

# Graph Theory

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Graphs

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# Graphs

# Seven Bridges of Königsberg



Is it possible to walk through the city crossing each of the seven bridges once and only once?

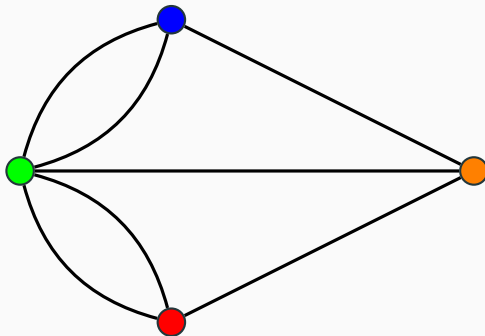
[www.nature.com/nbt/journal/v29/n11](http://www.nature.com/nbt/journal/v29/n11)



- Born 1707 in Basel, Switzerland.
- Euler's identity:  $e^{i\pi} + 1 = 0$ .
- Solved the Bridges of Königsberg problem.
- It's not possible to cross all bridges once and once only.

[https://en.wikipedia.org/wiki/Leonhard\\_Euler](https://en.wikipedia.org/wiki/Leonhard_Euler)

# Graph of Königsberg



## Definition

A *graph* consists of a finite set  $V$  and a set  $E$  of 2-subsets of  $V$ .

**Vertices** – the elements of the set  $V$  are called vertices.

**Edges** – the elements of  $E$  are called edges.

$G = (V, E)$  – this is the way we write the graph  $G$  consists of the vertex set  $V$  and the edge set  $E$ .

$$V = \{ \textit{Green}, \textit{Blue}, \textit{Orange}, \textit{Red} \}$$

$$E = \{ \\ \{ \textit{Green}, \textit{Blue} \}, \{ \textit{Green}, \textit{Blue} \}, \{ \textit{Green}, \textit{Red} \}, \\ \{ \textit{Green}, \textit{Red} \}, \{ \textit{Blue}, \textit{Orange} \}, \{ \textit{Green}, \textit{Orange} \}, \\ \{ \textit{Red}, \textit{Orange} \} \\ \}$$



## Adjacency list

Green	Blue	Orange	Red
Blue	Green	Blue	Green
Orange	Orange	Green	Orange
Red		Red	

# Defining different types of graphs

## Our definition of a graph

The definition given above for a graph is not consistent with looped edges, directed edges or repeated edges. We only need to make small changes to the definition of a graph to allow for directed edges and repeated edges.

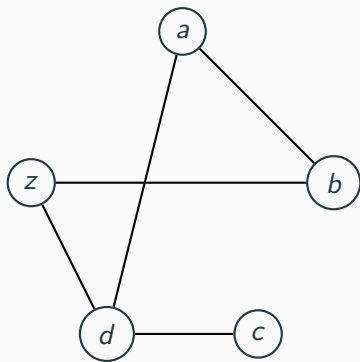
**Repeated edges** are edges that start and end at the same vertices.

**Directed edges** are edges where a direction is added.

**Looped edges** begin and end at the same vertex.

The application will determine the definition we want to use.

## A better example



### Exercise

Determine the vertex set, edge set and adjacency list of this graph.

[global.oup.com/booksites/content/9780198507185/](http://global.oup.com/booksites/content/9780198507185/)

# Degree of a vertex

## Definition

The degree of a vertex is the number of edges that contain it.



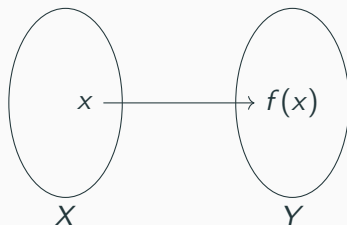
The degree of the vertex  $a$  is 2.

## Exercise

For each of the vertices on the previous slide, determine its degree.

## Definition

Suppose that  $X$  and  $Y$  are sets. We say we have a function  $f$  from  $X$  to  $Y$  if for each  $x$  in  $X$  we can specify a unique element in  $Y$ , which we denote by  $f(x)$ .

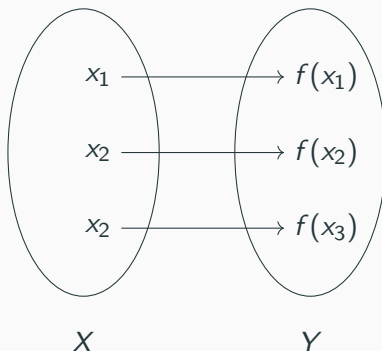


# Bijections

## Definition

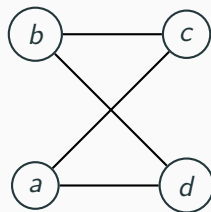
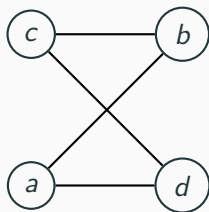
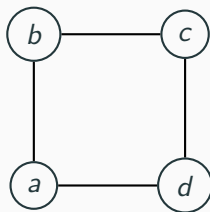
A bijection is function  $f$  from a set  $X$  to a set  $Y$  where both of the following are true:

- every  $y$  in  $Y$  is a value  $f(x)$  for at most one  $x$  in  $X$ .
- every  $y$  in  $Y$  is a value  $f(x)$  for at least one  $x$  in  $X$ .



## Definition

Two graphs  $G_1$  and  $G_2$  are said to be isomorphic when there is a bijection  $\alpha$  for the vertex set  $V_1$  of  $G_1$  to the vertex set  $V_2$  of  $G_2$  such that  $\{\alpha(x), \alpha(y)\}$  is an edge of  $G_2$  if and only if  $(x, y)$  is an edge of  $G_1$ .



# Sum of degrees

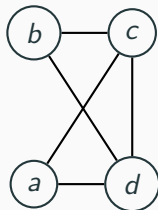
## Theorem

*The sum of the degrees of the vertices of a graph  $G = (V, E)$  is equal to twice the number of edges:*

$$\sum_{v \in V} \delta(v) = 2|E|$$

## Proof.

The degree  $\delta(v)$  of a vertex  $v$  is equal to the number of edges incident on it. Every edge is incident on two vertices. So every edge contributes 1 to the degrees of two distinct vertices. Therefore every edge contributes 2 to the sum total of the degrees of all the vertices.  $\square$

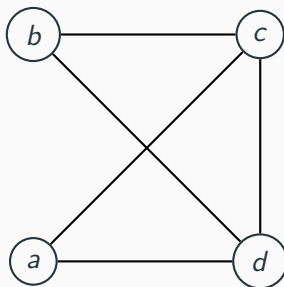




# Handshaking lemma

## Definition

A vertex is an odd vertex if its degree is odd, and it is an even vertex if its degree is even. The set of all odd vertices is denoted  $V_o$  and the set of all even vertices is denoted  $V_e$ .



## Exercise

Which of the above vertices are even, and which are odd?

# Handshaking lemma

## Lemma

*The number of odd vertices  $|V_o|$  in a graph is even.*

## Proof.

The sets  $V_o$  and  $V_e$  are disjoint (i.e. they don't have any elements in common.) Also, every vertex is either in  $V_o$  or  $V_e$ . Therefore  $V = V_o \cup V_e$  and  $|V| = |V_o| + |V_e|$ .

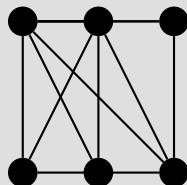
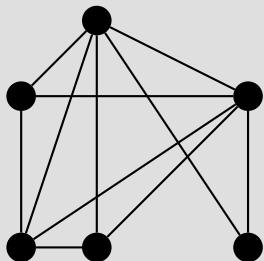
Furthermore:

$$\sum_{v \in V_o} \delta(v) + \sum_{v \in V_e} \delta(v) = 2|E|$$

Both  $2|E|$  and  $\sum_{v \in V_e} \delta(v)$  are even, so  $\sum_{v \in V_o} \delta(v)$  must be. Since  $\delta(v)$  is odd for every  $v$  in  $V_o$ , this must mean that  $|V_o|$  is even. □

## Exercise

Determine if these two graphs are isomorphic.



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# Paths and Cycles

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