Graphs

ian.mcloughlin@gmit.ie

Seven Bridges of Königsberg

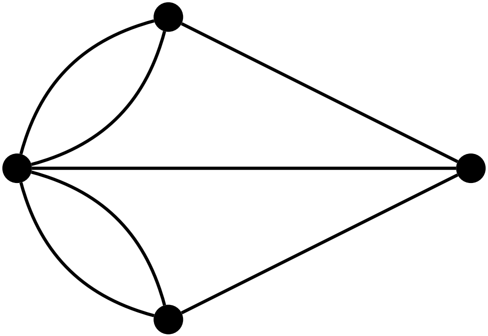
Is it possible to walk through the city crossing each of the seven bridges exactly once?

www.nature.com/nbt/journal/v29/n11

Leonhard Euler

- Born 1707 in Basel, Switzerland.
- Euler's identity: $\mathrm{e}^{i\pi} + 1 = 0$.
- Solved the Bridges of Königsberg.
- It's not possible.

Königsberg (multi)graph



Graphs

Definition

A *graph* consists of a finite set V and a set E of 2-subsets of V.

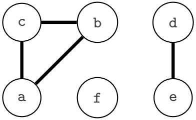
2-subset – a subset with two elements.

Vertices – the elements of the set V are called vertices.

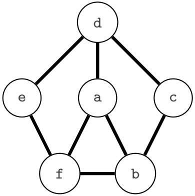
Edges – the elements of E are called edges.

$G = (V, E)$ – this is the way we write the graph G consists of the vertex set V and the edge set E.

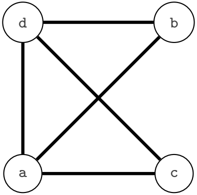
Drawings of graphs



G_1: 6 vertices, 4 edges



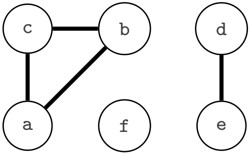
G_2: 6 vertices, 8 edges



G_3: 4 vertices, 5 edges

G_4: 1 vertex, 0 edges

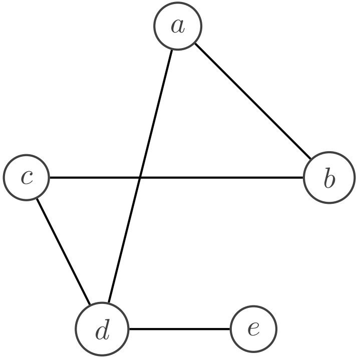
Drawings are not definitions



$$V = \{a, b, c, d, e, f\}$$

$$E = \{\{a, b\}, \{a, c\}, \{b, c\}, \{d, e\}\}$$

A better example



Exercise

Determine the vertex set, edge set and adjacency list of this graph.

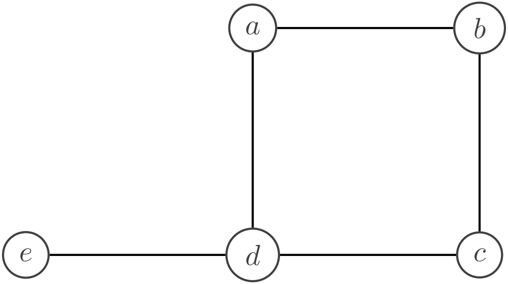
global.oup.com/booksites/content/9780198507185/

Adjanceny list

a	b	c	d	e
b	a	b	a	d
d	c	d	c	
			e	

Useful (sometimes) for representing graphs in code.

Another better example

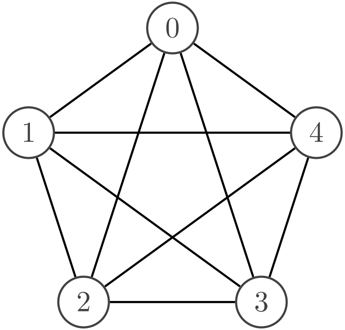


Exercise

Determine the vertex set, edge set and adjacency list of this graph.

global.oup.com/booksites/content/9780198507185/

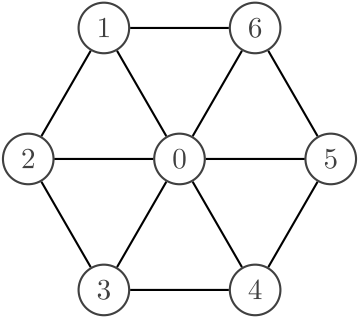
Complete graph K_n



Exercise

Determine the vertex set, edge set and adjacency list of K_5.

Wheel graph W_n



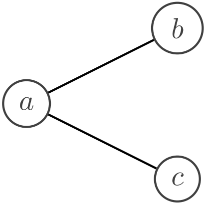
Exercise

Determine the vertex set, edge set and adjacency list of W_6.

Degree of a vertex

Definition

The degree of a vertex is the number of edges that contain it.



The degree of the vertex a is 2.

Exercise

For each of the vertices on the previous slide, determine its degree.

Sum of degrees

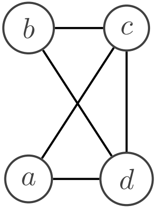
Theorem

The sum of the degrees of the vertices of a graph $G = (V, E)$ is equal to twice the number of edges:

$$\sum_{\mathrm{v} \in V} \delta(\mathrm{v}) = 2|E|$$

Proof.

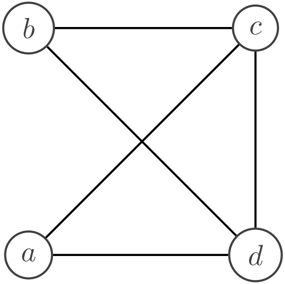
The degree $\delta(\mathrm{v})$ of a vertex v is equal to the number of edges incident on it. Every edge is incident on two vertices. So every edge contributes 1 to the degrees of two distinct vertices. Therefore every edge contributes 2 to the sum total of the degrees of all the vertices. $\square$



Even and odd vertices

Definition

A vertex is an odd vertex if its degree is odd, and it is an even vertex if its degree is even. The set of all odd vertices is denoted V_o and the set of all even vertices is denoted V_e.



Exercise

Which of the above vertices are even, and which are odd?

Handshaking lemma

Lemma

The number of odd vertices $|V_o|$ in a graph is even.

Proof.

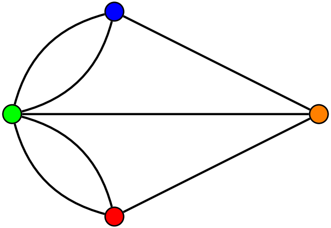
The sets V_o and V_e are disjoint (i.e. they don't have any elements in common.) Also, every vertex is either in V_o or V_e. Therefore $V = V_o \cup V_e$ and $|V| = |V_o| + |V_e|$.

Furthermore:

$$\sum_{\mathrm{v} \in V_o} \delta(\mathrm{v}) + \sum_{\mathrm{v} \in V_e} \delta(\mathrm{v}) = 2|E|$$

Both $2|E|$ and $\sum_{\mathrm{v} \in V_e} \delta(\mathrm{v})$ are even, so $\sum_{\mathrm{v} \in V_o} \delta(\mathrm{v})$ must be. Since $\delta(\mathrm{v})$ is odd for every v in V_o, this must mean that $|V_o|$ is even. $\square$

Sets of Königsberg



$$V = \{G, B, O, R\}$$

$$E = \{\{G, B\}, \{G, B\}, \{G, R\}, \{G, R\}, \{B, O\}, \{G, O\}, \{R, O\}\}$$

Not a graph by our definition

Note that the Bridges of Königsberg graph above is not a graph, due to the repeated edges. It's a multi-graph.

Defining different types of graphs

Our definition of a graph

The definition given above for a graph is not consistent with looped edges, directed edges or repeated edges. We only need to make small changes to the definition of a graph to allow for directed edges and repeated edges.

Repeated edges start and end at the same vertices.

Directed edges have a direction.

Looped edges begin and end at the same vertex.

The application will determine the definition we want to use.

Directed graph definition

Definition

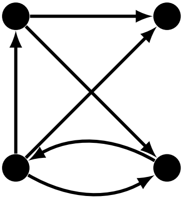
A *directed graph* (digraph) consists of a finite set V and a set E of 2-tuples (ordered pairs of elements) from V.

Looped edges are allowed in this definition. A single one per vertex.

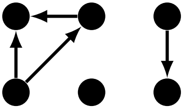
Multiple edges between the same start and end vertices are not allowed. However, two edges are allowed between every pair of vertices so long as they have opposite directions.

Direct edges use round brackets rather than curly braces: $E = \{(a, b) \mid a, b \in V\}$.

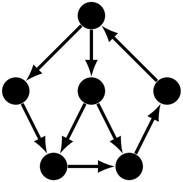
Example directed graphs



4 vertices, 6 directed edges



6 vertices, 4 directed edges



6 vertices, 8 directed edges

1 vertex, 0 directed edges

Multigraph definition

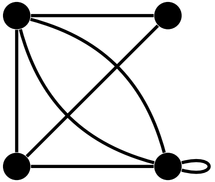
Definition

A *multigraph* consists of a finite set V and a multiset E of 2-multisubsets from V.

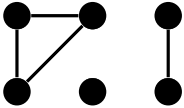
Multisets are like sets, but the same element can be in the set more than once.

Directed multigraphs are similar, but E is a set of 2-tuples of elements in V.

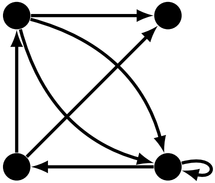
Example multigraphs



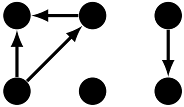
Multigraph



Still a multigraph



Directed multigraph



Still a directed multigraph