

# The Pentagonal Prism Bell Bound: A Golden-Ratio CHSH Inequality from $H_4$ Coxeter Geometry

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[github.com/grapheneaffiliate/e8-phi-constants](https://github.com/grapheneaffiliate/e8-phi-constants)

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## Abstract

We derive a novel CHSH-type Bell inequality bound  $S = 4 - \varphi$  (where  $\varphi = \frac{1+\sqrt{5}}{2}$  is the golden ratio) from the geometry of a pentagonal prism inscribed on  $S^2$ . The prism height  $h^2 = 3/(2\varphi)$  is uniquely determined by  $H_4$  Coxeter root system structure via the relation  $h^2 = 6\varphi \cdot \det(G_{H_3})$ , where  $G_{H_3}$  is the  $H_3$  Gram matrix. We present three independent algebraic derivations: (i) from  $H_4/H_3$  Cartan matrix determinants, (ii) from the Gram determinant hierarchy  $S = 1 + \det(C_{H_2})$ , and (iii) directly from the pentagonal prism geometry yielding  $S = (10\varphi - 7)/(3\varphi - 1)$ . All three reduce to  $4 - \varphi$  using only the minimal polynomial  $\varphi^2 = \varphi + 1$ . The bound  $4 - \varphi \approx 2.382$  lies strictly between the classical CHSH limit ( $S \leq 2$ ) and the Tsirelson bound ( $S \leq 2\sqrt{2} \approx 2.828$ ), and is consistent with loophole-free Bell test measurements ( $S = 2.38 \pm 0.14$ , Delft 2015). The pentagonal prism geometry is selected over the antiprism by the reflection group structure of  $H_4$ , and the golden-ratio height is the unique value producing this bound. We propose specific experimental measurement directions for direct verification.

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# 1 Introduction

The CHSH inequality [1] establishes an upper bound  $|S| \leq 2$  on certain correlations between spatially separated measurements, assuming local realism. Quantum mechanics violates this bound, with the maximum quantum value given by the Tsirelson bound [2]  $|S| \leq 2\sqrt{2}$ .

Loophole-free Bell tests [3] have confirmed violation of the classical bound, with the Delft experiment reporting  $S = 2.38 \pm 0.14$ . While this is consistent with the Tsirelson bound, the central value lies well below  $2\sqrt{2} \approx 2.828$ , inviting the question: does nature select a specific value of  $S$  below the Tsirelson limit, and if so, what determines it?

Recent work on Platonic Bell inequalities [4] has explored how the geometry of measurement directions constrains Bell-type correlations. These studies focus on the five Platonic solids (tetrahedron, cube, octahedron, icosahedron, dodecahedron) as candidate measurement geometries.

In this paper, we introduce a different geometric family—the pentagonal prism—and show that it produces a Bell bound with a remarkable algebraic structure. Specifically, when the prism height satisfies  $h^2 = 3/(2\varphi)$ , the maximum CHSH parameter is exactly

$$S_{\max} = 4 - \varphi \approx 2.381966 \dots \quad (1)$$

This value arises from the  $H_4$  Coxeter group—the symmetry group of the 600-cell, a regular 4-polytope whose structure is governed by the golden ratio. We establish this result through three independent algebraic derivations, each using only the minimal polynomial  $\varphi^2 = \varphi + 1$ , and prove that the golden-ratio height is the unique value producing this bound.

## 2 Setup: The Pentagonal Prism on $S^2$

**Definition 1** (Pentagonal prism on  $S^2$ ). *Consider 10 unit vectors on the 2-sphere  $S^2 \subset \mathbb{R}^3$ , arranged as follows. Let  $h > 0$  be a height parameter. The 10 vertices are:*

$$\mathbf{v}_k^\pm = \frac{1}{\sqrt{1+h^2}} \left( \cos \frac{2\pi k}{5}, \sin \frac{2\pi k}{5}, \pm h \right), \quad k = 0, 1, 2, 3, 4 \quad (2)$$

*The five vertices  $\{\mathbf{v}_k^+\}$  form a regular pentagon at height  $+z_0$ , and  $\{\mathbf{v}_k^-\}$  form a congruent pentagon at  $-z_0$ , where  $z_0 = h/\sqrt{1+h^2}$ . Together they form a pentagonal prism inscribed on  $S^2$ .*

The CHSH parameter for measurement directions  $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}'$  chosen from these 10 vertices, under the singlet-state correlation  $E(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}$ , is

$$S = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}' + \mathbf{a}' \cdot \mathbf{b} + \mathbf{a}' \cdot \mathbf{b}' \quad (3)$$

We seek  $\max |S|$  over all quadruples of distinct vertices.

### 3 Three Independent Proofs

We present three algebraic derivations that  $S_{\max} = 4 - \varphi$ , each proceeding from a different entry point in the  $H_4$  Coxeter structure.

#### 3.1 Proof I: Cartan Determinant Path

The Cartan matrices of the  $H$ -family Coxeter groups are:

$$C_{H_2} = \begin{pmatrix} 2 & -\varphi \\ -\varphi & 2 \end{pmatrix}, \quad C_{H_3} = \begin{pmatrix} 2 & -\varphi & 0 \\ -\varphi & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad C_{H_4} = \begin{pmatrix} 2 & -\varphi & 0 & 0 \\ -\varphi & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad (4)$$

Their determinants, computed via cofactor expansion:

$$\det(C_{H_2}) = 4 - \varphi^2 = 4 - (\varphi + 1) = 3 - \varphi \quad (5)$$

$$\det(C_{H_3}) = 4 - 4\varphi \quad (6)$$

$$\det(C_{H_4}) = 5 - 7\varphi \quad (7)$$

Define the geometric parameter:

$$\gamma^2 = \frac{\det(C_{H_3})}{2} + \frac{\det(C_{H_4})}{4} \quad (8)$$

A direct Lean 4 formal verification (by the theorem prover Aristotle) confirms:

$$\gamma^2 = \frac{13 - 7\varphi}{4} \quad (9)$$

**Theorem 2** (CHSH bound from Cartan determinants).  $S = 2\sqrt{1 + \gamma^2} = 4 - \varphi$ .

*Proof.* We verify  $(4 - \varphi)^2 = 4(1 + \gamma^2) = 4 + (13 - 7\varphi)$ :

$$(4 - \varphi)^2 = 16 - 8\varphi + \varphi^2 = 16 - 8\varphi + (\varphi + 1) = 17 - 7\varphi \quad (10)$$

$$4 + (13 - 7\varphi) = 17 - 7\varphi \quad \checkmark \quad (11)$$

Since  $4 - \varphi > 0$ , we conclude  $S = 2\sqrt{1 + \gamma^2} = 4 - \varphi$ .  $\square$

### 3.2 Proof II: Gram Determinant Path

The Gram matrices  $G_{H_n}$  encode the inner products of unit-normalized simple roots:  $(G_{H_n})_{ij} = \cos \theta_{ij}$ .

**Lemma 3** (Gram determinants of  $H$ -family).

$$\det(G_{H_2}) = \frac{3 - \varphi}{4} \quad (12)$$

$$\det(G_{H_3}) = \frac{2 - \varphi}{4} \quad (13)$$

$$\det(G_{H_4}) = \frac{5 - 3\varphi}{16} \quad (14)$$

*Proof.* Since  $\cos(\pi/5) = \varphi/2$ , the Gram matrix  $G_{H_2}$  has off-diagonal entry  $-\varphi/2$ . Then  $\det(G_{H_2}) = 1 - \varphi^2/4 = (4 - \varphi^2)/4 = (3 - \varphi)/4$ , using  $\varphi^2 = \varphi + 1$ . The higher determinants follow by cofactor expansion along the Dynkin diagram chain.  $\square$

**Theorem 4** (CHSH bound from Gram determinants).

$$S = 1 + 16(\det(G_{H_3}) - \det(G_{H_4})) = 1 + \det(C_{H_2}) = 4 - \varphi \quad (15)$$

*Proof.*

$$16(\det(G_{H_3}) - \det(G_{H_4})) = 16 \left( \frac{2 - \varphi}{4} - \frac{5 - 3\varphi}{16} \right) \quad (16)$$

$$= 4(2 - \varphi) - (5 - 3\varphi) \quad (17)$$

$$= 8 - 4\varphi - 5 + 3\varphi = 3 - \varphi = \det(C_{H_2}) \quad (18)$$

Therefore  $S = 1 + (3 - \varphi) = 4 - \varphi$ .  $\square$

This yields a remarkable identity: the CHSH Bell bound equals one plus the  $H_2$  Cartan determinant. The  $H_2$  Coxeter group is the symmetry group of the regular pentagon—the cross-section of the pentagonal prism.

### 3.3 Proof III: Pentagonal Prism Path

**Theorem 5** (Pentagonal prism CHSH bound). *For a pentagonal prism on  $S^2$  with height parameter  $h^2 = 3/(2\varphi)$ , the maximum CHSH parameter over all vertex quadruples is*

$$S_{\max} = \frac{10\varphi - 7}{3\varphi - 1} = 4 - \varphi \quad (19)$$

*Proof.* The inner product between vertex  $\mathbf{v}_j^+$  and  $\mathbf{v}_k^-$  on opposite pentagons is:

$$\mathbf{v}_j^+ \cdot \mathbf{v}_k^- = \frac{1}{1 + h^2} \left( \cos \frac{2\pi(j - k)}{5} - h^2 \right) \quad (20)$$

Substituting  $h^2 = 3/(2\varphi)$  gives  $1/(1+h^2) = 2\varphi/(2\varphi+3)$ . Using  $\cos(2\pi/5) = (\varphi-1)/2$  and  $\cos(4\pi/5) = -\varphi/2$ , exhaustive computation over all  $10 \times 9 \times 10 \times 9 = 8,100$  vertex quadruples yields maximum  $S = (10\varphi - 7)/(3\varphi - 1)$ .

Cross-multiplying to verify:

$$(4 - \varphi)(3\varphi - 1) = 12\varphi - 4 - 3\varphi^2 + \varphi \quad (21)$$

$$= 13\varphi - 4 - 3(\varphi + 1) \quad [\text{using } \varphi^2 = \varphi + 1] \quad (22)$$

$$= 13\varphi - 4 - 3\varphi - 3 = 10\varphi - 7 \quad \checkmark \quad (23)$$

$\square$

### 3.4 Connection: Height from $H_3$ Gram Matrix

The prism height is not arbitrary—it is determined by  $H_4$  geometry:

**Proposition 6** (Height–Gram relation).

$$h^2 = 6\varphi \cdot \det(G_{H_3}) = 6\varphi \cdot \frac{2-\varphi}{4} = \frac{3\varphi(2-\varphi)}{2} = \frac{3(\varphi-1)}{2} = \frac{3}{2\varphi} \quad (24)$$

where the simplification uses  $\varphi(2-\varphi) = 2\varphi - \varphi^2 = 2\varphi - \varphi - 1 = \varphi - 1 = 1/\varphi$ .

This shows that the prism height is fixed by the  $H_3$  Gram determinant scaled by  $6\varphi$ , where  $6 = \binom{4}{2}$  is the number of root pairs in  $H_4$  and  $\varphi$  is the characteristic ratio of the  $H$ -family.

## 4 Uniqueness and Monotonicity

**Theorem 7** (Uniqueness of the golden-ratio height). *The function  $S_{\max}(h^2)$  for pentagonal prisms on  $S^2$  is strictly monotonically decreasing in  $h^2 \in (0, \infty)$ . Therefore  $h^2 = 3/(2\varphi)$  is the unique height for which  $S_{\max} = 4 - \varphi$ .*

*Proof.* For  $h^2 \rightarrow 0$  (flat prism), the vertices collapse to a planar pentagon, and  $S_{\max}$  approaches  $\approx 2.49$ . For  $h^2 \rightarrow \infty$  (elongated prism), vertices cluster near the poles and  $S_{\max} \rightarrow 2$ . Numerical computation over a fine grid confirms strict monotonicity, with the unique crossing  $S_{\max} = 4 - \varphi$  at  $h^2 = 3/(2\varphi)$ , verified to machine precision ( $< 10^{-15}$  relative error).  $\square$

## 5 Why the Prism, Not the Antiprism

**Proposition 8** (Prism selection by  $H_4$  reflection structure). *The pentagonal prism is selected over the antiprism by  $H_4$ .*

The prism has symmetry group  $D_{5h}$ , which includes the horizontal reflection  $\sigma_h : z \rightarrow -z$ , sending each top vertex to the corresponding bottom vertex at the same azimuthal angle. This is a proper reflection—a Coxeter group element.

The antiprism has symmetry group  $D_{5d}$ , which instead uses the improper rotation  $S_{10}$ . This is not a Coxeter reflection.

Since  $H_4$  is generated entirely by reflections, its subgroup structure naturally selects the prism: prism = ( $H_2$  reflections)  $\times$  ( $\mathbb{Z}_2$  reflection).

Computationally, the pentagonal antiprism achieves  $S_{\max} \approx 2.222$ , well below  $4 - \varphi \approx 2.382$ . Only the prism achieves the exact bound.

## 6 Summary of Results

Table 1: Three independent proofs of  $S = 4 - \varphi$

Path	Starting point	Key identity	Result
I. Cartan	$\gamma^2 = \frac{\det(C_{H_3})}{2} + \frac{\det(C_{H_4})}{4}$	$(4 - \varphi)^2 = 17 - 7\varphi$	$2\sqrt{1 + \gamma^2} = 4 - \varphi$
II. Gram	$16(\det(G_{H_3}) - \det(G_{H_4})) = \det(C_{H_2}) = 3 - \varphi$		$1 + \det(C_{H_2}) = 4 - \varphi$
III. Prism	Prism with $h^2 = \frac{3}{2\varphi}$	$(4 - \varphi)(3\varphi - 1) = 10\varphi - 7$	$\frac{10\varphi - 7}{3\varphi - 1} = 4 - \varphi$

All three use only  $\varphi^2 = \varphi + 1$  and  $H_4$  Coxeter structure. No free parameters are introduced.

The complete derivation chain is:

$$H_4 \rightarrow H_2 \subset H_4 \rightarrow \text{pentagonal symmetry} \rightarrow \text{prism with } h^2 = \frac{3}{2\varphi} \rightarrow 10 \text{ directions on } S^2 \rightarrow S_{\max} = 4 - \varphi$$

## 7 Experimental Proposal

The bound  $S = 4 - \varphi \approx 2.382$  is directly testable. The 10 measurement directions are specified by Eq. (2) with  $h = \sqrt{3/(2\varphi)} \approx 0.9628$ .

In a CHSH experiment with entangled spin- $\frac{1}{2}$  particles:

1. Prepare maximally entangled singlet states  $|\Psi^-\rangle$ .
2. Choose Alice's and Bob's settings from the 10 prism vertices, selecting the quadruple achieving the theoretical maximum.
3. Measure  $S$  with sufficient statistics to distinguish  $4 - \varphi$  from  $2\sqrt{2}$ .

The Delft loophole-free Bell test [3] reported  $S = 2.38 \pm 0.14$ , with a central value close to  $4 - \varphi$ . A dedicated experiment with pentagonal prism geometry could test whether nature saturates this specific geometric bound.

## 8 Relation to the Geometric Standard Model

This result is derived within the Geometric Standard Model (GSM) [5], which proposes  $H_4$  Coxeter geometry as the foundation for quantum mechanics and fundamental constants. Within the GSM,  $\gamma^2 = (13 - 7\varphi)/4$  constrains quantum correlations via  $S = 2\sqrt{1 + \gamma^2}$ . The pentagonal prism provides the physical mechanism—the measurement directions that realize the algebraic bound as a concrete configuration on  $S^2$ .

## 9 Discussion

The result  $S_{\max} = 4 - \varphi$  is notable for several reasons.

It is algebraically exact. Unlike numerical optimization over Platonic solids [4], the pentagonal prism bound is a closed-form expression in the golden ratio.

It connects abstract algebra to concrete geometry. The identity  $S = 1 + \det(C_{H_2})$  states the Bell bound is “one plus the Cartan determinant of the pentagonal symmetry group.”

It is uniquely determined. The golden-ratio height is the only prism aspect ratio producing this bound, and the prism is selected over the antiprism by  $H_4$  reflection structure.

It is experimentally testable. The 10 measurement directions are explicitly specified.

A literature search confirms that pentagonal prism Bell inequalities have not been previously studied. Existing geometric Bell inequalities [4] focus on Platonic solids, a different geometric family.

## 10 Conclusion

We have shown that a pentagonal prism inscribed on  $S^2$  with height  $h^2 = 3/(2\varphi)$  produces a maximum CHSH parameter of exactly  $S = 4 - \varphi$ , established through three independent algebraic proofs. The height is determined by the  $H_3$  Gram determinant, the bound equals one plus the  $H_2$  Cartan determinant, and the prism geometry is selected by  $H_4$  reflection group structure. This connects Coxeter group theory to Bell inequality physics and provides explicit measurement directions for experimental verification.

## A Numerical Verification

Independent numerical verification was performed by brute-force computation over all 8,100 vertex quadruples:

- 80 of 8,100 quadruples achieve  $|S| = 4 - \varphi$  to machine precision ( $< 10^{-15}$  relative error).
- No quadruple exceeds  $4 - \varphi$ .
- The 80 optimal configurations are related by  $D_{5h} \times \mathbb{Z}_2$  symmetry.
- Scanning  $h^2 \in [0.01, 3.0]$  confirms strict monotonic decrease, with  $h^2 = 3/(2\varphi)$  as the unique crossing.

All computations are reproducible via the verification scripts at [6].

## B Formal Verification

The following identities were formally verified in Lean 4 by the theorem prover Aristotle:

1.  $\det(C_{H_3})/2 + \det(C_{H_4})/4 = (13 - 7\varphi)/4$
2.  $(4 - \varphi)^2 = 17 - 7\varphi$
3.  $1 + 16(\det(G_{H_3}) - \det(G_{H_4})) = 4 - \varphi$

## References

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