Graph Theory

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Topics

Graphs

Paths and Cycles

Trees

Colouring

Sorting

Searching

Shortest Paths

Graphs

Seven Bridges of Königsberg

Is it possible to walk through the city crossing each of the seven bridges once and only once?

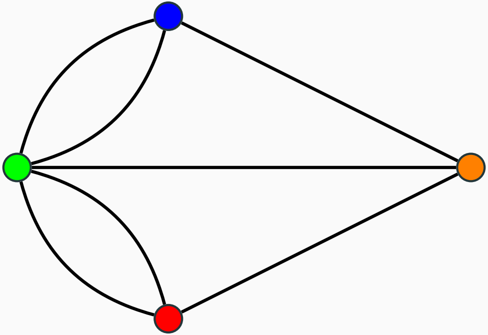
www.nature.com/nbt/journal/v29/n11

Leonhard Euler

- Born 1707 in Basel, Switzerland.
- Euler's identity: $\mathrm{e}^{i\pi} + 1 = 0$.
- Solved the Bridges of Königsberg problem.
- It's not possible to cross all bridges once and once only.

https://en.wikipedia.org/wiki/Leonhard_Euler

Graph of Königsberg



Graph definition

Definition

A *graph* consists of a finite set V and a set E of 2-subsets of V.

Vertices – the elements of the set V are called vertices.

Edges – the elements of E are called edges.

$G = (V, E)$ – this is the way we write the graph G consists of the vertex set V and the edge set E.

Sets of Königsberg

$$V = \{Green, Blue, Orange, Red\}$$

$$\begin{aligned}
E = \{ & \\
& \{Green, Blue\}, \{Green, Blue\}, \{Green, Red\}, \\
& \{Green, Red\}, \{Blue, Orange\}, \{Green, Orange\}, \\
& \{Red, Orange\} \\
\} &
\end{aligned}$$

Adjanceny list

Green	Blue	Orange	Red
Blue	Green	Blue	Green
Orange	Orange	Green	Orange
Red		Red	

Defining different types of graphs

Our definition of a graph

The definition given above for a graph is not consistent with looped edges, directed edges or repeated edges. We only need to make small changes to the definition of a graph to allow for directed edges and repeated edges.

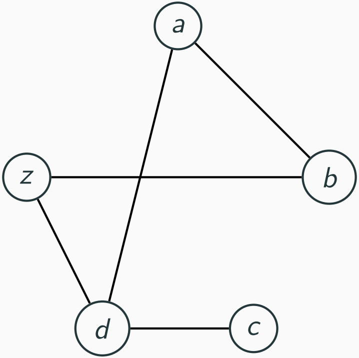
Repeated edges are edges that start and end at the same vertices.

Directed edges are edges where a direction is added.

Looped edges begin and end at the same vertex.

The application will determine the definition we want to use.

A better example



Exercise

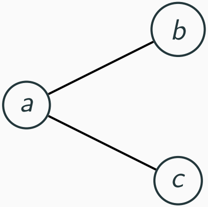
Determine the vertex set, edge set and adacency list of this graph.

global.oup.com/booksites/content/9780198507185/

Degree of a vertex

Definition

The degree of a vertex is the number of edges that contain it.



The degree of the vertex a is 2.

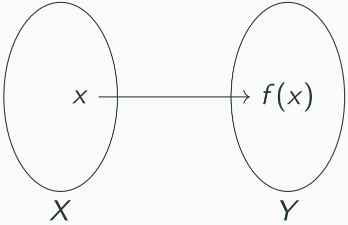
Exercise

For each of the vertices on the previous slide, determine its degree.

Functions

Definition

Suppose that X and Y are sets. We say we have a function f from X to Y if for each x in X we can specify a unique element in Y, which we denote by $f(x)$.

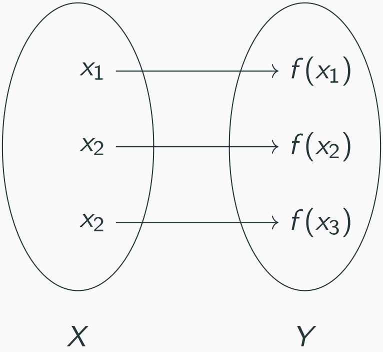


Bijections

Definition

A bijection is function f from a set X to a set Y where both of the following are true:

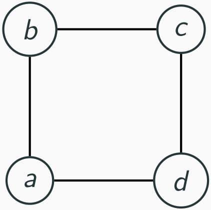
- every y in Y is a value $f(x)$ for at most one x in X.
- every y in Y is a value $f(x)$ for at least one x in X.

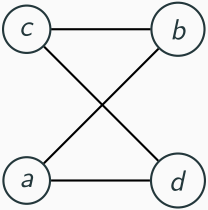


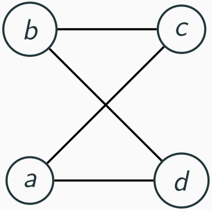
Isomorphism

Definition

Two graphs G_1 and G_2 are said to be isomorphic when there is a bijection α for the vertex set V_1 of G_1 to the vertex set V_2 of G_2 such that $\{\alpha(x), \alpha(y)\}$ is an edge of G_2 if and only if (x, y) is an edge of G_1.







Sum of degrees

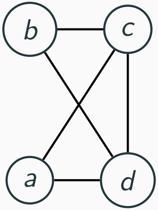
Theorem

The sum of the degrees of the vertices of a graph $G = (V, E)$ is equal to twice the number of edges:

$$\sum_{\mathrm{v} \in V} \delta(\mathrm{v}) = 2|E|$$

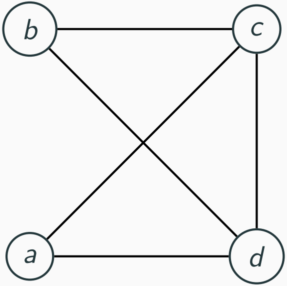
Proof.

The degree $\delta(\mathrm{v})$ of a vertex v is equal to the number of edges incident on it. Every edge is incident on two vertices. So every edge contributes 1 to the degrees of two distinct vertices. Therefore every edge contributes 2 to the sum total of the degrees of all the vertices. □



Definition

A vertex is an odd vertex if its degree is odd, and it is an even vertex if its degree is even. The set of all odd vertices is denoted V_o and the set of all even vertices is denoted V_e.



Exercise

Which of the above vertices are even, and which are odd?

Handshaking lemma

Lemma

The number of odd vertices $|V_o|$ in a graph is even.

Proof.

The sets V_o and V_e are disjoint (i.e. they don't have any elements in common.) Also, every vertex is either in V_o or V_e. Therefore $V = V_o \cup V_e$ and $|V| = |V_o| + |V_e|$.

Furthermore:

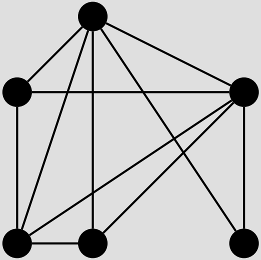
$$\sum_{\mathrm{v} \in V_o} \delta(\mathrm{v}) + \sum_{\mathrm{v} \in V_e} \delta(\mathrm{v}) = 2|E|$$

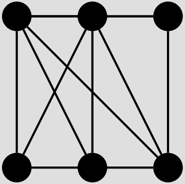
Both $2|E|$ and $\sum_{\mathrm{v} \in V_e} \delta(\mathrm{v})$ are even, so $\sum_{\mathrm{v} \in V_o} \delta(\mathrm{v})$ must be. Since $\delta(\mathrm{v})$ is odd for every v in V_o, this must mean that $|V_o|$ is even. □

Isomorphism: degrees

Exercise

Determine if these two graphs are isomorphic.





Paths and Cycles

Walks

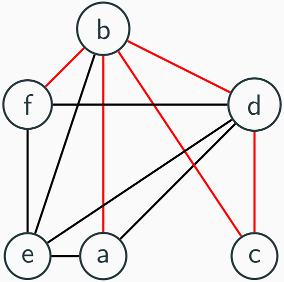
Definition

A *walk* in a graph is a sequence of vertices

$$v_1, v_2, \dots, v_k$$

such that v_i and v_{i+1} are adjacent for $1 \leq i < k$.

For instance, in the following graph a, b, c, d, b, f is a walk.

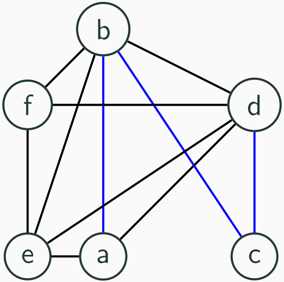


Paths

Definition

A *path* is a walk where each vertex is distinct.

For instance, in the following graph a, b, c, d is a path.

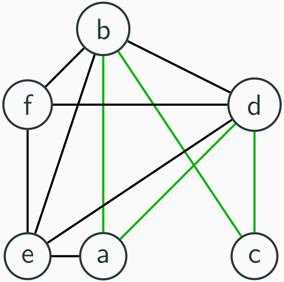


Cycles

Definition

A *cycle* is a walk where each vertex is distinct, except for the start and end vertices being equal.

For instance, in the following graph a, b, c, d, a is a path.

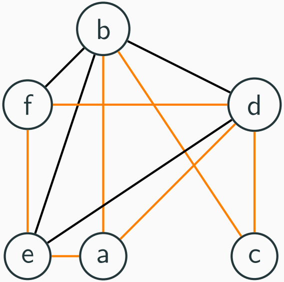


Hamiltonian cycles

Definition

A *Hamiltonian cycle* is a cycle that contains all of vertices of the graph.

For instance, in the following graph a, b, c, d, f, e, a is a path.



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