit no anti-self-dual connection.
Before we prove that, let's state a simple but important corollary of the above proposition

Corollary 4.5.1. Easy special cases

If $b_2^+(M) = 0$, then all 2-forms are anti-self-dual and anti-self-dual connections are anavoidable.
If $c_1(\mathcal{L}) \cdot c_1(\mathcal{L}) > 0$ then for every Riemannian metric on M, there are no anti-self-dual connections on $\mathcal{L} \to M$.

Proof. Let g be any metric on M. If there is an anti-self-dual connection $A \in \mathcal{A}(\mathcal{L})$ then its curvature $F_A \in i\Omega^2_-(M)$ is a harmonic form representing $c_1(\mathcal{L})$. We can calculate the self-intersection of $c_1(\mathcal{L})$ by the integral:

$$c_1(\mathcal{L}) \cdot c_1(\mathcal{L}) = \int_M F_A \wedge F_A = \int_M F_A \wedge (- * F_A) = - \int_M \underbrace{g(F_A, F_A)}_{=|\alpha|^2 \geq 0} dvol \leq 0$$

Thus, if $c_1(\mathcal{L}) \cdot c_1(\mathcal{L}) > 0$ then there cannot be any anti-self-dual connections on $\mathcal{L}$. □

We will now prove the transversality result. For the remainder of this section let $b_2^+ \geq 1$ and $b_2^- \geq 1$ (if $b_2^- = 0$, then there are only self-dual connections).

The space of conformal structures

Recall that a choice of volume form makes $(\Lambda^2(\mathbb{R}^4), \wedge)$ into a pseudo-euclidean space of signature 0. A conformal structure $[g] \in \mathfrak{G}(\mathbb{R}^4)$ on $\mathbb{R}^4$ singles out a maximal positive subspace $\Lambda^2_+ \subset \Lambda^2(\mathbb{R}^4)$ and a maximal negative subspace $\Lambda^2_- \subset \Lambda^2(\mathbb{R}^4)$. Remarkably, the converse is true and such a splitting of Λ^2 also defines a conformal structure on $\mathbb{R}^4$. Even more, because Λ^2_+ and Λ^2_- are wedge orthogonal, a conformal structure on $\mathbb{R}^4$ is in fact equivalent to maximal negative subspace and we will identify $[g]$ with $\Lambda^- := \Lambda^2_-$. More precisely, if we denote by $Gr_2(\mathbb{R}^4)$ the Grassmanian of 2-dimensional subspaces of $\mathbb{R}^4$ we have:

Proposition 4.5.2. The space of conformal structures on $\mathbb{R}^4$

The space of conformal structures $\mathfrak{G}(\mathbb{R}^4)$ is equal to the open subspace of maximal negative 2-dimensional subspaces $U_{\lambda^-} \subset Gr_2(\mathbb{R}^4)$.
Further, by fixing a $\Lambda_0^- \in \mathfrak{G} = U_{\lambda^-}$ one can describe any other maximal negative subspace $\Lambda^- \in U_{\lambda^-}$ as the graph of a linear map:

$$m : \Lambda_0^- \to \Lambda_0^+, \quad |m(\omega)| < |\omega| \text{ for all non-zero } \omega \in \Lambda_0^-$$

All of this carries over to manifolds: A conformal structure $g \in \mathfrak{G}(M)$ corresponds to a bundle splitting $\Lambda^2(M) = \Lambda^+ \oplus \Lambda^-$. Because on every fiber Λ^+ is the orthogonal wedge complement of Λ^- a conformal structure is equivalent to a choice of anti-self-dual subbundle Λ^-. Again, fixing a $\Lambda_0^- \in \mathfrak{G}$ one can describe any other anti-self-dual subbundle by a bundle map

$$m : \Lambda_0^- \to \Lambda_0^+, \quad |m_x(\omega_x)| < |\omega_x|, x \in M$$

proving:

Proposition 4.5.3. Tangent space of conformal structures on M

The tangent space of the space of conformal classes of M at $\Lambda_0^- \in \mathfrak{C}$ can be identified with the space of bundle homomorphisms (the underline to emphasize that we are dealing with bundle maps) :

$$T_{\Lambda_0^-}\mathfrak{C} = \underline{Hom}(\Lambda_0^-, \Lambda_0^+)$$

The space of anti-self-dual harmonic subspaces

The splitting of $\Lambda^2(M)$ gives a splitting of deRham cohomology $H^2_{dR}(M)$ and hence of the vector space of harmonic two forms $\mathcal{H}^2(M) = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$. What we are interested in is the space of $\mathcal{H}^2_-$'s that contain $c_1(\mathcal{L})$. This can be understood similarly to the space of maximal negative subspaces of $\Lambda^2(\mathbb{R}^n)$:

Proposition 4.5.4. The space of anti-self-dual harmonic subspaces

Let $Gr_{b_2^-}(\mathcal{H}^2)$ be the Grassmanian of b_2^--dimensional subspaces of $\mathcal{H}^2(M)$ and $U_{h^-} \subset Gr_{b_2^-}(\mathcal{H}^2)$ be the open subset of maximal negative subspaces with respect to the *intersection form* [a]. We will also call this open subspace the **space of anti-self-dual harmonic subspaces**. Fixing $\mathcal{H}_0^- \in U_{h^-}$ we can again describe any other element in U_{h^-} by a graph and identify the tangent space at $\mathcal{H}_0^-$ with:

$$T_{\mathcal{H}_0^-}U_{h^-} = Hom(\mathcal{H}_0^-, \mathcal{H}_0^+)$$

[a] Recall that the intersection form $\alpha, \beta \in H^2_{dR}(M)$ is given by $\int_M \alpha \wedge \beta$. This is analogous to the wedge product we used for $\Lambda^2(\mathbb{R}^4)$.

Now define the the space of off anti-self-dual harmonic subspaces containing $c_1(\mathcal{L})$

$$N_c := \{\mathcal{H}^- \in U_{h^-} | c_1(\mathcal{L}) \in H^-\} \subset U_{h^-}$$

We can then express the first goal outlined in the beginning of the section more precisely: We want to show that this a b_2^+-dimensional subspace inside U_{h^-}. Equivalently, the tangent space of N_c at any $\mathcal{H}_0^- \in N_c$ is a b_2^+-dimensional subspace of the tangent space of U_{h^-} at of $\mathcal{H}_0^-$. Consider the evaluation map

$$e_c : T_{\mathcal{H}_0^-}U_{h^-} = Hom(H_0^-, H_0^+) \to H_0^+, \quad f \mapsto f(c)$$

we can then express the tangent space as:

$$T_{\mathcal{H}_0^-}N_c = ker(e_c)$$

Because the evaluation map is surjective the dimension of the orthogonal complement is:

$$codim(T_{\mathcal{H}_0^-}N_c) = dim(T_{\mathcal{H}_0^-}U_{h^-}/T_{\mathcal{H}_0^-}N_c)$$

$$= dim(T_{\mathcal{H}_0^-}U_{h^-}/ker(e_c)) = dim(im(e_c)) = dim(\mathcal{H}_0^+) = b_2^+$$

Proposition 4.5.5. N_c is a codimension b_2^+ subspace

The space N_c off anti-self-dual harmonic subspaces containing $c_1(\mathcal{L})$ is codimension b_2^+ subspace of $U_{\langle^-} \subset Gr_{b_2^-}(\mathcal{H}^2)$.

The result

What is left is to prove that the map taking a conformal class $[g]$ to $\mathcal{H}^2_-(g)$ is transverse. Formally define:

$$P : \mathfrak{G} \to U_{h^-}, \quad (\Lambda^-_o \mapsto \Lambda^+_0) \mapsto (\mathcal{H}^-_0 \mapsto \mathcal{H}^+_0)$$

It's derivative is

$$TP : \underline{Hom}(\Lambda^-_0, \Lambda^+_0) \to Hom(\mathcal{H}^-_0, \mathcal{H}^+_0), \quad TP(m)(\omega) = pr_H(m(\omega))$$

where $pr_H : \Lambda^+ \to H^+$ is the orthogonal projection.

The final transversality result is

Theorem 4.5.1. Transversality result

The map P is transverse to $N_c = \{H^- \in U | c \in H^-\} \subset U \subset Gr_{b^2_-}\left(H^2(M)\right)$.

Proof. One has to prove that

$$e_c \circ DP : Hom(\Lambda^-, \Lambda^+) \to \mathcal{H}^+$$

is surjective. See [Don97] §4.3.3. and §4.3.5 □

This gives the desired result:

Corollary 4.5.2. Conditions for avoiding reducible solutions

If $b^+_2(M) \geq 1$, then for a generic metric, there are no reducible solutions.

Knowing that N_c is a codimension b^+_2-dimensional subspace it is not hard to believe that one can prove even more:

Theorem 4.5.2. Condition for avoiding reducible solution along path

If $b^+_2(M) \geq 2$, then for any two metrics g_0 and g_1 and every generic path g_t connecting them, there are no reducible solutions for all metrics g_t. The moduli space $\mathfrak{M}_\eta(\mathfrak{s})$ hence determines a well-defined cobordism class inside $\mathcal{C}(\mathfrak{s})$.

4.6 Compactness, Finiteness, and Regularity

This section will have a very analytical flavour. We will prove bounds on solutions to SW_η and use gauge transformations to increase the regularity of the equations. With these results we can then prove three important results: 1. the moduli space is always compact, 2. the moduli space consists only of smooth configurations and 3. there are only finitely many $spin^c$-structures for which the moduli space is non-empty, making the computation of the invariants tractable.

First, we need a few lemmas:

Lemma 4.11. ***Weitzenböck formula:***

$$\partial\!\!\!/^* \partial\!\!\!/ \psi = \nabla^* \nabla \psi + \frac{1}{4} scal \cdot \psi + \frac{1}{2} F^+ \cdot \psi$$

Lemma 4.12. *The following equality holds:*

$$\Delta(|\psi|^2) = -2|\nabla\psi|^2 + 2\langle \nabla^*\nabla\psi, \psi\rangle$$

where $\Delta = d^*d$ is the standard scalar laplacian which in local orthonormal frame coming from geodesic coordinates reads $\Delta \overset{local}{=} -\sum_k \partial_{e_k}\partial_{e_k}$.
If $|\psi(x)|^2$ has a local maxima at x_0 then $\Delta(|\psi|^2) \geq 0$.

Proof. We proof the equality using geodesic corrdiantes:

$$\Delta(|\psi|^2) = -\sum_k \partial_{e_k}\partial_{e_k}\langle\psi,\psi\rangle_{\mathbb{R}} = -\sum_k \partial_{e_k} 2\langle\nabla_{e_k}\psi,\psi\rangle = -2\sum_k \langle\nabla_{e_k}\nabla_{e_k}\psi,\psi\rangle + \langle\nabla_{e_k}\psi,\nabla_{e_k}\psi\rangle$$

where we used that connection is compatible with the metric. Using the well-known identity $\sum_k \nabla_{e_k}\nabla_{e_k}\psi = \nabla^*\nabla\psi$ and the fact that $\nabla\psi = e^k \otimes \nabla_{e_k}\psi$ proves the first part of the lemma. For the second part note that if $|\psi(x)|^2 = \langle\psi(x),\psi(x)\rangle$ takes a local maximum the exterior derivative of it is zero:

$$0 = d\big(\langle\psi(x_0),\psi(x_0)\rangle_{\mathbb{R}}\big) = 2\langle\nabla\psi(x_0),\psi(x_0)\rangle \Leftrightarrow \nabla\psi(x_0) = 0 \in T^*M \otimes \mathcal{W}^+$$

□

Lemma 4.13. ***Key estimate:*** *If $(\psi, A_{\mathcal{L}^2}) \in Sol(SW_\eta(g,\mathfrak{s}))^5$, then either $\psi = 0$ or*

$$|\psi|^2_\infty \leq max_{x\in M}\{-scal(x) + 2||\eta||_\infty\}$$

Proof. Let us assume that $\psi \neq 0$. By the above lemma 4.12 we have: $\Delta(|\psi|^2) \leq 2\langle\nabla^*\nabla\psi,\psi\rangle$. Using the Weitzenböck formula gives:

$$\Delta(|\psi|^2) \leq 2\langle \partial\!\!\!/^* \underbrace{\partial\!\!\!/\psi}_{=0},\psi\rangle - \frac{1}{2}scal\cdot|\psi|^2 - \langle F^+\cdot\psi,\psi\rangle = -\frac{1}{2}scal\cdot|\psi|^2 - \langle\underbrace{\sigma(\psi)\cdot\psi}_{=\frac{1}{2}|\psi|^2\psi},\psi\rangle - \langle\phi\cdot\psi,\psi\rangle$$

$$\leq -\frac{1}{2}scal(x)\cdot|\psi(x)|^2 - \frac{1}{2}|\psi(x)|^2 + |\phi\cdot\psi||\psi(x)| \leq -\frac{1}{2}scal(x)\cdot|\psi(x)|^2 - \frac{1}{2}|\psi(x)|^4 + |\phi|_\infty|\psi(x)|^2$$

where $|\phi|_\infty$ is the operator norm of $\phi\cdot \in End(\mathcal{W}^+)$. If $|\psi(x)|^2$ achieves it's global maxima at x_0, then by lemma 4.12, $\Delta(|\psi(x)|^2) \geq 0$ and thus:

$$0 \leq \Delta(|\psi(x_0)|^2) \leq \frac{|\psi(x_0)|^2}{2}\big(-scal(x_0) - |\psi(x_0)|^2 + 2|\phi|_\infty\big)$$

$$\Rightarrow \forall x \in M : |\psi(x)|^2 \leq |\psi(x_o)|^2 \leq \max_{p\in M}\{-scal(p) + 2|\phi|_\infty\}$$

□

Lemma 4.14. *We can gauge transform any configuration $c = (\psi, A_0 + a) \in \mathcal{C}$ to:*
- A configuration $c' = (\psi', A_0 + ia') \in \mathcal{C}$ such that $||P(a')||_2 \leq C$ where $P : L^2(\Lambda^1(M)) \to L^2(\Lambda^1(M))$ denotes the projection onto $H^1(M)$ and $C < \infty$.
- A configuration $c' = (\psi', A_0 + ia')$ with $a' \in ker(d^*)$ [4].
Hence, the Seiberg-Witten equations can be modified to:

$$\begin{cases} \partial\!\!\!/_{A_0}\psi_n = -\frac{1}{2}a_n\cdot\psi_n \\ (d^+ + d^*)a_n = \sigma(\psi_n) - F^+_{A_0} - \eta \\ ||Pa_n||_2 \leq C \end{cases}$$

Proof. We prove the second statement: Write $A = A_0 + ib \in \mathcal{A}(\mathcal{L}^2)$. Then use the hodge decomposition to write: $b = b_0 + df + d^*\beta$ where $b_0 \in ker(\Delta), f \in \Omega^0(M), \beta \in \Omega^2(M)$. We can now gauge transform $(\psi, A) \in Sol^2(SW_\eta)$:

$$(\psi, A)\cdot exp(\frac{i}{2}f) = \big(exp(-\frac{i}{2}f)\psi, A - idf\big) = \big(exp(-\frac{i}{2}f)\psi, A_0 + \underbrace{ib_0 + id^*\beta}_{:=a}\big)$$

□

Lemma 4.15. ***Sobolov embedding:*** *If $kp = n$ then there is a continuous embedding:*

$$W^{p,k} \hookrightarrow L^q, \quad \forall q \in [1,\infty)$$

Lemma 4.16. *The squaring map reduces the regularity by one:*

$$\sigma : \Gamma^{k+1,2}(\mathcal{W}^+) \to \Gamma^{k,2}(i\Lambda^2_+(M))$$

Lemma 4.17. *Let $D : \Gamma(E) \to \Gamma(F)$ be an elliptic operator and $P : \Gamma(E) \to ker(D)$ be the projection onto the kernel, then:*

$$||\eta - P\eta||_{k,p} \leq C||Da||_{n-1,p}$$

[4] This is often called coulomb gauge fixing as the same gauge fixing was used by physicists to solve maxwells equations for the coulomb electric monopole. See chapter 2

4.6.1 Compactness

We are now ready to prove:

Theorem 4.6.1. Compactness

Given a perturbation $\eta \in \Gamma^{m,2}(\Lambda^2(M))$ with $m \geq 4$, the moduli space $\mathfrak{M}^2_\eta(g, \mathfrak{s})$ is compact. More concretely, for any sequence of configurations

$$c_n \in Sol^2(SW_\eta),$$

there exists a sequence of gauge transformations

$$g_n \in \mathcal{G}^{3,2}$$

and a configuration

$$c \in Sol^2(SW_\eta)$$

such that, after passing to a subsequence, the gauge-transformed configurations converge:

$$c_n \cdot g_n \to c.$$

Proof. Recall from lemma 4.14 that the perturbed Seiberg–Witten equations can be rewritten as

$$\begin{cases} \partial\!\!\!/_{A_0}\psi_n = -\frac{1}{2}\, a_n \cdot \psi_n, \\ (d^+ + d^*)a_n = \sigma(\psi_n) - F^+_{A_0} - \eta, \\ \|Pa_n\|_2 \leq C. \end{cases}$$

We now establish a sequence of bounds:

1. **Bound on $\|a_n\|_{1,p}$.**
   The key estimate on ψ (see lemma 4.13) implies that $\|(d^+ + d^*)a_n\|_\infty$ is bounded. By lemma 4.17, we have
   $$\|a_n - Pa_n\|_{1,p} \leq C\|(d^+ + d^*)a_n\|_{0,p} \leq C \cdot \mathrm{vol}(M)^{\frac{1}{p}} \cdot \|(d^+ + d^*)a_n\|_\infty,$$
   which shows that $\|a_n - Pa_n\|_{1,p}$ is bounded. Since Pa_n lies in the finite-dimensional space $H^1(M)$, all Sobolev norms on Pa_n are equivalent. Together with the bound on $\|Pa_n\|_2$, this implies that $\|Pa_n\|_{k,p}$ is bounded for all k. Therefore, $\|a_n\|_{1,p}$ is bounded.

2. **Bound on $\|\psi_n\|_{1,p}$.**
   The key estimate, combined with the bound on $\|a_n\|_{1,p}$, yields that $\|a_n \cdot \psi_n\|_\infty$ is bounded. In view of the first equation,
   $$\|\partial\!\!\!/_{A_0}\psi_n\|_{0,p} = \left\| -\frac{1}{2}\, a_n \cdot \psi_n \right\|_{0,p},$$
   we deduce that $\|\partial\!\!\!/_{A_0}\psi_n\|_{0,p}$ is bounded. Then, by elliptic regularity, $\|\psi_n\|_{1,p}$ is bounded.

3. **Bound on $\|\psi_n\|_{2,p}$.**
   The previous bounds imply that
   $$\|\partial\!\!\!/_{A_0}\psi_n\|_{1,p} = \left\| -\frac{1}{2}\, a_n \cdot \psi_n \right\|_{1,p}$$
   is bounded. Elliptic regularity then yields that $\|\psi_n\|_{2,p}$ is bounded.

4. **Bound on $\|a_n\|_{2,p}$.**
   The second equation shows that $\|(d^+ + d^*)a_n\|_{1,p}$ is bounded. Consequently,
   $$\|a_n - Pa_n\|_{2,p} \leq C\, \|(d^+ + d^*)a_n\|_{1,p}$$
   is bounded. As before, the boundedness of $\|Pa_n\|_{k,p}$ implies that $\|a_n\|_{2,p}$ is bounded.

5. **Compactness.**
   Finally, the last 2 bounds imply with (the more general) sobolev embedding theorem that a subsequence of c_n converges weakly in $W^{2,p}$ and strongly in $W^{1,q}$ to a configuration $c \in \mathcal{C}^2(\mathfrak{s})$

This completes the proof of the compactness of $\mathfrak{M}^2_\eta(g, \mathfrak{s})$. □

4.6.2 Regularity

Even though, we extended to sobolev spaces, the following theorem shows that the equations have enough regularity so that the moduli space consists of only smooth objects.

Theorem 4.6.2. The moduli space consists of smooth objects

If the perturbation $\eta \in i\Gamma^{k,2}(\Lambda^2(M))$ is smooth that is if $\eta \in i\Omega^2(M)$, then the moduli space of SW monopoles consists of smooth configurations. More precisely for every $c \in Sol^2(SW_\eta(g,\mathfrak{s}))$ there is a $g \in \mathcal{G}^{3,2}(\mathfrak{s})$ s.t. $c \cdot g \in Sol^{k+1}(SW_\eta(g,\mathfrak{s}))$. Thus:

$$\mathfrak{M}^2_\eta(g,\mathfrak{s}) \cong \mathfrak{M}^{k+1}_\eta(g,\mathfrak{s}) \quad \forall k \geq 2$$

and if $k = \infty$, the moduli space consists of smooth configurations.

Proof. We have seen in lemma 4.14 that we can rewrite the Seiberg-Witten equations to

$$\begin{cases} \not{\partial}_{A_0}\psi = -\frac{1}{2}a \cdot \psi \\ (d^+ + d^*)a = \sigma(\psi) - F^+_{A_0} - \eta \end{cases}$$

Because $\not{\partial}_{A_0}$ and $d^+ + d^*$ are elliptic we can use the first equation to boost the regularity of ψ and the second to boost the regularity of a.

Note that because (by definition) $a, \psi \in W^{2,2}$ by the sobolev embedding theorem (see lemma 4.15), we have $a, \psi \in L^p = W^{0,p}$ for all $p \geq 1$ implying $\frac{1}{2}a \cdot \psi = \partial\!\!\!/ A_0 \psi \in W^{0,p}$. Using the first (elliptic) equation, we can increase the regularity of the spinnor field: $\psi \in W^{1,p}$. Then $\sigma(\psi) \in W^{0,p}$ and because $F^+_{A_0} \in i\Omega^2(M), \eta \in i\Gamma^{k,2}(\Lambda^2(M))$, also $(d^+ + d^*)a \in W^{0,p}$ which by ellitpicity implies $a \in W^{1,p}$. At this point we have seen that the equations boost the regularity from $W^{0,p}$ to $W^{1,p}$. We write down the implication chain once more showing that we can boost the regularity from $W^{m,p}$ to $W^{m+1,p}$ as long as $k \geq m$:

$$a, \psi \in W^{m,p} \Rightarrow \partial\!\!\!/ A_0 \psi \in W^{m,p}\psi \Rightarrow \psi \in W^{m+1,p} \Rightarrow \sigma(\psi) \in W^{m,p}$$

$$\overset{\eta \in W^{k,p}}{\Rightarrow} (d^+ + d^*)a \in W^{m,p} \Rightarrow a \in W^{m+1,p}$$

The iteration works until $k = m$ implying that $a, \psi \in W^{k+1,p}$.

□

4.6.3 Finiteness

In this section we want to prove that there is only a finite number of Seiberg-Witten moduli space, so that this becomes a tractable invariant.
To prove this we need another lemma:

Lemma 4.18. *If $(\psi, A) \in Sol^2(SW_\eta)$, then $||F^+_A||_2 \leq C < \infty$.*

Proof. If $(\psi, A) \in Sol(SW_\eta)$ then the Lichnerowiczs formula becomes:

$$\not{\partial}^* \underbrace{\not{\partial}\psi}_{=0} = \nabla^*\nabla\psi + \frac{1}{4}scal \cdot \psi + \frac{1}{2}F^+ \cdot \psi$$

$$= \nabla^*\nabla\psi + \frac{1}{4}scal \cdot \psi + \frac{1}{2}\big(\sigma(\psi) + \eta\big) \cdot \psi = \nabla^*\nabla\psi + \frac{1}{4}scal \cdot \psi + \frac{1}{4}|\psi|^2\psi + \frac{1}{2}\eta \cdot \psi$$

Taking the inner product with ψ on both sides:

$$0 = \underbrace{\langle \nabla^*\nabla\psi, \psi\rangle}_{=|\nabla\psi|^2} + \frac{1}{4}scal\langle\psi,\psi\rangle + \frac{1}{4}|\psi|^4 + \frac{1}{2}\langle\eta\cdot\psi,\psi\rangle$$

Now integrating over M and applying the cauchy schwarz inequality (three times) gives:

$$\frac{1}{4}\int|\psi|^4 \leq \frac{1}{4}\int|\psi|^4 + \int|\nabla\psi|^2 = \frac{1}{4}\int(-scal)|\psi|^2 + \frac{1}{2}\int\underbrace{-\langle\eta\cdot\psi,\psi\rangle}_{\leq|\eta|_\infty|\psi|^2}$$

$$\leq \frac{1}{4}(\int |scal|^2)^{\frac{1}{2}}(\int |\psi|^4)^{\frac{1}{2}} + \frac{1}{2}\Big(\int |\eta|^2_\infty\Big)^{\frac{1}{2}}\Big(\int |\psi|^4\Big)^{\frac{1}{2}}$$

$$\Rightarrow \Big(\int |\psi|^4\Big)^{\frac{1}{2}} \leq \Big(\int |scal|^2\Big)^{\frac{1}{2}} + 2\Big(\int |\eta|^2_\infty\Big)^{\frac{1}{2}}$$

Using $||F^+||_2 = ||\sigma(\psi)+\eta||_2 \leq ||\sigma(\psi)||_2 + ||\eta||_2$ and $|\sigma(\psi(x))| = \frac{1}{2\sqrt{2}}|\psi(x)|^2$ finally gives:

$$||F^+|| \leq \frac{1}{8}\Big(\int |\psi|^4\Big)^{\frac{1}{2}} + \Big(\int |\eta|^2\Big)^{\frac{1}{2}} \leq \frac{1}{8}\Big(\int |scal|^2\Big)^{\frac{1}{2}} + \frac{5}{4}\Big(\int |\eta|^2\Big)^{\frac{1}{2}}$$

□

Theorem 4.6.3. Finiteness

For at most finitely many $Spin^{\mathbb{C}}$ structures is $\mathfrak{M}$ is non-empty.

Proof. We will show that the self-intersection of the first chern class ie $c_1(\mathfrak{s})^2$ is bounded impyling that only finitely many first chern classes $c_1(\mathfrak{s}) \in H^2(M,\mathbb{Z})$ are possible. Since only finitely many $spin^c$-structures can have the same first chern class the result follows.
1. Lower bound: We are only interested in cases where the virtual dimension is positive, giving us a lower bound:

$$0 \leq dim(\mathfrak{M}_\eta(\mathfrak{s})) = \frac{1}{4}\big(c_1(\mathfrak{s})^2 - 2\chi(M) - 3sign(M)\big) \Leftrightarrow 2\xi(M) + 3sign(M) \leq c_1(\mathfrak{s})^2$$

2. Upper bound: Recall that $[F_A] = -2\pi i c_1(\mathfrak{s})$ and and thus $4\pi^2 c_1(\mathfrak{s})^2 = [F_A]\cdot[F_A] = \int_M F_A \wedge F_A$. It will thus suffices to find an upper bound for this integral. Note that:

$$\int_M \alpha\wedge\alpha = \int |\alpha^+|^2 - |\alpha^-|^2 = ||\alpha^+||^2 - ||\alpha^-||^2$$

which follows easily by splitting α in self-dual and anti-self dual component, using $\alpha^+\wedge\alpha^- = 0$ and $\alpha^\pm\wedge\alpha^\pm = \pm|\alpha^\pm|^2$. Applying this to F_A^+ together with lemma 4.18 gives us an upper bound:

$$c_1(\mathfrak{s})^2 = \frac{1}{4\pi^2}[F_A]\cdot[F_A] \leq \frac{1}{4\pi^2}||F_A^+||^2 \leq C < \infty$$

□

4.7 Orientation

In order for $\mathfrak{M}_\eta$ to define an integral homology class, we need it to be orientable. We will now sketch how one can define an orientation.

We have seen that if $b_2^+ \geq 1$, then in the generic case $H^0(c,\mathfrak{s})$ and $H^2(c,\mathfrak{s})$ vanish and the tangent space of the moduli space $\mathfrak{M}_\eta(g,\mathfrak{s})$ at $[c] = [\psi, A]$ can be identified with

$$T_{[c]}\mathfrak{M}_\eta = H^1(c,\mathfrak{s}) = ker(sw_{*c})/im(D_c\mathcal{G})$$

To define an orientation of this tangent space we can orient its determinant line and because of the vanishing of the other cohomology spaces this is equivalent to orienting the determinant line of the fundamental complex:

$$\big(\Lambda^{top}H(c,\mathfrak{s})^0\big)^* \otimes \big(\Lambda^{top}H^1(c,\mathfrak{s})\big) \otimes \big(\Lambda^{top}H^2(c,\mathfrak{s})\big)^*$$

How does this help? We will see that this line can be identified with the determinant line of a fredholm operator allowing us to deform the fredholm operator without changing this determinant line. The point of the deformation it, that it will be easy to define the orientation on it and that we can easily see how this extends to a trivial determinant line bundle over $\mathcal{B}^*$. We can thus consistently define an orientation of this deformed determinant line which can be identified with an orientation of the determinant lines of the original fundamental complexes for every $[c] \in \mathcal{B}^*$ which due to the vanishing of the cohomology groups can be identified with an orientation of $T_{[c]}\mathfrak{M}_\eta$.

Here are the two main lemmas:

Lemma 4.19. *Let $F : V \to W$ be fredholm operator. The* ***determinant line*** *is define by:*

$$det(F) := \left(\Lambda^{top}(ker(F))\right) \otimes \left(\Lambda^{top}(coker(F))\right)^*$$

Given a homotopy through fredholm operatos $F_t : V \to W$, we can define a real line bundle by $det(F_t) \mapsto t$ called the (real) ***determinant line bundle****. Since every bundle over a contractible space is trivial, we get an identification of F_0 with F_1.*

Lemma 4.20. *Given an elliptic complex:*

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

with negative dirac operator $\not{D}^- = a^* \oplus b : B \to A \oplus C$ (which is elliptic and thus fredholm), the ***determinant line*** *of the complex (defined by the left-hand side of the following expression) can be identified with the determinant line of the diarc operator:*

$$\left(\Lambda^{top}H^0\right)^* \otimes \left(\Lambda^{top}H^1\right) \otimes \left(\Lambda^{top}H^2\right)^* = det(\not{D}^-) = det(a^* \oplus b)$$

Let's apply those two lemmas to our fundament complex:

$$0 \to \underbrace{\mathfrak{g}(\mathfrak{s})}_{=i\Omega^0(M)} \xrightarrow{D_{c\cdot\mathcal{G}}} \underbrace{T_{(\psi,A_{\mathcal{L}^2})}\mathcal{C}}_{=\Gamma(\mathcal{W}^+)\oplus i\Omega^1(M)} \xrightarrow{sw_{*(\psi,A_{\mathcal{L}^2})}} \underbrace{T_{(0,\eta)}\mathcal{X}}_{=\Gamma(\mathcal{W}^-)\oplus i\Omega^2_+(M)} \to 0$$

The determinant line is $det(\not{D}^-) = det(D^*_{c\cdot\mathcal{G}} \oplus sw_{*c})$ and by lemma 4.19 does not change under deformation. In particular it will not change if we eliminate the 0th order term of our elliptic complex to get:

$$0 \to \underbrace{\mathfrak{g}(\mathfrak{s})}_{=i\Omega^0(M)} \xrightarrow{(0,2d)} \underbrace{T_{(\psi,A_{\mathcal{L}^2})}\mathcal{C}}_{=\Gamma(\mathcal{W}^+)\oplus i\Omega^1(M)} \xrightarrow{\not{\partial}\oplus d^+} \underbrace{T_{(0,0)}\mathcal{X}}_{=\Gamma(\mathcal{W}^-)\oplus i\Omega^2_+(M)} \to 0$$

The determinant line of this dirac operator is determined by

$$\tilde{H}^0(c,\mathfrak{s}) = H^0(M), \quad \tilde{H}^1(c,\mathfrak{s}) = ker(\not{\partial}) \oplus H^1(M), \quad \tilde{H}^2(c,\mathfrak{s}) = coker(\not{\partial}) \oplus H^2_+$$

where we used that $\Gamma(\Lambda^2_+)/im(d^+) = H^2_+ \oplus im(d^+)/im(d^+)$.

Proposition 4.7.1. Orientation of deformed determinant line

An orientation of the determinant line of the deformed and of the original fundamental complex, is determined by an orientation of $H^0(M), H^1(M)$ and $H^2_+(M)$.

Proof. Since $\not{\partial}$ is elliptic and $\mathbb{C}$-linear, both $ker(\not{\partial})$ and $coker(\not{\partial})$ are complex finite dimensional vector spaces which carry a cononical orientation. Together with an orientation of $H^0(M), H^1(M)$ and $H^2_+(M)$ this determines an orientation for all the cohomology spaces $\tilde{H}^i(c,\mathfrak{s})$. This determines an orientation of the determinant line of the deformed dirac complex which by lemma 4.19 can be identified with the determinant line of our original fundamental complex. □

We have thus oriented the determinant line of the fundamental complex at every $c \in \mathcal{C}$. We can now combine all the determinant lines to a real-line bundle:

$$\left(\Lambda^{top}\tilde{H}(c,\mathfrak{s})^0\right)^* \otimes \left(\Lambda^{top}\tilde{H}^1(c,\mathfrak{s})\right) \otimes \left(\Lambda^{top}\tilde{H}^2(c,\mathfrak{s})\right)^* \mapsto c$$

Because the kernel of the dirac operator $\not{\partial} = \not{\partial}^+$ and of it's adjoint $\not{\partial}^-$ are equivariant, meaning:

$$\not{\partial}_A\psi = 0 \Leftrightarrow \not{\partial}_{A,g}(\psi \cdot g) = 0$$

and all other components of the fiber (the cohomology classes of M) do not depend on c, the bundle:

$$\left(\Lambda^{top}\tilde{H}(c,\mathfrak{s})^0\right)^* \otimes \left(\Lambda^{top}\tilde{H}^1(c,\mathfrak{s})\right) \otimes \left(\Lambda^{top}\tilde{H}^2(c,\mathfrak{s})\right)^* \mapsto [c]$$

is trivial, hence:

Theorem 4.7.1. The moduli space ir orientbale

The moduli space of SW monopoles $\mathfrak{M}_\eta(g,\mathfrak{s})$ is orientatble for a generic choice of η and g.

Proof. We have already shown that the determinant line (of the deformed complex) over every point in $\mathcal{B}$ can be given a consistent orientation as the bundle that organizes them is trivial. This determinant line at $[c]$ is the same as the determinant line of the original fundamental complex and because $H^0(c,\mathfrak{s})$ and $H^2(c,\mathfrak{s})$ vanish, this gives a consistent orientation of $H^1(c,\mathfrak{s}) = T_{[c]}\mathfrak{M}$ at every point $[c] \in \mathcal{B}$. □

4.8 The invariants

As outlined in the beginning of this chapter one possibility to extract information from the moduli space $\mathfrak{M}_\eta$ is to evaluate cohomology classes of the ambient space $\mathcal{B}$ on the homology class determined by $[\mathfrak{M}_\eta] \in H_{dim(\mathfrak{M}_\eta}(\mathcal{B},\mathbb{Z})$. Assuming $b_2^+ \geq 1$, this is possible because the moduli space of monopoles is a smooth finite dimensional, orientable, compact manifold. One natural cohomology class on $\mathcal{B}$ comes from the following observation.

Lemma 4.21. *There is a natural S^1-bundle over $\mathcal{B}$*

$$S^1 \hookrightarrow \mathcal{B}^0 \to \mathcal{B}$$

where $\mathcal{B}^0 = \mathcal{C}(\mathfrak{s})/\mathcal{G}^0(\mathfrak{s})$ with $\mathcal{G}^0(\mathfrak{s}) = \{g \in \mathcal{G}(\mathfrak{s}) \big| g(p_0) = id \in U(1)\} \subset \mathcal{G}(\mathfrak{s})$ where $p_0 \in M$ is a fixed point. Note that $\mathcal{G}^0(\mathfrak{s})$ can be interpreted as the gauge group with fixed gauge at p_0 and that $\mathcal{G}(\mathfrak{s}) = \mathcal{G}^0(\mathfrak{s}) \times S^1$.
This bundle turns out to be contractible and is thus the universal bundle $\mathcal{E}_{S^1}\mathcal{G}(\mathfrak{s})$ with fiber S^1 over the classifying space $\mathcal{B}\mathcal{G}(\mathfrak{s}) = \mathcal{B}$.

Remark 4.1. *One can construct this bundle in another way that lends itself to generalizations to theories with non-abelian gauge groups (like the $SU(2)$ group in yang mills theory studies by donaldson):*
Consider the infinite dimensional principal bundle

$$\mathcal{G}(\mathfrak{s}) \hookrightarrow M \times \mathcal{C}(\mathfrak{s}) \to M \times \mathcal{B}(\mathfrak{s})$$

where $\mathcal{G}$ acts trivially on M. One can then build the trivial bundle $\underline{\mathbb{C}} := \mathbb{C} \times M \times \mathcal{C}(\mathfrak{s}) \to M \times \mathcal{C}(\mathfrak{s})$. The group of gauge transformations $\mathcal{G}(\mathfrak{s})$ acts freely on this bundle by $(z,p,c)\cdot g = (g^{-1}(p)z, p, c\cdot g)$. It thus descends to a complex line bundle:

$$\big(M \times \mathcal{C}(\mathfrak{s})\big) \times_{\mathcal{G}} \mathbb{C} \to M \times \mathcal{B}(\mathfrak{s})$$

This bundle has by construction the same transition functions as the original one but the action of $g \in \mathcal{G}(\mathfrak{s})$ on $[(p,c),z] \in \big(M \times \mathcal{C}(\mathfrak{s})\big) \times_{\mathcal{G}} \mathbb{C}$ is:

$$[(p,c),z]\cdot g = [(p,c)\cdot g, z] = [(p,c), g^{-1}(p)z]$$

Note that if we restrict this bundle to $p_0 \times \mathcal{B}(\mathfrak{s})$ then the action of $g_0 \in \mathcal{G}_0(\mathfrak{s}) \subset \mathcal{G}(\mathfrak{s})$is trivial:

$$[c,z]\cdot g = [c\cdot g, z] = [c, g^{-1}(p_0)z] = [c,z]$$

and write $\mathcal{G}(\mathfrak{s}) \ni g = g_0 e^{i\theta}$ the action becomes:

$$[c,z]\cdot g = [c,z]\cdot g_0 e^{i\theta} = [c, g^{-1}(p_0)z] = [c, e^{-i\theta}z]$$

showing that this is the same bundle as the one in lemma 4.21.

Having found a natural cohomology class $\mathfrak{u} := c_1(\mathcal{B}^0) \in H^2(\mathcal{B},\mathbb{Z})$, we can evaluate it on $[\mathfrak{M}_\eta]$ to define the seiberg-witen invaruiants:

Definition 4.8.1. Seiberg-Witten Invariants

Let $k := dim(\mathfrak{M}_\eta(g, \mathfrak{s}))$
- If k is even dimensional ie if $b_1(M) + b_2^+(M)$ is odd, then we can define:

$$SW_M(\mathfrak{s}) = \langle \mathfrak{u}^{\frac{k}{2}}, [\mathfrak{M}_\eta(\mathfrak{s})] \rangle = \int_{\mathfrak{M}_\eta(\mathfrak{s})} \mathfrak{u}^{\frac{k}{2}}$$

If $b_2^+(M) \geq 2$ we have already mentioned that any two generic metrics can be connected by a path that avoids reducible solutions, such that the moduli spaces are cobordant and the fundamental class $[\mathfrak{M}_\eta(\mathfrak{s})] \in H_k(\mathcal{B}(\mathfrak{s}))$ thus independent of the chosen (generic) metric. The invariants can thus be understood as a map $SW_M : \{spin^c\text{-structures on M}\} \to \mathbb{Z}$.
If $b_2^+(M) = 1$ a path connecting two generic metrics might touch a metric with reducible solutions creating a cobordism between $\mathfrak{M}_1$ and $\mathfrak{M}_2 \cup \pm\mathbb{C}P^m$ and thus changing the invariant by ± 1.
- If k is odd we define

$$SW_M(\mathfrak{s}) = 0$$

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