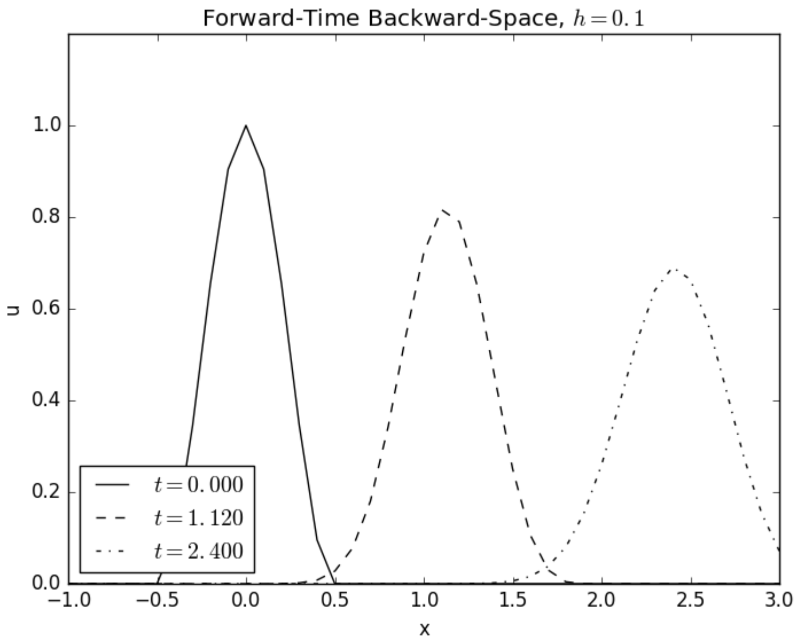
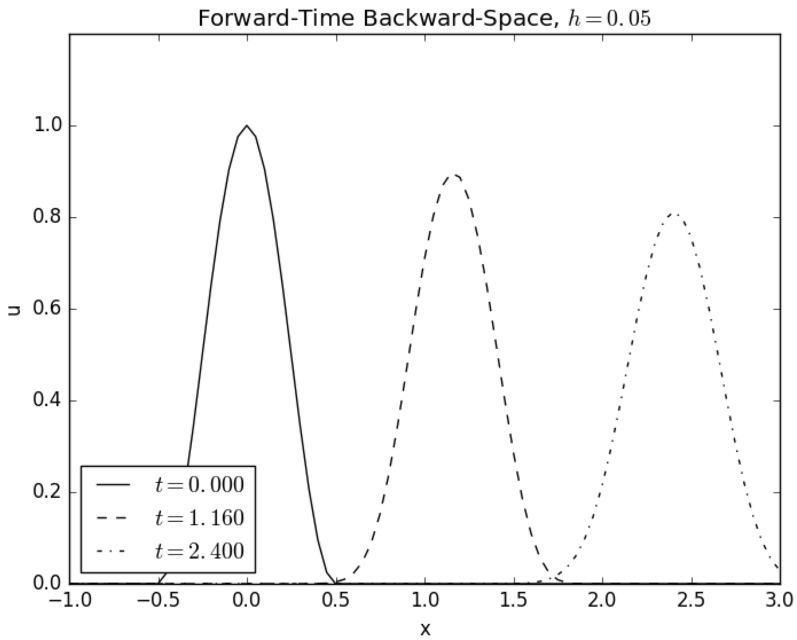
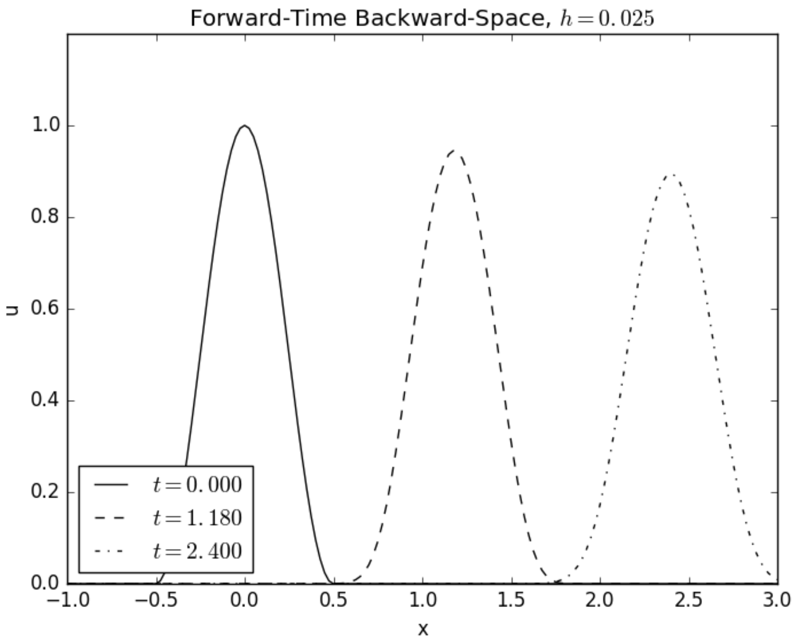
1 C1: Finite Difference Methods

1.1 Forward-Time Backward-Space

As can be seen in the following plots, for the coarse grid ($h = 0.1$), the wave loses nearly half its amplitude by the end of the simulation ($t = 2.4$). The case of $h = 0.1$ is not a good approximation. As h approaches zero, the approximation approaches the exact solution–namely, the wave at $t = 2.4$ appears to approach the shape of the original wave translated by $at = 2.4$. The forward-time backward-space scheme provides a good approximation, provided that h and k are taken sufficiently small.

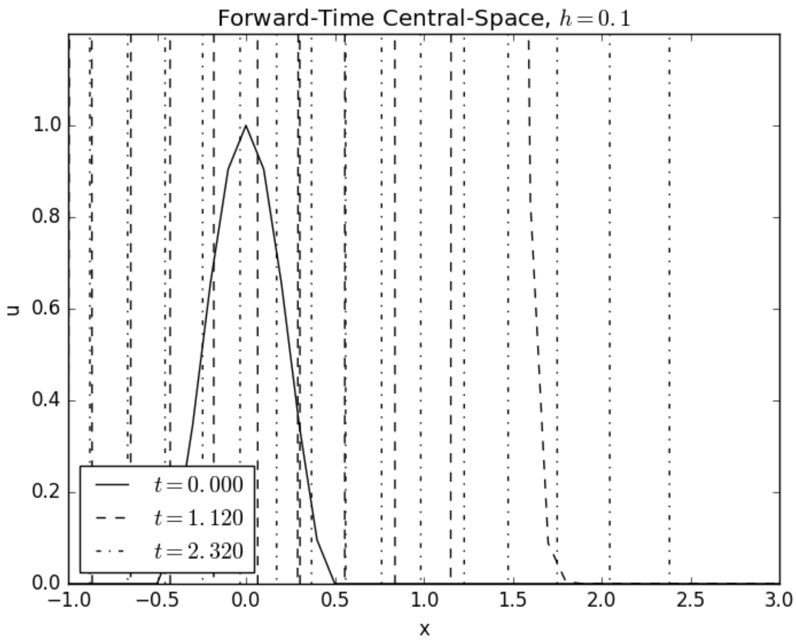

Forward-Time Backward-Space, $h = 0.1$
u
x
$t = 0.000$
$t = 1.120$
$t = 2.400$

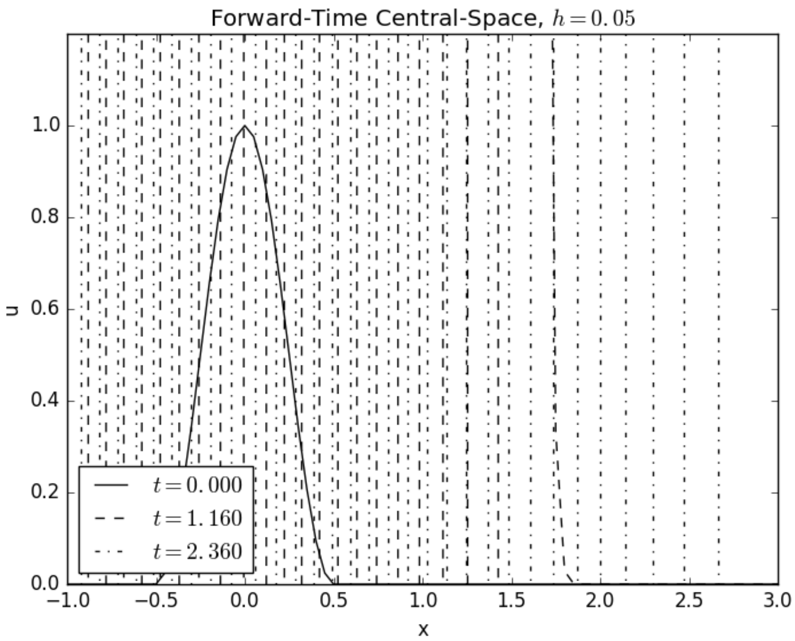

Forward-Time Backward-Space, $h = 0.05$
u
x
$t = 0.000$
$t = 1.160$
$t = 2.400$

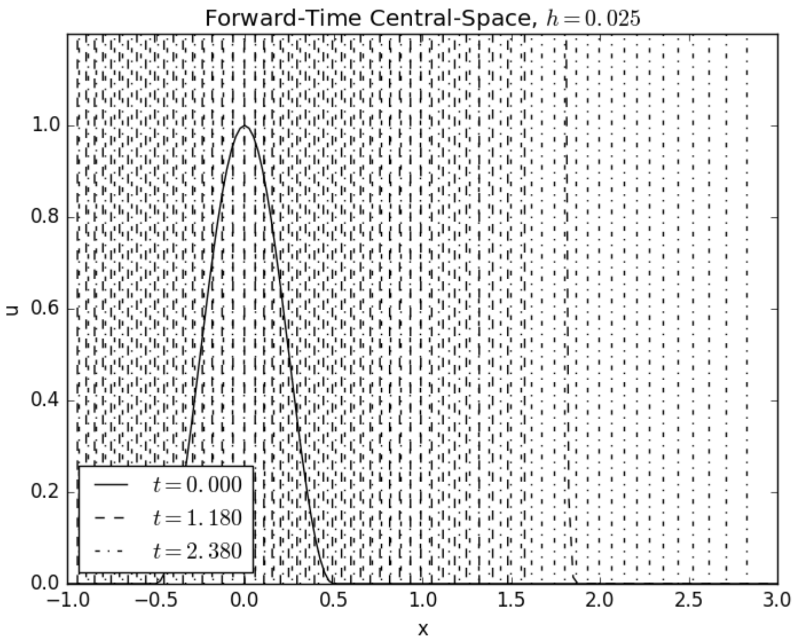


1.2 Forward-Time Central-Space

The results for the forward-time central-space scheme are crazy and unstable.

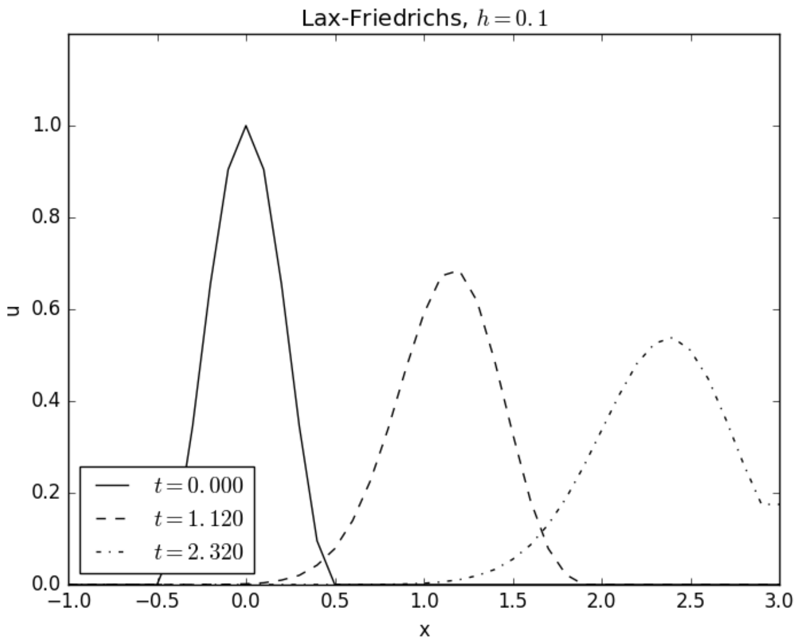

Forward-Time Central-Space, $h = 0.1$
u
x
$t = 0.000$
$t = 1.120$
$t = 2.320$

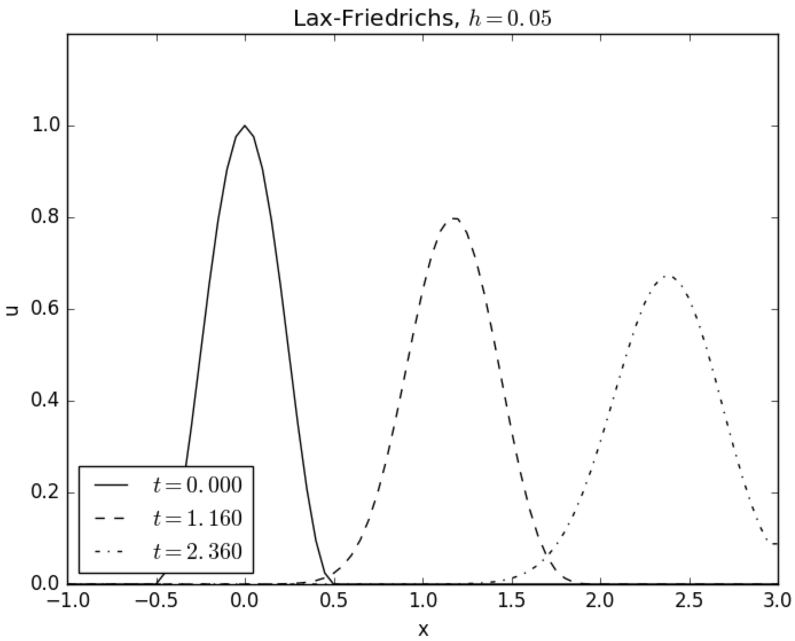

Forward-Time Central-Space, $h = 0.05$
u
x
$t = 0.000$
$t = 1.160$
$t = 2.360$

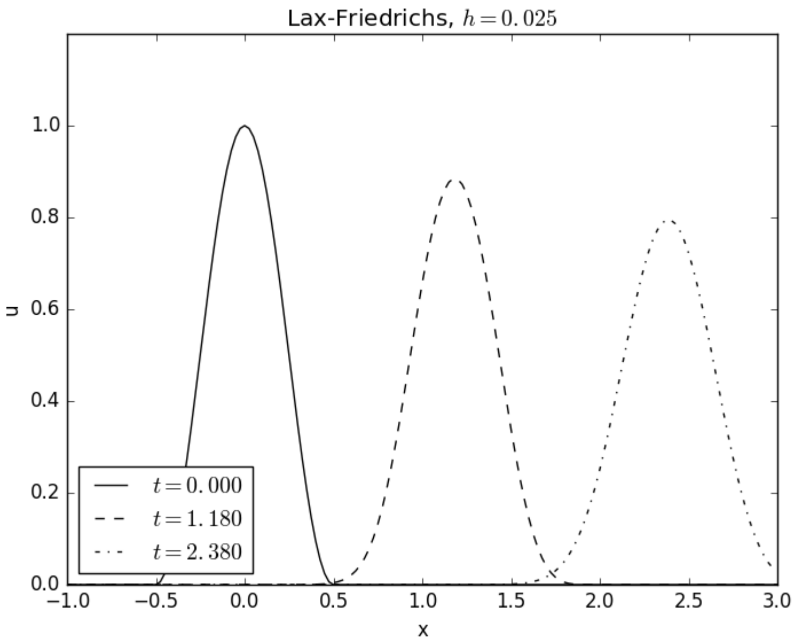


1.3 Lax-Friedrichs

In comparison to the forward-time backward-space and forward-time central-space schemes, the Lax-Friedrichs scheme appears to perform worse in terms of both accuracy of the approximation and capturing the physics of the problem. For the cases of $h = 0.1$ and $h = 0.05$, the wave amplitude has dropped appreciably and the wave has spread out in a non-physical way. Only at $h = 0.025$ did the approximation appear to be sufficient. Note, however, that the Lax-Friedrichs scheme with $h = 0.025$ was less accurate than FTBS and FTCS.

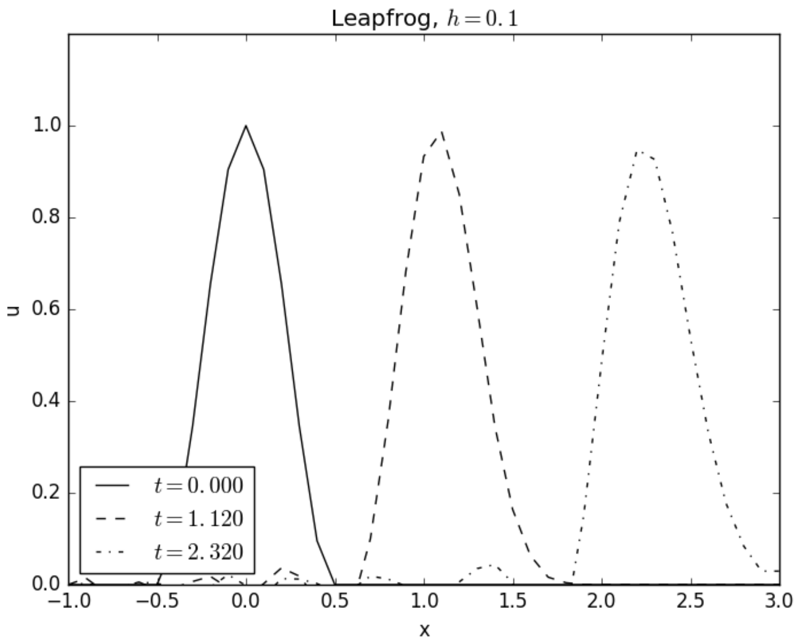

Lax-Friedrichs, $h = 0.1$
u
x
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$t = 0.000$
$t = 1.120$
$t = 2.320$

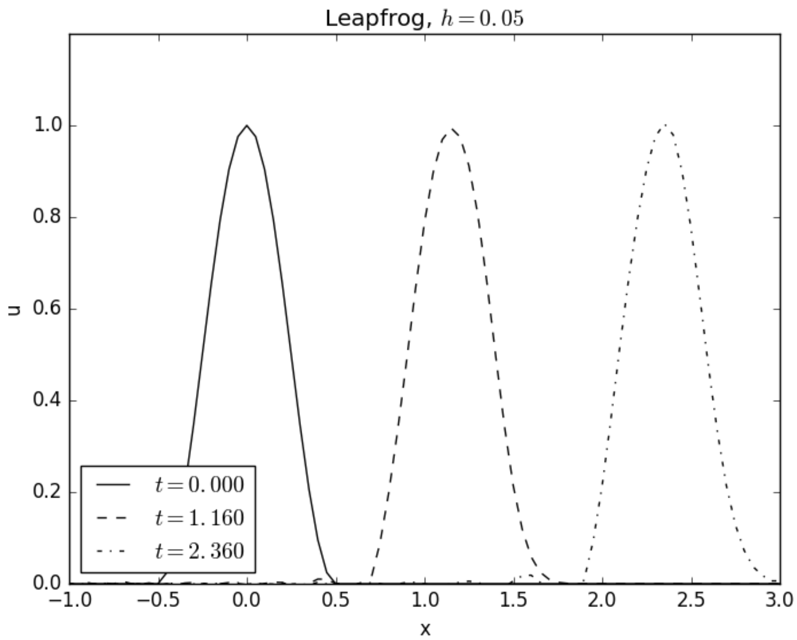

Lax-Friedrichs, $h = 0.05$
u
x
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$t = 0.000$
$t = 1.160$
$t = 2.360$

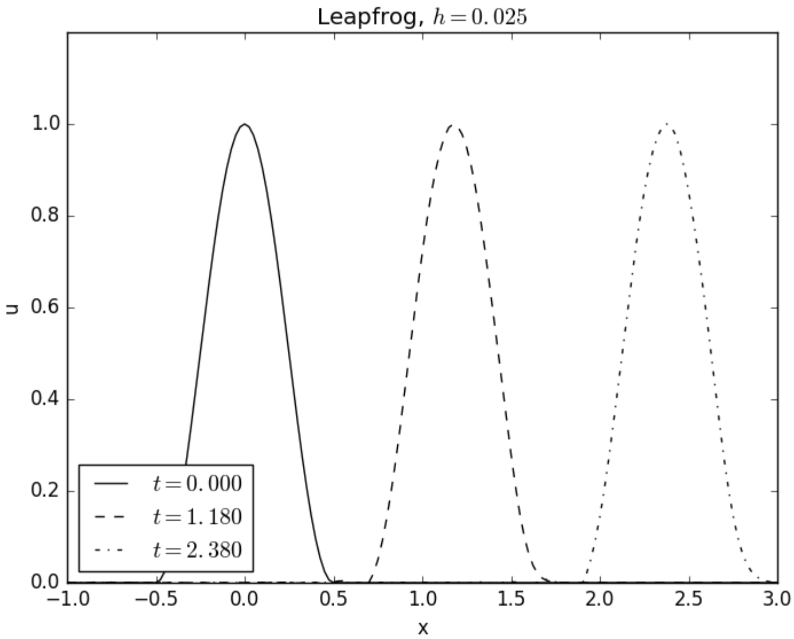


1.4 Leapfrog

The leapfrog scheme did in a loss of amplitude or spreading of the initial wave, even on the coarse grid ($h = 0.1$). In all cases, it provided a reasonable approximation. However, for the case of $h = 0.1$, there is a trail of non-physical of oscillations behind the wave at $t = 2.4$; and the wave is starting to spread toward the front of the wave. The trail of oscillations are negligible at $h = 0.025$ and the overall approximation is very good.

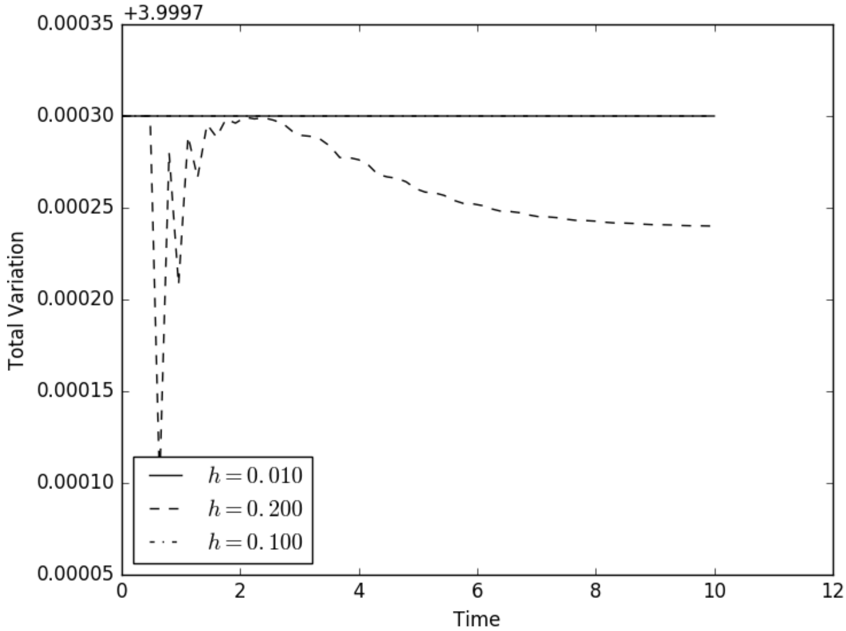

Leapfrog, $h = 0.1$
u
x
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$t = 0.000$
$t = 1.120$
$t = 2.320$


Leapfrog, $h = 0.05$
u
x
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$t = 0.000$
$t = 1.160$
$t = 2.360$

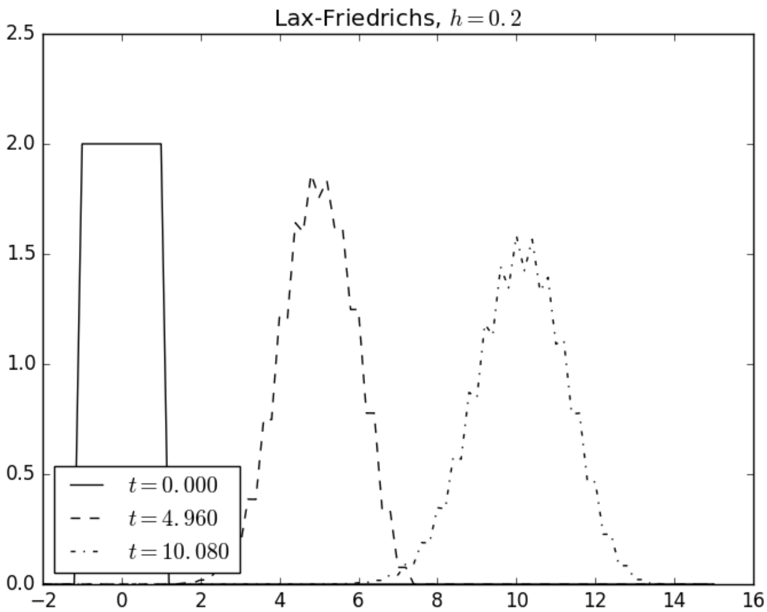


2 C2: Lax-Friedrichs

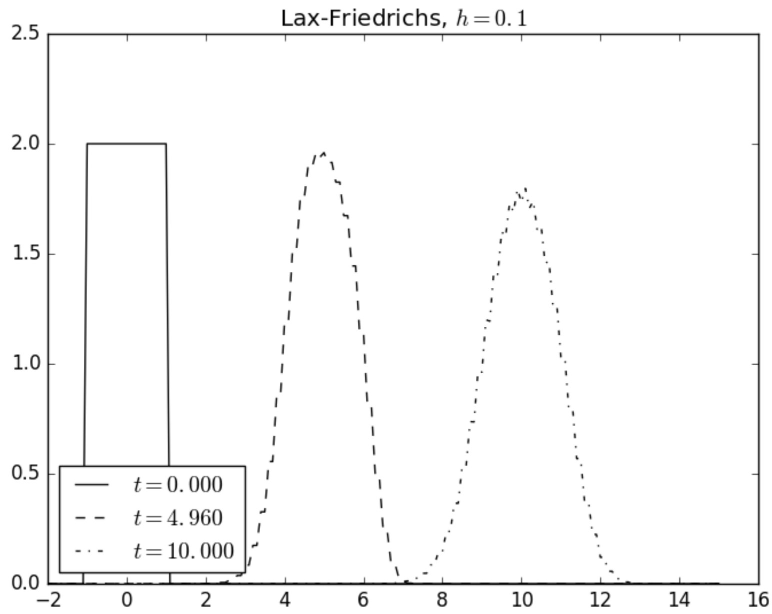
The figure below shows the total variation in time for $h = \frac{1}{5}, \frac{1}{10}$ and $h = \frac{1}{100}$. As h shrinks ($h = \frac{1}{10}, \frac{1}{100}$), the total variation tends the hold steady at 4.0. For $h = \frac{1}{5}$, it can be seen that the total variation varies in time, and does not hold steady at 4.0.

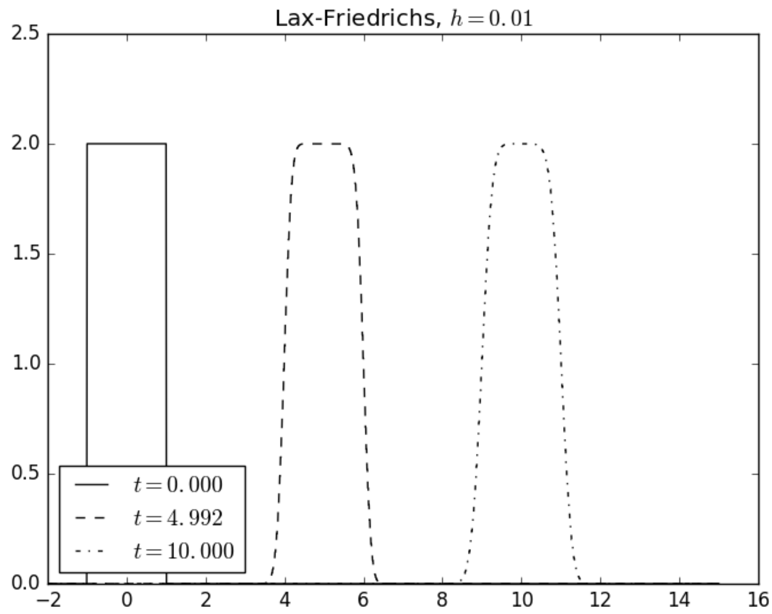


+3.9997
Total Variation
Time
$h=0.010$
$h=0.200$
$h=0.100$



Lax-Friedrichs, $h=0.2$
$t=0.000$
$t=4.960$
$t=10.080$





3 Source Code

```
using PyPlot;

u0(x) = if abs(x) <= 0.5;
  (cos(pi * x))^2;
else
  0;
end;

const x0 = -1.0;
const x1 = 3.0;

const t0 = 0.0;
const t1 = 2.4;

const lambda = 0.8;

for h in [1./10.; 1./20.; 1./40.]

  println("Forward-Time Backward-Space, h = $h");

  xs = collect(x0:h:(x1+eps()));
  k = h * lambda;
  ts = collect(t0:k:(t1+eps()));
  us = zeros(length(xs), length(ts));
  us[:, 1] = map(u0, xs); # initial conditions

  for n in 1:length(ts)-1, m in 2:length(xs)
    us[m, n+1] = us[m, n] - lambda * (us[m, n] - us[m-1, n]);
  end

  plot(xs, us[:, 1], "k-"; label=@sprintf("\$t = %.3f\$", ts[1]));
  plot(xs, us[:, div(length(ts), 2)], "k--"; label=@sprintf("\$t = %.3f\$",
                                                   ts[div(length(ts), 2)]));
  plot(xs, us[:, end], "k-."; label=@sprintf("\$t = %.3f\$", ts[end]));
  legend(; loc=3);
  title("Forward-Time Backward-Space, \$h = $h\$");
  xlabel("x");
  ylabel("u");
  ylim([0.0; 1.2]);
  savefig("forward-time_backward-space_h-$h.png");
  clf();
end

for h in [1./10.; 1./20.; 1./40.]

  println("Forward-Time Central-Space, h = $h");

  xs = collect(x0:h:x1);
  k = h * lambda;
  ts = collect(t0:k:t1);
  us = zeros(length(xs), length(ts));
  us[:, 1] = map(u0, xs); # initial conditions

  for n in 1:length(ts)-1
    for m in 2:length(xs)-1
      us[m, n+1] = us[m, n] - lambda * (us[m, n] - us[m-1, n]);
    end
    us[end, n+1] = us[end-1, n+1];
  end

  plot(xs, us[:, 1], "k-"; label=@sprintf("\$t = %.3f\$", ts[1]));
  plot(xs, us[:, div(length(ts), 2)], "k--"; label=@sprintf("\$t = %.3f\$",
                                                   ts[div(length(ts), 2)]));
  plot(xs, us[:, end], "k-."; label=@sprintf("\$t = %.3f\$", ts[end]));
  legend(; loc=3);
  title("Forward-Time Central-Space, \$h = $h\$");
  xlabel("x");
  ylabel("u");
  ylim([0.0; 1.2]);
  savefig("forward-time_central-space_h-$h.png");
```

```
  clf();
end

for h in [1./10.; 1./20.; 1./40.]

  println("Lax-Friedrichs, h = $h");

  xs = collect(x0:h:x1);
  k = h * lambda;
  ts = collect(t0:k:t1);
  us = zeros(length(xs), length(ts));
  us[:, 1] = map(u0, xs); # initial conditions

  for n in 1:length(ts)-1
    for m in 2:length(xs)-1
      us[m, n+1] = (0.5 * (us[m+1, n] + us[m-1, n]) -
                    0.5 * lambda * (us[m+1, n] - us[m-1, n]));
    end
    us[end, n+1] = us[end-1, n+1];
  end

  plot(xs, us[:, 1], "k-"; label=@sprintf("\$t = %.3f\$", ts[1]));
  plot(xs, us[:, div(length(ts), 2)], "k--"; label=@sprintf("\$t = %.3f\$",
                                                   ts[div(length(ts), 2)]));
  plot(xs, us[:, end], "k-."; label=@sprintf("\$t = %.3f\$", ts[end]));
  legend(; loc=3);
  title("Lax-Friedrichs, \$h = $h\$");
  xlabel("x");
  ylabel("u");
  ylim([0.0; 1.2]);
  savefig("lax-friedrichs_h-$h.png");
  clf();
end

for h in [1./10.; 1./20.; 1./40.]

  println("Leapfrog, h = $h");

  xs = collect(x0:h:x1);
  k = h * lambda;
  ts = collect(t0:k:t1);
  us = zeros(length(xs), length(ts));
  us[:, 1] = map(u0, xs); # initial conditions

  # one step of FTCS
  for m in 2:length(xs)-1
    us[m, 2] = us[m, 1] - lambda * (us[m, 1] - us[m-1, 1]);
  end
  us[end, 2] = us[end-1, 2];

  for n in 2:length(ts)-1
    for m in 2:length(xs)-1
      us[m, n+1] = us[m, n-1] - lambda * (us[m+1, n] - us[m-1, n]);
    end
    us[end, n+1] = us[end-1, n+1];
  end

  plot(xs, us[:, 1], "k-"; label=@sprintf("\$t = %.3f\$", ts[1]));
  plot(xs, us[:, div(length(ts), 2)], "k--"; label=@sprintf("\$t = %.3f\$",
                                                   ts[div(length(ts), 2)]));
  plot(xs, us[:, end], "k-."; label=@sprintf("\$t = %.3f\$", ts[end]));
  legend(; loc=3);
  title("Leapfrog, \$h = $h\$");
  xlabel("x");
  ylabel("u");
  ylim([0.0; 1.2]);
  savefig("leapfrog_h-$h.png");
  clf();
end
```

```julia
function total_variation(us::Vector{Float64})
  sum = 0.0;
  for j=2:length(us)-1
    sum += abs(us[j+1] - us[j]);
  end
  return sum;
end

all_vars = Dict();
for h in [1./5.; 1./10.; 1./100.]

  println("Lax-Friedrichs, h = $h");

  xs = collect(-2.0:h:(15.0+0.8*h));
  k = h * lambda;
  ts = collect(0.0:k:(10.0+0.8*k));
  us = zeros(length(xs), length(ts));
  us[:, 1] = map(x -> (abs(x) <= 1) ? 2.0 : 0.0, xs); # initial conditions
  vars = zeros(length(ts));
  vars[1] = total_variation(us[:, 1]);

  for n in 1:length(ts)-1
    for m in 2:length(xs)-1
      us[m, n+1] = (0.5 * (us[m+1, n] + us[m-1, n]) -
                    0.5 * lambda * (us[m+1, n] - us[m-1, n]));
    end
    us[end, n+1] = us[end-1, n+1];
    println("t = $(ts[n+1]), TV = $(total_variation(us[:, n+1]))");
    vars[n+1] = total_variation(us[:, n+1]);
  end

  @show sum(vars) / length(vars);
  plot(ts, vars);
  title("Lax-Friedrichs, \$h = $h\$");
  xlabel("Time");
  ylabel("Total Variation");
  savefig("lax-friedrichs-TV_h-$h.png");
  clf();

  plot(xs, us[:, 1], "k-"; label=@sprintf("\$t = %.3f\$", ts[1]));
  plot(xs, us[:, div(length(ts), 2)], "k--"; label=@sprintf("\$t = %.3f\$",
                                                      ts[div(length(ts), 2)]));
  plot(xs, us[:, end], "k-."; label=@sprintf("\$t = %.3f\$", ts[end]));
  legend(; loc=3);
  title("Lax-Friedrichs, \$h = $h\$");
  ylim([0.0; 2.5]);
  savefig("lax-friedrichs-c2_h-$h.png");
  clf();

  all_vars[h] = (copy(ts), copy(vars));
end

linetypes = ["k-", "k--", "k-."];
for ((h, data), linetype) in zip(all_vars, linetypes)
  plot(data[1], data[2], linetype; label=@sprintf("\$h = %.3f\$", h));
end
legend(; loc=3);
xlabel("Time");
ylabel("Total Variation");
savefig("lax-friedrichs-TV_all.png");
```