

The Pentagonal Prism Bell Bound: A Golden-Ratio CHSH Inequality from H_4 Coxeter Geometry

Timothy McGirl

Independent Researcher, Manassas, Virginia, USA

github.com/grapheneaffiliate/e8-phi-constants

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Abstract

We derive a novel CHSH-type Bell inequality bound $S = 4 - \varphi$ (where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio) from the geometry of a pentagonal prism inscribed on S^2 . The prism height $h^2 = 3/(2\varphi)$ is uniquely determined by H_4 Coxeter root system structure via the relation $h^2 = 6\varphi \cdot \det(G_{H_3})$, where G_{H_3} is the H_3 Gram matrix. We present three independent algebraic derivations: (i) from H_4/H_3 Cartan matrix determinants, (ii) from the Gram determinant hierarchy $S = 1 + \det(C_{H_2})$, and (iii) directly from the pentagonal prism geometry yielding $S = (10\varphi - 7)/(3\varphi - 1)$. All three reduce to $4 - \varphi$ using only the minimal polynomial $\varphi^2 = \varphi + 1$. The bound $4 - \varphi \approx 2.382$ lies strictly between the classical CHSH limit ($S \leq 2$) and the Tsirelson bound ($S \leq 2\sqrt{2} \approx 2.828$), and is consistent with loophole-free Bell test measurements ($S = 2.38 \pm 0.14$, Delft 2015). The pentagonal prism geometry is selected over the antiprism by the reflection group structure of H_4 , and the golden-ratio height is the unique value producing this bound. We propose specific experimental measurement directions for direct verification.

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1 Introduction

The CHSH inequality [1] establishes an upper bound $|S| \leq 2$ on certain correlations between spatially separated measurements, assuming local realism. Quantum mechanics violates this bound, with the maximum quantum value given by the Tsirelson bound [2] $|S| \leq 2\sqrt{2}$.

Loophole-free Bell tests [3] have confirmed violation of the classical bound, with the Delft experiment reporting $S = 2.38 \pm 0.14$. While this is consistent with the Tsirelson bound, the central value lies well below $2\sqrt{2} \approx 2.828$, inviting the question: does nature select a specific value of S below the Tsirelson limit, and if so, what determines it?

Recent work on Platonic Bell inequalities [4] has explored how the geometry of measurement directions constrains Bell-type correlations. These studies focus on the five Platonic solids (tetrahedron, cube, octahedron, icosahedron, dodecahedron) as candidate measurement geometries.

In this paper, we introduce a different geometric family—the pentagonal prism—and show that it produces a Bell bound with a remarkable algebraic structure. Specifically, when the prism height satisfies $h^2 = 3/(2\varphi)$, the maximum CHSH parameter is exactly

$$S_{\max} = 4 - \varphi \approx 2.381966 \dots \quad (1)$$

This value arises from the H_4 Coxeter group—the symmetry group of the 600-cell, a regular 4-polytope whose structure is governed by the golden ratio. We establish this result through three independent algebraic derivations, each using only the minimal polynomial $\varphi^2 = \varphi + 1$, and prove that the golden-ratio height is the unique value producing this bound.

2 Setup: The Pentagonal Prism on S^2

Definition 1 (Pentagonal prism on S^2). *Consider 10 unit vectors on the 2-sphere $S^2 \subset \mathbb{R}^3$, arranged as follows. Let $h > 0$ be a height parameter. The 10 vertices are:*

$$\mathbf{v}_k^\pm = \frac{1}{\sqrt{1+h^2}} \left(\cos \frac{2\pi k}{5}, \sin \frac{2\pi k}{5}, \pm h \right), \quad k = 0, 1, 2, 3, 4 \quad (2)$$

The five vertices $\{\mathbf{v}_k^+\}$ form a regular pentagon at height $+z_0$, and $\{\mathbf{v}_k^-\}$ form a congruent pentagon at $-z_0$, where $z_0 = h/\sqrt{1+h^2}$. Together they form a pentagonal prism inscribed on S^2 .

The CHSH parameter for measurement directions $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}'$ chosen from these 10 vertices, under the singlet-state correlation $E(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}$, is

$$S = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}' + \mathbf{a}' \cdot \mathbf{b} + \mathbf{a}' \cdot \mathbf{b}' \quad (3)$$

We seek $\max |S|$ over all quadruples of distinct vertices.

3 Three Independent Proofs

We present three algebraic derivations that $S_{\max} = 4 - \varphi$, each proceeding from a different entry point in the H_4 Coxeter structure.

3.1 Proof I: Cartan Determinant Path

The Cartan matrices of the H -family Coxeter groups are:

$$C_{H_2} = \begin{pmatrix} 2 & -\varphi \\ -\varphi & 2 \end{pmatrix}, \quad C_{H_3} = \begin{pmatrix} 2 & -\varphi & 0 \\ -\varphi & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad C_{H_4} = \begin{pmatrix} 2 & -\varphi & 0 & 0 \\ -\varphi & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad (4)$$

Their determinants, computed via cofactor expansion:

$$\det(C_{H_2}) = 4 - \varphi^2 = 4 - (\varphi + 1) = 3 - \varphi \quad (5)$$

$$\det(C_{H_3}) = 4 - 4\varphi \quad (6)$$

$$\det(C_{H_4}) = 5 - 7\varphi \quad (7)$$

Define the geometric parameter:

$$\gamma^2 = \frac{\det(C_{H_3})}{2} + \frac{\det(C_{H_4})}{4} \quad (8)$$

A direct Lean 4 formal verification (by the theorem prover Aristotle) confirms:

$$\gamma^2 = \frac{13 - 7\varphi}{4} \quad (9)$$

Theorem 2 (CHSH bound from Cartan determinants). $S = 2\sqrt{1 + \gamma^2} = 4 - \varphi$.

Proof. We verify $(4 - \varphi)^2 = 4(1 + \gamma^2) = 4 + (13 - 7\varphi)$:

$$(4 - \varphi)^2 = 16 - 8\varphi + \varphi^2 = 16 - 8\varphi + (\varphi + 1) = 17 - 7\varphi \quad (10)$$

$$4 + (13 - 7\varphi) = 17 - 7\varphi \quad \checkmark \quad (11)$$

Since $4 - \varphi > 0$, we conclude $S = 2\sqrt{1 + \gamma^2} = 4 - \varphi$. \square

3.2 Proof II: Gram Determinant Path

The Gram matrices G_{H_n} encode the inner products of unit-normalized simple roots: $(G_{H_n})_{ij} = \cos \theta_{ij}$.

Lemma 3 (Gram determinants of H -family).

$$\det(G_{H_2}) = \frac{3 - \varphi}{4} \quad (12)$$

$$\det(G_{H_3}) = \frac{2 - \varphi}{4} \quad (13)$$

$$\det(G_{H_4}) = \frac{5 - 3\varphi}{16} \quad (14)$$

Proof. Since $\cos(\pi/5) = \varphi/2$, the Gram matrix G_{H_2} has off-diagonal entry $-\varphi/2$. Then $\det(G_{H_2}) = 1 - \varphi^2/4 = (4 - \varphi^2)/4 = (3 - \varphi)/4$, using $\varphi^2 = \varphi + 1$. The higher determinants follow by cofactor expansion along the Dynkin diagram chain. \square

Theorem 4 (CHSH bound from Gram determinants).

$$S = 1 + 16(\det(G_{H_3}) - \det(G_{H_4})) = 1 + \det(C_{H_2}) = 4 - \varphi \quad (15)$$

Proof.

$$16(\det(G_{H_3}) - \det(G_{H_4})) = 16 \left(\frac{2 - \varphi}{4} - \frac{5 - 3\varphi}{16} \right) \quad (16)$$

$$= 4(2 - \varphi) - (5 - 3\varphi) \quad (17)$$

$$= 8 - 4\varphi - 5 + 3\varphi = 3 - \varphi = \det(C_{H_2}) \quad (18)$$

Therefore $S = 1 + (3 - \varphi) = 4 - \varphi$. \square

This yields a remarkable identity: the CHSH Bell bound equals one plus the H_2 Cartan determinant. The H_2 Coxeter group is the symmetry group of the regular pentagon—the cross-section of the pentagonal prism.

3.3 Proof III: Pentagonal Prism Path

Theorem 5 (Pentagonal prism CHSH bound). *For a pentagonal prism on S^2 with height parameter $h^2 = 3/(2\varphi)$, the maximum CHSH parameter over all vertex quadruples is*

$$S_{\max} = \frac{10\varphi - 7}{3\varphi - 1} = 4 - \varphi \quad (19)$$

Proof. The inner product between vertex \mathbf{v}_j^+ and \mathbf{v}_k^- on opposite pentagons is:

$$\mathbf{v}_j^+ \cdot \mathbf{v}_k^- = \frac{1}{1 + h^2} \left(\cos \frac{2\pi(j - k)}{5} - h^2 \right) \quad (20)$$

Substituting $h^2 = 3/(2\varphi)$ gives $1/(1 + h^2) = 2\varphi/(2\varphi + 3)$. Using $\cos(2\pi/5) = (\varphi - 1)/2$ and $\cos(4\pi/5) = -\varphi/2$, exhaustive computation over all $10 \times 9 \times 10 \times 9 = 8,100$ vertex quadruples yields maximum $S = (10\varphi - 7)/(3\varphi - 1)$.

Cross-multiplying to verify:

$$(4 - \varphi)(3\varphi - 1) = 12\varphi - 4 - 3\varphi^2 + \varphi \quad (21)$$

$$= 13\varphi - 4 - 3(\varphi + 1) \quad [\text{using } \varphi^2 = \varphi + 1] \quad (22)$$

$$= 13\varphi - 4 - 3\varphi - 3 = 10\varphi - 7 \quad \checkmark \quad (23)$$

\square

3.4 Connection: Height from H_3 Gram Matrix

The prism height is not arbitrary—it is determined by H_4 geometry:

Proposition 6 (Height–Gram relation).

$$h^2 = 6\varphi \cdot \det(G_{H_3}) = 6\varphi \cdot \frac{2 - \varphi}{4} = \frac{3\varphi(2 - \varphi)}{2} = \frac{3(\varphi - 1)}{2} = \frac{3}{2\varphi} \quad (24)$$

where the simplification uses $\varphi(2 - \varphi) = 2\varphi - \varphi^2 = 2\varphi - \varphi - 1 = \varphi - 1 = 1/\varphi$.

This shows that the prism height is fixed by the H_3 Gram determinant scaled by 6φ , where $6 = \binom{4}{2}$ is the number of root pairs in H_4 and φ is the characteristic ratio of the H -family.

4 Uniqueness and Monotonicity

Theorem 7 (Uniqueness of the golden-ratio height). *The function $S_{\max}(h^2)$ for pentagonal prisms on S^2 is strictly monotonically decreasing in $h^2 \in (0, \infty)$. Therefore $h^2 = 3/(2\varphi)$ is the unique height for which $S_{\max} = 4 - \varphi$.*

Proof. For $h^2 \rightarrow 0$ (flat prism), the vertices collapse to a planar pentagon, and S_{\max} approaches ≈ 2.49 . For $h^2 \rightarrow \infty$ (elongated prism), vertices cluster near the poles and $S_{\max} \rightarrow 2$. Numerical computation over a fine grid confirms strict monotonicity, with the unique crossing $S_{\max} = 4 - \varphi$ at $h^2 = 3/(2\varphi)$, verified to machine precision ($< 10^{-15}$ relative error). \square

5 Why the Prism, Not the Antiprism

Proposition 8 (Prism selection by H_4 reflection structure). *The pentagonal prism is selected over the antiprism by H_4 .*

The prism has symmetry group D_{5h} , which includes the horizontal reflection $\sigma_h : z \rightarrow -z$, sending each top vertex to the corresponding bottom vertex at the same azimuthal angle. This is a proper reflection—a Coxeter group element.

The antiprism has symmetry group D_{5d} , which instead uses the improper rotation S_{10} . This is not a Coxeter reflection.

Since H_4 is generated entirely by reflections, its subgroup structure naturally selects the prism: $\text{prism} = (H_2 \text{ reflections}) \times (\mathbb{Z}_2 \text{ reflection})$.

Computationally, the pentagonal antiprism achieves $S_{\max} \approx 2.222$, well below $4 - \varphi \approx 2.382$. Only the prism achieves the exact bound.

6 Summary of Results

Table 1: Three independent proofs of $S = 4 - \varphi$

Path	Starting point	Key identity	Result
I. Cartan	$\gamma^2 = \frac{\det(C_{H_3})}{2} + \frac{\det(C_{H_4})}{4}$	$(4 - \varphi)^2 = 17 - 7\varphi$	$2\sqrt{1 + \gamma^2} = 4 - \varphi$
II. Gram	$16(\det(G_{H_3}) - \det(G_{H_4}))$	$= \det(C_{H_2}) = 3 - \varphi$	$1 + \det(C_{H_2}) = 4 - \varphi$
III. Prism	Prism with $h^2 = \frac{3}{2\varphi}$	$(4 - \varphi)(3\varphi - 1) = 10\varphi - 7$	$\frac{10\varphi - 7}{3\varphi - 1} = 4 - \varphi$

All three use only $\varphi^2 = \varphi + 1$ and H_4 Coxeter structure. No free parameters are introduced.

The complete derivation chain is:

$$H_4 \rightarrow H_2 \subset H_4 \rightarrow \text{pentagonal symmetry} \rightarrow \text{prism with } h^2 = \frac{3}{2\varphi} \rightarrow 10 \text{ directions on } S^2 \rightarrow S_{\max} = 4 - \varphi$$

7 Experimental Proposal

The bound $S = 4 - \varphi \approx 2.382$ is directly testable. The 10 measurement directions are specified by Eq. (2) with $h = \sqrt{3/(2\varphi)} \approx 0.9628$.

In a CHSH experiment with entangled spin- $\frac{1}{2}$ particles:

1. Prepare maximally entangled singlet states $|\Psi^-\rangle$.
2. Choose Alice's and Bob's settings from the 10 prism vertices, selecting the quadruple achieving the theoretical maximum.
3. Measure S with sufficient statistics to distinguish $4 - \varphi$ from $2\sqrt{2}$.

The Delft loophole-free Bell test [3] reported $S = 2.38 \pm 0.14$, with a central value close to $4 - \varphi$. A dedicated experiment with pentagonal prism geometry could test whether nature saturates this specific geometric bound.

8 Relation to the Geometric Standard Model

This result is derived within the Geometric Standard Model (GSM) [5], which proposes H_4 Coxeter geometry as the foundation for quantum mechanics and fundamental constants. Within the GSM, $\gamma^2 = (13 - 7\varphi)/4$ constrains quantum correlations via $S = 2\sqrt{1 + \gamma^2}$. The pentagonal prism provides the physical mechanism—the measurement directions that realize the algebraic bound as a concrete configuration on S^2 .

9 Discussion

The result $S_{\max} = 4 - \varphi$ is notable for several reasons.

It is algebraically exact. Unlike numerical optimization over Platonic solids [4], the pentagonal prism bound is a closed-form expression in the golden ratio.

It connects abstract algebra to concrete geometry. The identity $S = 1 + \det(C_{H_2})$ states the Bell bound is “one plus the Cartan determinant of the pentagonal symmetry group.”

It is uniquely determined. The golden-ratio height is the only prism aspect ratio producing this bound, and the prism is selected over the antiprism by H_4 reflection structure.

It is experimentally testable. The 10 measurement directions are explicitly specified.

A literature search confirms that pentagonal prism Bell inequalities have not been previously studied. Existing geometric Bell inequalities [4] focus on Platonic solids, a different geometric family.

10 Conclusion

We have shown that a pentagonal prism inscribed on S^2 with height $h^2 = 3/(2\varphi)$ produces a maximum CHSH parameter of exactly $S = 4 - \varphi$, established through three independent algebraic proofs. The height is determined by the H_3 Gram determinant, the bound equals one plus the H_2 Cartan determinant, and the prism geometry is selected by H_4 reflection group structure. This connects Coxeter group theory to Bell inequality physics and provides explicit measurement directions for experimental verification.

A Numerical Verification

Independent numerical verification was performed by brute-force computation over all 8,100 vertex quadruples:

- 80 of 8,100 quadruples achieve $|S| = 4 - \varphi$ to machine precision ($< 10^{-15}$ relative error).
- No quadruple exceeds $4 - \varphi$.
- The 80 optimal configurations are related by $D_{5h} \times \mathbb{Z}_2$ symmetry.
- Scanning $h^2 \in [0.01, 3.0]$ confirms strict monotonic decrease, with $h^2 = 3/(2\varphi)$ as the unique crossing.

All computations are reproducible via the verification scripts at [6].

B Formal Verification

The following identities were formally verified in Lean 4 by the theorem prover Aristotle:

1. $\det(C_{H_3})/2 + \det(C_{H_4})/4 = (13 - 7\varphi)/4$
2. $(4 - \varphi)^2 = 17 - 7\varphi$
3. $1 + 16(\det(G_{H_3}) - \det(G_{H_4})) = 4 - \varphi$

References

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