Comment on Snizkho et *al.*, Phys. Rev. Resarch 4, 023179 (2021)

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In a recent paper published in Phys. Rev. Research by Snizkho et *al.* [1], the authors explore two timely concepts at the forefront of quantum mechanics. Namely, quantum weak measurements and topological phase transitions. In particular, they address the measurement-induced geometric phase which emerges following an infinite sequence of weak measurements. Moreover, they outline a general framework for treating measurement-induced phase factors, and apply it to a broad class of measurements. Nevertheless, we argue that the general framework proposed by the authors strongly relies on a continuous limit approximation which does not hold in the limiting cases of infinite measurement strength, and infinite measurement back-action, as explored by the authors.

More specific, for the continuous measurement limit, an analytic expression for the measurement-induced geometric phase is obtained by first considering the explicit form of the Kraus decomposition, given by:

$$\langle\psi_0|M^{(0)}_{N-1}...M^{(0)}_1|\psi_0\rangle = \langle\uparrow|\delta R(M^{(0)}\delta R)^{N-1}|\uparrow\rangle, \quad (1)$$

where $|\psi_0\rangle$ is the initial state, δR is a matrix independent of the measurement index $k = 1, .., N-1$, and $M^{(0)}$ is a 2×2 diagonal Kraus operator, whose diagonal elements are $(1, \mathrm{Exp}(-2\frac{C+iA}{N}) + O(\frac{1}{N^2}))$ [1]. The calculation of the matrix power $(M^{(0)}\delta R)^{N-1}$ is performed by expressing the product $(M^{(0)}\delta R)$ in its diagonal basis, in the form VDV^{-1}, where $D = (I + \frac{\Lambda}{N})^N$, and V is the corresponding matrix for the change of basis.

For the continuous-measurement limit ($N\to\infty$), the authors invoke the approximation $(I + \frac{\Lambda}{N})^N \approx e^{\Lambda}$, which is of course only valid if Λ is finite, as it can be easily demonstrated following L'Hopitals rule. Doing some basic matrix manipulation, it can be shown that for the specific Kraus operators considered in [1], Λ matrix takes the form:

$$\Lambda = \begin{bmatrix} -C - i(\pi + A) - \tau & 0 \\ 0 & -C - i(\pi + A) + \tau \end{bmatrix}, \quad (2)$$

where C is the measurement strength, A is the non-Hermitian component of the measurement backaction [1], and τ is a function of C, A, and the polar angle θ, of the form $\tau = \sqrt{Z^2 - \pi^2\sin\theta^2}$, with $Z = (C + iA + i\pi\cos\theta)$. From these derivations, the authors obtain an analytic expression for the geometric phase, of the form:

$$\sqrt{P}e^{\chi_{\rm geom}} = e^{i(\cos\theta 1)Z}(\cosh\tau + Z\sinh\tau/\tau), \quad (3)$$

recovering the analytic expression in [2]. For ease of notation we consider a fixed measurement orientation $d = +$. In what follows, the authors explore the limiting cases $A\to\infty$ and $C\to\infty$, by expanding the logarithmic of Eq. (3), at large A and large C [1].

For the case of $A\to\infty$, they obtain :

$$\chi_{\rm geom} = \pi(\cos\theta - 1) + \frac{\pi^2\sin\theta^2}{2A} \quad (4)$$

$$-\frac{\pi^2\sin\theta^2}{4A^2}[e^{2C}\sin 2A + 2\pi\cos\theta - 2\pi\cos\theta] + O(A^{-3}) \quad (5)$$

while for the case of $C\to\infty$ they obtain:

$$\chi_{\rm geom} = \pi(\cos\theta - 1) + \frac{\pi^2\sin\theta^2}{2C^2}[A + \pi\cos\theta] + O(C^{-3}). \quad (6)$$

Note that for the case of $C\to\infty$ and $A\to\infty$ the geometric phase diverges (Eq. 6).Nevertheless, it should noted that due to the linear dependence of Λ on C and A (Eq. (2)), it turns out that $\Lambda\to\infty$, when $C\to\infty$ or $A\to\infty$. Therefore, in such limiting cases where Λ is unbounded, the approximation $(I + \frac{\Lambda}{N})^N \approx e^{\Lambda}$ does not hold, thus Eq. (3) cannot be used.

REFERENCES

[1] Kyrylo Snizhko, Nihal Rao, Parveen Kumar, and Yuval Gefen, *Weak-measurement-induced phases and dephasing: Broken symmetry of the geometric phase*, Phys. Rev. Research **3**, 043045 (2021).

[2] Valentin Gebhart, Kyrylo Snizhko, Thomas Wellens, Andreas Buchleitner, Alessandro Romito, and Yuval Gefen, *Topological transition in measurement-induced geometric phases*, PNAS **117**, 5706 (2020).