Asian Options for Crypto

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1 Introduction

Since Black and Scholes (1973) presented their first major paper, Option Pricing has been one of the most dynamic areas of finance. To derive the option pricing formula, Black and Scholes assumed that the market followed ideal conditions where there are no transaction costs, borrowing and lending rates are equal, stock pays no dividends, the underlying asset price follows a geometric Brownian motion, short term interest rates are known and that the option can be exercised only at the expiration date.

However, since then, there have been subsequent demonstrations where emphasis has been laid to ease the conditions presented in the paper by Black and Scholes and thereby develop the existing model in a way that it represents the financial markets closely than the previous ones. Merton (1973) developed a model which could price options even when the interest rate is stochastic, the option is exercised prior to expiration and the underlying stock pays dividends. Following that, Thorp (1973), Merton and Samuelson (1974), Cox and Ross (1975) and Ingersoll (1975), all of them presented their models which helped develop the existing one in a way that it could represent the day-to-day financial markets in a better way.

After all this development, the fundamental assumption under Black and Scholes' model was that the underlying stock price had to follow diffusion process, which meant that for every small interval, the stock price could only vary by a small amount. However, this does not capture the discontinuities (or jumps) that are evident in the stock price due to factors that dictate financial markets. As a result, incorporating jump stochastic process alongside the diffusion process in option pricing has been an area that has showed a lot of interest, starting in 1973, when Merton presented the first paper for Option pricing with discontinuous returns.

This paper, as a result, focuses on pricing options using diffusion and jump processes. However, specifically, we will only focus on a particular type of option, called Asian options for a particular market, i.e., Crypto Market. By definition, Asian Option is an option whose payoff depends on the average price over a certain period of time. It is a path-dependent option and as a result the value of the option at expiration depends on the path by which the stock price arrived at its eventual price at expiration.

This paper develops the model for pricing Asian Options of Crypto Indexes and currencies in three steps. In the first step, we test if we can observe normality in the data. Following that, there is an attempt to show the presence of jumps in the crypto indexes and currencies. As part of second step, we develop jump parameters. In the final step, we show that the crypto data satisfies the underlying assumptions to fit a jump-diffusion model and thus Asian options on crypto data can be priced using this. Finally, to validate our model, we compare the trends we observe in prices obtained using this model with that obtained using Bates model when used for pricing Asian options on crude oil futures.

2 Model fitting and testing normality in the underlying data

We first try to fit the crypto data using the Geometric Brownian Motion model, which is the simplest diffusion model. Here, we perform parameter estimation and testing for goodness-of-fit on the data.

2.1 Parameter Estimation

The Geometric Brownian Motion (GBM) is given by: $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$, where B_t is Standard Brownian Motion. At each given time, the GBM has a lognormal distribution. So, we define $r_i := log\frac{S(t_i)}{S(t_i - 1)}$ such that it follows a normal distribution with estimators of mean and variance defined as $\hat{m} := (\hat{\mu} - \frac{1}{2}\hat{\sigma}^2)\Delta t$ and $\hat{v} := \hat{\sigma}^2 \Delta t$. Then, we use Maximum Likelihood Estimation to find estimators of $\hat{m},\ \hat{v}$: Finally, we deduce the estimators of $\theta = (\mu,\ \sigma^2) :\ \hat{\sigma}^2 = \frac{\hat{v}}{\Delta t},\ \hat{\mu} = \frac{\hat{\sigma^2}}{2} + \frac{1}{\Delta t}\sum \frac{r_i}{n}$.

The estimators are given by:

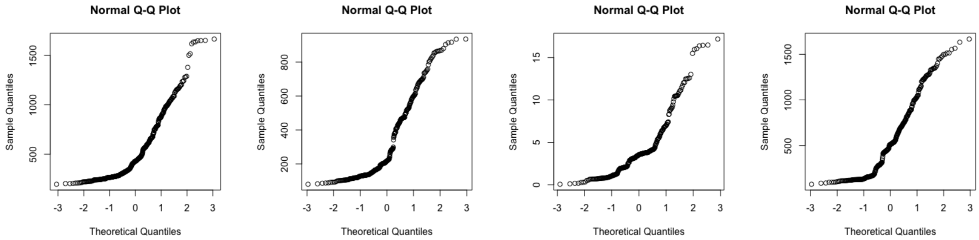
Table 1

Dataset	$\hat{\mu}$	$\hat{\sigma^2}$
Fund	0.002797723	0.003234769
Bitcoin	0.007072325	0.004312415
Ethereum	-0.0030339808	0.003368448
Ripple	-0.001084996	0.003450279
BitcoinCash	-0.001605029	0.006179305
Litecoin	-7.541072e-05	3.813981e-03
EOS	-0.002702388	0.004724703
Monero	-0.003210876	0.003286628

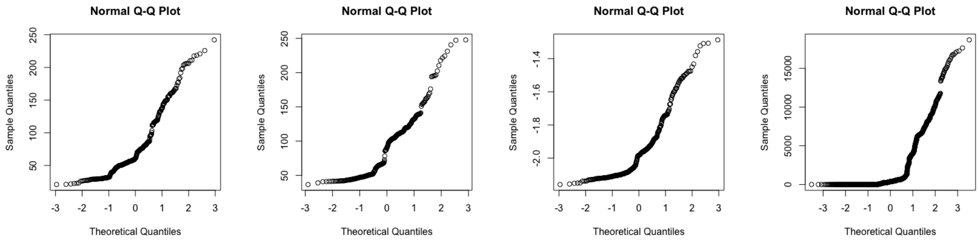
2.2 Test for goodness-of-fit

To test for the normality assumption given by Geometric Brownian model, we plot the QQ-plots of each asset and then perform the Shapiro-Wilk test. However, as QQ-plots don't give more quantitative information regarding how the data is deviated from normal distribution, a formal normality test is also applied to provide more specific information. In the following figure (Figure 1) are the QQ-plots.

Figure 1



(a) Fund (b) Ethereum (c) EOS (d) Bitcoin Cash



(e) Litecoin (f) Monero (g) Ripple (h) Bitcoin

Here, we observe from the plots that the distribution is not quite normal. The common pattern of these QQ-plots is that they all have heavy tails, or in other words the kurtosis is too large. Also, this is consistent with the conclusion given by Shapiro-Wilk normality test. So, normality does not seem to be a good assumption, and we should try other distribution assumptions. In Shapiro-Wilk test, the test statistic, $W = \frac{(\sum_{i=1}^{n} a_i x_{(i)})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} (0 < W < 1)$, represents two ways of calculating the variance.

Null hypothesis H_0: The sample is normally distributed, if $p - value > 0.05$ in which case normality can be assumed. The results given by Shapiro-Wilk test are in Table 2, which imply that we may want to reject the null hypothesis.

Table 2

Dataset	TestStatistics	P-value
Fund	0.952	7.45e-11
Bitcoin	0.8226	<2.2e-16
Ethereum	0.95432	1.48e-08
Ripple	0.94333	7.17e-10
BitcoinCash	0.90645	2.36e-13
Litecoin	0.93843	2.09e-10
EOS	0.95931	7.58e-07
Monero	0.96361	2.71e-06

Now that we observe that we can not use the normality assumption, we try to show the presence of jumps.

3 Testing the presence of jumps

In this section, we apply a statistical test in order to determine whether jumps are present in different crypto asset price datasets. This test is originally developed by Yacine Ait-Sahalia and Jean Jacod in thier paper on Testing for Jumps in a Discretely observed Process.

3.1 Some necessary setups

Let X denote the process governing the underlying data where $\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$, and $\Delta_n = \frac{T}{n}$, is the time increment in observed path. $\hat{A} := \frac{\Delta_n^{1-\frac{p}{2}}}{m_p} \sum_{i=1}^{[\frac{t}{\Delta_n}]} |\Delta_i^n X|^p I(\Delta_i^n X \leq \alpha \Delta_n^{\omega})$ with $m_p = \pi^{-\frac{1}{2}} 2^{\frac{p}{2}} \Gamma(\frac{p+1}{2})$, as well as $\hat{B}$, defined in following paragraph, are 2 different measures of variability of the data. The test hypothesis is H_0: There are no jumps.

3.2 Basic idea behind the test

The test statistic is $\hat{S}(p,k,\Delta_n)_t = \frac{\hat{B}(p,k\Delta_n)_t}{\hat{B}(p,\Delta_n)_t}$, where $\hat{B}(p,\Delta_n)_t := \sum_{i=1}^{[\frac{t}{\Delta_n}]} |\Delta_i^n X|^p$. Under H_0, the rejection area is $C_{n,t}^c := \left\{ \hat{S} < c_{n,t}^c = k^{\frac{p}{2}-1} - z_\alpha \sqrt{\hat{V}_{n,t}^c} \right\}$ where $k \geq 2,\ p > 3$ are constant parameters, $\hat{V}_{n,t}^c := \frac{\Delta_n M(p,k) \hat{A}(2p,\Delta_n)_t}{\hat{A}(p,\Delta_n)_t}$. For simplicity, we fix $p = 4,\ k = 2,\ \omega = 0.47,\ \alpha = 4 * volatility$ as presented in the paper.

Here is the intuition behind the construction of $\hat{S}$: It takes ratio of two measures of variability,

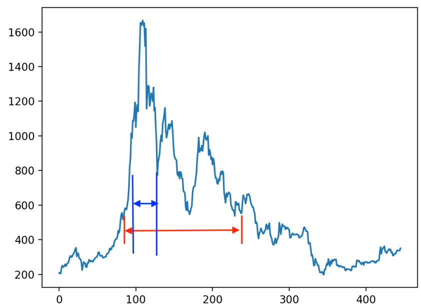


Figure 2

the denominator (represented in blue) includes the contribution of jumps, while the numerator (in red) is robust to jumps. To determine whether or not to reject H_0, there are two more criterion other than the rejection region: first, if $\alpha > 1$ then reject null hypothesis directly. Since α is generally 3 to 5 times volatility (although, here we fix $\alpha = 4$), so big α implies large volatility. Secondly, we also reject if $c_{n,t}^c < 0$, since it is a sign of large $\hat{V}_{n,t}^c$.

And the flow chart in Figure 3 discribes the total process of this test:

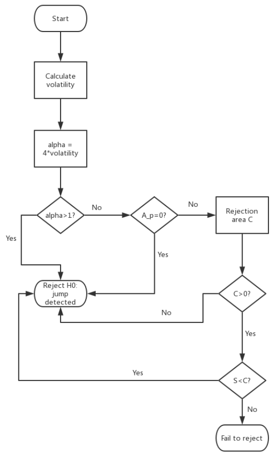


Figure 3

3.3 Conculsion

Based on our tests, jumps are detected in all of the crypto assets (namely Fund, Bitcoin, Ethereum, Ripple, BitcoinCash, EOS, Litecoin, Monero).
Now that we observe jumps in the crypto data, we try developing jump parameters.

4 Developing jump parameters

To create a jump model or a jump diffusion model for future market prediction, computations of parameter estimators are needed from historical market data. Traditional Cumulative Sum method and Block Average method can be use to estimate jump frequency, but they do not give us any information about jump amplitude. Sequential Average method can help us get distribution of jump amplitude. Here, we use Block Average method and Cumulative Sum method to estimate jump frequency, while we use Sequential Average method to estimate jump amplitude.

4.1 Jump Frequency Estimator

Cumulative Sum method is a sequential analysis technique. This method involves adding the difference between data and sample mean to previous term. Following sequence is computed to get:
$S_0 = 0$
$S_i = S_{i-1} + x_i - \mu$

After computation of this new sequence S, we look for V-shape in this sequence. V-shape indicates that any element in sequence S is larger or smaller than its neighbor elements. If we observe a V-shape, we can confirm presence of a jump in the data set.

Block Average method divides data into groups of fixed size. Following that, it compares difference of group average to a threshold limit. If the absolute value of difference between two consecutive groups exceed the threshold, we can claim presence of a jump in the data set.

Here, we use historical data of past 10 years to evaluate jump parameters, and define the group size in Block Average method to be 5. This method give us a result of 255 jumps, happening in past 10 years, out of 2531 data point.

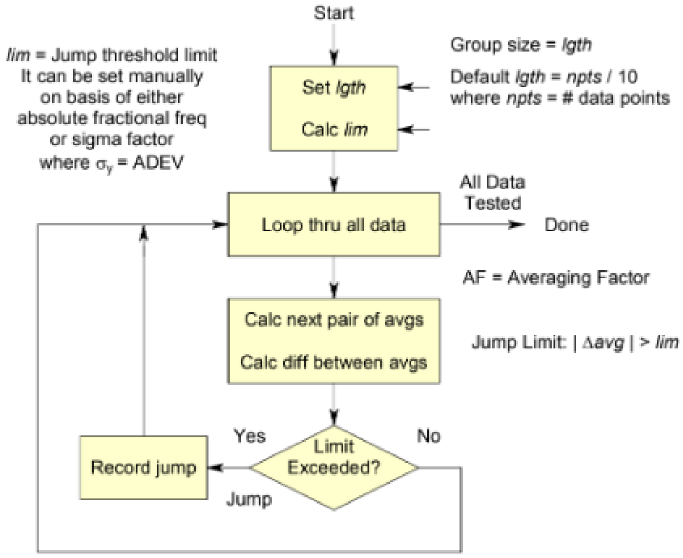


Figure 4

4.2 Jump Amplitude Estimator

Sequential Average method divides data into block, and searches for jumps sequentially. Sequential Average method is able to locate jumps closely, so it creates a chance of analysing jump amplitude.

In Sequential Average method, we calculate the jump threshold limit base on Allan deviation, and apply the limit to average of elements within a group, with stated group size. Looping through all data, the first element that exceeds the limit will be considered as a suspected jump. If the suspected jump point also exceeds the limit of average of the group following after suspected jump point, the point would be confirmed to be a jump.

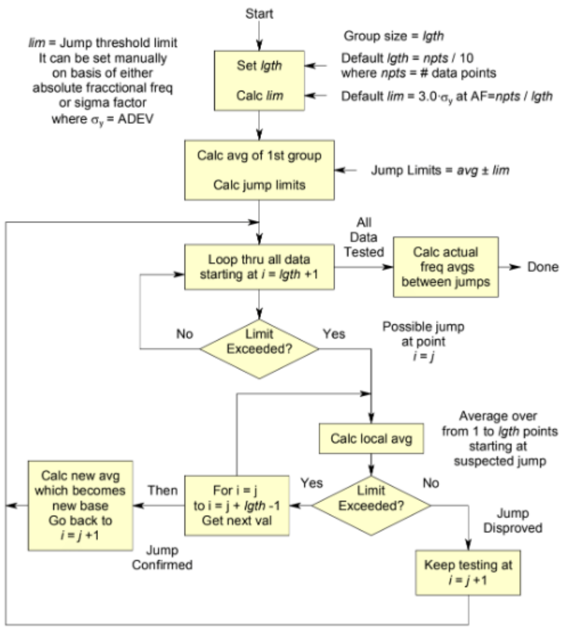


Figure 5

Here, we use historical data of past 10 years to evaluate jump parameters for crude oil future, and define the limit parameter to be half of the data size. Graph below shows the distribution of absolute jump amplitude:

Mean = 0.031814444733628246
Variance = 0.030649858994438124
Kurtosis = 6.7760074253936615

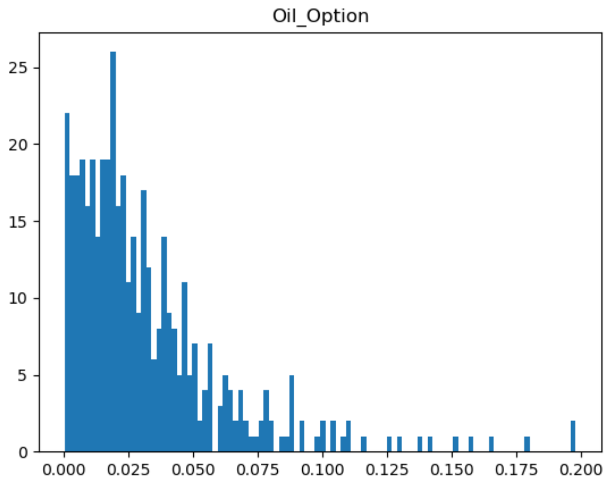


Figure 6

The distribution of jump amplitude shown on QQ-plot:

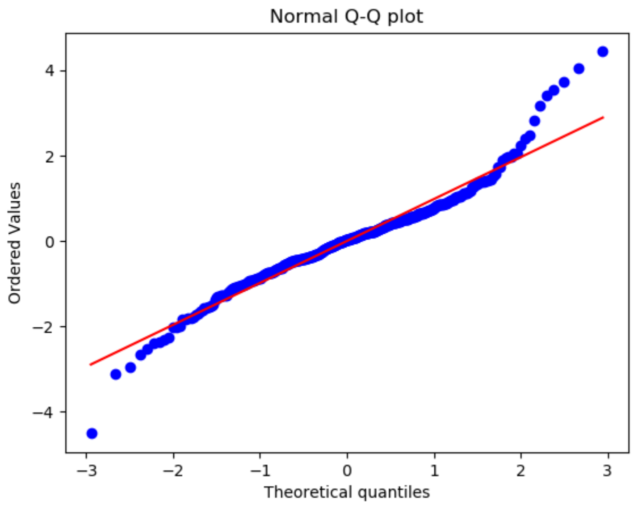


Figure 7

Similarly for Crypto currency:
Bitcoin:

Mean = 0.10368530617916927
Variance = 0.11026390602978582
Kurtosis = 22.490438145742445

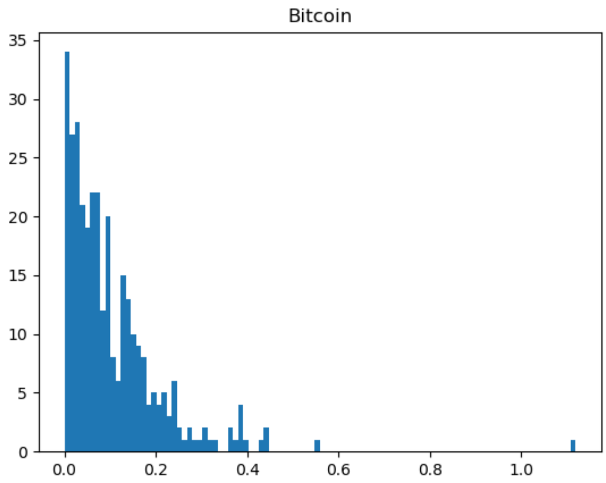


Figure 8

The distribution of jump amplitude shown on QQ-plot:

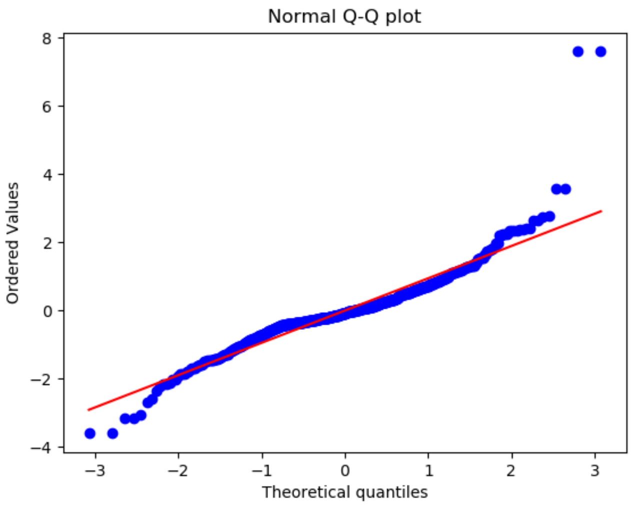


Figure 9

Fund:

$$\text{Mean} = 0.06486075524211993$$
$$\text{Variance} = 0.05227290461346189$$
$$\text{Kurtosis} = -0.4400244001603264$$

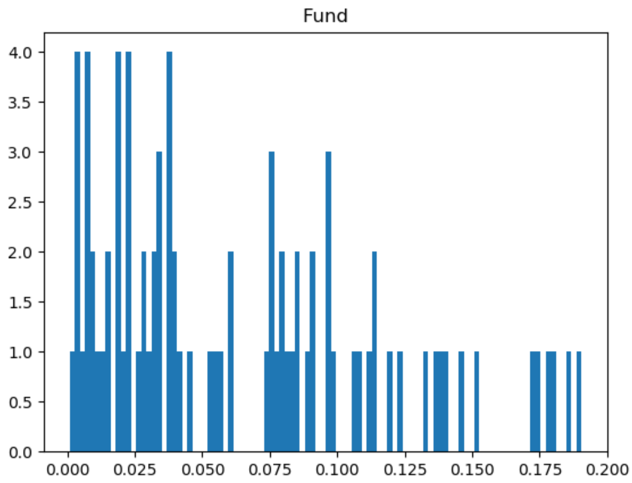


Figure 10

The distribution of jump amplitude shown on QQ-plot:

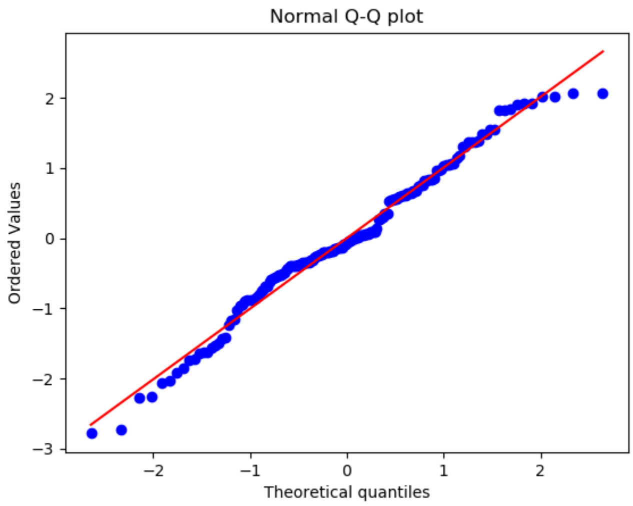


Figure 11

Ethereum:

Mean = 0.05640716986397925
Variance = 0.047582584406260876
Kurtosis = 0.4918708446448661

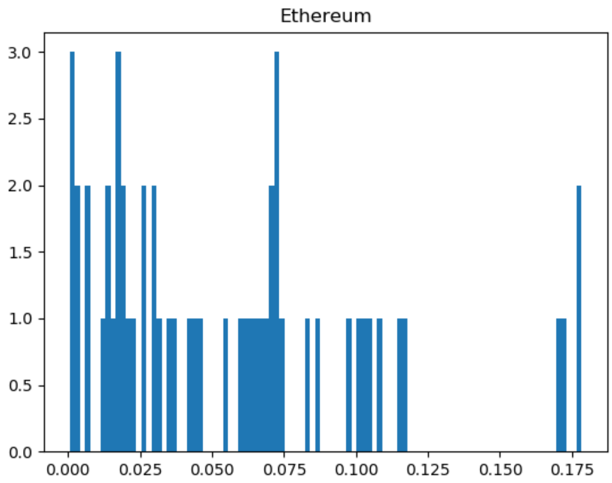


Figure 12

The distribution of jump amplitude shown on QQ-plot:

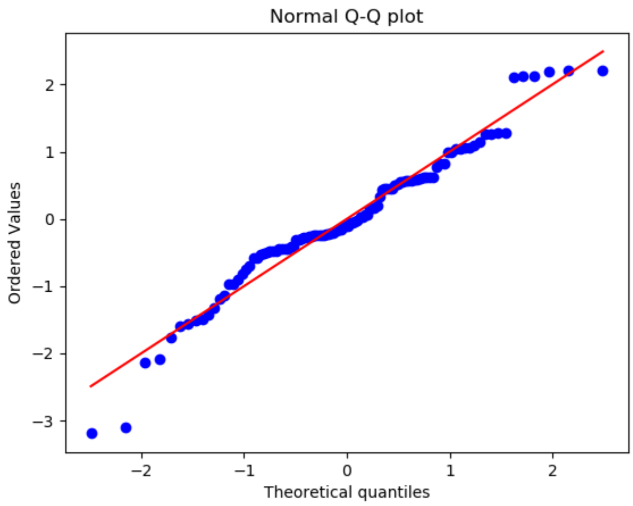


Figure 13

Litecoin:

Mean = 0.06867618778510798
Variance = 0.06700034669333206
Kurtosis = 4.010396928332448

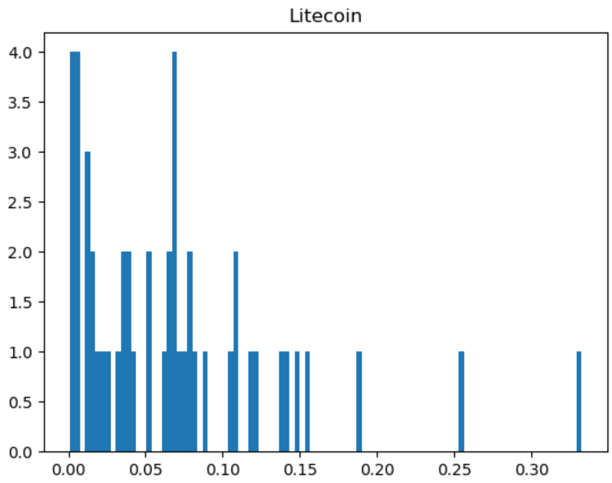


Figure 14

The distribution of jump amplitude shown on QQ-plot:

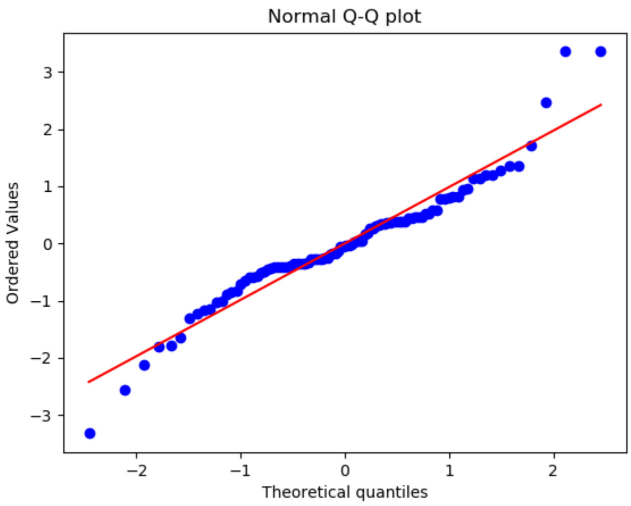


Figure 15

Based on the Normal Q-Q plot plotted above, we can say that the distribution of jump amplitude closely fits to the Normal distribution. It is therefore reasonable to conclude that the sizes of jump in Crude Oil Asian Option follow Normal distribution.

5 Model Validation

Based on the above results, we observed that we can use Merton's Jump Diffusion model to price Asian Options on Crypto currencies as their data satisfy all the underlying assumptions. However, as part of validation, as we do not have Asian Option prices on Crypto currencies, we price Asian Options on commodities and compare the trends observed in prices with those obtained from Bates model for crude oil options in the paper by I. Kyriakou, P.K. Pouliasis and N.C. Papapostolou. While doing that, there are couple observations/assumptions which were taken into account before performing this.

Although there would be a difference in pricing due to the stochastic nature of volatility in Bates model as compared to Merton's Diffusion model, we perform these as we are more interested in observing the trends in pricing rather than exact prices with uncertainty associated. Secondly, although in this paper we attempt to price Asian Options on Crypto currencies, we validate our pricing model by pricing Asian Options on commodities as those are the ones whose prices calibrated using Bates model were available.

There are few reasons why for Crypto data, Asian Options make sense. Asian options, whose payoff is dependent on the average price of the underlying asset during a pre-specified time window, are popular for the risk management. These are cheaper than plain vanilla options due to the lower volatility. Arithmetic averages prevent momentarily wild fluctuations which have been observed in Crypto currencies since they were introduced.

Now, to price this, deriving a closed form formula can be extremely hard due to the existence of jump component, or in most context a closed form expression is unnecessary in terms of application purpose. Instead, Monte Carlo simulation is applied to solve option pricing problems. Monte Carlo simulation considers a large variety of possibilities and helps eliminate uncertainty by enabling us to vary risk assumptions under all parameters and thus model a range of possible outcomes. Here, we will use the Monte Carlo method to price Crypto Asian options. Since there are evidence of jumps exist in the underlying asset prices, we use Merton's Jump-Diffusion model described in (1):

$$S_t = J \cdot S_0 e^{(r-\sigma^2/2+\lambda k)\tau+\sigma X(t)} \tag{1}$$

where S_0 is the initial stock price, S_t is the stock price at time t, r is the risk free interest rate, σ is the volatility, λ and k are jump parameters namely the jump frequency and jump mean (to be estimated), $\tau = T/n$ with T being time to maturity (we take it to be 1 year) and n being time steps. J is the jump component for this model, $J = \sum_{i=1}^{N_t} Y_i$, where N_t is a Poisson process with intensity λ and $Y_i \sim N(k - \frac{1}{2}\lambda^2, \lambda)$ are the jump sizes; $X(t)$ is a normal random variable $N(0, t)$.

Before running the simulator to price these options on crude oil futures, we observe that there are systematic departures from the benchmark lognormal process which describe the crude oil price dynamics. As a result, we use the Merton's diffusion model overlaid with price jumps of random magnitude and arrival. To calibrate the model, we use the prices of liquid WTI futures American vanilla options trading on the NYMEX division of the CME Group.

We begin our validation by first extracting the required data. As we are comparing our model to the Bates model provided in the paper by I. Kyriakou, P.K. Pouliasis and N.C. Papapostolou, we begin estimating our model, using market quotes of NYMEX WTI futures American options trading on the NYMEX division of the CME Group. here, both futures and options are quoted in US dollars and cents per barrel (bbl). They are traded for all deliveries (consecutive months) within the current year and the next five years. The June and December contracts are listed beyond the sixth year whereas additional months are added on an annual basis after the December contract expires, so that an additional June and a December contract are added nine years forward and the consecutive months in the sixth calendar year are filled in. Each futures contract is traded until the close of business on the third business day prior to the 25th calendar day of the month preceding the delivery month. If the 25th calendar day of the month is a non-business day, the trading ceases on the third business day prior to the business day preceding the 25th calendar day of the month preceding the delivery month. For each option contract, the trading ceases on the third business day prior to the business day preceding the termination of the trading in the underlying futures contract.

As showed in their paper, the raw data set we use here consists of settlement mid-point call and put option prices, open interest and daily volume for all available American vanilla options on WTI futures on February 3, 2015. Following that, we collect data for the first six nearby maturities, ranging from 42 days to maturity (April 2015) to 195 days to maturity (September 2015) with an average open interest and trading volume of 229,185 and 20,458, respectively. Beyond six months to expiry, liquidity is concentrated on the contracts expiring in March, June, September and December, whereas beyond one year, liquidity is concentrated on the contracts with December expiry. Thus, among the remaining contracts, we choose the first two contracts with expiration in December 2015 (287 days to maturity) and March 2016 (379 days to maturity) with an open interest of 16,556 and 67,822 and a trading volume of 402 and 300 contracts, respectively. This way, we are making sure that the options we are pricing are liquid enough.

Next, in order to test how good our simulator simulates the option price, we run the simulator on crude oil price data and compared the results to the data given in Jumps and stochastic volatility in crude oil prices and advances in average option pricing. In the last part of this paper there are data presenting some important paramater estimations as well as the simulated results from our model. We will also use a 95% CI from above paper (Bates model) as a benchmark.

Once we have the data, we now get the jump parameters for this data set. We need the jump parameters as we are using Merton's jump diffusion model. As part of the model, stock price returns are a mixture of stochastic differential equations to describe the motions of continuous sample path stochastic processes and stochastic differential equations for Poisson driven processes.

Specifically, the Stochastic differential equation conditional on S(t)=S, is given by:

$$\frac{dS}{S} = (\alpha - \lambda k)dt + \sigma dZ + dq$$

where α is the instantaneous expected return of the stock, σ^2 is the instantaneous variance of the return, given that Poisson event does not occur. Moreover, dZ is the standard Gauss-Wiener Process, q(t) is an independent Poisson process.

From sections before, jump parameters and volatility are computed. With parameters publish to public, such as risk free rate, original price, and time step, the price of Crude Oil Asian option can be estimate using Merton Model. Following are estimated Crude Oil Asian option price at February 3 2015, using Merton Model.

spot price = 53.86
risk free rate = 0.00023
volatility = 0.0336
time steps = 12
dividend = 0
jump frequency = 255 / 2500
jump mean = 0.031814
jump st.dev. = 0.03065

Monte Carlo Simulation give following result with 1000 scenarios:

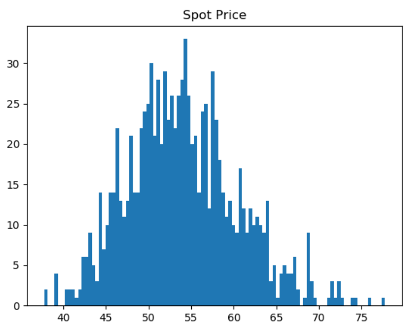


Figure 16

The average profit of buying Crude Oil Asian option in 1000 scenarios is 2.9194992496774086
Similarly, from February to April the average profit of buying Crude Oil Asian option in 1000 scenarios is 2.8066568189575953.

spot price = 54.95
risk free rate = 0.00047
volatility = 0.032265638
time steps = 12
dividend = 0
jump frequency = 255 / 2500
jump mean = 0.031814
jump st.dev. = 0.03065
From February to May the average profit of buying Crude Oil Asian option in 1000 scenarios is

2.783310312576093

spot price = 56.11
risk free rate = 0.00072
volatility = 0.0296628764020579
time steps = 12
dividend = 0
jump frequency = 255 / 2500
jump mean = 0.031814
jump st.dev. = 0.03065

(1; 2; 3; 4; 5; 6; 7)

References

[1] R. C. Merton, "Option pricing when underlying stock returns are discontinuous," *Journal of financial economics*, vol. 3, no. 1-2, pp. 125–144, 1976.

[2] J. Hain, "Comparison of common tests for normality," *Retrieved from www. statistik-mathematik. uni-wuerzburg. de/.../da hain final. pdf*, 2010.

[3] Y. Aït-Sahalia, J. Jacod *et al.*, "Testing for jumps in a discretely observed process," *The Annals of Statistics*, vol. 37, no. 1, pp. 184–222, 2009.

[4] E. Bayraktar and H. Xing, "Pricing asian options for jump diffusion," *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, vol. 21, no. 1, pp. 117–143, 2011.

[5] P. Carr and L. Wu, "What type of process underlies options? a simple robust test," *The Journal of Finance*, vol. 58, no. 6, pp. 2581–2610, 2003.

[6] W. Riley, "Algorithms for frequency jump detection," *Metrologia*, vol. 45, no. 6, p. S154, 2008.

[7] I. Kyriakou, P. K. Pouliasis, and N. C. Papapostolou, "Jumps and stochastic volatility in crude oil prices and advances in average option pricing," *Quantitative Finance*, vol. 16, no. 12, pp. 1859–1873, 2016.