# Algorithmic Graph Theory – Chapter1

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# 1.1 Graphs and Digraphs

# Basic Definition

- A graph $G = (V, E)$ is an ordered pair of finite sets.
- Elements of $V$ are called vertices or nodes.
- Elements of $E \subseteq V^{(2)}$ are called edges or arcs.
- $|V|$ is called the order of G.
- $|E|$ is called the size of G.

# Basic Definition

- $v_1$ and $v_2$ are adjacent vertices if $(v_1, v_2) \in E$ is an edge of a graph $G = (V, E)$.
- $(v_1, v_2)$ or $v_1 v_2$ is incident with the vertices $v_1$ and $v_2$.

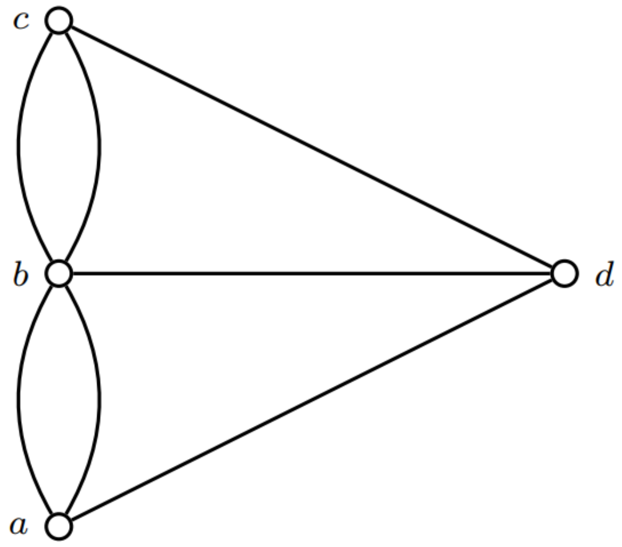



Figure 1.1: The seven bridges of Königsberg puzzle.

# Digraph

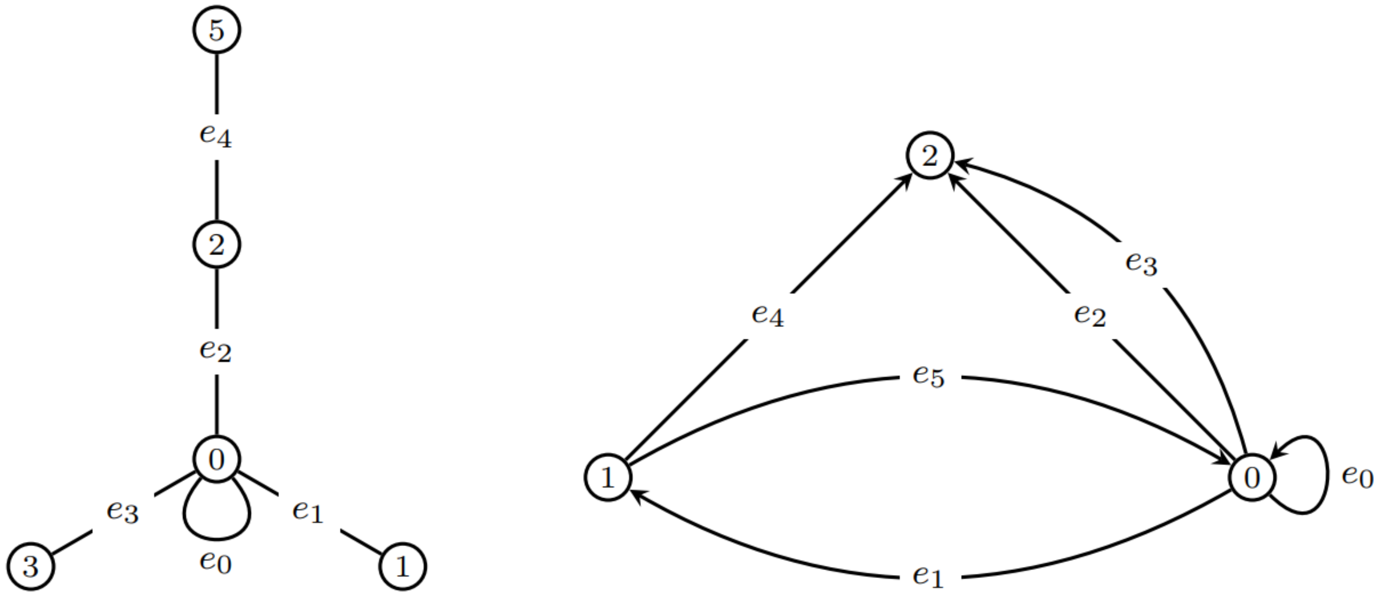
- A directed graph or digraph G is a graph each of whose edges is directed.
- directed edge $uv$: tail vertex $u$ -> head vertex $v$
- The set of edges is a subset of the ordered pairs $V \times V$.

- indegree id($v$): the number of edges such that $v$ is the head.
- outdegree od($v$): the number of edges such that $v$ is the tail.
- degree deg($v$): id($v$) + od($v$), the number of edges incident to that vertex (self-loop counts 2)

- $G$ is undirected => $uv = vu$
- G is directed => $uv \neq vu$

# Multigraphs

- There are multiple edges between a pair of vertices.




(a) (b)

Figure 1.5: A graph $G$ and digraph $H$ with a loop and multi-edges.

# Weighted Graph

- Has a real number called the weight.
- vertex: $(v, w_v)$, vertex weight $w_v \in \mathbb{R}$
- edge: $(w_e, u, v)$, edge weight $w_e \in \mathbb{R}$

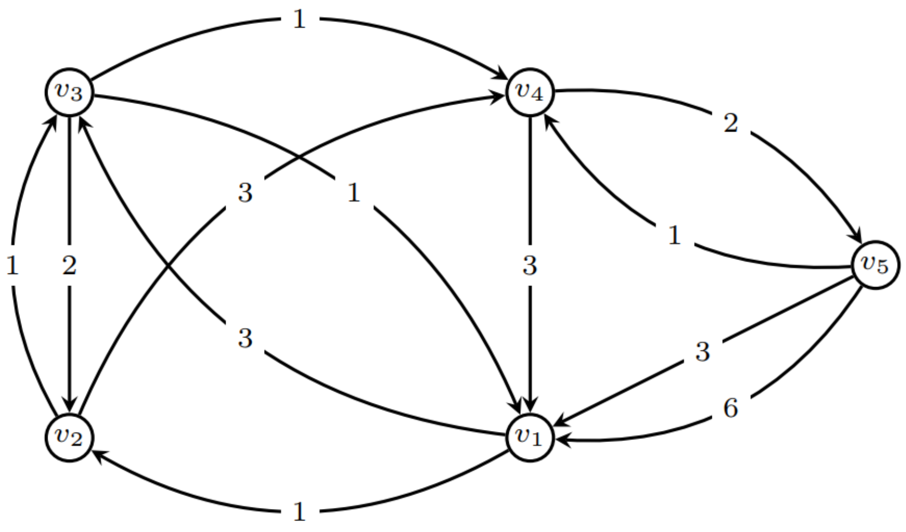



Figure 1.6: An example of a weighted multigraph.

# Weighted Graph

- weighted indegree: the sum of the weights of edges going into $v$.
- weighted outdegree: the sum of the weights of edges going out of $v$.
- weighted degree: weighted indegree + weighted outdegree, the sum of the weights of edges incident to that vertex

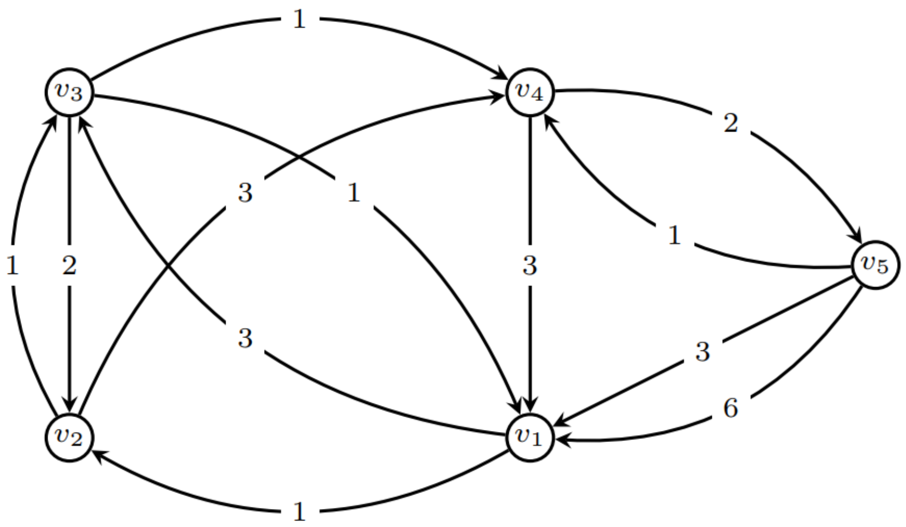
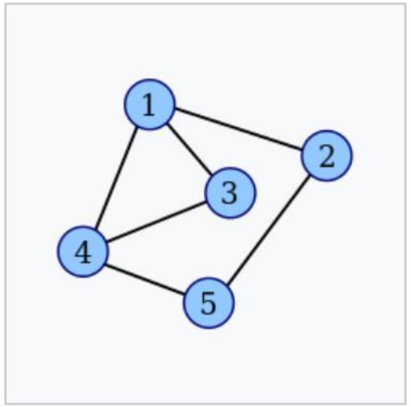


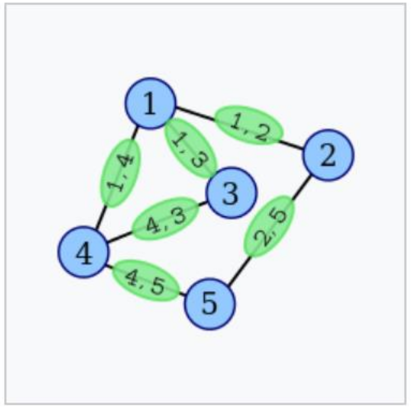

Figure 1.6: An example of a weighted multigraph.

# Line Graph

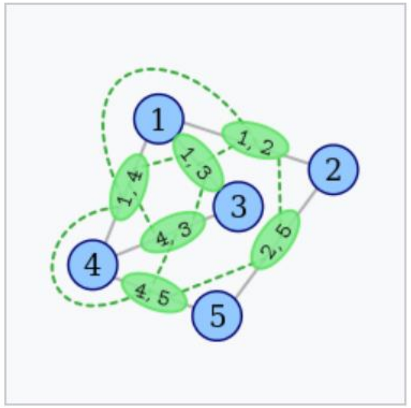
- The line graph $\mathcal{L}(G)$ is the multidigraph whose vertices are the edges of $G$ and whose edges are the edges are $(e, e')$ where $h(e) = t(e')$ (for $e, e' \in E$).
- $h(e)$: the head of the edge $e$
- $t(e)$: the tail of the edge $e$




Graph G




Vertices in L(G) constructed from edges in G

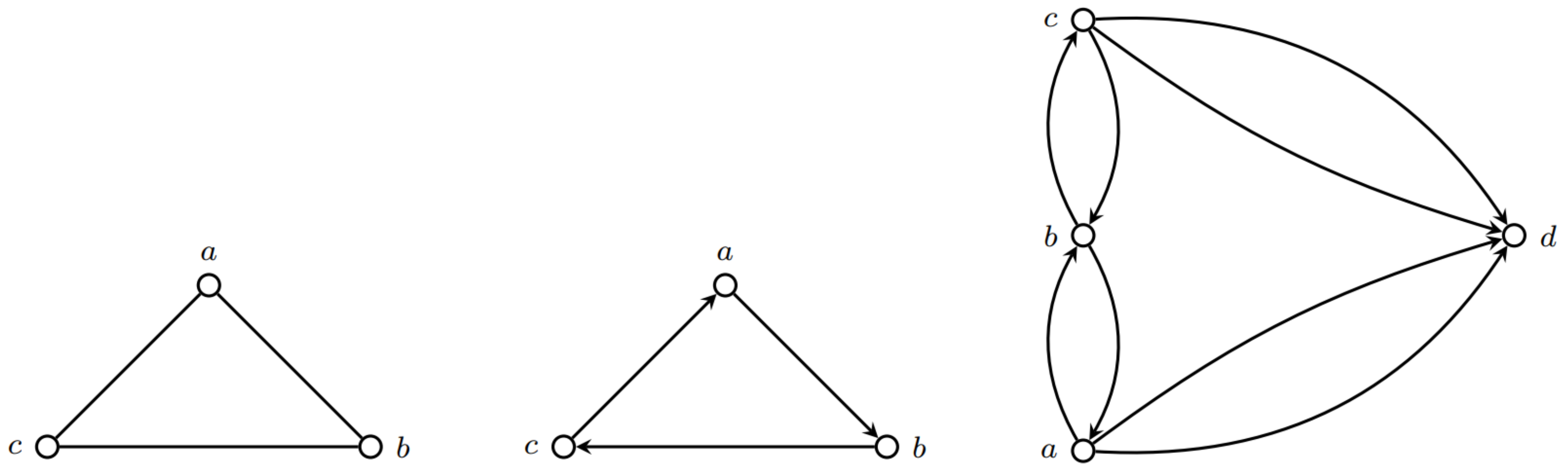



Added edges in L(G)


The line graph L(G)

# Simple Graphs

- no self-loops and no multiple edges




(a) Simple graph. (b) Digraph. (c) Multidigraph.

Figure 1.7: A simple graph, its digraph version, and a multidigraph.

# Simple Graphs

- $\deg(v) = |\mathrm{adj}(v)|$
- isolated vertex: $\deg(v) = 0$
- pendant vertex: $\deg(v) = 1$

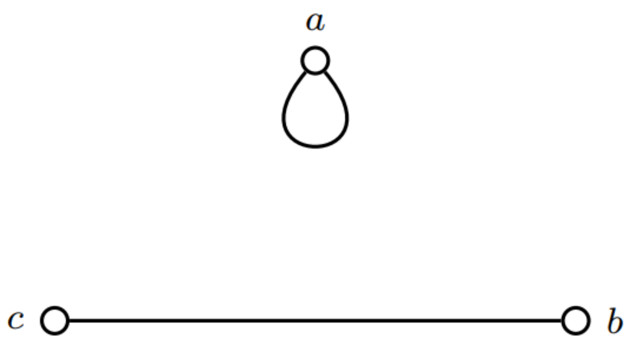



Figure 1.4: A figure with a self-loop.

# Simple Graphs

- minimum degree $\delta(G)$
- maximum degree $\Delta(G)$
- $r$-regular graph is a regular graph ($\delta(G) = \Delta(G)$) each of whose vertices has degree $r$.

- $\sum_{v \in V} \deg(v) = 2|E|$ (i.e., nonnegative and even)
- It's because each edge $e = v_1 v_2$ is incident with two vertices, so $e$ is counted twice.

- If $G$ is an $r$-regular graph having $n$ vertices and $m$ edges, then $m = rn/2$.

# Simple Graphs

- A graph $G$ contains an even number of vertices with odd degrees.
- $V_e$: even degree vertices, $V_o$: odd degree vertices

That is, $V = V_e \cup V_o$ and $V_e \cap V_o = \emptyset$. From Theorem 1.9, we have

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_e} \deg(v) + \sum_{v \in V_o} \deg(v) = 2|E|$$

which can be re-arranged as

$$\sum_{v \in V_o} \deg(v) = \sum_{v \in V} \deg(v) - \sum_{v \in V_e} \deg(v).$$

As $\sum_{v \in V} \deg(v)$ and $\sum_{v \in V_e} \deg(v)$ are both even, their difference is also even.

# 1.2 Subgraphs and Other Graph Types

# Definition

- $H$ is subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

# Walk

- A $u$-$v$ walk is an alternating sequence of vertices and edges starting with $u$ and ending at $v$.
- Consecutive vertices and edges are incident.
- A walk $W : v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n$ where each edge $e_i = v_{i-1}v_i$ and the length $n$ refers to the number of (not necessarily distinct) edges in the walk.
- If the graph has no multiple edges, then usually a walk $W : v_0, v_1, v_2, \dots, v_{n-1}, v_n$.
- Allowed to have repeated vertices and edges in a walk.

# Walk

- e.g., $a$-$e$ walk: $a, b, c, b, e$ (length 4)

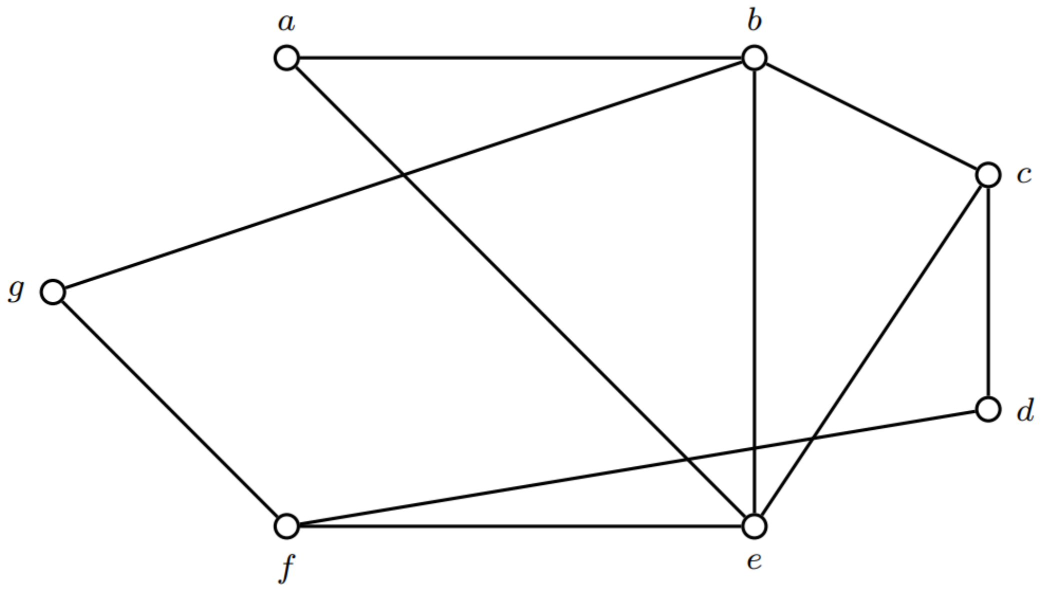



Figure 1.9: Walking along a graph.

# Trail

- A walk with no repeating edges.
- e.g., $a$-$b$ walk: $a, b, c, d, f, g, b$ (no repeated edges, but one repeated vertex)

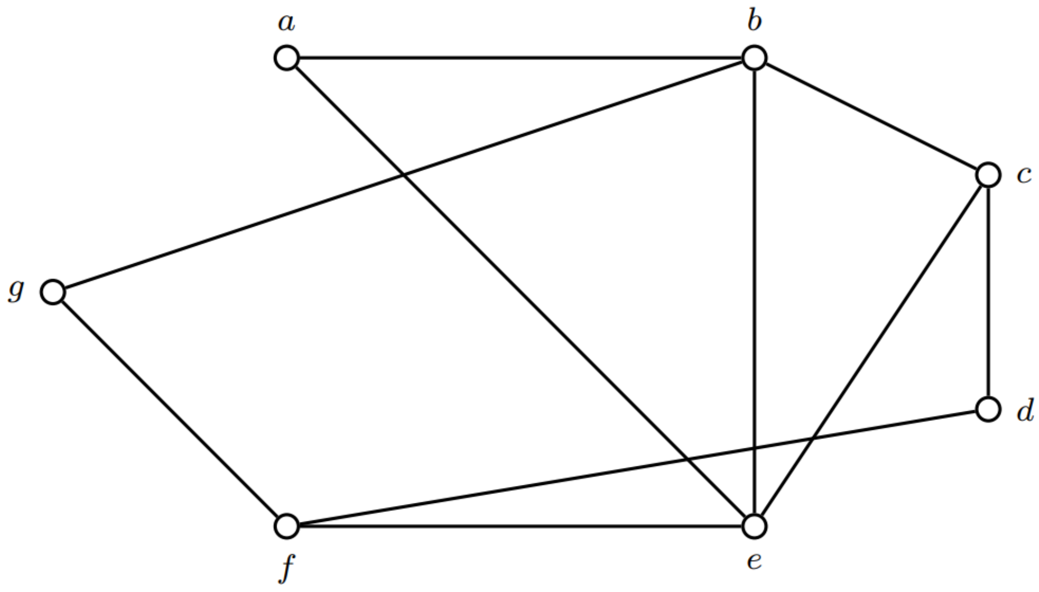



Figure 1.9: Walking along a graph.

# Path

- A walk with no repeating vertices except for possible the first and last.
- It is also a trail (no repeating vertices => no repeating edges).

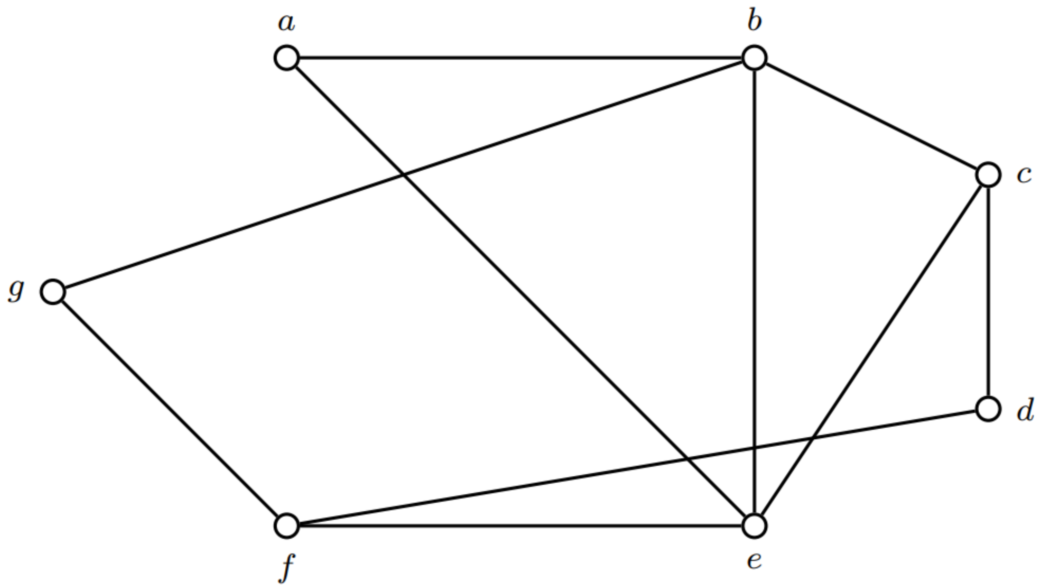



Figure 1.9: Walking along a graph.

# Closed

- closed walk, closed trail, closed path or cycle: start and end vertices are the same.
- e.g., closed path: $a, b, c, e, a$

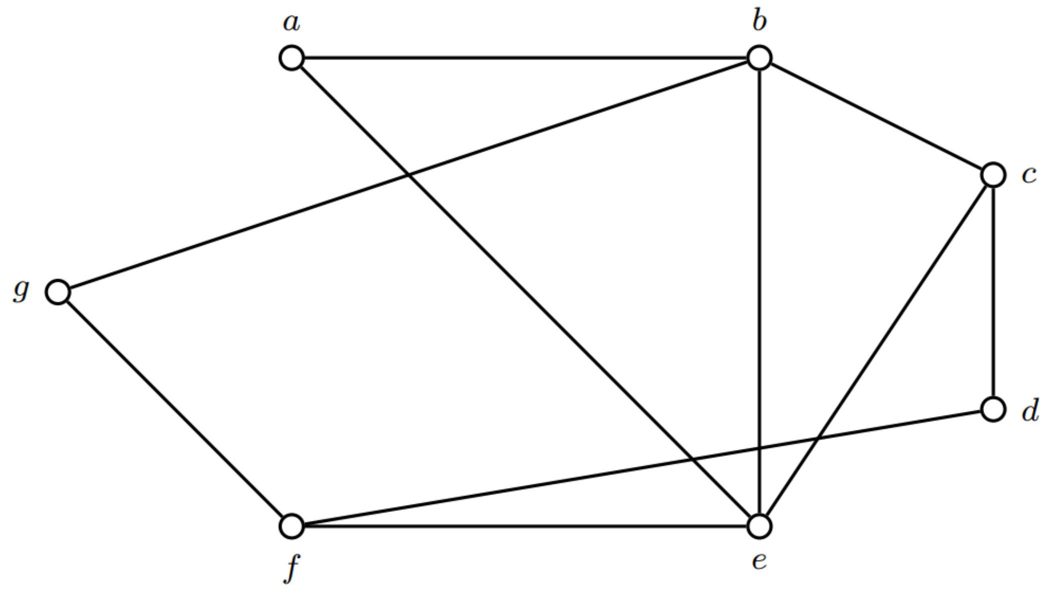



Figure 1.9: Walking along a graph.

# Features

- Let $G$ be a simple (di)graph of order $n = |V|$. Any path in $G$ has length at most $n - 1$.
- The length of the shortest cycle in a graph is called the girth.
- Every $u$-$v$ walk in a graph contains a $u$-$v$ path.

# Connectivity

- A graph is said to be connected if for every pair of distinct vertices $u, v$ there is a $u$-$v$ path joining them.
- geodesic path or shortest path: $u$-$v$ path of minimum length

- component: connected subgraph such that it isn't proper subgraph of any connected subgraph, which is maximal connected subgraph.
- Any connected graph is its own component.
- The number of connected components $\omega(G)$.

- Suppose that exactly two vertices of a graph have odd degree. Then those two vertices are connected by a path.

# Tree

- Has some form of hierarchy.
- Undirected graph that is connected and has no cycles.

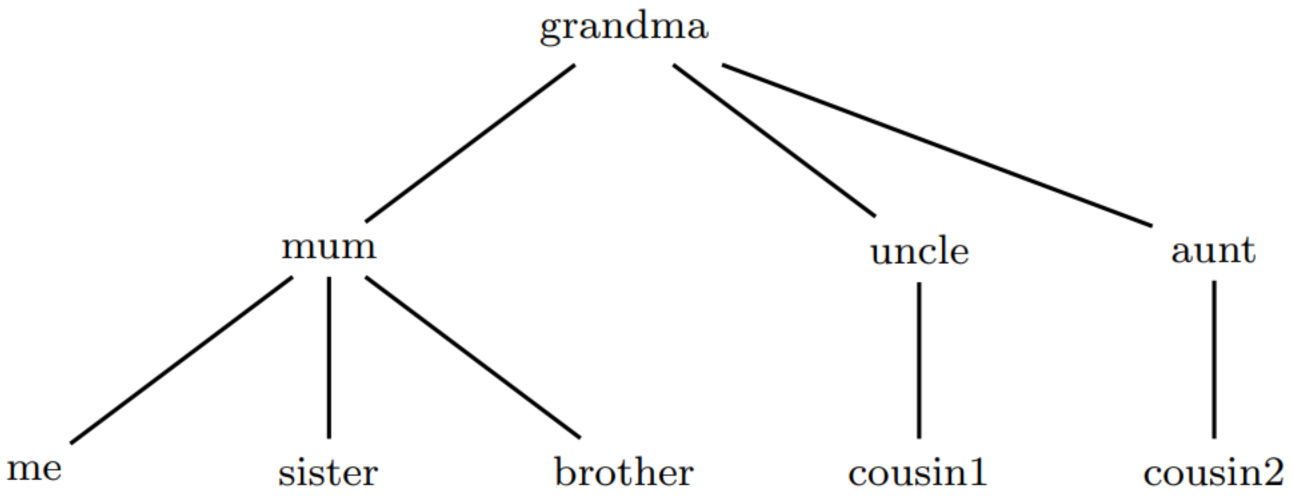
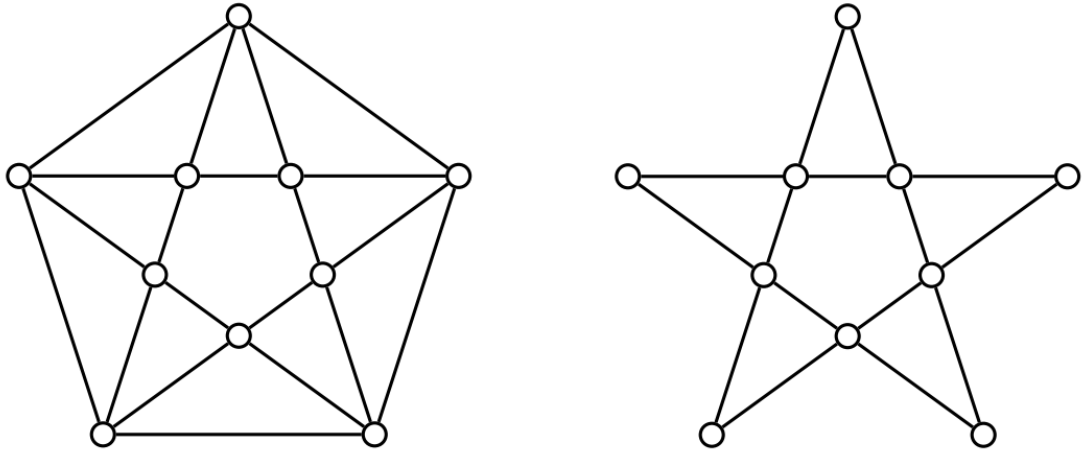



Figure 1.11: A family tree.

# Spanning Subgraph

- $H$ is a subgraph of $G$ and $V(H) = V(G)$.
- To obtain a spanning subgraph from a given graph, delete edges from the graph.

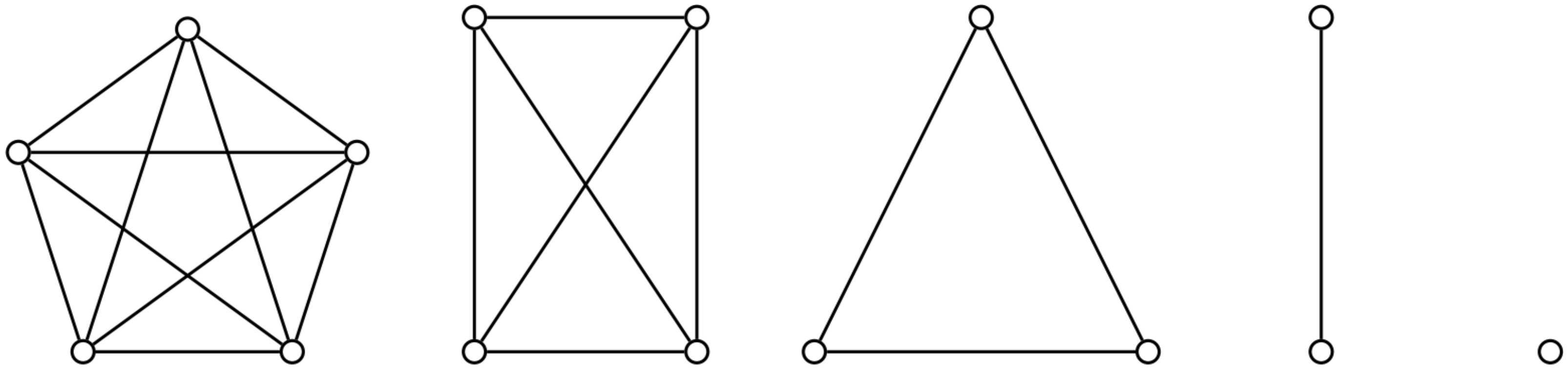


(a) (b)

Figure 1.12: A graph and one of its subgraphs.

# Complete Graph

- complete graph $K_n$: any two distinct vertices are adjacent.
- $|V(K_n)| = n$, $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$
- For any graph $G$ with $n$ vertices, $|E(G)| \leq \frac{n(n-1)}{2}$

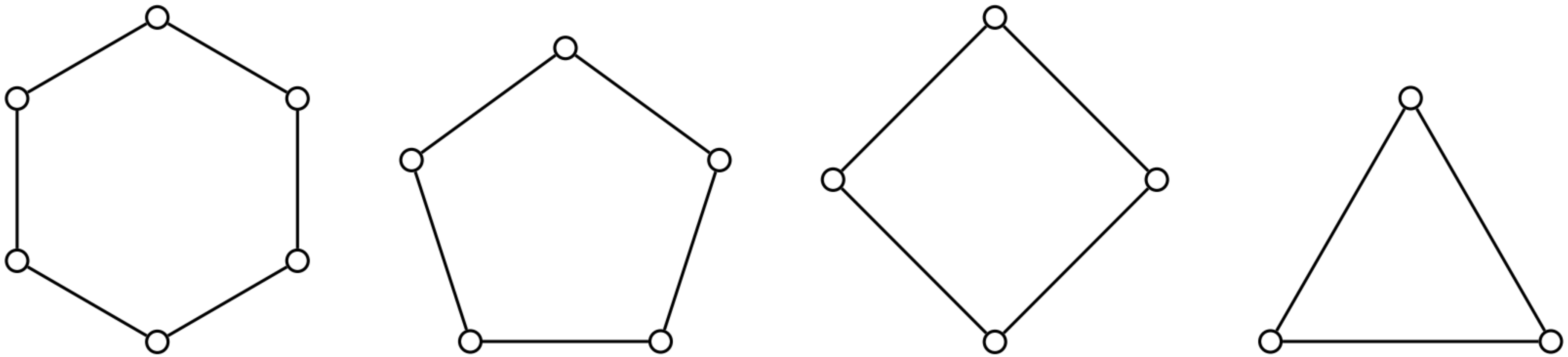


(a) $K_5$ (b) $K_4$ (c) $K_3$ (d) $K_2$ (e) $K_1$

Figure 1.13: Complete graphs $K_n$ for $1 \leq n \leq 5$.

# Cycle Graph

- The cycle graph $C_n$ on $n \geq 3$ vertices is the connected 2-regular graph.



(a) $C_6$ (b) $C_5$ (c) $C_4$ (d) $C_3$

Figure 1.14: Cycle graphs $C_n$ for $3 \leq n \leq 6$.

# Path Graph

- The path graph $P_n$ on $n \geq 3$ vertices is a spanning subgraph of $C_n$ obtained by deleting one edge.
- Otherwise, $P_1 = K_1$, $P_2 = K_2$.

# Bipartite Graph

- A graph with at least two vertices such that $V(G)$ can be split into two disjoint subsets $V_1$ and $V_2$, both nonempty.
- Every edge $uv \in E(G)$ is such that $u \in V_1$ and $v \in V_2$, or $v \in V_1$ and $u \in V_2$.
- A graph is bipartite if and only if it has no odd cycles.

# Bipartite Graph

- complete bipartite graph $K_{m,n}$: vertex set portioned into two nonempty disjoint sets $V_1$ and $V_2$ with $|V_1| = m$ and $|V_2| = n$.
- Any vertex in $V_1$ is adjacent to each vertex in $V_2$.
- Where $m = 1$ then $K_{1,n}$ is called the star graph.

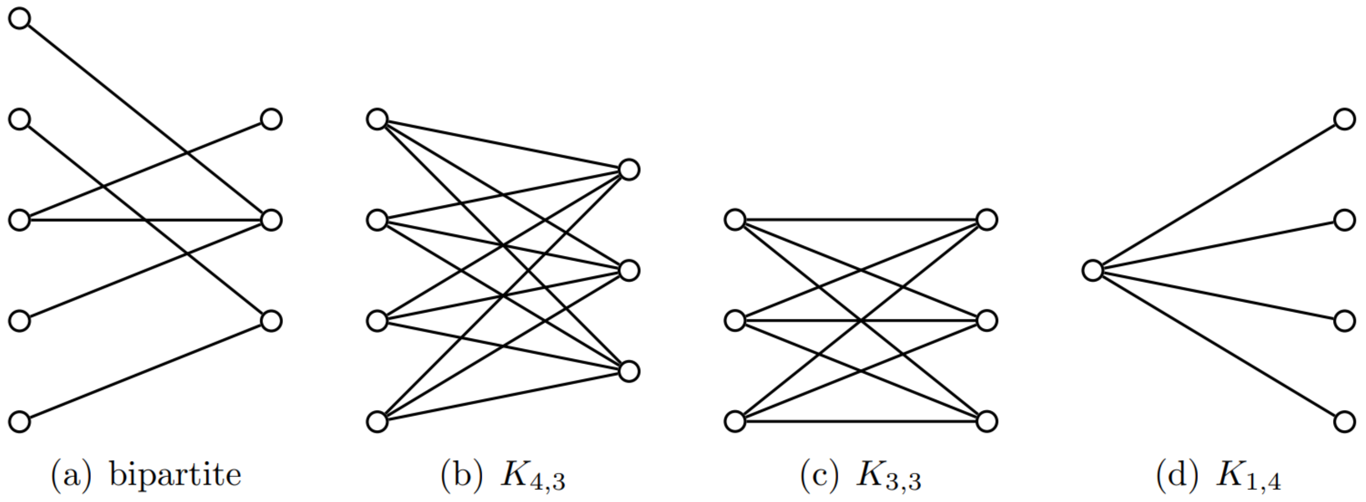



Figure 1.18: Bipartite, complete bipartite, and star graphs.

# 1.3 Representing Graphs as Matrices

# Matrix

An $m \times n$ matrix $A$ can be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \hdotsfor{4} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The positive integers $m$ and $n$ are the row and column dimensions of $A$, respectively. The entry in row $i$ column $j$ is denoted $a_{ij}$. Where the dimensions of $A$ are clear from context, $A$ is also written as $A = [a_{ij}]$.

# Adjacency Matrix

Let $G$ be an undirected graph with vertices $V = \{v_1, \ldots, v_n\}$ and edge set $E$. The *adjacency matrix* of $G$ is the $n \times n$ matrix $A = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E, \\ 0, & \text{otherwise.} \end{cases}$$

The adjacency matrix of $G$ is also written as $A(G)$. As $G$ is an undirected graph, then $A$ is a symmetric matrix. That is, $A$ is a square matrix such that $a_{ij} = a_{ji}$.

# Adjacency Matrix

Now let $G$ be a directed graph with vertices $V = \{v_1, \ldots, v_n\}$ and edge set $E$. The $(0, -1, 1)$-*adjacency matrix* of $G$ is the $n \times n$ matrix $A = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E, \\ -1, & \text{if } v_j v_i \in E, \\ 0, & \text{otherwise.} \end{cases}$$

# Adjacency Matrix

More generally, if $G$ is an undirected multigraph with edge $e_{ij} = v_iv_j$ having multiplicity $w_{ij}$, or a weighted graph with edge $e_{ij} = v_iv_j$ having weight $w_{ij}$, then we can define the (weighted) *adjacency matrix* $A = [a_{ij}]$ by

$$a_{ij} = \begin{cases} w_{ij}, & \text{if } v_iv_j \in E, \\ 0, & \text{otherwise.} \end{cases}$$

# Bipartite Case

- The matrix $A'$ has the form:

$$A' = \begin{bmatrix} \mathbf{0} & B \\ B^T & \mathbf{0} \end{bmatrix}$$

- where $\mathbf{0}$ is a zero matrix, $B$ is a reduced adjacency matrix.

# Tanner Graph

- bipartite graph
- $m$ rows: $V_1$
- $n$ columns: $V_2$
- In fact, it's the reduced adjacency matrix.

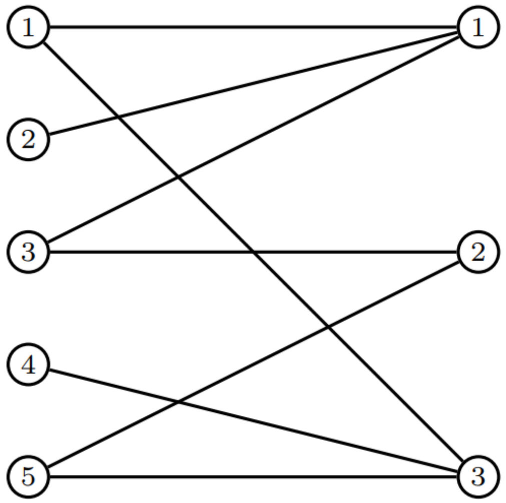



Figure 1.20: A Tanner graph.

# Incidence Matrix

Let $G$ be a digraph with edge set $E = \{e_1, \ldots, e_m\}$ and vertex set $V = \{v_1, \ldots, v_n\}$. The *incidence matrix* of $G$ is the $n \times m$ matrix $B = [b_{ij}]$ defined by

$$b_{ij} = \begin{cases} -1, & \text{if } v_i \text{ is the tail of } e_j, \\ 1, & \text{if } v_i \text{ is the head of } e_j, \\ 2, & \text{if } e_j \text{ is a self-loop at } v_i, \\ 0, & \text{otherwise.} \end{cases} \tag{1.10}$$

Each column of $B$ corresponds to an edge and each row corresponds to a vertex. The definition of incidence matrix of a digraph as contained in expression (1.10) is applicable to digraphs with self-loops as well as multidigraphs.

# Incidence Matrix

**Theorem 1.26.** *The incidence matrix of an undirected graph $G$ is related to the adjacency matrix of its line graph $L(G)$ by the following theorem:*

$$A(L(G)) = D(G)^T D(G) - 2I_n \ ,$$

*where $A(L(G))$ is the adjacency matrix of the line graph of $G$.*

# Laplacian Matrix

The *degree matrix* of a graph $G = (V, E)$ is an $n \times n$ diagonal matrix $D$ whose $i$-th diagonal entry is the degree of the $i$-th vertex in $V$. The *Laplacian matrix* $\mathcal{L}$ of $G$ is the difference between the degree matrix and the adjacency matrix:

$$\mathcal{L} = D - A.$$

In other words, for an undirected unweighted simple graph, $\mathcal{L} = [\ell_{ij}]$ is given by

$$\ell_{ij} = \begin{cases} -1, & \text{if } i \neq j \text{ and } v_i v_j \in E, \\ d_i, & \text{if } i = j, \\ 0, & \text{otherwise}, \end{cases}$$

where $d_i = \deg(v_i)$ is the degree of vertex $v_i$.

# Distance Matrix

- The $n \times n$ matrix $[d(v_i, v_j)]$ is the distance matrix of G.
- distance (or geodesic distance) $d(v, w)$: the number of edges in a shortest path connecting them
- It’s important to understand the “connectivity”.

# 1.4 Isomorphic Graphs

# Definition

- Structurally equivalent (fundamentally same).
- If $G$ and $H$ are isomorphic, $G \cong H$.
- Two graphs $G$ and $H$ are isomorphic if there is a bijection $f : V(G) \to V(H)$ such that whenever $uv \in E(G)$ then $f(u)f(v) \in E(H)$.
- If $u$ and $v$ are adjacent in $G$, then their counterparts in $V(H)$ are also adjacent in $H$.

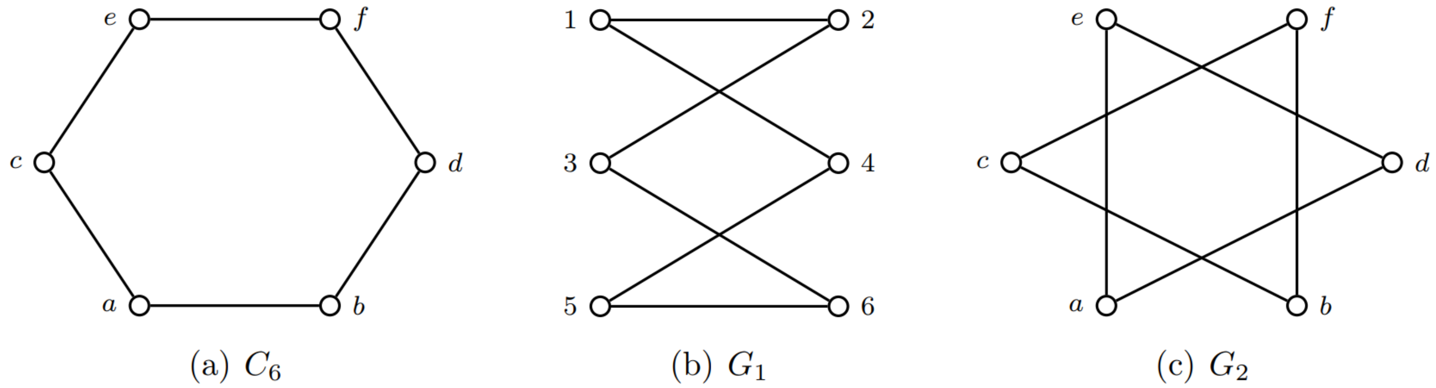



Figure 1.24: Isomorphic and nonisomorphic graphs.

# Features

- order and size equal
- $\deg(v) = \deg\big(f(v)\big)$ for all $v \in G$
- Adjacencies are preserved
- Shortest paths are preserved, if $v_1, v_2, \ldots, v_k$ is a shortest path, then $f(v_1), f(v_2), \ldots f(v_k)$ is also a shortest path.

# Lexicographical Ordering

- Like the way words are arranged in a dictionary (alphabetically).
- e.g., “apple” < “banana”, [1, 2, 3] < [2, 1, 1]

- In the case of matrix, $A_1 < A_2$ if the list of entries of $A_1$ is less than or equal to the list of entries of $A_2$ in the lexicographical ordering.
- The list of entries is obtained by concatenating the entries of the matrix, row-by-row.
- For example:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} < \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

# Permutation Equivalent

- If there is a permutation matrix $P$ such that $A_1 = PA_2P^{-1}$.
- $A_1$ is the same as $A_2$ after a suitable re-ordering of the rows and columns.

- Graphs $G_1$ and $G_2$ are isomorphic if and only if $A_1$ is permutation equivalent to $A_2$.
- canonical label: the lexicographically maximal element of the permutation equivalence class of the adjacency matrix of $G$
- To check if isomorphic, simply check if their canonical labels are equal.

# Permutation Equivalent

Algorithm 1.1: Computing graph isomorphism using canonical labels.

**Input**: Two undirected simple graphs $G_1$ and $G_2$, each having $n$ vertices.
**Output**: True if $G_1 \cong G_2$; False otherwise.

```
for i ← 1, 2 do
    A_i ← adjacency matrix of G_i
    p_i ← permutation equivalence class of A_i
    A'_i ← lexicographically maximal element of p_i
if A'_1 = A'_2 then
    return True
return False
```

# References


- [1] Algorithmic Graph Theory
- [2] Line graph - Wikipedia