1 First Draft

2 The Wizard Problem

A wizard attempts to cast haste every t seconds. If haste is cast then the wizard can attempt casts every t/r seconds over the next nt/r seconds. The probability of casting haste is p. Haste effects do not stack, but the duration of a haste effect can be reset if haste cast again. In the limit as $t \to \infty$, what is the rate of cast attempts?

3 Transition Matrix for Wizard

Given constants n, p, we have n different states the wizard can be in. They can either not be under a haste effect, or they have been under the latest haste effect for kt/r seconds where $k = 0, ..., n-1$. From this we define an $n \times n$ transition matrix P as follows:

$P_{1,1} = 1 - p$,
$P_{n,1} = 1 - p$,
$P_{k,1} = 0$ for $k = 2, ..., n-1$,
$P_{k,2} = p$, and
$P_{k,k+1} = 1 - p$ for $k = 2, ..., n-1$.

The column vector C_k, with $C_k = \langle P_{1,k}, ..., P_{n,k} \rangle^T$, is the vector of the k-th column of P.

4 Limiting Distribution for States

Proposition 1: The limiting distribution for P, given constants n, p, is $\langle (1-p)^{n-1}, p, (1-p)p, (1-p)^2 p, ..., (1-p)^{n-2} p \rangle$. We'll name this w, and its k-th component w_k.

Proof: We will show that the components of w sum to one:
(1) $w_1 + w_2 + w_3 + ... + w_n = (1-p)^{n-1} + p + (1-p)p + ... + (1-p)^{n-2}p$
$= (1-p)^{n-1} + p(1 + (1-p) + (1-p)^2 + ...(1-p)^{n-2}))$
$= (1-p)^{n-1} + p\frac{1-(1-p)^{n-1}}{p}$
$= (1-p)^{n-1} + 1 - (1-p)^{n-1}$
$= 1$.

We must show that $w \cdot C_k = w_k$ for $k = 1, ..., n$. For $k = 1$ we have:
$w \cdot C_1 = \langle (1-p)^{n-1}, ..., (1-p)^{n-2}p \rangle \cdot \langle 1-p, 0, ..., 0, 1-p \rangle^T$
$= (1-p)^n + (1-p)^{n-1}p$
$= (1-p)^{n-1}(1-p) + (1-p)^{n-1}p$
$= (1-p)^{n-1}$
$= w_1$.

When $k = 2$ we have:
$w \cdot C_2 = w_1 p + w_2 p + ... + w_n p$
$= p \sum_{i=1}^{n} w_i$

$= p$
$= w_2$.
We established that $\sum_{i=1}^{n} w_i$ with the equality in (1). When $k > 2$ we have:
$w \cdot C_k = w_{k-1}(1-p)$
$= p(1-p)^{k-3}(1-p)$
$= p(1-p)^{k-2}$
$= w_k$. ■ (check this calculation. look into writing style / justifications)

5 Average Cast Rate

The limiting distribution w provides the average frequency of each state, ignoring how long each one lasts. We can weight each state by how long it lasts. The first state lasts t seconds, while all the others last t/r seconds. We then take $w \cdot u$ where $u = \langle t, t/r, t/r, ..., t/r\rangle$ to get:
$t(1-p)^{n-1} + t/r(p + p(1-p) + p(1-p)^2 + ... + p(1-p)^{n-2})$
$= t(1-p)^{n-1} + t/r(1-(1-p)^{n-1})$
$= t/r + [(tr-t)/r](1-p)^{n-1}$.

We could ask about the case for when the haste effect lasts for a duration that is not an integer multiple of t/r. Suppose instead that the haste effect lasts for T seconds. This leads to new weights u' which are the same as those for $u = \langle u'_1, ..., u'_n\rangle$ except that $u'_n = (t/r)(1 - T//(t/r)) + t(T//(t/r))$, where $a//b$ denotes the remainder of a divided by b. If $T = nt/r$ then $u'_n = u_n$ as before, but otherwise the time spent in this state is a weighted average of time spent under the haste effect and time spent outside the haste effect. Let $Q = T//(t/r)$ for readability. We can recalculate the average casts per second:
$w \cdot u'$
$= t(1-p)^{n-1} + t/r(p + p(1-p) + p(1-p)^2 + ... + p(1-p)^{n-3}) + u'_n p(1-p)^{n-2})$
$= t(1-p)^{n-1} + t/r(1-(1-p)^{n-2}) + u'_n p(1-p)^{n-2}$
$= t(1-p)^{n-1} + t/r - t/r(1-p)^{n-2} + [(t/r)(1-Q) + tQ]p(1-p)^{n-2}$
$= t/r + t(1-p)^{n-1} - t/r(1-p)^{n-2} + t/r[p(1-p)^{n-2}] + [tQ - (t/r)Q]p(1-p)^{n-2}$
$= t/r + t(1-p)^{n-1} - t/r(1-p)^{n-1} + ((tr-t)/r)Qp(1-p)^{n-2}$
$= t/r + [(tr-t)/r](1-p)^{n-1} + [(tr-t)/r]Qp(1-p)^{n-2}$
$= t/r + [(tr/-t)/r]((1-p)^{n-1} + (T//(t/r))p(1-p)^{n-2})$.

We can see that this formula agrees with the other whenever T is divisible by t/r.