The Geometry of Compact 2-Manifolds

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1 Introduction

The torus $\mathbb{T}^2$ and sphere $\mathbb{S}^2$ are two commonly studied 2-manifolds in differential geometry. While the torus might seem like a simple geometric object, the doughnut shaped figure, it does not have to be the case: the torus could be constructed as a product of two circles as well, which lives in $\mathbb{R}^4$ and is in fact a flat one (i.e. the Cliford torus). An important difference between the torus and sphere is that the former admits a flat version while the latter does not, in any representation, the sphere always have to curve somewhere. In the first two section of this essay, this fact will be examined through the Guass-Bonnet Theorem. To do so, will first construct the torus and sphere using simplicial complexes in order to derive their Euler characteristics. Then examine the Guassian curvature, which is related to Euler characteristic by the Guass-Bonnet Theorem. Finally, the majority part of this essay will be focused on various embedding problems of compact 2-manifolds, where by an embedding here we always refer to a tamed embedding (i.e. it is homeomorphic to a polygon or polyhedral in $\mathbb{R}^3$). Examples would range from the flat torus to Klein bottle, and from there we would extend the discussion to the general compact 2-manifolds to explore a sufficient condition for embeddability into $\mathbb{R}^3$.

2 Constructing the Surfaces With Simplicial Complexes

This section is focused on simplicial complexes, which is going to be used in constructing the surfaces, and explore the Eurler characteristics of $\mathbb{T}^2$ and $\mathbb{S}^2$. The simplicial complexes is a special kind of cell complex. In geometry, a simplices is the generalization of the notion of a triangle or tetrahedron to arbitrary dimensions. Formally, let $\{v_0, ..., v_k\}$ be an affinely independent set of $k+1$ points in $\mathbb{R}^n$, the simplices spanned by them is the set

$$[v_0, ..., v_k] \coloneqq \left\{ \sum_{i=0}^{k} t_i v_i \mid t_i \geq 0, \sum_{i=0}^{k} t_i = 1 \right\}$$

with the subspace topology. A simplicial complex K is a set of simplices that satisfies:

1. Every face of a simplices from K is also in K.

2. For $\sigma_1, \sigma_2 \in K$, if $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a face for both σ_1 and σ_2.

A simplicial k-complexes K is a simplicial complexes in which the largest dimension of a simplices in it equals k. We can use the simplicial 2-complexes to build up surfaces for the torus and sphere as following: the vertices with same letters in the square torus are identified (like A

and A_1); while in the sphere, the surface is divided into a spherical triangulation mesh with $2n$ geodesic triangles. It is easy to verify that they form valid complexes. Such a process is also called a triangulation.

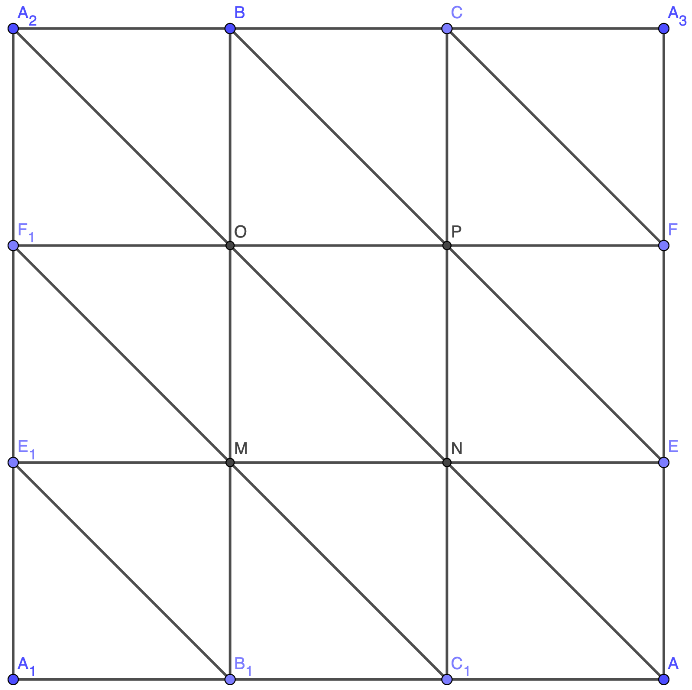

Figure 1: Triangulation of torus

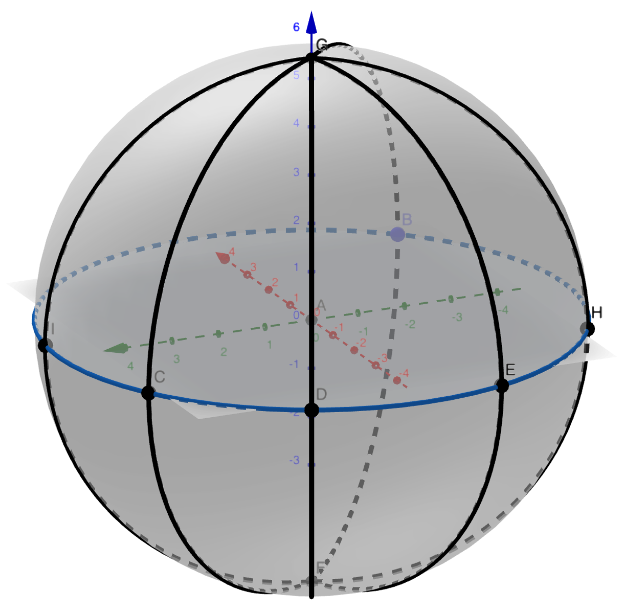

Figure 2: Triangulation of sphere

The Euler characteristics are therefore:

$$\chi(\mathbb{T}^2) = V + F - E = 9 + 18 - 27 = 0$$

$$\chi(\mathbb{S}^2) = V + F - E = (n+2) + 2n - (3n+1) = 1$$

3 Gaussian Curvature

The Gauss-Bonnet theorem relates the topological invariant, Eurler characteristics, to the geometric property curvature. It states that for a compact 2-manifold M with boundary ∂M, let K be the Gaussian curvature of M and k_g be the geodesic curvature of ∂M, then

$$\int_M K dA + \int_{\partial M} k_g ds = 2\pi\chi(M)$$

In the cases of $\mathbb{T}^2$ and $\mathbb{S}^2$, $\partial M = \emptyset$, therefore

$$\int_{\mathbb{T}^2} K_{\mathbb{T}^2} dA = 0$$

$$\int_{\mathbb{S}^2} K_{\mathbb{S}^2} dA = 2\pi > 0$$

This shows that $\mathbb{T}^2$ has 0 total Guassian curvature, while $\mathbb{S}^2$ has a positive total curvature. As it known to all, the plane has constant 0 curvature. Therefore, there is a flat represenatation of the torus, but the sphere can never be represented in such way.

4 Embeddings of the Flat Torus

Next, we consider the problem of trying to visualize the square flat torus in $\mathbb{R}^3$, in other words, to realize an embedding for the flat torus. Here, an embedding refers to a homeomorphism onto its image. Under such definition, it is easy to realize that the usual doughnut shaped torus is an embedding. An embedding into $\mathbb{R}^3$ should not have self-intersections, for example the image of a Klein bottle in $\mathbb{R}^3$ is not an embedding.

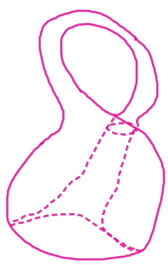

Figure 3: This is not an embedding of K

An isometric embedding is a smooth embedding that perserves the Riemannian metric. It turns out that there is no such a nicely smooth way to do so and the doughnut torus is not an isometric embedding, because the meridians on the surface are always stretched while bending the cylinder into a torus. Essentially, this is due to the difference in the curvature between the torus and sphere. A proof of the non-existence of a smooth isometric embedding for the torus goes the following way:

Proof. Assume for a moment that there exists such an embedding. Then the image of the flat torus would be a compact subset of $\mathbb{R}^3$, because the image of a compact space under continuous map is compact. So it is bounded, it can be bounded by a sphere. We can find a sphere that encloses the image and has a common point p with the image.

Now consider the local neighbourhood around p, let U_p, V_p denote the open neighbourhood of p in the image and sphere respectively. The Guassian curvature is positive everywhere in V_p, while it is everywhere 0 in U_p as the flat torus has everywhere vanishing curvature, and the curvature is perserved under isometric embedding according to Theorema Egregium. Therefore, at the point p the Gauss curvature must be zero and strictly positive at the same time, which gives a contradiction.

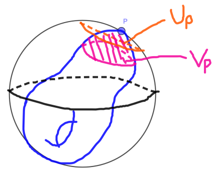

Figure 4: Ilustration of the proof

Despite the non-existence of a smooth isometric embedding, the Nash-Kuiper Theorem ensures that there exists an isometric embedding for square flat torus which is not infinitely differentiable. To realize this, a particular interesting example is constructed, and this method somehow manages to bypass the curvature (the full details of this example is presented in [1]): perturb the torus by adding waves in the direction of the meridians (like an accordion), with large amplitude on the inside and small amplitude on the outside. This is called a corrugation. If this perturbation is well designed, we can manage so that the longitudes now all have the same length. But now the perturbed latitudes have varying lengths, so need to do the same thing by adding small waves in another direction, getting all latitudes to have the same length again. Finally by iterating this procedure in a way to make the embedding converges in the C^1 topology to a flat embedded torus.

Here is a brief discription of the corrugating process, in one-dimensional senario: To begin with, a corrugation takes an initial smooth curve $f_0 : [0, 1] \to \mathbb{R}^3$ and creates a new curve f, which has its speed as given by the function $r : [0, 1] \to \mathbb{R}$ with $r > \|f_0'\|_{\mathbb{R}^3}$. Now consider that we have a one parameter family of loops $h : [0, 1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ satisfying the isometric condition $\|h(t, u)\|_{\mathbb{R}^3} = r(t)$, $\forall (t, u) \in [0, 1] \times \mathbb{R}/\mathbb{Z}$, and the barycentric condition

$$\forall t \in [0, 1], f_0'(t) = \int_0^1 h(t, u)du$$

This last condition expresses the derivative $f_0'(t)$ as the barycenter of the loop $h(t, \star)$. One then chooses the number N of oscillations of the corrugated map f and set

$$f(t) := f_0(0) + \int_0^1 h(u, Nu)du$$

Now if we set

$$h(t, u) = r(t)e^{i\alpha(t)\cos(2\pi u)}$$

and f_0 is one-dimensional, then the signed curvature measure of f:

$$\mu = kds = k(t)\,\|f'(t)\|_{\mathbb{R}^3}\,dt$$

and that of the initial curve $\mu_0 = k_0 ds$ would have the following relation:

$$\mu = \mu_0 + [\alpha' \cos(2\pi N t) - 2\pi N \alpha \sin(2\pi N t)]dt$$

Therefore, we can see that the corrugation modifies the curvature in sines and cosines with frequency N. This one-dimensional corrugation is extended to two-dimensions where the full details are described in [1], and by applying corrugations in each direction we finally achieve an isometric embedding. Here is a 3D model of the resulting image of the embedded flat torus, given in [2]:

Figure 5: 3D model of the flat torus embedding

5 Embeddings of Other Compact 2-Manifolds

The previous section shows that we could isometrically embed the square flat torus into $\mathbb{R}^3$, using a function that is not smooth and get a resulting image as a perturbation of corrugations. Next, a natural question is, do other 2-manifolds like $\mathbb{R}P^2$ admit an embedding into $\mathbb{R}^3$? For $\mathbb{R}P^2$ the answer is no, as a direct consequence of the Link Appearing theorem and its lemma, discribed in [3]:

Theorem 1 (Link Appearing Theorem)**.** *Any complete 6-graph in* $\mathbb{R}^3$ *contains a pair of disjoint cycles that form a non-trivial link.*

Lemma 2. *For any embedding of a Mobius band* M *in* $\mathbb{R}^3$*, the boundary* ∂M *and the meridian* C *of* M *forms a non-trivial link.*

Here a link refers to an embedding of a pair of circles in $\mathbb{R}^3$, and a link is trivial if one of the two curves bounds a disc in $\mathbb{R}^3$ that is dijoint from the other one. Otherwise, the link is called non-trivial. Using the previous lemma, we can easily show that $\mathbb{R}P^2$ is not embedable into $\mathbb{R}^3$:

Proof. Suppose $\mathbb{R}P^2$ is embedded in $\mathbb{R}^3$. Identify $\mathbb{R}P^2$ as the space abtained from antipodal identification on a sphere $\mathbb{S}^2$. In a previous assignment we showed that $\mathbb{R}P^2$ can be abtained by gluing the boundaries of a Mobius band and a disc. Therefore, removing an open disc D from the surface of $\mathbb{R}P^2$ gives a Mobius band M, as shown in the figure below. Then, the boundary ∂M and the meridian C of M form together a non-trivial link. Therefore, C and the disc D must intersect each other, yeilding a contradiction.

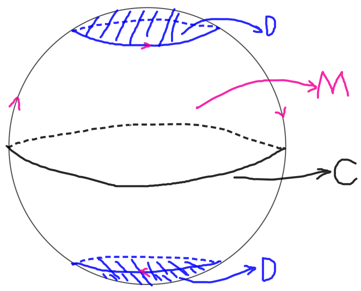

Figure 6: $\mathbb{R}P^2$ is not embeddable into $\mathbb{R}^3$

Likewise, we could also prove that the Klein bottle K is not embeddable into $\mathbb{R}^3$ using the above lemma:

Proof. The Klein bottle can be decomposed into two Mobius bands M_1 and M_2, with their boundaries $\partial M_1, \partial M_2$ identified. Let C_1, C_2 be the meridians of M_1, M_2 respectively. Then C_1 forms non-trivial links with the discs bounded by ∂M_1 and ∂M_2 (denoted as D_1, D_2), and since $\partial M_1 = \partial M_2$ the linking number of C_1 and D_1 would be even. However, from Link Appearing theorem we know that this linking number must be odd, contradiction.

6 Conclusion

In the previous sections, it is shown that the flat torus admits an (C^1 isometric) embedding into $\mathbb{R}^3$ while $\mathbb{R}P^2$ is not embeddable in $\mathbb{R}^3$. In general, Whitney embedding theorem states that any smooth real n-manifold can be smoothly embedded into $\mathbb{R}^{2n}$, and this result is extended by putting restrictions on manifolds: in fact, any compact orientable n-manifold embeds in $\mathbb{R}^{2n-1}$. Therefore, a sufficient condition for a compact 2-manifold to be embeddable in $\mathbb{R}^3$ is to be orientable. Notice that this is only a sufficient condition for embedability, it is not necessary, as we know that the Mobius band is not orientable but still embeds into $\mathbb{R}^3$.

[1, 3, 2, 4]

References

[1] V. Borrelli, S. Jabrane, F. Lazarus, and B. Thibert, "Flat tori in three-dimensional space and convex integration," *Proceedings of the National Academy of Sciences*, vol. 109, no. 19, pp. 7218–7223, 2012.

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[4] J. Lee, *Introduction to topological manifolds.* Springer Science & Business Media, 2010, vol. 202.