Comment on Snizkho *et al.*, Phys. Rev. Resarch 4, 023179 (2021)

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In a recent paper Snizkho *et al.* [Phys. Rev. Research **4**, 02179 (2021)] outline a general framework for treating measurement-induced phase factors, and apply it to a broad class of measurements. We argue that such general framework relies on a continuous-limit approximation which does not hold in the limiting cases of infinite measurement strength and infinite measurement backaction, as explored by the authors. In addition, we argue that a detailed experimental scheme, which could justify the origin and independence of the measurement parameters C and A, is not properly presented. Furthermore, the claim of topological phase transition for the case of non-Hermitian Kraus operators appears in direct contradiction with a previous paper published by the same authors [2].

In a recent paper published in Phys. Rev. Research by Snizkho et *al.* [1], the authors explore two timely concepts at the forefront of quantum mechanics. Namely, quantum weak measurements and topological phase transitions. In particular, they address the measurement-induced geometric phase which emerges following an infinite sequence of weak measurements. Moreover, they outline a general framework for treating measurement-induced phase factors, and apply it to a broad class of measurements. Nevertheless, we argue that the general framework proposed by the authors strongly relies on a continuous limit approximation which does not hold in the limiting cases of infinite measurement strength, and infinite measurement backaction, as explored by the authors. In addition, we argue that a detailed experimental scheme, which could justify the origin and independence of the measurement parameters C and A, is not properly presented. Furthermore, the claim of topological phase transition for the case of non-Hermitian Kraus operators appears in direct contradiction with a previous paper published by the same authors [2].

More specific, for the continuous measurement limit, an analytic expression for the measurement-induced geometric phase is obtained by first considering the explicit form of the Kraus decomposition, given by:

$$\langle\psi_0|M_{N-1}^{(0)}...M_1^{(0)}|\psi_0\rangle = \langle\uparrow|\delta R(M^{(0)}\delta R)^{N-1}|\uparrow\rangle, \quad (1)$$

where $|\psi_0\rangle$ is the initial state, δR is a matrix independent of the measurement index $k = 1, .., N-1$, and $M^{(0)}$ is a 2×2 diagonal Kraus operator, whose diagonal elements are $(1, \mathrm{Exp}(-2\frac{C+iA}{N}) + O(\frac{1}{N^2}))$ [1]. The calculation of the matrix power $(M^{(0)}\delta R)^{N-1}$ is performed by expressing the product $(M^{(0)}\delta R)$ in its diagonal basis, in the form VDV^{-1}, where $D = (I + \frac{\Lambda}{N})^N$, and V is the corresponding matrix for the change of basis.

For the continuous-measurement limit ($N \to \infty$), the authors invoke the approximation $(I + \frac{\Lambda}{N})^N \approx e^{\Lambda}$, which is of course only valid if Λ is finite, as it can be easily demonstrated following L'Hopitals rule [3]. Doing some basic matrix manipulation, it can be shown that for the specific Kraus operators considered in [1], the Λ matrix takes the form:

$$\Lambda = \begin{bmatrix} -C - i(\pi + A) - \tau & 0 \\ 0 & -C - i(\pi + A) + \tau \end{bmatrix}, \quad (2)$$

where C is the measurement strength, A is the non-Hermitian component of the measurement backaction [1], and τ is a function of C, A, and the polar angle θ, of the form $\tau = \sqrt{Z^2 - \pi^2 \sin\theta^2}$, with $Z = (C + iA + i\pi\cos\theta)$. From these derivations, the authors obtain an analytic expression for the geometric phase, of the form:

$$\sqrt{P}e^{\chi_{\mathrm{geom}}} = e^{i\pi(\cos\theta - 1) - Z}(\cosh\tau + Z\sinh\tau/\tau), \quad (3)$$

recovering the analytic expression reported in [2] for $A = 0$. For ease of notation we consider a fixed measurement orientation $d = +$.

In what follows, the authors explore the limiting cases $A \to \infty$ and $C \to \infty$, by expanding the logarithmic of Eq. (3), at infinite A and infinite C [1].

For the case of $A \to \infty$, they obtain :

$$\chi_{\mathrm{geom}} = \pi(\cos\theta - 1) + \frac{\pi^2\sin\theta^2}{2A} \quad (4)$$

$$-\frac{\pi^2\sin\theta^2}{4A^2}[e^{2C}\sin(2A + 2\pi\cos\theta) - 2\pi\cos\theta] + O(A^{-3}) \quad (5)$$

while for the case of $C \to \infty$ they obtain:

$$\chi_{\mathrm{geom}} = \pi(\cos\theta - 1) + \frac{\pi^2\sin\theta^2}{2C^2}[A + \pi\cos\theta] + O(C^{-3}). \quad (6)$$

Nevertheless, it should noted that due to the linear dependence of Λ on C and A (Eq. (2)), it follows that

$\Lambda \to \infty$, when $C \to \infty$ or $A \to \infty$. Therefore, in such limiting cases where Λ diverges, the approximation $(I + \frac{\Lambda}{N})^N \approx e^\Lambda$ does not hold, thus Eq. (3) cannot be used.

The topological nature of the phase transition studied by the authors strongly relies on the Hermiticity of the Kraus operators involved in the measurement process itself. In particular, for the case of photons transmitted by a sequence of imperfect polarizers, as analyzed in Ref. [2], it is clearly stated:

"The polarizers must be carefully designed such that no additional phase difference between the two polarizations is accumulated by the light passing through a polarizer. This is particularly important because the topological nature of the transition investigated is protected by the hermiticity of the Kraus operators M_k"

Thus, it appears that there is a fundamental contradiction between the claims of topological phase transition for the case of non-Hermitian Kraus operators in Ref. [1], and the restrictions previously imposed in Ref. [2], by the same authors. This contradiction is not ellaborated by Snizkho *et al.* in any detail.

In addition, Snizkho *et al.* [1] bring forth a mysterious dynamical phase factor arising during the measurement process. In particular they claim to demonstrate:

"We demonstrate that weak-measurement-induced phases generically involve both geometrical and dynamical components"

Nevertheless, we argue that it is only realistic to take such dynamical phase factors into account, as considered for instance in Ref. [2] where the authors readily introduce a dynamical phase factor $e^{i\gamma}$, as trivially stated in [2]:

"In both setups, we assume an extra phase difference, $e^{i\gamma}$ produced by means other than measurements."

Note that such dynamical phase factor is fundamental in order to manipulate the visibility of the Mach-Zehnder interferomenter, and be able to detect any phase difference at all. Furthermore, in order to properly distinguish geometric from dynamical components in the acquired phase, the authors should have also analyzed the Berry connection and the Berry curvature, for which the dynamical component of the phase would simply vanish.

Finally, not only the authors do not explain the apparent contradictions with Ref. [2] regarding the topological nature of the phase transition, they do not even mention explicitly the actual experimental device that could be modeled by the non-Hermitian Kraus operator proposed in their paper. For the case of photons, an imperfect elliptical polarizer could in principle lead to the non-Hermitian Kraus operator proposed in Ref. [1]. Nevertheless, in this case the elliptic component (A) cannot be manipulated independently of the linear component (C). In particular, the limit $C \to \infty$, corresponding to a perfect polarizer and a projective measurement, can only be achieved if $A = 0$.

REFERENCES

[1] Kyrylo Snizhko, Nihal Rao, Parveen Kumar, and Yuval Gefen, *Weak-measurement-induced phases and dephasing: Broken symmetry of the geometric phase*, Phys. Rev. Research **3**, 043045 (2021).

[2] Valentin Gebhart, Kyrylo Snizhko, Thomas Wellens, Andreas Buchleitner, Alessandro Romito, and Yuval Gefen, *Topological transition in measurement-induced geometric phases*, PNAS **117**, 5706 (2020).

[3] Visit our GitHub repository for an annotated derivation of L'Hopitals rule `https://github.com/grapts/geometric-phases`