Graph Theory

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Topics

Graphs

Seven Bridges of Königsberg

Is it possible to walk through the city crossing each of the seven bridges once and only once?

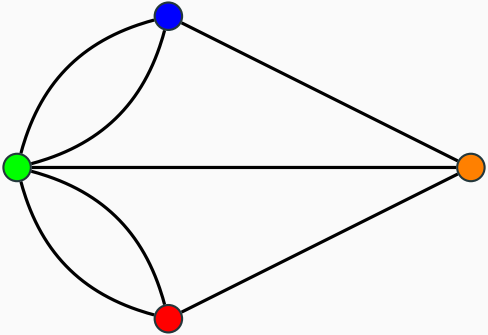
www.nature.com/nbt/journal/v29/n11

Leonhard Euler

- Born 1707 in Basel, Switzerland.
- Euler's identity: $\mathrm{e}^{i\pi} + 1 = 0$.
- Solved the Bridges of Königsberg problem.
- It's not possible to cross all bridges once and once only.

https://en.wikipedia.org/wiki/Leonhard_Euler

Graph of Königsberg



Graph definition

Definition

A *graph* consists of a finite set V and a set E of 2-subsets of V.

Vertices – the elements of the set V are called vertices.

Edges – the elements of E are called edges.

$G = (V, E)$ – this is the way we write the graph G consists of the vertex set V and the edge set E.

Sets of Königsberg

$$V = \{Green, Blue, Orange, Red\}$$

$$\begin{aligned}E = \{ & \\ & \{Green, Blue\}, \{Green, Blue\}, \{Green, Red\}, \\ & \{Green, Red\}, \{Blue, Orange\}, \{Green, Orange\}, \\ & \{Red, Orange\} \\ \} & \end{aligned}$$

Adjanceny list

Green	Blue	Orange	Red
Blue	Green	Blue	Green
Orange	Orange	Green	Orange
Red		Red	

Defining different types of graphs

Our definition of a graph

The definition given above for a graph is not consistent with looped edges, directed edges or repeated edges. We only need to make small changes to the definition of a graph to allow for directed edges and repeated edges.

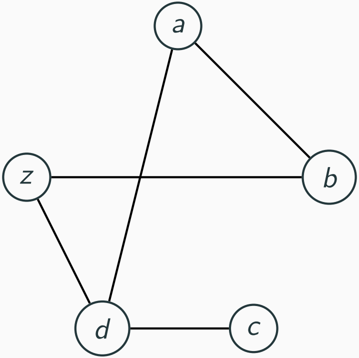
Repeated edges are edges that start and end at the same vertices.

Directed edges are edges where a direction is added.

Looped edges begin and end at the same vertex.

The application will determine the definition we want to use.

A better example



Exercise

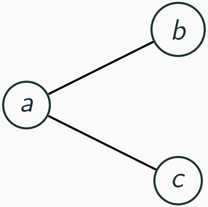
Determine the vertex set, edge set and adacency list of this graph.

global.oup.com/booksites/content/9780198507185/

Degree of a vertex

Definition

The degree of a vertex is the number of edges that contain it.



The degree of the vertex a is 2.

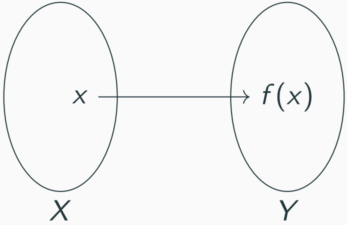
Exercise

For each of the vertices on the previous slide, determine its degree.

Functions

Definition

Suppose that X and Y are sets. We say we have a function f from X to Y if for each x in X we can specify a unique element in Y, which we denote by $f(x)$.

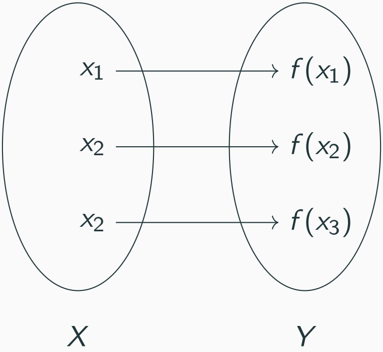


Bijections

Definition

A bijection is function f from a set X to a set Y where both of the following are true:

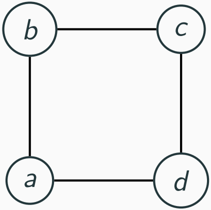
- every y in Y is a value $f(x)$ for at most one x in X.
- every y in Y is a value $f(x)$ for at least one x in X.

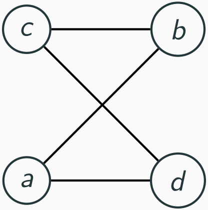


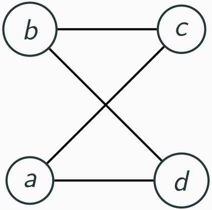
Isomorphism

Definition

Two graphs G_1 and G_2 are said to be isomorphic when there is a bijection α for the vertex set V_1 of G_1 to the vertex set V_2 of G_2 such that $\{\alpha(x), \alpha(y)\}$ is an edge of G_2 if and only if (x, y) is an edge of G_1.







Sum of degrees

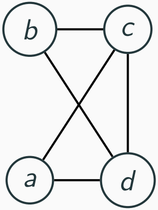
Theorem

The sum of the degrees of the vertices of a graph $G = (V, E)$ is equal to twice the number of edges:

$$\sum_{\mathrm{v} \in V} \delta(\mathrm{v}) = 2|E|$$

Proof.

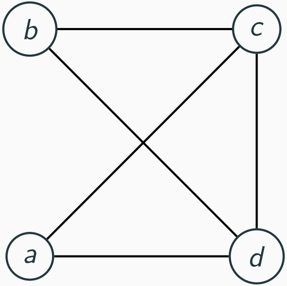
The degree $\delta(\mathrm{v})$ of a vertex v is equal to the number of edges incident on it. Every edge is incident on two vertices. So every edge contributes 1 to the degrees of two distinct vertices. Therefore every edge contributes 2 to the sum total of the degrees of all the vertices. □



Handshaking lemma

Definition

A vertex is an odd vertex if its degree is odd, and it is an even vertex if its degree is even. The set of all odd vertices is denoted V_o and the set of all even vertices is denoted V_e.



Exercise

Which of the above vertices are even, and which are odd?

Handshaking lemma

Lemma

The number of odd vertices $|V_o|$ in a graph is even.

Proof.

The sets V_o and V_e are disjoint (i.e. they don't have any elements in common.) Also, every vertex is either in V_o or V_e. Therefore $V = V_o \cup V_e$ and $|V| = |V_o| + |V_e|$.

Furthermore:

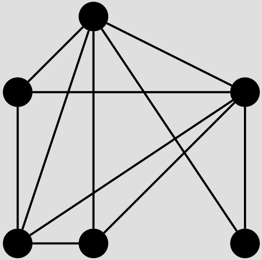
$$\sum_{\mathrm{v} \in V_o} \delta(\mathrm{v}) + \sum_{\mathrm{v} \in V_e} \delta(\mathrm{v}) = 2|E|$$

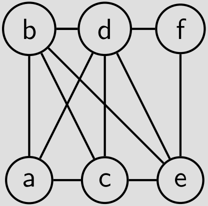
Both $2|E|$ and $\sum_{\mathrm{v} \in V_e} \delta(\mathrm{v})$ are even, so $\sum_{\mathrm{v} \in V_o} \delta(\mathrm{v})$ must be. Since $\delta(\mathrm{v})$ is odd for every v in V_o, this must mean that $|V_o|$ is even. $\square$

Isomorphism: degrees

Exercise

Determine if these two graphs are isomorphic.





Paths and Cycles

Walks

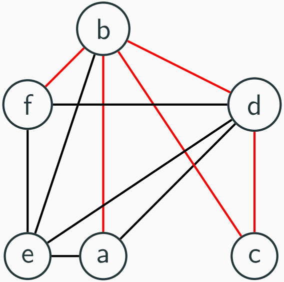
Definition

A *walk* in a graph is a sequence of vertices

$$v_1, v_2, \ldots, v_k$$

such that v_i and v_{i+1} are adjacent for $1 \leq i < k$.

For instance, in the following graph a, b, c, d, b, f is a walk.

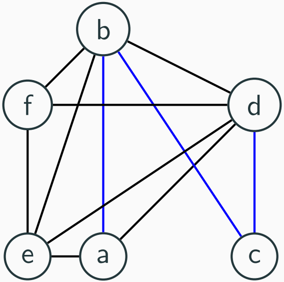


Paths

Definition

A *path* is a walk where each vertex is distinct.

For instance, in the following graph a, b, c, d is a path.

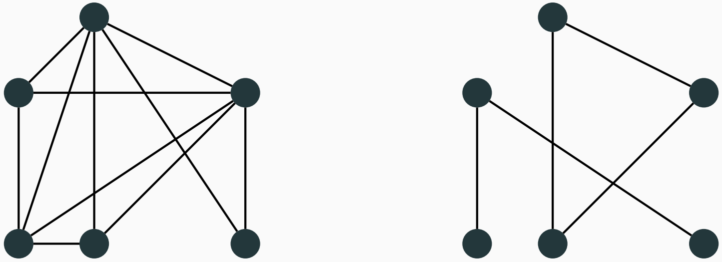


Connected graphs

Definition

A graph is *connected* if there is a path between each pair of vertices.

For instance, the graph on the left below is connected, but the one on the right is not.

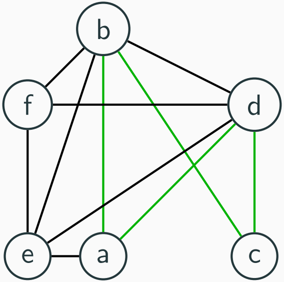


Cycles

Definition

A *cycle* is a walk where each vertex is distinct, except for the start and end vertices being equal.

For instance, in the following graph a, b, c, d, a is a cycle.

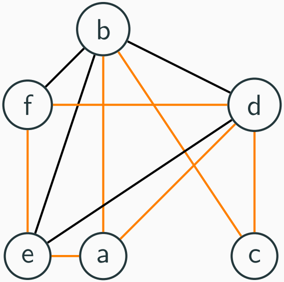


Hamiltonian cycles

Definition

A *Hamiltonian cycle* is a cycle that contains all of vertices of the graph.

For instance, in the following graph a, b, c, d, f, e, a is a Hamiltonian cycle.

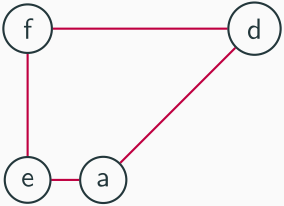


Eulerian walk

Definition

An *Eulerian walk* is a walk which uses each edge of the graph exactly once.

For instance, in the following graph a, d, f, e, a is an Eulerian walk.



Trees

Colouring

Sorting

Searching

Shortest Paths