Lemma 1. *Given a timestamp list $T = [t_1, t_2, \cdots, t_n]$ ($n \geq 1$) and an integer $t \geq 1$, let $T' \subseteq T$ be a sublist no shorter than t whose frequency is equal to the t-frequency $F_t(T)$, there exists an integer $i \in [1, n-t+1]$ such that $T' = [t_i, t_{i+1}, \cdots, t_{i+|T'|-1}]$.*

Proof. Given a $T' = [t_j, ..., t_k]$ which is a sublist of T, and the elements in T' do not have consecutive subscripts in T, we can calculate that the local frequency derived from T' is $\frac{|T'|}{t_k - t_j + 1}$, however, if we we insert missing elements between t_k and t_j into T, denoted by T'', such that the subscripts of the elements are consecutive in T. Then, the local frequency derived from T'' is $\frac{|T''|}{t_k - t_j + 1}$ which is obviously greater because T'' have more elements. $\square$

Lemma 2. *Given a timestamp list $T = [t_1, t_2, \cdots, t_n]$ ($n \geq 1$) and an integer $t \geq 1$, $F_t(T)$ is the reciprocal of the minimum slope of all t-lines between $p'_i = (i-1, t_i - 1) \in P_{start}$ and $p_j = (j, t_j) \in P_{end}$ with $j \geq i$.*

Proof. Based on Lemma 1, the elements in T' must have continuous subscripts in T, so $F_t(u, v) = \{max(\frac{|T'|}{t_k - t_j + 1}) | |T'| \geq t\} = \{max(\frac{j-i+1}{t_j - t_i + 1}) | j - i + 1 \geq t\} = \{\frac{1}{min(\frac{t_j - (t_i - 1)}{j - (i-1)})} | j - i + 1 \geq t\} = \{\frac{1}{min(\mathrm{slp}(p'_i, p_j))} | j - i + 1 \geq t\}$. $\square$

Lemma 3. *For an end point p_j and an integer t, any start point p'_i with $1 \leq i \leq j - t + 1$ such that $\mathrm{slp}(p'_i, p_j) = \mathrm{mslp}(\cdot, p_j)$ must satisfy that $p'_i \in \mathrm{UCC}_j$.*

Proof. For the p'_j which is not in the upper convex curve and the two points next to it p'_i and p'_k which are in the curve, based on $\mathrm{slp}(p'_i, p'_j) \leq \mathrm{slp}(p'_j, p'_k)$ we consider three cases for the end point p_j. Firstly, if $\mathrm{slp}(p'_j, p_j) \geq \mathrm{slp}(p'_j, p'_k)$, then $\mathrm{slp}(p'_i, p_j) \leq \mathrm{slp}(p'_j, p_j)$. Secondly, if $\mathrm{slp}(p'_i, p'_j) \leq \mathrm{slp}(p'_j, p_j) < \mathrm{slp}(p'_j, p'_k)$, then both $\mathrm{slp}(p'_i, p_j)$ and $\mathrm{slp}(p'_k, p_j)$ are less than $\mathrm{slp}(p'_j, p_j)$. Third, if $\mathrm{slp}(p'_j, p_j) < \mathrm{slp}(p'_i, p'_j)$, then $\mathrm{slp}(p'_k, p_j) < \mathrm{slp}(p'_j, p_j)$. In either case, $\mathrm{slp}(p'_j, p_j)$ is not the minimum. $\square$

Lemma 4. *The slopes of the t-lines from points in UCC_j to endpoint p_j decreases monotonically first and then increases. (In boundary cases, it may be monotonically decreasing or increasing).*

Proof. The slope of p_j to UCC_j can be denoted by $k = \frac{y_j - y}{x_j - x}$, its derivative is $k' = \frac{k - \frac{dy}{dx}}{(x_j - x)^3} = \{\frac{k - \mathrm{slp}(p'_i, p'_{i+1})}{(x_j - x)^3} | 1 \leq i < |T|\}$. Consider three cases, firstly, if the initial k is greater than $\mathrm{slp}(p'_i, p'_{i+1})$, i.e., $\mathrm{slp}(p'_1, p_j) > \mathrm{slp}(p'_1, p'_2)$, then k is monotonically increasing because $\mathrm{slp}(p'_i, p'_{i+1})$ is decreasing which means k' is always greater than 0. In this case, p'_1 is OSP. Secondly, if $\mathrm{slp}(p'_1, p_j) \leq \mathrm{slp}(p'_1, p'_2)$, thus k decreases initially until it reaches a certain point p'_i where $\mathrm{slp}(p'_i, p_j) > \mathrm{slp}(p'_i, p'_{i+1})$, and after that k starts to increase. In this case, p'_i is OSP. Third, if there is no point p'_i such that $\mathrm{slp}(p'_i, p_j) > \mathrm{slp}(p'_i, p'_{i+1})$, then k is monotonically decreasing, and $p'_{|T|}$ is OSP. $\square$

Lemma 5. *Those start points other than p'_{j-t+1} that do not exist in UCC_{j-1} cannot exist in UCC_j.*

Proof. Considering end points p_j and p_{j-1}, since timestamps in T are sorted incrementally, thus p_j is to the upper right of p_{j-1}, which means the proof in Lemma 3 of p_{j-1} can be applied to p_j. $\square$

Lemma 6. *When iterating to a new end point p_j, points before its OSP can not be the start point to obtain the minimum slope, which means that they should be removed from UCC.*

Proof. Suppose that the OSP of p_j is p'_l, and the OSP of a new end point $p_{j'}$ (after p_j) is $p'_{l'}$, while the x-value of $p'_{l'}$ is less than that of p'_l. According to the property of optimal start point and the definition of upper convex curve, there is $\mathrm{slp}(p'_{l'}, p_{j'}) > \mathrm{slp}(p'_{l'}, p'_{l'+1}) > \mathrm{slp}(p'_{l-1}, p'_l) > \mathrm{slp}(p'_l, p_j)$, which means $p'_{l'}$ can not be the start point to obtain the minimum slope. $\square$

Lemma 7. *Given a temporal graph $\mathcal{G}$, $\mathcal{C}^k_{t',f'} \subseteq \mathcal{C}^k_{t,f}$ if $f' \geq f$ and $t' \geq t$.*

Proof. Assume a vertex $u \in \mathcal{C}^k_{t',f'}$, we have $|\mathcal{N}_{t,f}(u)| \geq |\mathcal{N}_{t',f'}(u)| \geq k$ if $f' \geq f$ and $t' \geq t$. Thus, u must also be present in $\mathcal{C}^k_{t,f}$, namely, $\mathcal{C}^k_{t,f}$ contains all vertices in $\mathcal{C}^k_{t',f'}$. $\square$

Lemma 8. *Given a vertex v and an integer k, we have* $\mathrm{cf}(v, k, t) \geq \mathrm{cf}(v, k, t')$ *if $t < t'$.*

Proof. We prove by contradiction. Supposing that $t < t'$ and there exists a vertex v satisfying $\mathrm{cf}(v, k, t) < \mathrm{cf}(v, k, t')$. We have $v \in \mathcal{C}^k_{t',\mathrm{cf}(v,k,t')}$ and $v \notin \mathcal{C}^k_{t,\mathrm{cf}(v,k,t')}$ since $\mathrm{cf}(v, k, t) < \mathrm{cf}(v, k, t')$. However, $\mathcal{C}^k_{t',\mathrm{cf}(v,k,t')} \subseteq \mathcal{C}^k_{t,\mathrm{cf}(v,k,t')}$ due to $t < t'$ which is a contradiction. $\square$

Lemma 9. *Given k and t, let $N^k_t(u)$ be the set of neighbors of a vertex u such that for each $v \in N^k_t(u)$ we have* $\mathrm{cf}(v, k, t) \geq \mathrm{cf}(u, k, t-1)$ *and* $F_t(u, v) \geq \mathrm{cf}(u, k, t-1)$. *Consequently,* $\mathrm{cf}(u, k, t) < \mathrm{cf}(u, k, t-1)$ *if and only if* $|N^k_t(u)| < k$.

Proof. ($\Rightarrow$) We prove by contradiction, suppose $\mathrm{cf}(u, k, t) < \mathrm{cf}(u, k, t-1)$ and $|N^k_t(u)| \geq k$, then u have more than k neighbors v that satisfying $\mathrm{cf}(v, k, t) \geq \mathrm{cf}(u, k, t-1)$ and $F_t(u, v) \geq \mathrm{cf}(u, k, t-1)$, which means $u \in \mathcal{C}^k_{t,\mathrm{cf}(u,k,t-1)}$. Based on the definition of core frequency, we have $\mathrm{cf}(u, k, t-1) \leq \mathrm{cf}(u, k, t)$, which contradicts to the assumption.

($\Leftarrow$) We also prove by contradiction, suppose that $|N^k_t(u)| < k$ and $\mathrm{cf}(u, k, t) \geq \mathrm{cf}(u, k, t-1)$. Firstly, $u \in \mathcal{C}^k_{t,\mathrm{cf}(u,k,t-1)}$ because $\mathrm{cf}(u, k, t) \geq \mathrm{cf}(u, k, t-1)$. However, vertex u have less than k neighbors who have more than k $(t, \mathrm{cf}(u, k, t-1))$-neighbors due to $|N^k_t(u)| < k$, which is a contradiction. $\square$

Lemma 10. *Given t and k,* $\mathrm{cf}(u, k, t)$ *equals to the maximum float $f \leq 1$ such that the number of neighbor vertices (denoted by v) of u, which satisfy* $\mathrm{cf}(v, k, t) \geq f$ *and $F_t(u, v) \geq f$, is no less than k.*

Proof. We consider the value of f' in two cases. First, if $f' > f$, then u does not exist in $\mathcal{C}^k_{t,f'}$ due to the maximality of f, so

$\mathrm{cf}(u,k,t) \neq f'$. Second, if $f' < f$, Since it does not satisfy the maximality in the definition of core frequency, $\mathrm{cf}(u,k,t) \neq f'$. □

Lemma 11. *For any skyline (k,t,f)-core, f is a core frequency of at least one vertex, and thus is preserved in the CF-Index.*

Proof. Assuming there is a (k,t,f)-core, whose f is not preserved in the CF-Index, then there must exist a vertex v that satisfies $\mathrm{cf}(v,k,t) > f$, so $(k,t,\mathrm{cf}(v,k,t))$-core dominates (k,t,f)-core. □

Lemma 12. *For any skyline (k,t,f)-core, we have $t = t'-1$ for a pair (t',f') preserved in the partition of k.*

Proof. For two consecutive entries in the partition of k and v, (t_1,f_1) and (t_2,f_2), since for any $t_1 \leq t < t_2-1$, core frequency satisfies $\mathrm{cf}(v,k,t) = \mathrm{cf}(v,k,t_2-1) = f_1$, so (k,t_2-1,f_1)-core dominates any other (k,t,f_1)-core, which means $t = t_2-1, t' = t_2, f = f_1, f' = f_2$ in the Lemma. □

Lemma 13. *Let $Skyline_k$ contain all pairs of (t,f). If a triple (k_1,t_1,f_1) is dominated by another (k_2,t_2,f_2), it can only be one of the two cases: 1) $k_1 = k_2, t_1 < t_2, f_1 = f_2$. 2) $k_1 < k_2, t_1 = t_2, f_1 = f_2$.*

Proof. Consider the following cases, and we will prove in turn the reasons why they should not be taken into account. 1) $k_1 = k_2, t_1 = t_2, f_1 \leq f_2$ 2) $k_1 < k_2, t_1 = t_2, f_1 < f_2$ 3)$k_1 < k_2, t_1 < t_2, f_1 \leq f_2$ For the first case, since during the traversal of CF-index to obtain the candidate skyline tuples in the algorithm, for fixed k and t, only the largest f is retained, therefore case 1 does not exist. For the second case, assuming this situation exists, then there must exist a point u that satisfies $\mathrm{cf}(u,k_2,t_2) = f_2$. Due to the monotonicity of core frequency, we have $\mathrm{cf}(u,k_1,t_1) \geq f_2$, which contradicts the assumptions of case 2. For the third case, we similarly assume that it holds, then there must exist a point u that satisfies $\mathrm{cf}(u,k_2,t_2) = f_2$ and $\mathrm{cf}(u,k_1,t_2) = f_3$ in which $f_3 \geq f_2$. So there exists a tuple (k_1,t_4,f_4) in the candidate skyline tuples that satisfies $t_4 \geq t_2 > t_1, f_4 \geq f_3 \geq f_1$, which means (k_1,t_1,f_1) have been removed in the first situation of the algorithm. □