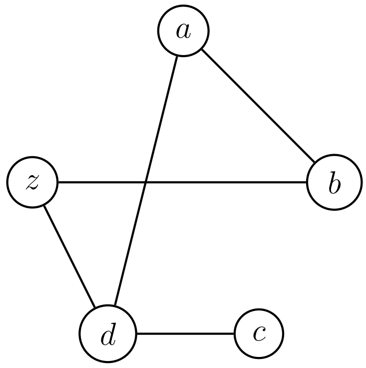
1. Consider the following graph [1].



   (a) Determine the vertex set.
   (b) Determine the edge set.
   (c) Determine the adjacency list.
   (d) For each of the vertices, determine the degree.

Solution:

(a) $\{a, b, c, d, z\}$.

(b) $\{\{a, b\}, \{a, d\}, \{b, z\}, \{c, d\}\{d, z\}\}$.

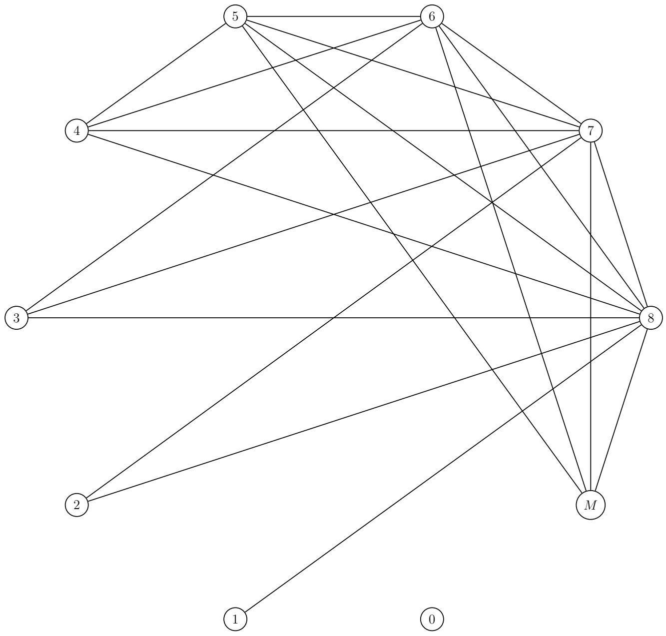
(c)

a	b	d	
b	a	z	
c	d		
d	a	c	z
z	b	d	

(d) $a : 2, b : 2, c : 1, d : 3, z : 2$.

2. Professor McBrain and his wife April give a party at which there are four other married couples. Some pairs of people shake hands when they meet, but naturally no couple shake hands with each other. At the end of the party the Professor asks everyone else how many people they have shaken hands with, and he receives nine different answers [1].
   (a) Draw a graph representing the handshakes exchanged at the party.
   (b) How many people shook hands with April?

Solution:



We construct a graph whose vertices are the people at the party, and there is an edge (x, y) whenever x and y shook hands. Since there are nine people apart from Professor McBrain, and the maximum number of handshakes in which any one person can be involved is eight, it follows that the nine different answers received by the Professor must be 0, 1, 2, 3, 4, 5, 6, 7, 8. We denote the vertices by these numbers and use M for McBrain himself. So we have a pictorial representation. Now, vertex 8 is joined to all the other vertices except one, which must therefore represent the spouse of 8. This vertex must be 0, since it is certainly not joined to 8 (or any other vertex, for that matter). Thus 8 and 0 are a married couple, and 8 is joined to 1, 2, ..., 7 and M. In particular 1 is joined to 8 and this is the only edge from 1. Hence vertex 7 is not joined to 0 and 1 (only), and the spouse of 7 must be 1, since 0 is married to 8. Continuing in the same way, we see that 6 and 2, and 5 and 3 are married couples. It follows that M and 4 are married, so vertex 4 represents April, who shook hands with four people.

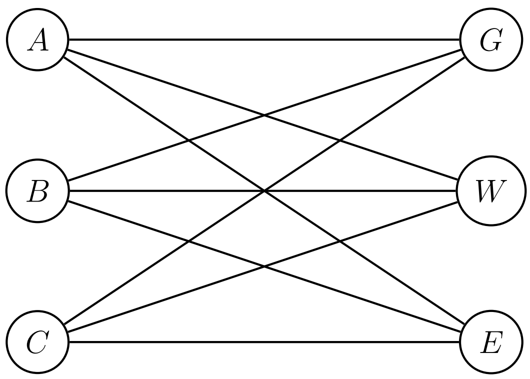
3. Three houses A, B, C each has to be connected to the gas, water and electricity supplies: G, W, E. Write down the adjacency list for the graph which represents this problem, and construct a pictorial representation of it. Can you find a picture in which the lines representing the edges do not cross? [1]

Solution:

Adjacency list:

A	G	E	W
B	G	E	W
C	G	E	W
G	A	B	C
E	A	B	C
W	A	B	C

Pictorial representation:



It's impossible to draw the graph without some edges crossing.

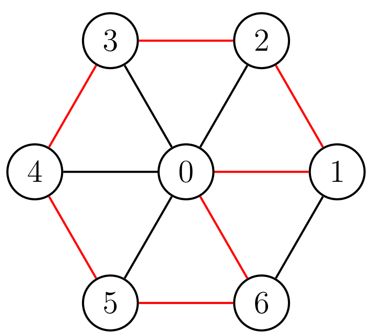
4. The pathways in a formal garden are to be laid out in the form of a wheel graph W_n whose vertex set is $V = \{0, 1, 2, \ldots, n\}$ and whose edges are:

$$\{0, 1\}, \{0, 2\}, \ldots, \{0, n\},$$
$$\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}, \{n, 1\}$$

Describe a route around the pathways which starts and ends at vertex 0 and visits every vertex once only [1].

Solution:

Say, for n = 6, we could draw the graph this way:



One such path is given in red.

5. For each positive integer n we define the complete graph K_n to be the graph with n vertices in which each pair of vertices is adjacent [1].
   (a) How many edges has K_n?
   (b) For which values of n can you find a pictorial representation of K_n with the property that the lines representing the edges do not cross?

Solution:

(a) There are two formulas we can derive here.
First:

- Every edge is connected to every other.
- Every edge looks like $\{a, b\}$ where a and b are vertices.
- There are n choices for a, and then $n-1$ for b.
- So the total number of edges seems to be $n(n-1)$.
- However, we've now counted the edge $\{a, b\}$ twice – once as $\{a, b\}$ and once as $\{b, a\}$.
- So the actual number of edges in K_n is:

$$\frac{n(n-1)}{2}$$

Second:

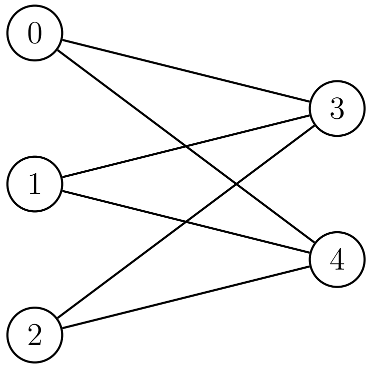
- Move through the vertices one by one.
- The first vertex is connected to $n-1$ others.
- The second vertex is connected to $n-2$ others, excluding the first vertex.
- The third vertex is connected to $n-3$ others, excluding the first and second.
- And so on.
- So the actual number of edges in K_n is:

$$(n-1)+(n-2)+\ldots+2+1$$

(b) You can easily draw K_1, K_2, K_3 and K_4 without edges crossing. However K_5 cannot be drawn without edges crossing and K_5 is a contained in K_n for all $n > 5$. So the answer is: for $n \in \{1, 2, 3, 4\}$.

6. A 3-cycle in a graph is a set of three mutually adjacent vertices. Construct a graph with five vertices and six edges which contains no 3-cycles [1].

Solution: There are lots of possibilities. Here's one:

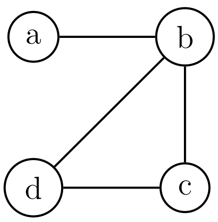


7. Consider the following two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where:

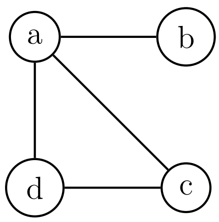
$$\begin{aligned} V_1 &= \{a, b, c, d\} \\ E_1 &= \{\{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}\} \\ V_2 &= \{1, 2, 3, 4\} \\ E_2 &= \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\}\} \end{aligned}$$

(a) Draw a picture of each of the graphs.

Solution:



G_1



G_2

(b) Determine a bijection between the vertex sets that preserves the edges.

Solution: There are lots of bijections between V_1 and V_2. However, not all of them preserve the edges. For instance the following bijection **will not** work:

$$f(a) = 1; f(b) = 2; f(c) = 3; f(d) = 4$$

When we apply f to the edges in this case:

$$\begin{aligned} f(E_1) &= \{\{f(a), f(b)\}, \{f(b), f(c)\}, \{f(b), f(d)\}, \{f(c), f(d)\}\} \\ &= \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\} \\ &\neq E_2. \end{aligned}$$

Here's an answer to the question – a bijection that will work:

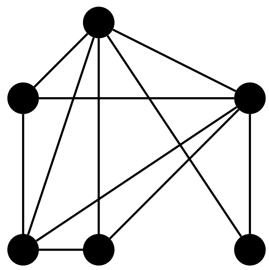
$$f(a) = 2; f(b) = 1; f(c) = 3; f(d) = 4$$

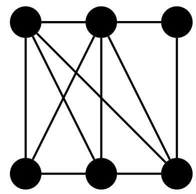
Then:

$$\begin{aligned} f(E_1) &= \{\{f(a), f(b)\}, \{f(b), f(c)\}, \{f(b), f(d)\}, \{f(c), f(d)\}\} \\ &= \{\{2, 1\}, \{1, 3\}, \{1, 4\}, \{3, 4\}\} \\ &= E_2. \end{aligned}$$

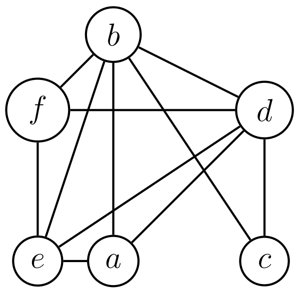
Remember that sets don't have order, so $\{1, 2\} = \{2, 1\}$.

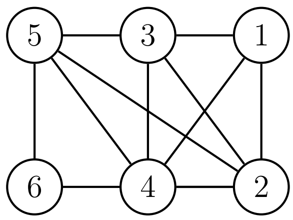
8. Determine if the following two graphs are isomorphic.





Solution: The first step here is to label the vertices and give the graphs names. Then we can work on finding a bijection.





$G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$

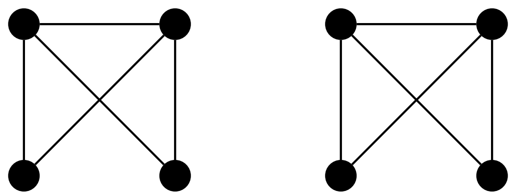
It's worth checking first if there are the same number of vertices and edges in the two graphs. If not, they can't be isomorphic. They both have 6 vertices, and 11 edges.

Next let's check that we have the same number of vertices in each graph with a given degree. If we don't, they can't be isomorphic. The degrees are:

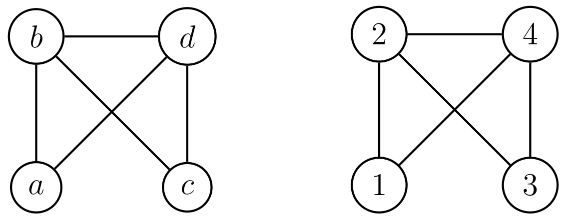
G_1	G_2
$\delta(a) = 2$	$\delta(1) = 3$
$\delta(b) = 5$	$\delta(2) = 4$
$\delta(c) = 2$	$\delta(3) = 4$
$\delta(d) = 5$	$\delta(4) = 5$
$\delta(e) = 4$	$\delta(5) = 4$
$\delta(f) = 3$	$\delta(6) = 2$

So we can see that the graphs cannot be isomorphic, since every vertex in G_1 would have to map to a vertex of equal degree in G_2. This isn't possible since, for instance, we have two vertices of degree 2 in G_1, and only one in G_2.

9. Determine three different isomorphisms between the following two graphs:



Solution: Again, label the vertices.

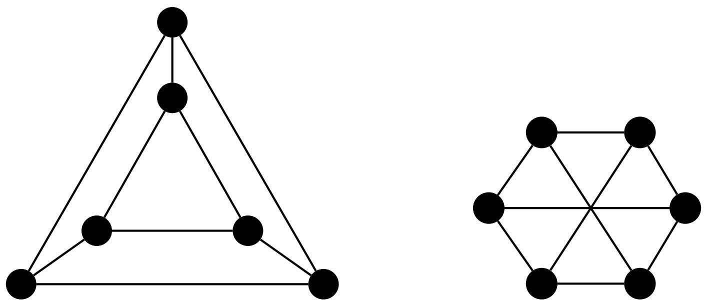


Here are three isomorphisms that work:

Ismorphism 1: $f(a) = 1; f(b) = 2; f(c) = 3; f(d) = 4$
Ismorphism 2: $g(a) = 3; g(b) = 2; g(c) = 1; g(d) = 4$
Ismorphism 3: $h(a) = 1; h(b) = 4; h(c) = 3; h(d) = 2$

You should check that these preserve the edges. There are more isomorphisms, and there are bijections that are not isomorphisms. You could easily list them all. In an exam, you should always show that the isomorphism preserves the edges by mapping all of the edges one by one.

10. Prove that these graphs are not isomorphic [1]:



> **Solution:** The graphs both have six vertices of degree three, so we can't rule out an isomorphism based on degrees of vertices. However, the graph on the left has two 3-cycles. The one on the right has none. Isomorphic graphs have the same number of 3-cycles, or n-cycles for any n for that matter. You should convince yourself of that fact.

11. Find an isomorphism between the graphs defined by the following adjacency lists [1].

a	b	c	d	e	f	g	h	i	j
b	a	b	c	d	a	b	c	d	e
e	c	d	e	a	h	i	j	f	g
f	g	h	i	j	i	j	f	g	h

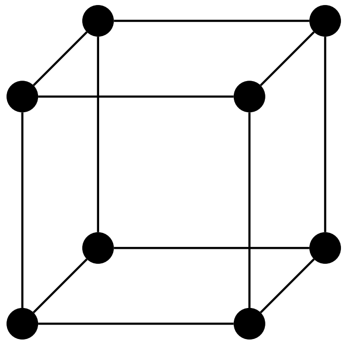
0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	0	1	0	2	6
5	0	1	2	3	4	4	3	5	7
7	6	8	7	6	8	9	9	9	8

> **Solution:** There are lots, we just have to find one. We can do this by trial and error – try a bijection between the vertices of the first graph and the second graph and see if it works. Then adapt it if it doesn't work.
>
> One that will work is:
>
> $$f(\mathrm{a}) = 4; f(\mathrm{b}) = 3; f(\mathrm{c}) = 2; f(\mathrm{d}) = 1; f(\mathrm{e}) = 6;$$
> $$f(\mathrm{f}) = 5; f(\mathrm{g}) = 7; f(\mathrm{h}) = 8; f(\mathrm{i}) = 0; f(\mathrm{j}) = 9;$$
>
> The point of this question is to highlight how hard finding an isomorphism generally is.

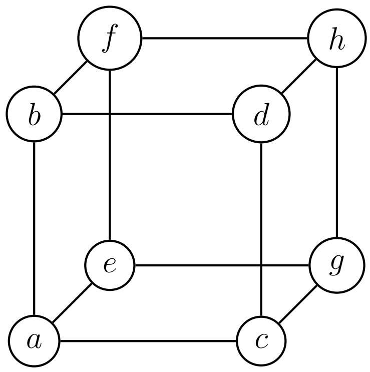
12. Show that the graph given by the following adjacency list [1]:

000	001	010	011	100	101	110	111
001	000	011	010	000	100	111	110
010	011	000	001	110	111	100	101
100	101	110	111	101	001	010	011

is isomorphic to the following graph:



Solution: Label the graph as follows:



In the adjacency list, every vertex is adjacent to, and only to, the vertices that differ from it in exactly one digit. Treat the three digits as coordinates in 3D space, so 101 is 1 in the x direction, 0 in the y direction and 1 in the z direction. Then the following bijection is an isomorphism:

$$f(\mathrm{a}) = 000; f(\mathrm{b}) = 010; f(\mathrm{c}) = 100; f(\mathrm{d}) = 110;$$
$$f(\mathrm{e}) = 001; f(\mathrm{f}) = 011; f(\mathrm{g}) = 101; f(\mathrm{h}) = 111;$$

There are many other isomorphisms. Try, for instance, rotating the cube using one of its symmetries.

References

[1] N. Biggs. *Discrete Mathematics.* Oxford science publications. OUP Oxford, 2002.