

Graph Theory

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Graphs

Paths and Cycles

Trees

Colouring

Sorting

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Shortest Paths

Graphs

Seven Bridges of Königsberg



Is it possible to walk through the city crossing each of the seven bridges once and only once?

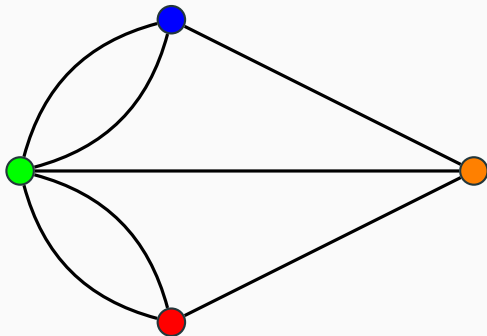
www.nature.com/nbt/journal/v29/n11



- Born 1707 in Basel, Switzerland.
- Euler's identity: $e^{i\pi} + 1 = 0$.
- Solved the Bridges of Königsberg problem.
- It's not possible to cross all bridges once and once only.

https://en.wikipedia.org/wiki/Leonhard_Euler

Graph of Königsberg



Definition

A *graph* consists of a finite set V and a set E of 2-subsets of V .

Vertices – the elements of the set V are called vertices.

Edges – the elements of E are called edges.

$G = (V, E)$ – this is the way we write the graph G consists of the vertex set V and the edge set E .

$$V = \{ \textit{Green}, \textit{Blue}, \textit{Orange}, \textit{Red} \}$$

$$E = \{ \\ \{ \textit{Green}, \textit{Blue} \}, \{ \textit{Green}, \textit{Blue} \}, \{ \textit{Green}, \textit{Red} \}, \\ \{ \textit{Green}, \textit{Red} \}, \{ \textit{Blue}, \textit{Orange} \}, \{ \textit{Green}, \textit{Orange} \}, \\ \{ \textit{Red}, \textit{Orange} \} \\ \}$$

Adjacency list

Green	Blue	Orange	Red
Blue	Green	Blue	Green
Orange	Orange	Green	Orange
Red		Red	

Defining different types of graphs

Our definition of a graph

The definition given above for a graph is not consistent with looped edges, directed edges or repeated edges. We only need to make small changes to the definition of a graph to allow for directed edges and repeated edges.

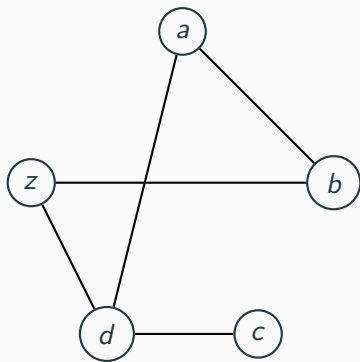
Repeated edges are edges that start and end at the same vertices.

Directed edges are edges where a direction is added.

Looped edges begin and end at the same vertex.

The application will determine the definition we want to use.

A better example



Exercise

Determine the vertex set, edge set and adjacency list of this graph.

global.oup.com/booksites/content/9780198507185/

Degree of a vertex

Definition

The degree of a vertex is the number of edges that contain it.



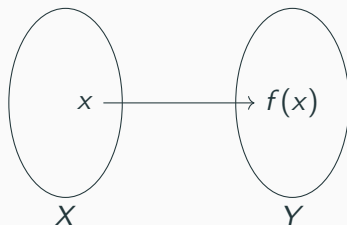
The degree of the vertex a is 2.

Exercise

For each of the vertices on the previous slide, determine its degree.

Definition

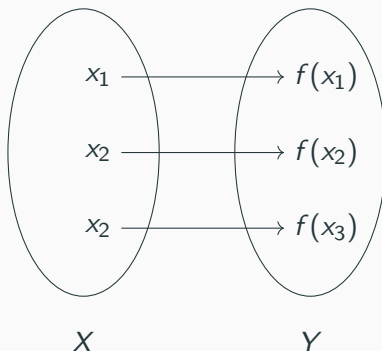
Suppose that X and Y are sets. We say we have a function f from X to Y if for each x in X we can specify a unique element in Y , which we denote by $f(x)$.



Definition

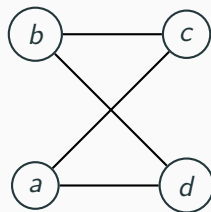
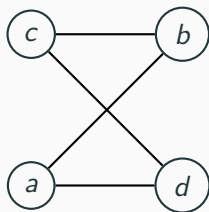
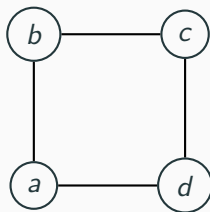
A bijection is function f from a set X to a set Y where both of the following are true:

- every y in Y is a value $f(x)$ for at most one x in X .
- every y in Y is a value $f(x)$ for at least one x in X .



Definition

Two graphs G_1 and G_2 are said to be isomorphic when there is a bijection α for the vertex set V_1 of G_1 to the vertex set V_2 of G_2 such that $\{\alpha(x), \alpha(y)\}$ is an edge of G_2 if and only if (x, y) is an edge of G_1 .



Sum of degrees

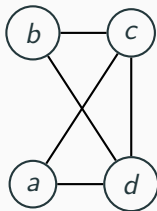
Theorem

The sum of the degrees of the vertices of a graph $G = (V, E)$ is equal to twice the number of edges:

$$\sum_{v \in V} \delta(v) = 2|E|$$

Proof.

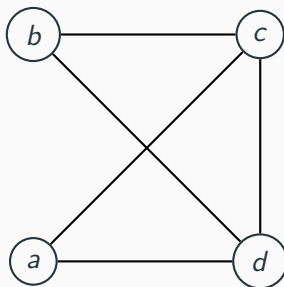
The degree $\delta(v)$ of a vertex v is equal to the number of edges incident on it. Every edge is incident on two vertices. So every edge contributes 1 to the degrees of two distinct vertices. Therefore every edge contributes 2 to the sum total of the degrees of all the vertices. \square



Handshaking lemma

Definition

A vertex is an odd vertex if its degree is odd, and it is an even vertex if its degree is even. The set of all odd vertices is denoted V_o and the set of all even vertices is denoted V_e .



Exercise

Which of the above vertices are even, and which are odd?

Lemma

The number of odd vertices $|V_o|$ in a graph is even.

Proof.

The sets V_o and V_e are disjoint (i.e. they don't have any elements in common.) Also, every vertex is either in V_o or V_e . Therefore $V = V_o \cup V_e$ and $|V| = |V_o| + |V_e|$.

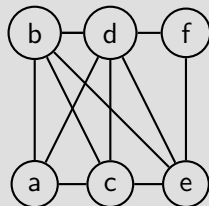
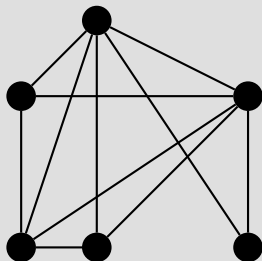
Furthermore:

$$\sum_{v \in V_o} \delta(v) + \sum_{v \in V_e} \delta(v) = 2|E|$$

Both $2|E|$ and $\sum_{v \in V_e} \delta(v)$ are even, so $\sum_{v \in V_o} \delta(v)$ must be. Since $\delta(v)$ is odd for every v in V_o , this must mean that $|V_o|$ is even. \square

Exercise

Determine if these two graphs are isomorphic.



Paths and Cycles

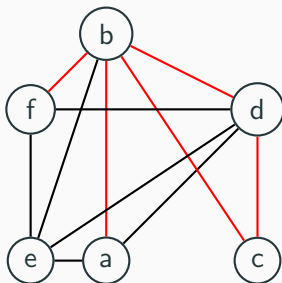
Definition

A *walk* in a graph is a sequence of vertices

$$v_1, v_2, \dots, v_k$$

such that v_i and v_{i+1} are adjacent for $1 \leq i < k$.

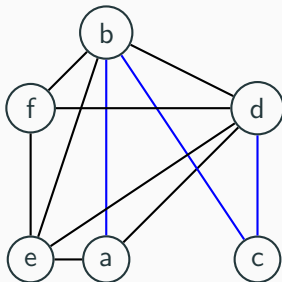
For instance, in the following graph a, b, c, d, b, f is a walk.



Definition

A *path* is a walk where each vertex is distinct.

For instance, in the following graph a, b, c, d is a path.

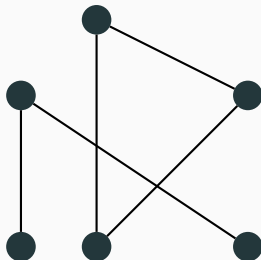
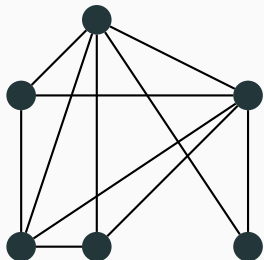


Connected graphs

Definition

A graph is *connected* if there is a path between each pair of vertices.

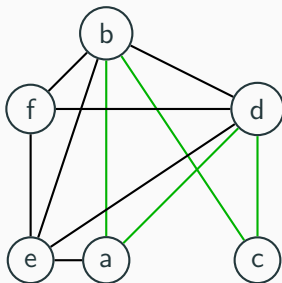
For instance, the graph on the left below is connected, but the one on the right is not.



Definition

A *cycle* is a walk where each vertex is distinct, except for the start and end vertices being equal.

For instance, in the following graph a, b, c, d, a is a path.

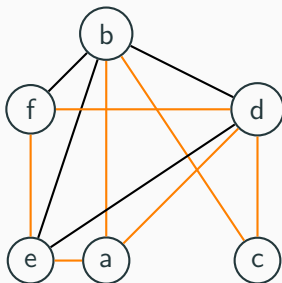


Hamiltonian cycles

Definition

A *Hamiltonian cycle* is a cycle that contains all of vertices of the graph.

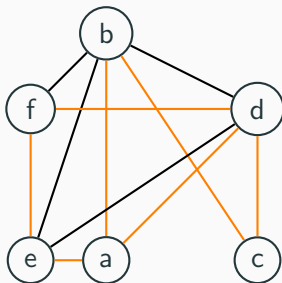
For instance, in the following graph a, b, c, d, f, e, a is a Hamiltonian cycle.



Definition

An *Eulerian walk* is a walk which uses each edge of the graph exactly once.

For instance, in the following graph a, b, c, d, f, e, a is a path.



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